Chapter 10

Quadratic Equations

CHAPTER HIGHLIGHTS

- *Quadratic Equations*
- *Finding the Roots by Factorization*
- *Finding the Roots by Using the Formula*
- *Sum and Product of Roots of a Quadratic Equation*
- *Nature of the Roots*
- *Signs of the Roots*
- *Constructing a Quadratic Equation*
- *Maximum or Minimum Value of a Quadratic Expression*

Quadratic Equations

'If a variable occurs in an equation with all positive integer powers and the highest power is two, then it is called a Quadratic Equation (in that variable)'.

In other words, a second degree polynomial in x equated to zero will be a quadratic equation. For such an equation to be a quadratic equation, the co-efficient of x^2 should not be zero.

The most general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ (and a, b, c are real).

Some examples of quadratic equations are

$$x^2 - 5x + 6 = 0 \quad (1)$$

$$x^2 - x - 6 = 0 \quad (2)$$

$$2x^2 + 3x - 2 = 0 \quad (3)$$

$$2x^2 + x - 3 = 0 \quad (4)$$

Like a first degree equation in x has one value of x satisfying the equation, a quadratic equation in x will have TWO values of x that satisfy the equation. The values of x that satisfy the equation are called the ROOTS of the equation. These roots may be real or imaginary.

For the above-given four quadratic equations, the roots are as given below:

Equation (1) : $x = 2$ and $x = 3$
Equation (2) : $x = -2$ and $x = 3$
Equation (3) : $x = 1/2$ and $x = -2$
Equation (4) : $x = 1$ and $x = -3/2$

In general, the roots of a quadratic equation can be found out in two ways.

1. By factorizing the expression on the left-hand side of the quadratic equation.
2. By using the standard formula.

All the expressions may not be easy to factorize whereas applying the formula is simple and straightforward.

Finding the Roots by Factorisation

If the quadratic equation $ax^2 + bx + c = 0$ can be written in the form $(x - \alpha)(x - \beta) = 0$, then the roots of the equation are α and β.

To find the roots of a quadratic equation, we should first write it in the form of $(x - \alpha)(x - \beta) = 0$, i.e. the left-hand side $ax^2 + bx + c$ of the quadratic equation $ax^2 + bx + c = 0$ should be factorized into two factors.

For this purpose, we should go through the following steps. We will understand these steps with the help of the equation $x^2 - 5x + 6 = 0$, which is the first of the four quadratic equations we looked at as examples earlier.

1. First write down b (the co-efficient of x) as the sum of two quantities whose product is equal to ac.

In this case, –5 has to be written as the sum of two quantities whose product is 6. We can write –5 as (–3) + (–2) so that the product of (–3) and (–2) is equal to 6.

2. Now rewrite the equation with the 'bx' term split in the aforementioned manner.
   In this case, the given equation can be written as $x^2 - 3x - 2x + 6 = 0$.
3. Take the first two terms and rewrite them together after taking out the common factor between the two of them. Similarly, the third and fourth terms should be rewritten after taking out the common factor between the two of them. In other words, you should ensure that what is left from the first and the second terms (after removing the common factor) is the same as that left from the third and the fourth terms (after removing their common factor).
   In this case, the equation can be rewritten as $x(x - 3) - 2(x - 3) = 0$; Between the first and second terms as well as the third and fourth terms, we are left with $(x - 3)$ is a common factor.
4. Rewrite the entire left-hand side to get the form $(x - \alpha)(x - \beta)$.
   In this case, if we take out $(x - 3)$ as the common factor, we can rewrite the given equation as $(x - 3)(x - 2) = 0$.
5. Now, α and β are the roots of the given quadratic equation.
   $\therefore$ For $x^2 - 5x + 6 = 0$, the roots of the equation are 3 and 2.

For the other three quadratic equations given earlier as examples, let us see how to factorize the expression and get the roots.

For equation (2), i.e. $x^2 - x - 6 = 0$, the co-efficient of x, which is –1 can be rewritten as (–3) + (+2) so that their product is –6, which is equal to ac (1 multiplied by –6). Then, we can rewrite the equation as $(x - 3)(x + 2) = 0$ giving us the roots as 3 and –2.

For equation (3), i.e. $2x^2 + 3x - 2 = 0$, the co-efficient of x, which is 3 can be rewritten as (+4) + (–1) so that their product is –4, which is the value of ac (–2 multiplied by 2). Then, we can rewrite the equation as $(2x - 1)(x + 2) = 0$, giving the roots as 1/2 and –2. For equation (4), i.e. $2x^2 + x - 3 = 0$, the co-efficient of x which is 1 can be rewritten as (+3) + (–2) so that their product is –6 which is equal to ac (2 multiplied by –3). Then, we can rewrite the given equation as $(x - 1)(2x + 3) = 0$, giving us the roots as 1 and –3/2.

