Question Set 13

DIFFERENTIAL EQUATIONS

(Marks with option : 08)

13.1 FORMATION OF DIFFERENTIAL EQUATION

Solved Examples	2 marks each

Ex. 1. Find the order and degree of the D.E. :

(1) $\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right) + y = 2\sin x$

(2) $\sqrt{1 + \frac{1}{\left(\frac{dy}{dx}\right)^2}} = \left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$

(3) $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0.$

Solution :

(1) The given D.E. is $\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right) + y = 2\sin x$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 1,

$\therefore$ the given D.E. is of order **2** and degree **1.**

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(2) The given D.E. is $\sqrt{1 + \frac{1}{\left(\frac{dy}{dx}\right)^2}} = \left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$

On squaring both sides, we get

$$1 + \frac{1}{\left(\frac{dy}{dx}\right)^2} = \left(\frac{d^2y}{dx^2}\right)^3$$

$$\therefore \left(\frac{dy}{dx}\right)^2 + 1 = \left(\frac{d^2y}{dx^2}\right)^3 \cdot \left(\frac{dy}{dx}\right)^2$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 3.

$\therefore$ order = **2** and degree = **3.**

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(3) The given D.E. is $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$.

$\therefore$ order $=$ **2**

Since this D.E. cannot be expressed as a polynomial in differential coefficients, the degree is **not defined.**

Ex. 2. Form the differential equations by eliminating the arbitrary constants from the following equations :

(1) $\boldsymbol{y = c^2 + \frac{c}{x}}$

(2) $\boldsymbol{x^3 + y^3 = 4ax}$

(3) $\boldsymbol{y = Ae^{5x} + Be^{-5x}}$

(4) $\boldsymbol{y^2 = (x+c)^3}$**.**

Solution :

(1) $y = c^2 + \frac{c}{x}$ ... (1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(c^2 + \frac{c}{x}\right) = 0 + c\left(-\frac{1}{x^2}\right)$$

$$\therefore \frac{dy}{dx} = -\frac{c}{x^2}$$

$$\therefore c = -x^2\frac{dy}{dx}$$

Substituting the value of c in (1), we get

$$y = \left(-x^2\frac{dy}{dx}\right)^2 + \frac{1}{x}\left(-x^2\frac{dy}{dx}\right)$$

$$\therefore y = x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx}$$

$$\therefore x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} - y = 0$$

This is the required D.E.

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(2) The given equation is $x^3 + y^3 = 4ax$ ... (1)

Differentiating w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 4a$$

Substituting the value of $4a$ in (1), we get

$$x^3+y^3=\left(3x^2+3y^2\frac{dy}{dx}\right)x$$

$$\therefore\ x^3+y^3=3x^3+3xy^2\frac{dy}{dx}$$

$$\therefore\ 2x^3-y^3+3xy^2\ \frac{dy}{dx}=0$$

This is the required D.E.

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(3) $y=Ae^{5x}+Be^{-5x}$ ... (1)

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx}=Ae^{5x}\times 5+Be^{-5x}\times(-5)$$

$$\therefore\ \frac{dy}{dx}=5Ae^{5x}-5Be^{-5x}$$

and $\dfrac{d^2y}{dx^2}=5Ae^{5x}\times 5-5Be^{-5x}\times(-5)$

$$=25Ae^{5x}+25Be^{-5x}=25(Ae^{5x}+Be^{-5x})$$

$$=25y$$ ... [By (1)]

$$\therefore\ \frac{d^2y}{dx^2}-25y=0$$

This is the required D.E.

