2 Matrices

Let's Study.

2.1 Elementry transformations.

2.2 Inverse of a matrix

2.2.1 Elementary transformation Method

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2.3.1 Method of Inversion

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Let's learn.

A matrix of order mxm is a square arrangement of m^2 elements. The corresponding determinant of the same elements, after expansion is seen to be a value which is an element itself.

In standard XI, we have studies the types of matrices and algebra of matrices namely addition, subtraction, multiplication of two matrices.

The matrices are useful in almost every branch of science. Many problems in Statistics are expressed in terms of matrices. Matrices are also useful in Economics, Operation Research. It would not be an exaggeration to say that the matrices are the language of atomic Physics.

Hence, it is necessary to learn the uses of matrices with the help of **elementary transformations** and the **inverse of a matrix**.

2.1 Elementary Transformation :

Let us first understand the meaning and applications of elementary transformations.

The elementary transformation of a matrix are the six operations, three of which are due to row and three are due to column.

They are as follows :

(a) Interchange of any two rows or any two columns. If we interchange the i^{th} row and the j^{th} row of a matrix then after this interchange the original matrix is transformed to a new matrix.

This transformation is symbolically denoted as $R_i \leftrightarrow R_j$ or R_{ij}.

The similar transformation can be due to two columns say $C_k \leftrightarrow C_i$ or C_{ki}.

(Recall that R and C symbolically represent the rows and columns of a matrix.)

For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $R_1 \leftrightarrow R_2$ gives the new matrix $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $C_1 \leftrightarrow C_2$ gives the new matrix $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

Note that $A \neq \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $\neq \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ but we write $A \sim \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $A \sim \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

Note : The symbol ~ is read as equivalent to.

(b) Multiplication of the elements of any row or column by a non-zero scalar :

If k is a non-zero scalar and the row R_i is to be multiplied by constant k then we multiply every element of R_i by the constant k and symbolically the transformation is denoted by kR_i or $R_i \rightarrow kR_i$.

For example, if $A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$ then $R_2 \rightarrow 4R_2$ gives $A \sim \begin{bmatrix} 0 & 2 \\ 12 & 16 \end{bmatrix}$

Similarly, if any column of a matrix is to be multiplied by a constant then we multiply every element of the column by the constant. It is denoted as kC_l or $C_l \rightarrow kC_l$.

For example, if $A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$ then $C_l \rightarrow -3\ C_l$ gives $A \sim \begin{bmatrix} 0 & 2 \\ -9 & 4 \end{bmatrix}$

Can you say that $A = \begin{bmatrix} 0 & 2 \\ 12 & 16 \end{bmatrix}$ or $A = \begin{bmatrix} 0 & 2 \\ -9 & 4 \end{bmatrix}$

(c) Adding the scalar multiples of all the elements of any row (column) to corresponding elements of any other row (column).

If k is a non-zero scalar and the k-multiples of the elements of R_i (C_i) are to be added to the elements of R_j (C_j) then the transformation is symbolically denoted as $R_j \rightarrow R_j + kR_i$, $C_j \rightarrow C_j + kC_i$

For example, if $A = \begin{bmatrix} -1 & 4 \\ 2 & 5 \end{bmatrix}$ and k = 2 then $R_l \rightarrow R_l + 2R_2$ gives

$$A \sim \begin{bmatrix} -1+2(2) & 4+2(5) \\ 2 & 5 \end{bmatrix}$$

$$\text{i.e.} \quad A \sim \begin{bmatrix} 3 & 14 \\ 2 & 5 \end{bmatrix}$$

(Can you find the transformation of A using $C_2 \rightarrow C_2 + (-3)\ C_1$?)

Note (1) : After the transformation , $R_j \rightarrow R_j + kR_i$, R_i remains the same as in the original matrix. Similarly, with the transformation , $C_j \rightarrow C_j + kC_i$, C_i remains the same as in the original matrix.

Note (2) : After the elementary transformation, the matrix obtained is said to be equivalent to the original matrix.

Ex. 1 : If $A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$, apply the transformation $R_1 \leftrightarrow R_2$ on A.

Solution :

$$\text{As} \quad A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ gives

$$A \sim \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$$

Ex. 2 : If $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix}$, apply the transformation $C_1 \rightarrow C_1 + 2C_3$.

Solution :

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

$C_1 \rightarrow C_1 + 2C_3$ gives

$$A \sim \begin{bmatrix} 1+2(2) & 0 & 2 \\ 2+2(4) & 3 & 4 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 5 & 0 & 2 \\ 10 & 3 & 4 \end{bmatrix}$$

Ex. 3 : If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 5 \end{bmatrix}$, apply $R_1 \leftrightarrow R_2$ and then $C_1 \rightarrow C_1 + 2C_3$ on A.

Solution :

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ gives

$$A \sim \begin{bmatrix} 3 & -2 & 5 \\ 1 & 2 & -1 \end{bmatrix}$$

Now $C_1 \to C_1 + 2C_3$ gives

$$A \sim \begin{bmatrix} 3+2(5) & -2 & 5 \\ 1+2(-1) & 2 & -1 \end{bmatrix} \quad \therefore \; A \sim \begin{bmatrix} 13 & -2 & 5 \\ -1 & 2 & -1 \end{bmatrix}$$

Exercise

Apply the given elementary transformation on each of the following matrices.

