Chapter 12

Introduction To Three Dimensional Geometry

Exercise 12.1

Question 1: A point is on the x-axis. What are its y-coordinates and z-coordinates?

Answer 1:

If a point is on the x-axis, then its y-coordinates and z-coordinates are zero.

Question 2: A point is in the XZ-plane. What can you say about its y-coordinate?

Answer 2:

If a point is in the XZ plane, then its y-coordinate is zero.

Question 3: Name the octants in which the following points lie:

(1, 2, 3), (4, –2, 3), (4, –2, –5), (4, 2, –5), (–4, 2, –5), (–4, 2, 5), (–3, –1, 6), (2, –4, –7)

Answer 3:

The x-coordinate, y-coordinate, and z-coordinate of point (1, 2, 3) are all positive.

Therefore, this point lies in octant I.

The x-coordinate, y-coordinate, and z-coordinate of point (4, –2, 3) are positive, negative, and positive respectively. Therefore, this point lies in octant IV.

The x-coordinate, y-coordinate, and z-coordinate of point (4, –2, –5) are positive, negative, and negative respectively. Therefore, this point lies in octant VIII.

The x-coordinate, y-coordinate, and z-coordinate of point (4, 2, –5) are positive, positive, and negative respectively. Therefore, this point lies in octant V.

The x-coordinate, y-coordinate, and z-coordinate of point (–4, 2, –5) are negative, positive, and negative respectively. Therefore, this point lies in octant VI.

The x-coordinate, y-coordinate, and z-coordinate of point (–4, 2, 5) are negative, positive, and positive respectively. Therefore, this point lies in octant II.

The x-coordinate, y-coordinate, and z-coordinate of point (–3, –1, 6) are negative, negative, and positive respectively. Therefore, this point lies in octant III.

The x-coordinate, y-coordinate, and z-coordinate of point (2, –4, –7) are positive, negative, and negative respectively. Therefore, this point lies in octant VIII.

Question 4: Fill in the blanks:

(i) The x-axis and y-axis taken together determine a plane known as________.

(ii) The coordinates of points in the XY-plane are of the form ________.

(iii) Coordinate planes divide the space into ______ octants.

Answer 4:

(i) The x-axis and y-axis taken together determine a plane known as xy - plane.

(ii) The coordinates of points in the XY-plane are of the form (x, y, 0).

(iii) Coordinate planes divide the space into eight octants.

Exercise 12.2

Question 1: Find the distance between the following pairs of points:

(i) (2, 3, 5) and (4, 3, 1) (ii) (–3, 7, 2) and (2, 4, –1)

(iii) (–1, 3, –4) and (1, –3, 4) (iv) (2, –1, 3) and (–2, 1, 3)

Answer 1:

The distance between points P (x_1, y_1, z_1) and P (x_2, y_2, z_2) is given by

PQ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

(i) Distance between points (2, 3, 5) and (4, 3, 1)

= $\sqrt{(4-2)^2 + (3-3)^2 + (1-5)^2}$

= $\sqrt{(2)^2 + (0)^2 + (-4)^2}$

= $\sqrt{4 + 16}$ = $\sqrt{20}$ = $2\sqrt{5}$

(ii) Distance between points (–3, 7, 2) and (2, 4, –1)

= $\sqrt{(2-(-3))^2 + (4-7)^2 + (-1-2)^2}$

= $\sqrt{(5)^2 + (-3)^2 + (-3)^2}$

= $\sqrt{25 + 9 + 9}$ = $\sqrt{43}$

(iii)Distance between points (–1, 3, –4) and (1, –3, 4)

= $\sqrt{(1-(-1))^2 + (-3-3)^2 + (4-(-4))^2}$

= $\sqrt{(2)^2 + (-6)^2 + (8)^2}$

= $\sqrt{4 + 36 + 64}$ = $\sqrt{104} = 2\sqrt{26}$

(iv)Distance between points (2, –1, 3) and (–2, 1, 3)

= $\sqrt{(-2-2)^2 + (1-(-1))^2 + (3-3)^2}$

$= \sqrt{(4)^2 + (2)^2 + (0)^2}$

$= \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$

Question 2: Show that the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1) are collinear.

Answer 2:

Let points (–2, 3, 5), (1, 2, 3), and (7, 0, –1) be denoted by P, Q, and R respectively.

Points P, Q, and R is collinear if they lie on a line.