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(4) $y^2=(x+c)^3$ ... (1)

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx}=3\,(x+c)^2\cdot(1)=3\,(x+c)^2$$

$$\therefore\ (x+c)^2=\frac{2y}{3}\cdot\frac{dy}{dx}$$

$$\therefore\ (x+c)^6=\left(\frac{2y}{3}\cdot\frac{dy}{dx}\right)^3$$

$$\therefore\ (y^2)^2=\frac{8y^3}{27}\cdot\left(\frac{dy}{dx}\right)^3$$ ... [By (1)]

$$\therefore\ 27y^4=8y^3\left(\frac{dy}{dx}\right)^3$$

$$\therefore\ 27y=8\left(\frac{dy}{dx}\right)^3$$

$\therefore\ 8\left(\frac{dy}{dx}\right)^3-27y=0$

This is the required D.E.

Ex. 3. Verify that :

(1) $y=a+\frac{b}{x}$ **is a solution of** $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$.

(2) $x^2+y^2=r^2$ **is a solution of the D.E.** $y=x\frac{dy}{dx}+r\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

(3) $y\sec x=\tan x+c$ **is a solution of D.E.** $\frac{dy}{dx}+y\tan x=\sec x$.

(4) $y=\log x+c$ **is a solution of the differential equation** $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$.

(5) $y=Ae^x+Be^{-2x}$ **is a solution of the D.E.** $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$.

Solution :

(1) $y=a+\frac{b}{x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=0+b\left(-\frac{1}{x^2}\right)=-\frac{b}{x^2}$$

$$\therefore\ x^2\frac{dy}{dx}=-b$$

Differentiating again w.r.t. x, we get

$$x^2\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(x^2)=0$$

$$\therefore\ x^2\frac{d^2y}{dx^2}+\frac{dy}{dx}\times 2x=0$$

$$\therefore\ x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$

Hence, $y=a+\frac{b}{x}$ is a solution of the D.E.

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0.$$

(2) $x^2 + y^2 = r^2$.

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0. \qquad \therefore\ x + y\frac{dy}{dx} = 0 \qquad \therefore\ \frac{dy}{dx} = \frac{-x}{y}$$

$$\text{RHS} = x\frac{dy}{dx} + r\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = x\left(\frac{-x}{y}\right) + r\sqrt{1 + \left(\frac{-x}{y}\right)^2}$$

$$= \frac{-x^2}{y} + r\sqrt{1 + \frac{x^2}{y^2}} = \frac{-x^2}{y} + r\sqrt{\frac{y^2 + x^2}{y^2}} = \frac{-x^2}{y} + r\sqrt{\frac{r^2}{y^2}}$$

$$= \frac{-x^2}{y} + r\cdot\left(\frac{r}{y}\right) = \frac{-x^2}{y} + \frac{r^2}{y} = \frac{r^2 - x^2}{y} = \frac{y^2}{y} = y = \text{LHS}$$

Thus, $y = x\frac{dy}{dx} + r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ is a differential equation whose solution is $x^2 + y^2 = r^2$.

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(3) $y\sec x = \tan x + c$

Differentiating w.r.t. x, we get

$$y\frac{d}{dx}(\sec x) + \sec x\frac{dy}{dx} = \frac{d}{dx}(\tan x) + \frac{d}{dx}(c)$$

$$\therefore\ y\sec x\tan x + \sec x\frac{dy}{dx} = \sec^2 x + 0$$

Dividing throughout by $\sec x$, we get

$$y\tan x + \frac{dy}{dx} = \sec x \qquad \text{... } [\because\ \sec x \neq 0]$$

$\therefore$ $y\sec x = \tan x + c$ is the general solution of the D.E.

$$\frac{dy}{dx} + y\tan x = \sec x.$$

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(4) $y = \log x + c$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{x} + 0 = \frac{1}{x}$$

$$\therefore\ x\frac{dy}{dx} = 1$$

Differentiating again w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}\times 1 = 0 \qquad \therefore\ x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

This shows that $y = \log x + c$ is a solution of the D.E. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