Finding the Roots by Using the Formula

If the quadratic equation is $ax^2 + bx + c = 0$, then we can use the standard formula given below to find out the roots of the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The roots of the four quadratic equations we took as examples can be taken and their roots found out by using the aforementioned formula. The student is advised to check it out for himself/herself that the roots can be obtained by using this formula also.

Sum and Product of Roots of a Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$, the sum of the roots and the product of the roots can be given by the following:

Sum of the roots = $-b/a$

Product of the roots = c/a

These two rules will be very helpful in solving problems on quadratic equation.

Nature of the Roots

We mentioned already that the roots of a quadratic equation with real co-efficients can be real or complex. When the roots are real, they can be equal or unequal. All this will depend on the expression $b^2 - 4ac$. Since $b^2 - 4ac$ determines the nature of the roots of the quadratic equation, it is called the 'DISCRIMINANT' of the quadratic equation.

If $b^2 - 4ac > 0$, then the roots of the quadratic equation will be real and distinct.

If $b^2 - 4ac = 0$, the roots are real and equal.

If $b^2 - 4ac < 0$, then the roots of the quadratic equation will be complex conjugates.

Thus, we can write down the following about the nature of the roots of a quadratic equation when a, b, and c are all *rational*.

when $b^2 - 4ac < 0$	the roots are complex and unequal
when $b^2 - 4ac = 0$	the roots are rational and equal
when $b^2 - 4ac > 0$ and a perfect square	the roots are rational and unequal
when $b^2 - 4ac > 0$ but not a perfect square	the roots are irrational and unequal

Whenever the roots of the quadratic equation are irrational, (a, b, c being rational), they will be of the form $a + \sqrt{b}$ and $a - \sqrt{b}$, i.e. whenever $a + \sqrt{b}$ is one root of a quadratic equation, then $a - \sqrt{b}$ will be the second root of the quadratic equation and vice versa.

Signs of the Roots

We can comment on the signs of the roots, i.e. whether the roots are positive or negative, based on the sign of the sum of the roots and the product of the roots of the quadratic equation. The following table will make clear the relationship between the sum and the product of the roots and the signs of the roots themselves.

Sign of product of the roots	Sign of sum of the roots	Sign of the roots
+ ve	+ ve	Both the roots are positive.
+ ve	– ve	Both the roots are negative.
– ve	+ ve	The numerically larger root is positive and the other root is negative.
– ve	– ve	The numerically larger root is negative and the other root is positive.

Constructing a Quadratic Equation

We can build a quadratic equation in the following three cases:

1. When the roots of the quadratic equation are given.
2. When the sum of the roots and the product of the roots of the quadratic equation are given.
3. When the relation between the roots of the equation to be framed and the roots of another equation is given.

If the roots of the quadratic equation are given as α and β, the equation can be written as

$$(x - \alpha)(x - \beta) = 0 \quad \text{i.e., } x^2 - x(\alpha + \beta) + \alpha\beta = 0$$

If p is the sum of the roots of the quadratic equation and q is the product of the roots of the quadratic equation, then the equation can be written as $x^2 - px + q = 0$.

Maximum or Minimum Value of a Quadratic Expression

An equation of the type $ax^2 + bx + c = 0$ is called a quadratic equation. An expression of the type $ax^2 + bx + c$ is called a 'quadratic expression'. The quadratic expression $ax^2 + bx + c$ takes different values as x takes different values.

As x varies from $-\infty$ to $+\infty$, (i.e. when x is real), the quadratic expression $ax^2 + bx + c$

1. Has a minimum value whenever $a > 0$ (i.e., a is positive). The minimum value of the quadratic expression is $(4ac - b^2)/4a$ and it occurs at $x = -b/2a$.
2. Has a maximum value whenever $a < 0$ (i.e. a is negative). The maximum value of the quadratic expression is $(4ac - b^2)/4a$ and it occurs at $x = -b/2a$.

Solved Examples

Example 1

Find the roots of the equation $11x^2 - 37x + 30 = 0$.

Solution

We have to write –37 as the sum of two parts whose product should be equal to $(11) \times (30)$

$$(-22) + (-15) = -37 \text{ and } (-22)(-15) = 11 \times 30$$

Therefore, $11x^2 - 37x + 30 = 0$

$\Rightarrow \quad 11x^2 - 22x - 15x + 30 = 0$

$\Rightarrow \quad 11x(x - 2) - 15(x - 2) = 0$

$\Rightarrow \quad (11x - 15)(x - 2) = 0 \; x = \frac{15}{11}$ or 2.