1. $A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$, $R_1 \leftrightarrow R_2$. 2. $B = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 5 & 4 \end{bmatrix}$ $R_1 \to R_1 - R_2$.

3. $A = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$, $C_1 \leftrightarrow C_2$; $\quad B = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$ $R_1 \leftrightarrow R_2$.

   What do you observe?

4. $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $2C_2$ $\quad B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 5 \end{bmatrix}$, $-3R_1$.

   Find the addition of the two new matrices.

5. $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$, $3R_3$ and then $C_3 + 2C_2$.

6. $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$, $C_3 + 2C_2$ and then $3R_3$.

   What do you conclude from ex. 5 and ex. 6?

7. Use suitable transformation on $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ to convert it into an upper triangular matrix.

8. Convert $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ into an identity matrix by suitable row transformations.

9. Transform $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$ into an upper triangular matrix by suitable column transformations.

2.2 Inverse of a matrix :

Definition : If A is a square matrix of order m and if there exists another square matrix B of the same order such that AB = BA = I, where I is the identity matrix of order m, then B is called as the inverse of A and is denoted by A^{-1}.

Using the notation A^{-1} for B we get the above equation as $AA^{-1} = A^{-1}A = I$. Hence, using the same definition we can say that A is also the inverse of B.

$$\therefore \quad B^{-1} = A$$

For example, if $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ then $AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = I_2$$

$$\text{and} \quad BA = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\therefore \quad B = A^{-1} \quad \text{and} \quad A = B^{-1}$$

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, can you find a matrix X such that AX = I ? Justify the answer.

This example illustrates that for the existence of such a matrix X, the necessary condition is $|A| \neq 0$, i.e. A is a non-singular matrix.

Note that -

(1) Every square matrix A of order $m \times m$ has its corresponding determinant; det A = |A|

(2) A matrix is said to be invertible if its inverse exists.

(3) A square matrix A has inverse if and only if $|A| \neq 0$

Uniqueness of inverse of a matrix

It can be proved that if A is a square matrix where $|A| \neq 0$ then its inverse, say A^{-1}, is unique.

Theorem : Prove that if A is a square matrix and its inverse exists then it is unique.

Proof : Let, 'A' be a square matrix of order 'm' and let its inverse exist.

Let, if possible, B and C be the two inverses of A.

Therefore, by definition of inverse AB = BA = I and AC = CA = I.

Now consider $B = BI = B(AC)$

$\therefore\ B = (BA)C = IC$

$\therefore\ B = C$

Hence $B = C$ i.e. the inverse is unique.

The inverse of a matrix (if it exists) can be obtained by using two methods.

(i) Elementary row or column transformation

(ii) Adjoint method

We now study these methods.

2.2.1 Inverse of a nonsingular matrix by elementary transformation :

By definition of inverse of A, if A^{-1} exists then $AA^{-1} = A^{-1}A = I$.

Let us consider the equation $AA^{-1} = I$. Here A is the given matrix of order m and I is the identity matrix of order 'm'. Hence the only unknown matrix is A^{-1}. Therefore, to find A^{-1}, we have to first convert A into I. This can be done by using elementary transformations.

Here we note that whenever any elementary row transformation is to be applied on the product AB = C of two matrices A and B, it is enough to apply it only on the prefactor, A. B remains unchanged. And apply the same row transformation to C.

For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix}$ then $AB = \begin{bmatrix} 1 & 10 \\ 1 & 20 \end{bmatrix} = C$ (say)

Now if we require C to be transformed to a new matrix by $R_1 \leftrightarrow R_2$ then $C \sim \begin{bmatrix} 1 & 20 \\ 1 & 10 \end{bmatrix}$

If the same transformation is used for A then $A \sim \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and B remains unchanged,

then the product $AB = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 20 \\ 1 & 10 \end{bmatrix}$ = as required. (Verify the product.)

Hence, the equation $AA^{-1} = I$ can be transformed into an equation of the type $IA^{-1} = B$, by applying same series of row transformations on both the sides of the equation.

However, if we start with the equation $A^{-1}A = I$ (which is also true by the definition of inverse) then the transformation of A should be due to the column transformation. Apply column transformation to post factor and other side, where as prefactor remains unchanged.

Thus, starting with the equation $AA^{-1} = I$, we perform a series of row transformations on both sides of the equation, so that 'A' gets transformed to I. Thus,

$$A \quad A^{-1} = I$$

$\downarrow$ Row Transformations $\qquad \downarrow$ Row Transformations

$$I \quad A^{-1} = B$$

$$\therefore\ A^{-1} = B$$

and for the equation $A^{-1}A = I$, we use a series of column transformations. Thus

$$\begin{array}{ccc} A^{-1}\,A & = & I \\ \downarrow \text{Column Transformation} & & \downarrow \text{Column Transformation} \\ A^{-1}\,I & = & B \\ \therefore\ A^{-1} & = & B \end{array}$$

Now if A is a given matrix of order '3' and it is nonsingular then we consider

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

For reducing the above matrix to

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the suitable row transformations are as follows :

(1) Reduce a_{11} to '1'.
(2) Then, reduce a_{21} and a_{31} to '0'.
(3) Reduce a_{22} to '1'.
(4) Then, reduce a_{12} and a_{32} to '0'.
(5) Reduce a_{33} to '1'.
(6) Then, reduce a_{13} and a_{23} to '0'.