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(5) Differentiating $y = Ae^x + Be^{-2x}$ twice w.r.t. x, we get

$$\frac{dy}{dx} = Ae^x - 2Be^{-2x}$$

and $\dfrac{d^2y}{dx^2} = Ae^x + 4Be^{-2x}$

$$\therefore \text{ LHS} = \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y$$

$$= (Ae^x + 4Be^{-2x}) + (Ae^x - 2Be^{-2x}) - 2(Ae^x + Be^{-2x})$$

$$= Ae^x + 4Be^{-2x} + Ae^x - 2Be^{-2x} - 2Ae^x - 2Be^{-2x}$$

$$= 0 = \text{RHS}$$

This shows that $y = Ae^x + Be^{-2x}$ is a solution of the D.E. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0.$

Examples for Practice	2 marks each

1. Find the order and degree of the D.E. :

(1) $\dfrac{dy}{dx} = \dfrac{2\sin x + 3}{\left(\dfrac{dy}{dx}\right)}$

(2) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + x = \sqrt{1 + \dfrac{d^3y}{dx^3}}$

(3) $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 8\dfrac{d^2y}{dx^2}$

(4) $\left(\dfrac{d^3y}{dx^3}\right)^{\frac{1}{2}} \cdot \left(\dfrac{dy}{dx}\right)^{\frac{1}{3}} = 20$

(5) $x + \dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{d^2y}{dx^2}\right)^2}$

(6) $e^{\left(\frac{dy}{dx}\right)} + \dfrac{dy}{dx} = x.$

2. Write the degree of the differential equation

$(y''')^2 + 3\,(y'') + 3xy' + 5y = 0.$ ***(March '22) (1 mark)***

3. Form the differential equations by eliminating the arbitrary constants from the following equations :

(1) $(x - a)^2 + y^2 = 1$

(2) $y = a + \dfrac{a}{x}$

(3) $y = (c_1 + c_2x)\,e^x$

(4) $y = Ae^{3x} + Be^{-3x}$

(5) $y = A\cos 2x + B\sin 2x.$

4. If $y = e^{ax}$, show that $x\dfrac{dy}{dx} = y\log y.$

ANSWERS

1. (1) order = 1, degree = 2 (2) order = 3, degree = 1

(3) order = 2, degree = 2 (4) order = 3, degree = 3

(5) order = 2, degree = 1 (6) order = 1, degree is not defined.

2. 2

3. (1) $y^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=1$ (2) $x\,(x+1)\frac{dy}{dx}+y=0$

(3) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$ (4) $\frac{d^2y}{dx^2}=9y$

(5) $\frac{d^2y}{dx^2}+4y=0.$

Solved Examples	3 marks each

Ex. 4. Form the differential equations by eliminating the arbitrary constants from the following equations :

(1) $y=e^{-2x}(A\cos x+B\sin x)$ **(2)** $Ax^2+By^2=1$

(3) $y=A\cos(\log x)+B\sin(\log x)$ **(4)** $y=c_1e^{2x}+c_2e^{5x}$.

Solution :

(1) $y=e^{-2x}(A\cos x+B\sin x)$

$\therefore\ e^{2x}\cdot y=A\cos x+B\sin x$ ... (1)

Differentiating w.r.t. x, we get

$$e^{2x}\cdot\frac{dy}{dx}+y\cdot e^{2x}\times 2=A(-\sin x)+B\cos x$$

$$\therefore\ e^{2x}\left(\frac{dy}{dx}+2y\right)=-A\sin x+B\cos x$$

Differentiating again w.r.t. x, we get

$$e^{2x}\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}\right)+\left(\frac{dy}{dx}+2y\right)\cdot e^{2x}\times 2=-A\cos x+B(-\sin x)$$

$$\therefore\ e^{2x}\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2\frac{dy}{dx}+4y\right)=-(A\cos x+B\sin x)$$

$$\therefore\ e^{2x}\left(\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y\right)=-e^{2x}\cdot y \quad \text{... [By (1)]}$$

$$\therefore\ \frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=-y$$

$$\therefore\ \frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$$

This is the required D.E.