Example 2

Discuss the nature of the roots of the equation $8x^2 - 2x - 4 = 0$.

Solution

For the quadratic equation $ax^2 + bx + c = 0$, the nature of the roots is given by the discriminant $b^2 - 4ac$.

Discriminant of $8x^2 - 2x - 4 = 0$ is

$$(-2)^2 - 4(8)(-4) = 132.$$

Since the discriminant is positive but not a perfect square, the roots of the equation are irrational and unequal.

Example 3

If the sum of the roots of the equation $Rx^2 + 5x - 24 = 0$ is 5/11, then find the product of the roots of that equation.

Solution

For a quadratic equation $ax^2 + bx + c = 0$, the sum of the roots is $(-b/a)$ and the product of the roots is (c/a).

The sum of the roots of the equation

$Rx^2 + 5x - 24 = 0$ is $\left(\frac{-5}{R}\right)$, which is given as $\frac{5}{11}$

$\therefore R = -11$

In the given equation, product of the roots $= \frac{-24}{R} = \frac{-24}{-11} = +\frac{24}{11}$.

Example 4

Find the value of k, so that the roots of $6x^2 - 12x - k = 0$ are reciprocals of each other.

Solution

If the roots of the equation are reciprocals of each other, then the product of the roots should be equal to 1.

$\Rightarrow \quad \frac{-k}{6} = 1.$

Therefore $k = -6$.

Example 5

If $4 + \sqrt{7}$ is one root of a quadratic equation with rational co-efficients, then find the other root of the equation.

Solution

When the co-efficients of a quadratic equation are rational and the roots are irrational, they occur only in pairs like

$p \pm \sqrt{q}$ i.e., if $p+\sqrt{q}$ is one root, then the other root of the equation will be $p-\sqrt{q}$. So, in this case, the other root of the equation will be $4-\sqrt{7}$.

Example 6

Form a quadratic equation with rational co-efficients, one of whose roots is $5+\sqrt{6}$.

Solution

If $5+\sqrt{6}$ is one root, then the other root is $5-\sqrt{6}$ (because the co-efficients are rational).

The sum of the roots $= 5+\sqrt{6}+5-\sqrt{6} = 10$.

The product of the roots $= (5+\sqrt{6})(5-\sqrt{6}) = 25-6 = 19$.

Thus the required equation is $x^2 - 10x + 19 = 0$.

Example 7

If the price of each book goes up by ₹5, then Atul can buy 20, books less for ₹1200. Find the original price and the number of books Atul could buy at the original price.

Solution

Let the original price of each book be x.

Then the new price of each book will be $x + 5$.

The number of books that can be bought at the original price $= \dfrac{1200}{x}$

The number of books that can be bought at the new price $= \dfrac{1200}{x+5}$

Given that Atul gets 20 books less at new price, i.e. $\dfrac{1200}{x} - \dfrac{1200}{x+5} = 20$

$$\Rightarrow \quad \frac{60}{x} - \frac{60}{x+5} = 1$$

$$\Rightarrow \quad \frac{60(x+5-x)}{x^2+5x} = 1$$

$$\Rightarrow \quad 300 = x^2 + 5x$$

$$\Rightarrow \quad x^2 + 5x - 300 = 0$$

$$\Rightarrow \quad (x+20)(x-15) = 0$$

$$\Rightarrow \quad x = -20 \quad \text{or} \quad 15$$

As the price cannot be negative, the original price is ₹15.

Example 8

If α and β are the roots of the equation $x^2 - 3x - 180 = 0$ such that $\alpha < \beta$, then find the values of

(i) $\alpha^2 + \beta^2$ (ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ (iii) $\alpha - \beta$

Solution

From the given equation, we get $\alpha + \beta = 3$ and $\alpha\beta = -180$

(i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (3)^2 - 2(-180) = 369$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha+\beta}{\alpha\beta} = \dfrac{+3}{-180} = \dfrac{-1}{60}$

(iii) $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

$$\Rightarrow \quad \alpha - \beta = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$$

$$= \pm\sqrt{(+3)^2 - 4(-180)} = \pm\sqrt{9+720}$$

$$= \pm\sqrt{729} = \pm 27 \text{; as } \alpha < \beta,\ \alpha - \beta = -27.$$

Example 9

If $\sqrt{x+4} + \sqrt{x+8} = 7$, then find the value of x.