PQ $= \sqrt{(1 - (-2))^2 + (2 - 3)^2 + (3 - 5)^2}$

$= \sqrt{(3)^2 + (-1)^2 + (2)^2}$

$= \sqrt{9 + 1 + 4} = \sqrt{14}$

QR $= \sqrt{(7 - 1)^2 + (0 - 2)^2 + (-1 - 3)^2}$

$= \sqrt{(6)^2 + (-2)^2 + (-4)^2}$

$= \sqrt{36 + 4 + 16} = \sqrt{56} = 2\sqrt{14}$

PR $= \sqrt{(7 - (-2))^2 + (0 - 3)^2 + (-1 - 5)^2}$

$= \sqrt{(9)^2 + (-3)^2 + (-6)^2}$

$= \sqrt{81 + 9 + 36}$

$= \sqrt{126} = 3\sqrt{14}$

Here, PQ + QR $= \sqrt{14} + 2\sqrt{14} = 3\sqrt{14}$PR

Hence, points P (–2, 3, 5), Q (1, 2, 3), and R (7, 0, –1) are collinear.

Question 3: Verify the following:

(i) (0, 7, –10), (1, 6, –6) and (4, 9, –6) are the vertices of an isosceles triangle.

(ii) (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) are the vertices of a right angled triangle.

(iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram.

Answer 3:

(i) Let points (0, 7, –10), (1, 6, –6), and (4, 9, –6) be denoted by A, B, and C respectively.

AB = $\sqrt{(1-0)^2+(6-7)^2+(-6+10)^2}$

= $\sqrt{(1)^2+(-1)^2+(4)^2}$

= $\sqrt{1+1+8}$

= $\sqrt{10}$ = $3\sqrt{2}$

BC = $\sqrt{(4-1)^2+(9-6)^2+(-6+6)^2}$

= $\sqrt{(3)^2+(3)^2+(0)^2}$

= $\sqrt{9+9}=\sqrt{18}=3\sqrt{2}$

CA = $\sqrt{(0-4)^2+(7-9)^2+(-10+6)^2}$

= $\sqrt{(-4)^2+(-2)^2+(-4)^2}$

= $\sqrt{16+4+16}=\sqrt{36}$

Here, AB = BC ≠ CA

Thus, the given points are the vertices of an isosceles triangle

(ii)Let (0, 7, 10), (–1, 6, 6), and (–4, 9, 6) be denoted by A, B, and C respectively.

AB = $\sqrt{(-1-0)^2+(6-7)^2+(6-10)^2}$

= $\sqrt{(-1)^2+(-1)^2+(-4)^2}$

= $\sqrt{1+1+16} = \sqrt{18}$

= $3\sqrt{2}$

BC = $\sqrt{(-4+1)^2+(9-6)^2+(6-6)^2}$

= $\sqrt{(-3)^2+(3)^2+(0)^2}$

= $\sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

CA = $\sqrt{(0+4)^2+(7-9)^2+(10-6)^2}$

= $\sqrt{(4)^2+(-2)^2+(4)^2}$

= $\sqrt{16+4+16} = \sqrt{36} = 6$

Now, $AB^2 + BC^2 = \left(3\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2 = 18 + 18 = 30 = AC^2$

Therefore, by Pythagoras theorem, ABC is a right triangle.

Hence, the given points are the vertices of a right-angled triangle.

(iii) Let (–1, 2, 1), (1, –2, 5), (4, –7, 8), and (2, –3, 4) be denoted by A, B, C, and D respectively.

AB = $\sqrt{(1+1)^2+(-2-2)^2+(5-1)^2}$

= $\sqrt{4+16+16}$

= $\sqrt{36}$ = 6

BC = $\sqrt{(4-1)^2+(-7-2)^2+(8-5)^2}$

= $\sqrt{9+25+9} = \sqrt{43}$

CD = $\sqrt{(2-4)^2+(-3+7)^2+(4-8)^2}$

$= \sqrt{4 + 16 + 16}$

$= \sqrt{36}$

$= 6$

DA = $\sqrt{(-1-2)^2 + (2+3)^2 + (1-4)^2}$

$= \sqrt{9 + 25 + 9} = \sqrt{43}$

Here, AB = CD = 6, BC = AD = $\sqrt{43}$

Hence, the opposite sides of quadrilateral ABCD, whose vertices are taken in order, are equal.

Therefore, ABCD is a parallelogram.

Hence, the given points are the vertices of a parallelogram.