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(2) Differentiating $Ax^2+By^2=1$ ... (1)

twice w.r.t. x, we get

$$2Ax+2By\frac{dy}{dx}=0$$

i.e. $Ax+By\dfrac{dy}{dx}=0$ ... (2)

and $A+B\left[y\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}\cdot\dfrac{dy}{dx}\right]=0$

i.e. $A+B\left[y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2\right]=0$ ... (3)

These three equations in A and B are consistent.

$\therefore$ determinant of their consistency is zero.

$$\therefore\ \begin{vmatrix} x^2 & y^2 & 1 \\ x & y\dfrac{dy}{dx} & 0 \\ 1 & y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2 & 0 \end{vmatrix}=0$$

$$\therefore\ x^2(0-0)-y^2(0-0)+1\left[xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}\right]=0$$

$$\therefore\ xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

This is the required D.E.

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(3) $y=A\cos(\log x)+B\sin(\log x)$ ... (1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=-A\sin(\log x)\cdot\frac{d}{dx}(\log x)+B\cos(\log x)\cdot\frac{d}{dx}(\log x)$$

$$=\frac{-A\sin(\log x)}{x}+\frac{B\cos(\log x)}{x}$$

$$\therefore\ x\frac{dy}{dx}=-A\sin(\log x)+B\cos(\log x)$$

Differentiating again w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{-A\cos(\log x)}{x}-\frac{B\sin(\log x)}{x}$$

$$\therefore\ x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = -[A\cos(\log x)+B\sin(\log x)]$$

$$= -y \qquad \text{... [By (1)]}$$

$\therefore\ x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0$ is the required D.E.

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(4) $y=c_1e^{2x}+c_2e^{5x}$ ... (1)

Dividing both sides by e^{5x}, we get

$$e^{-5x}\cdot y=c_1e^{-3x}+c_2$$

Differentiating w.r.t. x, we get

$$e^{-5x}\cdot\frac{dy}{dx}+y\times e^{-5x}\times(-5)=c_1e^{-3x}\times(-3)+0$$

$$\therefore\ e^{-5x}\left(\frac{dy}{dx}-5y\right)=-3c_1e^{-3x}$$

Dividing both sides by e^{-3x}, we get

$$e^{-2x}\left(\frac{dy}{dx}-5y\right)=-3c_1$$

Differentiating w.r.t. x, we get

$$e^{-2x}\left(\frac{d^2y}{dx^2}-5\frac{dy}{dx}\right)+\left(\frac{dy}{dx}-5y\right)\cdot e^{-2x}(-2)=0$$

$$\therefore\ e^{-2x}\left(\frac{d^2y}{dx^2}-5\frac{dy}{dx}-2\frac{dy}{dx}+10y\right)=0$$

$$\therefore\ \frac{d^2y}{dx^2}-7\frac{dy}{dx}+10y=0$$

This is the required D.E.

Ex. 5. Form the differential equation of :

(1) the family of circles touching the X-axis at the origin.

(2) the hyperbola whose length of transverse and conjugate axes are half of that of the given hyperbola $\dfrac{x^2}{16}-\dfrac{y^2}{36}=k$.

Solution :

(1) Let C (0, k) be the centre of the circle touching the X-axis at the origin.

Then radius of the circle is k.

$\therefore$ equation of the circle is $(x-0)^2+(y-k)^2=k^2$

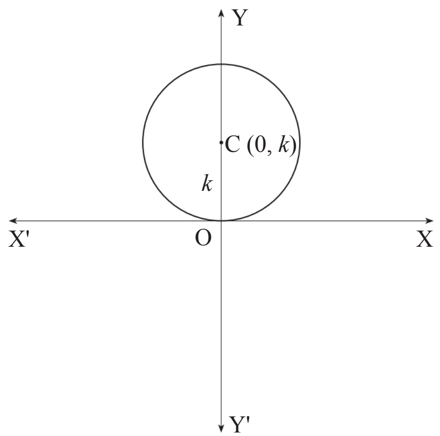
$\therefore\ x^2 + y^2 - 2ky + k^2 = k^2$

$\therefore\ x^2 + y^2 = 2ky$

$\therefore\ \dfrac{x^2}{y} + y = 2k$

where k is an arbitrary constant.