Solution

Given $\sqrt{x+4} + \sqrt{x+8} = 7$

Squaring on both sides, we get

$$x + 4 + x + 8 + 2\left(\sqrt{x+4}\sqrt{x+8}\right) = 49$$

$$\Rightarrow \quad 2x + 12 + 2\sqrt{x^2+12x+32} = 49$$

$$\Rightarrow \quad 2x - 37 = -2\sqrt{x^2+12x+32}$$

Squaring again on both sides, we have

$$(2x-37)^2 = 4(x^2 + 12x + 32)$$

$$\Rightarrow \quad 4x^2 - 148x + 1369 = 4x^2 + 48x + 128$$

$$\Rightarrow \quad 1241 = 196x$$

$$\Rightarrow \quad x = \frac{1241}{196}.$$

Example 10

If $4^{2x+1} + 4^{x+1} = 80$, then find the value of x.

Solution

Given $4^{2x+1} + 4^{x+1} = 80$

$$\Rightarrow \quad 4^{2x} \times 4 + 4^x \times 4 = 80$$

$$4^{2x} + 4^x = 20$$

Substituting $4^x = a$,

we get $a^2 + a = 20$

$$\Rightarrow \quad a^2 + a - 20 = 0$$

$$\Rightarrow \quad (a+5)(a-4) = 0$$

$$\Rightarrow \quad a = -5 \quad \text{or} \quad 4$$

If $4^x = -5$, there is no possible value for x as no power of 4 gives negative value.

If $4^x = 4$, then $x = 1$.

Exercises

Direction for questions 1 to 25: Select the correct alternative from the given choices.

1. The roots of the quadratic equation $2x^2 - 7x + 2 = 0$ are
   (A) Rational and unequal
   (B) Real and equal
   (C) Imaginary
   (D) Irrational

2. Find the nature of the roots of the quadratic equation $2x^2 + 6x - 5 = 0$.
   (A) Complex conjugates
   (B) Real and equal
   (C) Conjugate surds
   (D) Unequal and rational

3. Construct a quadratic equation whose roots are one third of the roots of $x^2 + 6x + 10 = 0$.
   (A) $x^2 + 18x + 90 = 0$ (B) $x^2 + 16x + 80 = 0$
   (C) $9x^2 + 18x + 10 = 0$ (D) $x^2 + 17x + 90 = 0$

4. A quadratic equation in x has its roots as reciprocals of each other. The co-efficient of x is twice the co-efficient of x^2. Find the sum of the squares of its roots.
   (A) 5 (B) 4 (C) 3 (D) 2

5. If one root of the quadratic equation $4x^2 - 8x + k = 0$, is three times the other root, find the value of k.
   (A) 3 (B) 9 (C) –3 (D) –6

6. The roots of the quadratic equation $(m - k + \ell)\, x^2 - 2mx + (m - \ell + k) = 0$ are
   (A) $1, \dfrac{\ell + m - k}{k + m - \ell}$ (B) $1, \dfrac{2m}{\ell + m - k}$
   (C) $1, \dfrac{k + m - \ell}{\ell + m - k}$ (D) $1, \dfrac{2k}{k - m + \ell}$

7. The expression $\dfrac{4ac - b^2}{4a}$ represents the maximum/minimum value of the quadratic expression $ax^2 + bx + c$. Which of the following is true?
   (A) It represents the maximum value when $a > 0$.
   (B) It represents the minimum value when $a < 0$.
   (C) Both (A) and (B)
   (D) Neither (A) nor (B)

8. Find the signs of the roots of the equation $x^2 + x - 420 = 0$.
   (A) Both are positive.
   (B) Both are negative.
   (C) The roots are of opposite signs with the numerically larger root being positive.
   (D) The roots are of opposite signs with the numerically larger root being negative.

9. If k is a natural number and $(k^2 - 3k + 2)(k^2 - 7k + 12) = 120$, find k.
   (A) 7 (B) 6 (C) 5 (D) 9

10. Both A and B were trying to solve a quadratic equation. A copied the co-efficient of x wrongly and got the roots of the equation as 12 and 6. B copied the constant term wrongly and got the roots as 1 and 26. Find the roots of the correct equation.
    (A) 6, 16 (B) −6, −16
    (C) 24, 3 (D) −3, −24

11. If the roots of the equation $(x - k_1)(x - k_2) + 1 = 0$, k_1 and k_2 are integers, then which of the following must be true?
    (A) k_1, k_2 are two consecutive integers
    (B) $k_2 - k_1 = 2$
    (C) $k_1 - k_2 = 2$
    (D) Either (B) or (C)