Question 4: Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, –1).

Answer 4:

Let P (x, y, z) be the point that is equidistant from points A (1, 2, 3) and B (3, 2, –1).

Accordingly, PA = PB

$= PA^2 = PB^2$

= (x – 1)2 + (y – 2)2 + (z – 3)2 = (x – 3)2 + (y – 2)2 + (z – 1)2

$\Rightarrow x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9 = x^2 - 6x + 9 + y^2 - 4y + 4 + z^2 + 2z + 1$

Here, AB = CD = 6, BC = AD =

$\Rightarrow -2x - 4y - 6z + 14 = -6x - 4y + 2z + 14$

$\Rightarrow -2x - 6z + 6x - 2z = 0$

$\Rightarrow 4x - 8z = 0$

$\Rightarrow$ x – 2z = 0

Thus, the required equation is x – 2z = 0.

Question 5: Find the equation of the set of points P, the sum of whose distances from A (4, 0, 0) and B (–4, 0, 0) is equal to 10.

Answer 5:

Let the coordinates of P be (x, y, z).

The coordinates of points A and B are (4, 0, 0) and (–4, 0, 0) respectively.

It is given that PA + PB = 10

$= \sqrt{(x-4)^2 + y^2 + z^2} + \sqrt{(x+4)^2 + y^2 + z^2} = 10$

$= \sqrt{(x-4)^2 + y^2 + z^2} = 10 - \sqrt{(x+4)^2 + y^2 + z^2}$

On squaring both sides, we obtain

$= (x-4)^2 + y^2 + z^2 = 100 - 20\sqrt{(x+4)^2 + y^2 + z^2} + (x+4)^2 + y^2 + z^2$

$= x^2 - 8x + 16 + y^2 + z^2 = 100 - 20\sqrt{x^2 + 8x + 16 + y2 + z2} + x^2 + 8x + 16 + y^2 + z^2$

$= 20\sqrt{x^2 + 8x + 16 + y2 + z2} = 100 + 16x$

$= 5\sqrt{x^2 + 8x + 16 + y^2 + z^2} = (25 + 4x)$

On squaring both sides again, we obtain

$25(x^2 + 8x + 16 + y^2 + z^2)$

$= 625 + 16x^2 + 200x$

$\Rightarrow 25x^2 + 200x + 400 + 25y^2 + 25z^2 = 625 + 16x^2 + 200x$

$\Rightarrow 9x^2 + 25y^2 + 25z^2 - 225 = 0$

Thus, the required equation is $9x^2 + 25y^2 + 25z^2 - 225 = 0$.

Exercise 12.3

Question 1: Find the coordinates of the point which divides the line segment joining the points (–2, 3, 5) and (1, –4, 6) in the ratio (i) 2:3 internally, (ii) 2:3 externally.

Answer 1:

(i) The coordinates of point R that divides the line segment joining points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2nz_1}{m+n}\right)$$

Let R (x, y, z) be the point that divides the line segment joining points (–2, 3, 5) and (1, –4, 6) internally in the ratio 2:3

x = $\frac{2(1)+3(-2)}{2+3}$, $y =$ $\frac{2(-4)+3(3)}{2+3}$ and z = $\frac{2(6)+3(5)}{2+3}$

i.e., x = $\frac{-4}{5}$, y = $\frac{1}{5}$, and z = $\frac{27}{5}$

Thus, the coordinates of the required point are $\left(\frac{-4}{5}, \frac{1}{5}, \frac{27}{5}\right)$

(ii) The coordinates of point R that divides the line segment joining points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) externally in the ratio m: n is

$$\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$$

Let R (x, y, z) be the point that divides the line segment joining points (2, 3, 5) and (1, –4, 6) externally in the ratio 2:3

x = $\frac{2(1)-3(-2)}{2-3}$, y = $\frac{2(-4)-3(3)}{2-3}$, and z = $\frac{2(6)-3(5)}{2-3}$

i.e., x = - 8, y = 17, and z = 3

Thus, the coordinates of the required point are (–8, 17, 3).

Question 2: Given that P (3, 2, – 4), Q (5, 4, –6) and R (9, 8, –10) are collinear. Find the ratio in which Q divides PR.

Answer 2:

Let point Q (5, 4, –6) divide the line segment joining points P (3, 2, –4) and R (9, 8, – 10) in the ratio k:1.