Differentiating w.r.t. x, we get



$$\frac{y \cdot \dfrac{d}{dx}(x^2) - x^2 \dfrac{dy}{dx}}{y^2} + \frac{dy}{dx} = 0$$

$\therefore\ y \times 2x - x^2 \dfrac{dy}{dx} + y^2 \dfrac{dy}{dx} = 0$

$\therefore\ 2xy = (x^2 - y^2)\dfrac{dy}{dx}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{2xy}{x^2 - y^2}$

This is the required D.E.

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(2) The equation of the hyperbola is

$\dfrac{x^2}{16} - \dfrac{y^2}{36} = k$, i.e. $\dfrac{x^2}{16k} - \dfrac{y^2}{36k} = 1$

Comparing this equation with $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, we get

$a^2 = 16k,\ b^2 = 36k$

$\therefore\ a = 4\sqrt{k},\ b = 6\sqrt{k}$

$\therefore\ l(\text{transverse axis}) = 2a = 8\sqrt{k}$

and $l(\text{conjugate axis}) = 2b = 12\sqrt{k}$

Let $2A$ and $2B$ be the lengths of the transverse and conjugate axes of the required hyperbola.

Then according to the given condition

$2A = a = 4\sqrt{k}$ and $2B = b = 6\sqrt{k}$

$\therefore\ A = 2\sqrt{k}$ and $B = 3\sqrt{k}$

$\therefore$ equation of the required hyperbola is

$\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1$

i.e. $\frac{x^2}{4k}-\frac{y^2}{9k}=1$

$\therefore$ $9x^2-4y^2=36k$, where k is an arbitrary constant.

Differentiating w.r.t. x, we get

$$9\times 2x-4\times 2y\frac{dy}{dx}=0$$

$$\therefore\ 9x-4y\frac{dy}{dx}=0$$

This is the required D.E.

Examples for Practice	3 marks each

1. Form the differential equations by eliminating the arbitrary constants from the following equations :

(1) $xy=Ae^x+Be^{-x}+x^2$

(2) $y=e^{-x}(A\cos 2x+B\sin 2x)$.

(3) $y=c_1e^{3x}+c_2e^{2x}$

(4) $xy=ae^{5x}+be^{-5x}$.

(5) $y^2=a(b-x)(b+x)$

(6) $(y-a)^2=4(x-b)$.

2. Form the differential equation of :

(1) all circles passing through the origin and having their centres on the X-axis.

(2) all lines which makes intercept 3 on X-axis. ***(March '22) (2 marks)***

(3) all parabolas whose axis is the X-axis.

ANSWERS

1. (1) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-xy+x^2-2=0$

(2) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=0$

(3) $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$

(4) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=25xy$

(5) $xy\frac{d^2y}{dx^2}+(x-y)\frac{dy}{dx}=0$

(6) $2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$.

2. (1) $y^2-x^2=2xy\frac{dy}{dx}$

(2) $y=(x-3)\frac{dy}{dx}$

(3) $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$.

13.2 SOLUTION OF DIFFERENTIAL EQUATION

1. Variable Separable Method

Solved Examples	2 marks each

Ex. 6. Solve the following differential equations :

(1) $\frac{dy}{dx} = 1 + x + y + xy$ (2) $y\frac{dy}{dx} + x = 0$. *(March '22)*

(3) $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$. *(Sept. '21)*

Solution :

(1) $\frac{dy}{dx} = 1 + x + y + xy$

$\therefore \frac{dy}{dx} = (1 + x) + y(1 + x) = (1 + x)(1 + y)$

$\therefore \frac{1}{1 + y} dy = (1 + x) dx$

Integrating, we get, $\int \frac{1}{1 + y} dy = \int (1 + x) dx$

$\therefore \log|1 + y| = x + \frac{x^2}{2} + c$

This is the general solution.