Remember that a similar working rule (but not the same) can be used if you are using column transformations.

Solved Examples

Ex. 1 : Find which of the following matrices are invertible

(i) $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ (ii) $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (iii) $C = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Solution :

(i) As $|A| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$

$\therefore$ A is singular and hence A is not invertible.

(ii) As $|B| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

$= \cos^2\theta + \sin^2\theta = 1 \neq 0$

$\therefore$ B is nonsingular,

$\therefore$ B is invertible.

(iii) $C = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$\therefore \quad |C| = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -12 \neq 0$

$\therefore$ C is nonsingular and hence C is invertible.

Ex. 2 : Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution :

As $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$

$\therefore \quad |A| \neq 0 \qquad \therefore \ A^{-1}$ exists.

Let $AA^{-1} = I$ (Here we can use only row transformation)

Using $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ becomes}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

Using $-\frac{1}{2}R_2$ we get

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Using $R_1 \to R_1 - 2R_2$

We get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

$\therefore \quad A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ (Verify the answer.)

Ex. 3 : Find the inverse of $A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ by using elementary row transformations.

Solution : Consider $|A| = 1 \neq 0$ $\therefore A^{-1}$ exists.

Now as row transformations are to be used we have to consider the equation $AA^{-1} = I$ and have to perform row transformations on A.

Consider $AA^{-1} = I$

i.e. $$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use $R_1 \leftrightarrow R_2$

i.e. $$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 6 \\ 2 & 2 & 5 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use $R_2 \rightarrow R_2 - 3R_1$ and $R_3 \rightarrow R_3 - 2R_1$

$$\therefore \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Now use $R_2 \rightarrow -R_2$

$$\therefore \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Use $R_1 \rightarrow R_1 - R_2$

$$\therefore \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Use $R_1 \rightarrow R_1 - 2R_3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\therefore IA^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad \therefore A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(You can verify that $AA^{-1} = I$)

Ex. 4 : Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ by elementary column transformation.

Solution :

As A^{-1} is required by column transformations therefore we have to consider $A^{-1}A = I$ and have to perform column transformations on A.

Consider

$$A^{-1}A = I$$

$$\therefore \quad A^{-1}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $C_2 \to C_2 - 3C_1$ and $C_3 \to C_3 - 3C_1$

$$\therefore \quad A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use $C_1 \to C_1 - C_2$

$$\therefore \quad A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -3 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use $C_1 \to C_1 - C_3$

$$\therefore \quad A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1}I = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

2.2.2 Inverse of a square matrix by adjoint method :

From the previous discussion about finding the inverse of a square matrix by elementary transformation it is clear that the method is elaborate and requires a series of transformations.

There is another method for finding the inverse and it is called as the inverse by the adjoint method. This method can be directly used for finding the inverse. However, for understanding this method you should know the definition of a minor, a co-factor and adjoint of the given matrix.

Let us first recall the definition of minor and co-factor of an element of a determinant.

Definition : Minor of an element a_{ij} of a determinant is the determinant obtained by deleting i th row and j th column in which the element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

(Can you find the order of the minor of any element of a determinant of order 'n'?)

Definition : Co-factor of an element a_{ij} of a determinant is given by $(-1)^{i+j} M_{ij}$, where M_{ij} is minor of the element a_{ij}. Co-factor of an element a_{ij} is denoted by A_{ij}.

Now for defining the adjoint of a matrix, we require the co-factors of the elements of the matrix.

Consider a matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$. Its corresponding determinant is $|A| = \begin{vmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{vmatrix}$

Here if we require the minor of the element '4', then it is $\begin{vmatrix} -2 & 3 \\ -8 & 9 \end{vmatrix} = -18+24 = 6$

Now as the element '4' belongs to 2nd row and 1st column,

using the notation we get $M_{21} = 6$

If further we require the co-factor of '4' then it is

$$= (-1)^{2+1} M_{21}$$
$$= (-1)(6)$$
$$= -6$$

Hence using notation, $A_{21} = -6$

Thus for any given matrix A, which is a square matrix, we can find the co-factors of all of its elements.

Definition :

The adjoint of a square matrix $A = [a_{ij}]_{m\times m}$ is defined as the transpose of the matrix $[A_{ij}]_{m\times m}$ where A_{ij} is the co-factor of the element a_{ij} of A, for all i and j, where $i, j = 1, 2, \ldots, m$.

The adjoint of the matrix A is denoted by adj A.