Therefore, by section formula,

(5, 4, -6) = $\left(\frac{k(9)+3}{k+1}, \frac{k(8)+2}{k+1}, \frac{k(-10)-4}{k+1}\right)$

= $\frac{9k+3}{k+1} = 5$

= 9k + 3 = 5k + 5

= 4k = 2

= k = $\frac{2}{4} = \frac{1}{2}$

Thus, point Q divides PR in the ratio 1:2.

Question 3: Find the ratio in which the YZ-plane divides the line segment formed by joining the points (–2, 4, 7) and (3, –5, 8).

Answer 3:

Let the YZ plane divide the line segment joining points (–2, 4, 7) and (3, –5, 8) in the ratio k:1.

Hence, by section formula, the coordinates of point of intersection are given by

$\left(\frac{k(3)-2}{k+1}, \frac{k(-5)+4}{k+1}, \frac{k(8)+7}{k+1}\right)$

On the YZ plane, the x-coordinate of any point is zero

= $\frac{3k-2}{k+1} = 0$

= 3k – 2 = 0

$= k = \frac{2}{3}$

Thus, the YZ plane divides the line segment formed by joining the given points in the ratio 2:3.

Question 4: Using section formula, show that the points A (2, –3, 4), B (–1, 2, 1) and C $(0, \frac{1}{3}, 2)$ are collinear.

Answer 4:

The given points are A (2, –3, 4), B (–1, 2, 1), and C $(0, \frac{1}{3}, 2)$.

Let P be a point that divides AB in the ratio k:1.

Hence, by section formula, the coordinates of P are given by

$$\left(\frac{k(-1)+2}{k+1}, \frac{k(2)-3}{k+1}, \frac{k(1)+4}{k+1}\right)$$

Now, we find the value of k at which point P coincides with point C.

By taking, $\frac{-k+2}{k+1} = 0$we obtain k = 2.

For k = 2, the coordinates of point p is $(0, \frac{1}{3}, 2)$.

i.e., C $(0, \frac{1}{3}, 2)$ is a point that divides AB externally in the ratio 2:1 and is the same as point P.

Hence, points A, B, and C are collinear.

Question 5: Find the coordinates of the points which trisect the line segment joining the points P (4, 2, –6) and Q (10, –16, 6).

Answer 5:

Let A and B be the points that trisect the line segment joining points P (4, 2, –6) and Q (10, –16, 6)

Point A divides PQ in the ratio 1:2. Therefore, by section formula, the coordinates of point A are given by

$$\left(\frac{1(10)+2(4)}{1+2}, \frac{1(-16)+2(2)}{1+2}, \frac{1(6)+2(-6)}{1+2}\right) = (6, -4, -2)$$

Point B divides PQ in the ratio 2:1. Therefore, by section formula, the coordinates of point B are given by

$$\left(\frac{2(10)+1(4)}{2+1}, \frac{2(-16)+1(2)}{2+1}, \frac{2(6)-1(6)}{2+1}\right) = (8, -10, 2)$$

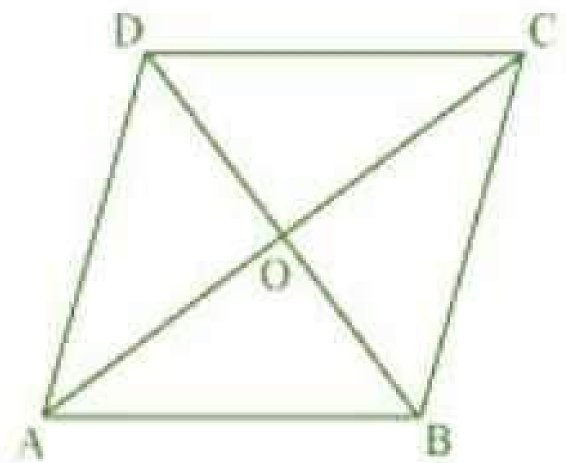
Thus, (6, –4, –2) and (8, –10, 2) are the points that trisect the line segment joining points P (4, 2, –6) and Q (10, –16, 6).

Miscellaneous Exercise

Question 1: Three vertices of a parallelogram ABCD are A (3, –1, 2), B (1, 2, –4) and C (–1, 1, 2). Find the coordinates of the fourth vertex.

Answer 1:

The three vertices of a parallelogram ABCD are given as A (3, –1, 2), B (1, 2, –4), and C (–1, 1, 2). Let the coordinates of the fourth vertex be D (x, y, z).



We know that the diagonals of a parallelogram bisect each other.