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(2) $y\frac{dy}{dx} + x = 0$

$\therefore y\frac{dy}{dx} = -x$

$\therefore y \, dy = -x \, dx$

Integrating both sides, we get

$\int y \, dy = -\int x \, dx$

$\therefore \frac{y^2}{2} = -\frac{x^2}{2} + c_1$

$\therefore y^2 = -x^2 + 2c_1$

$\therefore x^2 + y^2 = c$, where $c = 2c_1$

This is the general solution.

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(3) $\sec^2 x \cdot \tan y \, dx + \sec^2 y \cdot \tan x \, dy = 0$

$\therefore \frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$

Integrating, we get

$$\int \frac{\sec^2 x}{\tan x}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$$

$\therefore$ the general solution is

$\log|\tan x| + \log|\tan y| = \log c$, where $c_1 = \log c$

$\therefore$ $\log|\tan x \cdot \tan y| = \log c$

$\therefore$ $\tan x \cdot \tan y = c$

This is the general solution.

Examples for Practice	2 marks each

Solve the following differential equations :

1. (1) $\dfrac{dy}{dx} = x^2 y + y$ (2) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$

(3) $y - x\dfrac{dy}{dx} = 0$ (4) $y^3 - \dfrac{dy}{dx} = x^2\dfrac{dy}{dx}$.

2. (1) $\cos x \cos y\, dy - \sin x \sin y\, dx = 0$ (2) $\cos^2 y\, dx - \operatorname{cosec} x\, dy = 0$

(3) $\sec x\, dy + \operatorname{cosec} y\, dx = 0$ (4) $\tan y \dfrac{dy}{dx} = \sin(x+y) + \sin(x-y)$.

3. (1) $2e^{x+2y}\, dx - 3dy = 0$ (2) $\dfrac{dy}{dx} = x\sqrt{25-x^2}$

(3) $\dfrac{dy}{dx} = 4^{x+y}$ (4) $\log\left(\dfrac{dy}{dx}\right) = 2x + 3y$.

ANSWERS

1. (1) $\log|y| = \dfrac{x^3}{3} + x + c$ (2) $\tan^{-1} y = \tan^{-1} x + c$

(3) $x = cy$ (4) $2y^2 \tan^{-1} x + 1 = cy^2$.

2. (1) $\cos x \cdot \sin y = c$ (2) $\cos x + \tan y = c$

(3) $\sin x - \cos y = c$ (4) $\sec y + 2\cos x = c$.

3. (1) $4e^x + 3e^{-2y} = c$ (2) $3y + (25 - x^2)^{\frac{3}{2}} = c$

(3) $4^x + 4^{-y} = c$ (4) $3e^{2x} + 2e^{-3y} = c$.

Solved Examples	3 marks each

Ex. 7. Solve the following differential equations :

(1) $e^x \tan^2 y\, dx + (e^x - 1)\sec^2 y\, dy = 0$

(2) $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$

(3) $(x - y^2x)\, dx - (y + x^2y)\, dy = 0.$

Solution :

(1) $e^x \tan^2 y\, dx + (e^x - 1)\sec^2 y\, dy = 0$

$$\therefore \frac{e^x}{e^x - 1}\, dx + \frac{\sec^2 y}{\tan^2 y}\, dy = 0$$

On integrating, we get

$$\int \frac{e^x}{e^x - 1}\, dx + \int \frac{\sec^2 y}{\tan^2 y}\, dy = c \qquad \text{... (1)}$$

Put $e^x - 1 = t$ $\quad \therefore e^x\, dx = dt$

$$\therefore \text{(1) becomes, } \int \frac{1}{t}\, dt + \int \frac{1}{\cos^2 y} \times \frac{\cos^2 y}{\sin^2 y}\, dy = c$$

$$\therefore \int \frac{1}{t}\, dt + \int \operatorname{cosec}^2 y\, dy = c$$

$\therefore \log|t| + (-\cot y) = c$

$\therefore \log|e^x - 1| - \cot y = c$

This is the general solution.