For example, if A is a square matrix of order 3 × 3 then the matrix of its co-factors is

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

and the required adjoint of A is the transpose of the above matrix. Hence

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Ex. 1 : Find the co-factors of the elements of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution :

Here $a_{11} = 1 \quad \therefore \quad M_{11} = 4 \quad$ and $\quad A_{11} = (-1)^{1+1}(4) \quad = 4$

$a_{12} = 2 \quad \therefore \quad M_{12} = 3 \quad$ and $\quad A_{12} = (-1)^{1+2}(3) \quad = -3$

$a_{21} = 3 \quad \therefore \quad M_{21} = 2 \quad$ and $\quad A_{21} = (-1)^{2+1}(2) \quad = -2$

$a_{22} = 4 \quad \therefore \quad M_{22} = 1 \quad$ and $\quad A_{22} = (-1)^{2+2}(1) \quad = 1$

$\therefore$ the required co-factors are 4, -3, -2, 1.

Ex. 2 : Find the adjoint of matrix $A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$

Solution : Here $a_{11} = 2 \quad \therefore \quad M_{11} = 1$

$\therefore \quad A_{11} = (-1)^{1+1}(1) = 1$

$a_{12} = -3 \quad \therefore \quad M_{12} = 4$

$\therefore \quad A_{12} = (-1)^{1+2}(4) = -4$

$a_{21} = 4 \quad \therefore \quad M_{21} = -3$

$\therefore \quad A_{21} = (-1)^{2+1}(-3) = 3$

$a_{22} = 1 \quad \therefore \quad M_{22} = 2$

$\therefore \quad A_{22} = (-1)^{2+2}(2) = 2$

$\therefore$ the matrix $[A_{ij}]_{2\times 2} = \begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix}$

$\therefore \quad [A_{ij}]^T_{2\times 2} = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$

$\therefore \quad \text{adj } A = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$

Ex. 3 : Find the adjoint of matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}$

Solution : Here $a_{11} = 2 \quad \therefore \quad M_{11} = 0$

$\therefore \quad A_{11} = (-1)^{1+1} M_{11} = 0$

$a_{12} = 0 \quad \therefore \quad M_{12} = 8$

$\therefore \quad A_{12} = (-1)^{1+2} M_{12} = -8$

$a_{13} = -1 \quad \therefore \quad M_{13} = 4$

$\therefore \quad A_{13} = (-1)^{1+3} M_{13} = 4$

$a_{21} = 3 \quad \therefore \quad M_{21} = 1$

$\therefore \quad A_{21} = (-1)^{2+1} M_{21} = -1$

$a_{22} = 1 \quad \therefore \quad M_{22} = 3$

$\therefore \quad A_{22} = (-1)^{2+2} M_{22} = 3$

$a_{23} = 2 \quad \therefore \quad M_{23} = 2$

$\therefore \quad A_{23} = (-1)^{2+3} M_{23} = -2$

$a_{31} = -1 \quad \therefore \quad M_{31} = 1$

$\therefore \quad A_{31} = (-1)^{3+1} M_{31} = 1$

$a_{32} = 1 \quad \therefore \quad M_{32} = 7$

$\therefore \quad A_{32} = (-1)^{3+2} M_{32} = -7$

$a_{33} = 2 \quad \therefore \quad M_{33} = 2$

$\therefore \quad A_{33} = (-1)^{3+3} M_{33} = 2$

$$\therefore \quad \text{the matrix of co-factors is } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & -8 & 4 \\ -1 & 3 & -2 \\ 1 & -7 & 2 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = [A_{ij}]^T_{3\times 3} = \begin{bmatrix} 0 & -1 & 1 \\ -8 & 3 & -7 \\ 4 & -2 & 2 \end{bmatrix}$$

We know that a determinant can be expanded with the help of any row. For example, expansion by 2^{nd} row $a_{21} A_{21} + a_{22} A_{22} + ... + a_{2n} A_{2n} = |A|$.

But if we multiply the row by a different row of cofactors, then the sum is zero.

For example, $a_{21} A_{31} + a_{22} A_{32} + ... a_{2n} A_{3n} = 0$

This helps us to prove that $A^{-1} = \dfrac{\text{adj} A}{|A|}$

$$\therefore A.\text{adj } A = \begin{vmatrix} |A| & 0 & 0 & ... & 0 \\ 0 & |A| & 0 & ... & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & ... & |A| \end{vmatrix} = |A| \cdot I$$

$$\therefore \quad A^{-1} = \frac{\text{adj} A}{|A|}$$

Thus, if $A = [a_{ij}]_{m\times m}$ is a non-singular square matrix then its inverse exists and it is given by $A^{-1} = \dfrac{1}{|A|}(\text{adj} A)$

Think why A^{-1} does not exist if A is singular.

Ex.1. : If $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$, then find A^{-1} by the adjoint method.