Therefore, in parallelogram ABCD, AC and BD bisect each other.

∴ Mid-point of AC = Mid-point of BD

$= \left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right) = \left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-4}{2}\right)$

$= (1, 0, 2) = \left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-4}{2}\right)$

$= \frac{x+1}{2} = 1, \frac{y+2}{2} = 0$ and $\frac{z-4}{2}$

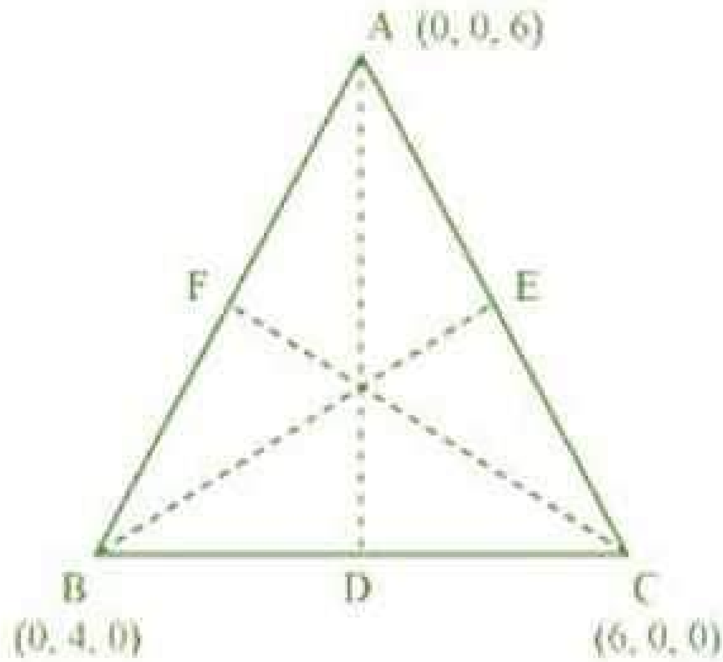
⇒ x = 1, y = –2, and z = 8

Thus, the coordinates of the fourth vertex are (1, –2, 8).

Question 2: Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0, 4, 0) and (6, 0, 0).

Answer 2:

Let AD, BE, and CF be the medians of the given triangle ABC.



Since AD is the median, D is the mid-point of BC.

∴Coordinates of point D $=\left(\frac{0+6}{2}, \frac{4+0}{2}, \frac{0+0}{2}\right)$ = (3, 2, 0)

AD $= \sqrt{(0-3)^2+(0-2)^2+(6-0)^2} = \sqrt{9+4+36} = \sqrt{49} = 7$

Since, BE is the median, E is the mid – point of AC

∴Coordinates of point E $= \left(\frac{0+6}{2}, \frac{0+0}{2}, \frac{6+0}{2}\right) = (3, 0, 3)$

BE $= \sqrt{(3-0)^2+(0-4)^2+(3-0)^2} = \sqrt{9+16+9} = \sqrt{34}$

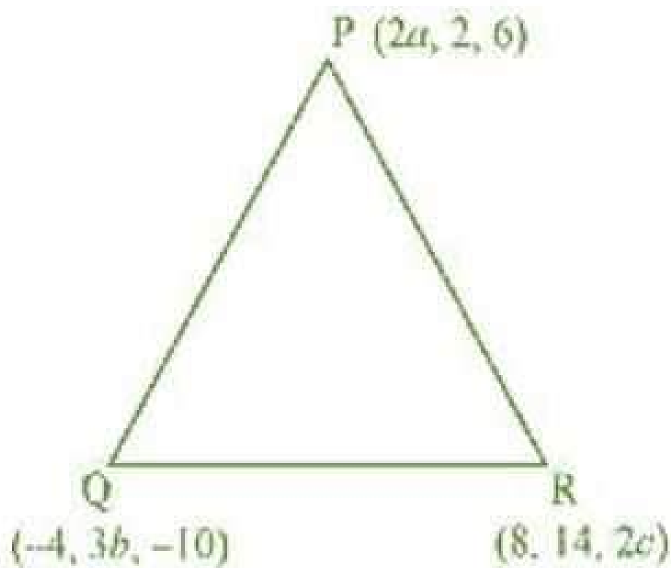
Since, CF is the median, F is the mid – point of AB

∴Coordinates of point F = $\left(\frac{0+0}{2}, \frac{0+4}{2}, \frac{6+0}{2}\right) = (0, 2, 3)$

Length of CF = $\sqrt{(6-0)^2 + (0-2)^2 + (0-3)^2} = \sqrt{36+4+9} = \sqrt{49} = 7$

Thus, the lengths of the medians of ΔABC are $7, \sqrt{34}$, and 7.