12. The roots of the equation $ax^2 + bx + c = 0$ are k less than those of the equation $px^2 + qx + r = 0$. Find the equation whose roots are k more than those of $px^2 + qx + r = 0$.
    (A) $ax^2 + bx + c = 0$
    (B) $a(x - 2k)^2 + b(x - 2k) + c = 0$
    (C) $a(x + 2k)^2 + b(x + 2k) + c = 0$
    (D) $a(x - k)^2 + b(x - k) + c = 0$

13. If one root of the equation $x^2 - 10x + 16 = 0$ is half of one of the roots of $x^2 - 4Rx + 8 = 0$. Find R such that both the equations have integral roots.
    (A) 1 (B) 2/3 (C) 3/2 (D) 4

14. If $x + y = 4$, find the maximum/minimum possible value of $x^2 + y^2$.
    (A) Minimum, 8 (B) Maximum, 8
    (C) Maximum, 16 (D) Minimum, 16

15. Find positive integral value(s) of p such that the equation $2x^2 + 8x + p = 0$ has rational roots.
    (A) 8 (B) 4
    (C) 6 (D) (A) or (C)

16. Two equations have a common root which is positive. The other roots of the equations satisfy $x^2 - 9x + 18 = 0$. The product of the sums of the roots of the two equations is 40. Find the common root.
    (A) 1 (B) 2 (C) 3 (D) 4

17. If one root of the equation $x^3 - 11x^2 + 37x - 35 = 0$ is $3 - \sqrt{2}$, then find the other two roots.
    (A) $5, 3 - \sqrt{2}$ (B) $-5, 3 + \sqrt{2}$
    (C) $5, 3 + \sqrt{2}$ (D) $-5, 3 - \sqrt{2}$

18. The roots [the values of x (and not $|x|$)] of the equation $|x|^2 + 6|x| - 55 = 0$ are α and β. One of the roots of $py^2 + qy + r = 0$ is $\alpha\beta$ times the other root. Which of the following can be concluded?
    (A) $25q^2 = -576pr$ (B) $25pr = -576q^2$
    (C) $25q^2 = 576pr$ (D) $25pr = 576q^2$

19. The sides of a right-angled triangle are such that the sum of the lengths of the longest and that of the shortest side is twice the length of the remaining side. Find the longest side of the triangle if the longer of the sides containing the right angle is 9 cm more than half the hypotenuse.

(A) 30 cm (B) 25 cm (C) 20 cm (D) 15 cm

20. Solve for x: $2\{3^{2(1+x)}\} - 4(3^{2+x}) + 10 = 0$

(A) $-1, \log_3\left(\frac{5}{3}\right)$ (B) $-1, \log_3 2$

(C) $-1, \frac{5}{3}$ (D) $-1, \log_3\left(\frac{3}{5}\right)$

21. If $\sqrt{x^2-2x-3}+\sqrt{x^2+5x-24}=\sqrt{x^2+7x-30}$, then find x.

(A) 2 (B) 3 (C) 4 (D) 6

22. Two software professionals Ranjan and Raman had 108 floppies between them. They sell them at different prices, but each receives the same sum. If Raman had sold his at Ranjan's price, he would have received ₹722 and if Ranjan had sold his at Raman's price, he would have received ₹578. How many floppies did Ranjan have?

(A) 51 (B) 57 (C) 68 (D) 40

23. The sum and product of the roots of a quadratic equation E are a and b, respectively. Find the equation whose roots are the product of first root of E and the square of the second root of E, and the product of the second root of E and the square of the first root of E.

(A) $x^2 - abx + b^3 = 0$ (B) $x^2 + abx + b^3 = 0$

(C) $x^2 + abx - b^3 = 0$ (D) $x^2 - abx - b^3 = 0$

24. Which of the following options represent(s) a condition for the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ to have exactly one common root, given that the roots of both the equations are real?

(A) $a - b = 1$ (B) $b - a = 1$

(C) $1 + a + b = 0$ (D) Either (A) or (B)

25. If the roots of $2x^2 + (4m + 1)x + 2(2m - 1) = 0$ are reciprocals of each other, find m.

(A) -1 (B) 0 (C) 1 (D) 3/4

Answer Keys

1. D	**2.** C	**3.** C	**4.** D	**5.** A	**6.** C	**7.** D	**8.** D	**9.** B	**10.** C
11. D	**12.** B	**13.** C	**14.** A	**15.** D	**16.** B	**17.** C	**18.** A	**19.** A	**20.** A
21. B	**22.** A	**23.** A	**24.** C	**25.** C					