Solution : For given matrix A, we get,

$M_{11} = 3, \qquad A_{11} = (-1)^{1+1}(3) = 3$

$M_{12} = 4, \qquad A_{12} = (-1)^{1+2}(4) = -4$

$M_{21} = -2, \qquad A_{21} = (-1)^{2+1}(-2) = 2$

$M_{22} = 2, \qquad A_{22} = (-1)^{2+2}(2) = 2$

$$\therefore \text{adj } A = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

$$\text{and } |A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6 + 8 = 14 \neq 0$$

$$\therefore \quad \text{using } A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

Ex. 2 : If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find A^{-1} by the adjoint method.

Solution : For the given matrix A

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = 1$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} = 1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = 1$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = -1$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} = 1$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{Now} \quad |A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 2(4-1) + 1(-2+1) + 1(1-2)$$

$$= 6 - 1 - 1$$

$$= 4$$

Therefore by using the formula for A^{-1}

$$A^{-1} = \frac{1}{|A|}(adj\, A)$$

$$\therefore \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Ex. 3 : If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, verify that A (adj A) = (adj A) A = |A| I.

Solution : For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A_{11} = (-1)^{1+1}(4) = 4$$

$$A_{12} = (-1)^{1+2}(3) = -3$$

$$A_{21} = (-1)^{2+1}(2) = -2$$

$$A_{22} = (-1)^{2+2}(1) = 1$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore \quad A(\text{adj } A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{... (i)}$$

$$(\text{adj } A) \,.\, A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-6 & 8-8 \\ -3+3 & -6+4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{... (ii)}$$

$$\text{and } \quad |A|\, I = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{... (iii)}$$

From (i), (ii) and (iii) we get, $A(\text{adj}A) = (\text{adj } A)\, A = |A|\, I$

(Note that this equation is valid for every nonsingular square matrix A)

Exercise

1. Find the co-factors of the elements of the following matrices

(i) $\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$

2. Find the matrix of co-factors for the following matrices

(i) $\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 3 \\ 0 & 3 & -5 \end{bmatrix}$

3. Find the adjoint of the following matrices.

(i) $\begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

verify that $A\,(\text{adj } A) = (\text{adj } A)\, A = |\,A\,|\, I$

5. Find the inverse of the following matrices by the adjoint method.

(i) $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

6. Find the inverse of the following matrices

(i) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Miscellaneous exercise 2 (A)

1. If A = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$ then reduce it to I_3 by using column transformations.

2. If A = $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ then reduce it to I_3 by using row transformations.

3. Check whether the following matrices are invertible or not

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3 \\ 10 & 15 \end{bmatrix}$

(v) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (vi) $\begin{bmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{bmatrix}$ (vii) $\begin{bmatrix} 3 & 4 & 3 \\ 1 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

(ix) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$

4. Find AB, if A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}$ Examine whether AB has inverse or not.

5. If A = $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is a nonsingular matrix then find A^{-1} by elementary row transformations.

Hence, find the inverse of $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

6. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and X is a 2×2 matrix such that $AX = I$, then find X.

7. Find the inverse of each of the following matrices (if they exist).

(i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & -10 \\ 2 & -7 \end{bmatrix}$ (vii) $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

(ix) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (x) $\begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ -1 & 3 & 0 \end{bmatrix}$

8. Find the inverse of $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

by (i) elementary row transformations
(ii) elementary column transformations

9. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ find AB and $(AB)^{-1}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

10. If $A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$, then show that $A^{-1} = \frac{1}{6}(A - 5I)$

11. Find matrix X such that AX = B, where

$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$

.

12. Find X, if AX = B where

$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

13. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 24 & 7 \\ 31 & 9 \end{bmatrix}$ then find matrix X such that AXB = C.

14. Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ by adjoint method.

15. Find the inverse of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ by adjoint method.

16. Find A^{-1} by adjoint method and by elementary transformations if $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$.

17. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ by elementary column transformations.

18. Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ by elementary row transformations.

19. Show with usual notations that for any matrix $A = [a_{ij}]_{3 \times 3}$

(i) $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$

(ii) $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = |A|$

20. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ then, find a matrix X such that XA= B.

2.3 Application of matrices :

In the previous discussion you have learnt the concept of inverse of a matrix. Now we intend to discuss the application of matrices for solving a system of linear equations.

For this we first learn to convert the given system of equations in the form of a matrix equation.

Consider the two linear equations, $2x + 3y = 5$ and $x - 4y = 9$. These equations can be written as shown below

$$\begin{bmatrix} 2x+3y \\ x-4y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

(Recall the meaning of equality of two matrices.)

Now using the definition of multiplication of matrices we can consider the above equation as

$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Now if we denote $\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} = A$, $\begin{bmatrix} x \\ y \end{bmatrix} = X$ and $\begin{bmatrix} 5 \\ 9 \end{bmatrix} = B$

then the above equation can be written as AX = B

In the equation AX = B, X is the column matrix of variables, A is the matrix of coefficients of variables and B is the column matrix of constants.

Note that if A is of order 2×2, X is of order 2×1, then B is of order 2×1.

Similarly, if there are three linear equations in three variables then as shown above they can be expressed as AX = B.

Find the respective orders of the matrices A, X and B in case of three equations in three variables.