Question 3: If the origin is the centroid of the triangle PQR with vertices P (2a, 2, 6), Q (–4, 3b, – 10) and R (8, 14, 2c), then find the values of a, b and c.



It is known that the coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), are.

$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

Therefore, coordinates of the centroid of ΔPQR

$= \left(\frac{2a-4+8}{3}, \frac{2+3b+17}{3}, \frac{6-10+2c}{3}\right) = \left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$

It is given that origin is the centroid of ΔPQR

$= (0, 0, 0) = \left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$

$= \frac{2a+4}{3} = 0, \frac{3b+16}{3} = 0$ and $\frac{2c-4}{3} = 0$

= a = - 2, b = - $\frac{16}{3}$ and c = 2

Thus, the respective values of a, b, and c are – 2, $-\frac{16}{3}$, and 2.

Question 4: Find the coordinates of a point on y-axis which are at a distance of 5√2 from the point P (3, –2, 5).

Answer 4:

If a point is on the y-axis, then x-coordinate and the z-coordinate of the point are zero.

Let A (0, b, 0) be the point on the y-axis at a distance of $5\sqrt{2}$ from point P (3, –2, 5).

Accordingly, AP = $5\sqrt{2}$

AP2 = 50

$= (3 – 0)^2 + (- 2 – b)^2 + (5 – 0)^2 = 50$

$= 9 + 4 + b^2 + 4b + 25 = 50$

$= b^2 + 4b – 12 = 0$

$= b^2 + 6b – 2b – 12 = 0$

= b (b + 6) – 2 (b + 6) = 0

= (b + 6) (b – 2) = 0

= b = - 6 or 2

Thus, the coordinates of the required points are (0, 2, 0) and (0, –6, 0).

Question 5: A point R with x-coordinate 4 lies on the line segment joining the point P (2, –3, 4) and Q (8, 0, 10). Find the coordinates of the point R.

[Hint suppose R divides PQ in the ratio k: 1. The coordinates of the point R are given by $\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$]

Answer 5:

The coordinates of points P and Q are given as P (2, –3, 4) and Q (8, 0, 10).

Let R divide line segment PQ in the ratio k:1.

Hence, by section formula, the coordinates of point R are given by

$$\left(\frac{k(8)+2}{k+1}, \frac{k(0)-3}{k+1}, \frac{k(10)+4}{k+1}\right) = \left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$$

It is given that the x-coordinate of point R is 4

$= \frac{8k+2}{k+1} = 4$

$= 8k + 2 = 4k + 4$

$= 4k = 2$

$= k = \frac{1}{2}$

Therefore, the coordinates of point R are $\left(4, \frac{-3}{\frac{1}{2}+1}, \frac{10\left(\frac{1}{2}\right)+4}{\frac{1}{2}+1}\right)$

Question 6: If A and B be the points (3, 4, 5) and (–1, 3, –7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

Answer 6:

The coordinates of points A and B are given as (3, 4, 5) and (–1, 3, –7) respectively.

Let the coordinates of point P be (x, y, z).

On using distance formula, we obtain

$PA^2 = (x - 3)^2 + (y - 4)^2 + (z - 5)^2$

$= x^2 + 9 - 6x + y^2 + 16 - 8y + z^2 + 25 - 10z$

$= x^2 - 6x + y^2 - 8y + z^2 - 10z + 50$

$PB^2 = (x + 1)^2 + (y - 3)^2 + (z + 7)^2$

$= x^2 + 2x + y^2 - 6y + z^2 + 14z + 59$

Now, if $PA^2 + PB^2 = k^2$, then

$(x^2 - 6x + y^2 - 8y + z^2 - 10z + 50) + (x^2 + 2x + y^2 - 6y + z^2 + 14z + 59)$
$= k^2$

$= 2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 = k^2$

$= 2(x^2 + y^2 + z^2 - 2x - 7y + 2z) = k^2 - 109$

$= x^2 + y^2 + z^2 - 2x - 7y + 2z = \frac{k^2-109}{2}$

Thus, the required equation is $= x^2 + y^2 + z^2 - 2x - 7y + 2z = \frac{k^2-109}{2}$