This matrix equation AX = B (in both the cases) can be used to find the values of the variables x and y or x, y and z as the case may be. There are two methods for this application which are namely

(i) method of inversion (ii) method of reduction

2.3.1 Method of inversion :

From the name of this method you can guess that here we are going to use the inverse of a matrix.

This can be done as follows :

Consider the three equations as

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

As explained in the beginning, they can be expressed as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \text{i.e. } AX = B.$$

Observe that the respective orders of A, X and B are 3×3, 3×1 and 3×1.

Now, if the solution of the three equations exists, then the matrix A must be nonsingular. Hence, A^{-1} exists. Therefore, A^{-1} can be found out either by transformation method or by adjoint method.

After finding A^{-1}, pre-multiply the matrix equation AX = B by A^{-1}

Thus we get,

$$A^{-1}(AX) = A^{-1}(B)$$
$$\text{i.e.} \quad (A^{-1}A)X = A^{-1}B$$
$$\text{i.e.} \quad IX = A^{-1}B$$
$$\text{i.e.} \quad X = A^{-1}B \quad \text{which gives the required solution.}$$

Solved Examples

Ex. 1 : Solve the equations 2x + 5y = 1 and 3x + 2y = 7 by the method of inversion.

Solution : Using the given equations we get the corresponding matrix equation as

$$\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e. $AX = B$, where $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

Hence, premultiplying the above matrix equation by A^{-1} , we get

$$(A^{-1}A)X = A^{-1}B$$

i.e. $IX = A^{-1}B$

i.e. $X = A^{-1}B$ ...... (i)

Now as $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$, $|A| = -11$ and adj $A = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$

$\therefore \quad A^{-1} = \dfrac{1}{|A|}(\text{adj } A)$

i.e. $A^{-1} = \dfrac{1}{-11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$

Hence using (i) we get

$$X = \frac{1}{-11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$X = \frac{1}{11}\begin{bmatrix} -2 & 5 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e. $\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{11}\begin{bmatrix} 33 \\ -11 \end{bmatrix}$

i.e. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Hence by equality of matrices we get $x = 3$ and $y = -1$.

Ex. 2 : Solve the following equations by the method of inversion

$x - y + z = 4,\ 2x + y - 3z = 0,\ x + y + z = 2.$

Solution : The required matrix equation is $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ i.e. $AX = B$

Hence, by premultiplying the equation by A^{-1}, we get,

i.e. $(A^{-1}A)X = A^{-1}B$

i.e. $IX = A^{-1}B$

i.e. $X = A^{-1}B$ ...... (i)

Now as $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, By definition, $\text{adj } A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$ and $|A| = 10$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

i.e. $$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Hence using (i)

$$X = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$X = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, by equality of matrices we get $x = 2$, $y = -1$ and $z = 1$

2.3.2 Method of reduction :

From the name of the method, it can be guessed that, the given equations are reduced to a certain form to get the solution.

Here also, we start by converting the given linear equation into matrix equation AX = B.

Then we perform the suitable row transformations on the matrix A.

Using the row transformations on A reduce it to an upper triangular matrix or lower triangular matrix or diagonal matrix.

The same row transformations are performed simultaneously on matrix B.

After this step we rewrite the equation in the form of system of linear equations. Now they are in such a form that they can be easily solved by elimination method. Thus, the required solution is obtained.

Solved Examples

Ex. 1 : Solve the equation $2x + 3y = 9$ and $y - x = -2$ using the method of reduction.

Solution : The given equations can be written as

$$2x + 3y = 9$$

and $$-x + y = -2$$

Hence the matrix equation is $\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$ (i.e. AX = B)

Now use $R_2 \to 2R_2 + R_1$

$\therefore$ We get $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$

We rewrite the equations as

$$2x + 3y = 9 \quad \ldots\ldots \text{(i)}$$
$$5y = 5 \quad \ldots\ldots \text{(ii)}$$

From (ii) $y = 1$ and using (i) we get $x = 3$

$\therefore$ $x = 3, y = 1$ is the required solution.

Ex. 2 : Solve the following equations by the method of reduction.

$x + 3y + 3z = 12$, $x + 4y + 4z = 15$ and $x + 3y + 4z = 13$.

Solution : The above equations can be written in the form AX = B

i.e. $$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 4 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \\ 13 \end{bmatrix}$$

using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

we get $$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

Again using $R_1 \to R_1 - 3R_2$ and $R_2 \to R_2 - R_3$ We get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Hence the required solution is $x = 3, y = 2, z = 1$. (verify)

Ex. 3 : Solve the following equations by the method of reduction.

$x + y + z = 1$, $2x + 3y + 2z = 2$ and $x + y + 2z = 4$.

Solution : The above equation can be written in the form AX = B as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

using $R_2 \to R_2 - R_3$ and $R_1 \to R_1 - \frac{1}{2} R_3$

we get $$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$$

Now using $R_1 \to R_1 - \frac{1}{4} R_2$ we get $\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ 4 \end{bmatrix}$

Note that here we have reduced the original matrix A to a lower triangular matrix. Hence we can rewrite the equations in their original form as

$$\frac{x}{4} = -\frac{1}{2} \quad \text{.........(i)} \quad \text{i.e.} \quad x = -2$$

$$x + 2y = -2 \quad \text{.........(ii)}$$

$\therefore \quad 2y = -2 + 2 = 0 \qquad \therefore \quad y = 0$

and $\quad x + y + 2z = 4$

$\therefore \quad 2z = 4 + 2 + 0$

$\therefore \quad 2z = 6$

$\therefore \quad z = 3$

$\therefore \quad x = -2, y = 0, z = 3$ is the required solution.

Ex. 4 : The cost of 2 books and 6 note books is Rs. 34 and the cost of 3 books and 4 notebooks is Rs. 31.

Using matrices, find the cost of one book and one note-book.

Solution : Let Rs. 'x' and ` Rs. 'y' be the costs of one book and one notebook respectively.

Hence, using the above information we get the following equations

$$2x + 6y = 34$$

and $\quad 3x + 4y = 31$

The above equations can be expressed in the form

$$\begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 34 \\ 31 \end{bmatrix} \quad \text{i.e. } AX = B$$

Now using $R_2 \to R_2 - \frac{3}{2} R_1$ we get

$$\begin{bmatrix} 2 & 6 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 34 \\ -20 \end{bmatrix}$$

As the above matrix 'A' is reduced to an upper triangular matrix, we can write the equations in their original form as $2x + 6y = 34$

and $\quad -5y = -20 \qquad \therefore y = 4$

and $\quad 2x = 34 - 6y = 34 - 24 \qquad \therefore 2x = 10 \qquad \therefore x = 5$

$\therefore$ the cost of a book is Rs. 5 and that of a note book is Rs. 4.

Exercise 2.3

1. Solve the following equations by inversion method.

(i) $x + 2y = 2, \quad 2x + 3y = 3$

(ii) $x + y = 4, \quad 2x - y = 5$

(iii) $2x + 6y = 8, \quad x + 3y = 5$

2. Solve the following equations by reduction method.

(i) $2x + y = 5, \quad 3x + 5y = -3$

(ii) $x + 3y = 2, \quad 3x + 5y = 4$

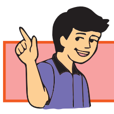
(iii) $3x - y = 1, \quad 4x + y = 6$

(iv) $5x + 2y = 4, \quad 7x + 3y = 5$

3. The cost of 4 pencils, 3 pens and 2 erasers is Rs. 60. The cost of 2 pencils, 4 pens and 6 erasers is Rs. 90, whereas the cost of 6 pencils, 2 pens and 3 erasers is Rs.70. Find the cost of each item by using matrices.

4. If three numbers are added, their sum is '2'. If 2 times the second number is subtracted from the sum of first and third number we get '8' and if three times the first number is added to the sum of second and third number we get '4'. Find the numbers using matrices.

5. The total cost of 3 T.V. sets and 2 V.C.R.s is Rs. 35000. The shop-keeper wants profit of 1000 per television and Rs. 500 per V.C.R. He can sell 2 T. V. sets and 1 V.C.R. and get the total revenue as Rs. 21,500. Find the cost price and the selling price of a T.V. sets and a V.C.R.



Let's Remember :

- If $A = [a_{ij}]_{m\times n}$ then A' or $A^T = [a_{ji}]_{n\times m}$
- If (i) A is symmetric then $A = A^T$ and (ii) if A is skew-symmetric then $-A = A^T$
- If A is a non singular matrix then $A^{-1} = \dfrac{1}{|A|}(\text{adj}\,A)$
- If A, B and C and three matrtices of the same order then

  (i) A + B = B + A (Commutative law of addition)

  (ii) (A + B) + C = A + (B + C) (Associative law for addition)

- If A, B and C are three matrices of appropriate orders so that the following products are defined then

  (i) (AB) C = A (BC) (Associative Law of multiplication)

  (ii) A(B + C) = AB + AC (Left Distributive Law)

  (iii) (A + B) C = AC + BC) (Right Distributive Law)

- The three types of elementary transformations are denoted as

  (i) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

  (ii) $R_i \to kR_j$ or $C_i \to kC_j$ (k is a scalar), $k \neq 0$

  (iii) $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$ (k is a scalar), $k \neq 0$

- If A and B are two square matrices of the same order such that AB = BA = I, then A and B are inverses of each other. A is denoted as B^{-1} and B is denoted as A^{-1}.

- For finding the inverse of A, if row transformations are to be used then we consider $AA^{-1} = I$ and if column transformations are to be used then we consider $A^{-1} A = 1$.

- For finding the inverse of any nonsingular square matrix, two methods can be used
  (i) Elementary transformation method (ii) Adjoint method.
- Using matrices, a system of linear equations can be solved. The two methods are
  (i) Method of inversion (ii) Method of reduction.
- If A is a non singular matrix then $A^{-1} = \frac{1}{|A|}(\text{adj } A)$
- A diagonal matrix is invertible iff all the elements in the diagonal are non - zero.
- The inverse of a diagonal matrix is a diagonal matrix.
- If A is a non singular matrix then det $A^{-1} = \frac{1}{|A|}$
- The adjoint of a square matrix $A = [a_{ij}]_{m\times m}$ is the transpose of the matrix $[A_{ij}]_{m\times m}$, where A_{ij} is the co-factor of the element a_{ij} of A.
- Every square matrix A can be expressed as the sum of symmetric and skew-symmetric matrix. i.e. $A=\frac{1}{2}[A+A^T]+\frac{1}{2}[A-A^T]$

Miscellaneous Exercise 2(B)

I) 1) Choose the correct alternative.

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, adj $A = \begin{bmatrix} 4 & a \\ -3 & b \end{bmatrix}$ then the values of a and b are,

A) a = – 2, b = 1 B) a = 2, b = 4
C) a = 2, b = –1 D) a = 1, b = –2

2) The inverse of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is

A) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ B) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ D) None of these

3) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and A(adjA) = k I then the value of k is .....

A) 1 B) –1
C) 0 D) –3

4) If $A = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix}$ then the adjoint of matrix A is

A) $\begin{bmatrix} -1 & 3 \\ -4 & 1 \end{bmatrix}$

B) $\begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$

C) $\begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$

D) $\begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$

5) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and A(adj A) = k I then the value of k is

A) 2

B) –2

C) 10

D) –10

6) If $A = \begin{bmatrix} \lambda & 1 \\ -1 & -\lambda \end{bmatrix}$ then A^{-1} does not exist if λ =

A) 0

B) ± 1

C) 2

D) 3

7) If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ then A^{-1} =

A) $\begin{bmatrix} 1/\cos\alpha & -1/\sin\alpha \\ 1/\sin\alpha & 1/\cos\alpha \end{bmatrix}$

B) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

C) $\begin{bmatrix} -\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

D) $\begin{bmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$

8) If $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where $\alpha \in R$ then $[F(\alpha)]^{-1}$ is =

A) F(– α)

B) $F(\alpha^{-1})$

C) F(2α)

D) None of these

9) The inverse of $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

A) I

B) A

C) A'

D) – I

10) The inverse of a symmetric matrix is -

A) Symmetric

B) Non-symmetric

C) Null matrix

D) Diagonal matrix

11) For a 2×2 matrix A, if $A(adjA) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ then determinant A equals

A) 20 B) 10

C) 30 D) 40

12) If $A' = -\frac{1}{2}\begin{bmatrix} 1 & -4 \\ -1 & 2 \end{bmatrix}$ then A =

a) $\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$

c) $\begin{bmatrix} 2 & -4 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$

II) 1) Solve the following equations by the methods of inversion.

(i) $2x–y = –2$

(ii) $x + y + z = 1, 2x + 3y + 2z = 2$ and $ax + ay + 2az = 4$

(iii) $5x – y + 4z = 5, 2x + 3y + 5z = 2$ and $5z – 2y + 6z = –1$

(iv) $2x + 3y = –5, 3d + y = 3$

(v) $x + y + z = –1, y + z = 2$ *and* $x + y – z = 3$

2) Express the following equation in matrix from and solve them by the method of reduction.

(i) $x – y + z = 1, 2x - y = 1, 3x + 3y – 4z = 2$

(ii) $x + y = 1, y + x =$

(iii) $2x– + z = 1, x + 2y + 3z = 8$ and $3x + y – 4z = 1$

(iv) $x + y + z = 6, 3x – y + 3z = 10$ and $5z + 5y – 4z = 3$

(v) $x + 2y + z = 8, 2x + 3y – z = 1$ and $3x – y – 2z = 5$

(vi) $x + 3y + 2z = 6, 3x – 2y + 5z = 5$ and $2x – 3y + 6z = 7$

3) The sum of three numbers is 6. If we multiply third number by 3 and add it to the second number we get 11.

By adding first and the third numbers we get a number which is double the second number. Use this information and find a system of linear equations. Find the three numbers using matrices.

4) The cost of 4 pencils, 3 pens and 2 books is Rs.150. The cost of 1 pencil, 2 pens and 3 books is Rs.125. The cos of 6 pencils, 2 pens and 3 books is Rs.175. Fild the cost of each item by using Matrices.

5) The sum of three numbers is 6. Thrice the third number when added to the first number gives 7.

On adding three times first number to the sum of second and third number we get 12. Find the three numbers by using Matrices.

6) The sum of three numbers is 2. If twice the second number is added to the sum of first and third number, we get o adding five times the first number to the sum of second and third we get 6. Find the three numbers by using matrices.

7) An amoun of Rs.5000 is invested in three types of investments, at interest rates 6.7, 7.7, 8% per annum respectively. The total annual income from these investimest is Rs.350/- If the total annual income from first two investment is Rs.70 more than the income from the third, find the amount of each investment using matrix method.

8) The sum of the costs of one ook each of Mathematics, Physics and Chemistry is Rs.210. Total cost of a mathematics book, 2 physics books, and a chemistry book is Rs. 240/- Also the total cost of a Mathematics book, 3 physics book and chemistry books is Rs. 300/-. Find the cost of each book, using Matrices.