---

(2) $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$ $\qquad \therefore y - x\dfrac{dy}{dx} = ay^2 + a\dfrac{dy}{dx}$

$\therefore y - ay^2 = x\dfrac{dy}{dx} + a\dfrac{dy}{dx}$ $\qquad \therefore y(1 - ay) = (x + a)\dfrac{dy}{dx}$

$$\therefore \frac{dx}{x + a} = \frac{dy}{y(1 - ay)}$$

$$\therefore \int \frac{dx}{x + a} - \int \frac{dy}{y(1 - ay)} = c_1 \qquad \text{... (1)}$$

$$\text{Now, } \int \frac{dy}{y(1 - ay)} = \int \frac{(1 - ay) + ay}{y(1 - ay)}\, dy = \int \left(\frac{1}{y} + \frac{a}{1 - ay}\right) dy$$

$$= \log|y| + a\,\frac{\log|1 - ay|}{-a} = \log|y| - \log|1 - ay| = \log\left|\frac{y}{1 - ay}\right|$$

$\therefore$ (1) gives, $\log|x+a| - \log\left|\dfrac{y}{1-ay}\right| = \log c$, where $c_1 = \log c$

$\therefore \log\left|\dfrac{(x+a)(1-ay)}{y}\right| = \log c \qquad \therefore \dfrac{(x+a)(1-ay)}{y} = c$

$\therefore (x+a)(1-ay) = cy$ is the general solution.

---

(3) $(x - y^2x)\,dx - (y + x^2y)\,dy = 0$

$\therefore x(1-y^2)dx - y(1+x^2)dy = 0$

$\therefore \dfrac{x}{1+x^2}dx - \dfrac{y}{1-y^2}dy = 0$

$\therefore \dfrac{2x}{1+x^2} - \dfrac{2y}{1-y^2}dy = 0$

Integrating both sides, we get

$$\int \frac{2x}{1+x^2}dx + \int \frac{-2y}{1-y^2}dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$$

$\therefore$ the general solution is

$\log|1+x^2| + \log|1-y^2| = \log c$, where $c_1 = \log c$

$\therefore \log|(1+x^2)(1-y^2)| = \log c$

$\therefore (1+x^2)(1-y^2) = c.$

Examples for Practice	3 marks each

Solve the following differential equations :

1. $y - x\dfrac{dy}{dx} = 3\left(1 + x^2\dfrac{dy}{dx}\right)$ **2.** $(1+y)^2 \tan^{-1}x\,dx + 2y(1+x^2)dy = 0$

3. $y\sqrt{1-x^2}\,dy + x\sqrt{1-y^2}\,dx = 0$

4. $e^{-x}\dfrac{dy}{dx} = y(1 + \tan x + \tan^2 x)$ **5.** $\dfrac{\cos^2 y}{x}dy + \dfrac{\cos^2 x}{y}dx = 0.$

ANSWERS

1. $(3x+1)(y-3) = cx$ **2.** $(\tan^{-1}x)^2 + \log|1+y^2| = c$

3. $\sqrt{1-x^2} + \sqrt{1-y^2} = c$ **4.** $\log|y| = e^x \tan x + c$

5. $2(x^2+y^2) + 2(x\sin 2x + y\sin 2y) + \cos 2y + \cos 2x + c = 0.$

2. General solution by substitution

Solved Examples	3 marks each

Ex. 8. Solve the following differential equations with the help of the substitutions shown against them :

(1) $1+\dfrac{dy}{dx}=\operatorname{cosec}(x+y)$

(2) $x+y\dfrac{dy}{dx}=\sec(x^2+y^2),\ x^2+y^2=u$ ***(March '22)***

(3) $\left(x\dfrac{dy}{dx}-y\right)e^{\frac{y}{x}}=x^2\cos x,\ y=vx$

(4) $\dfrac{dy}{dx}=(4x+y+1)^2,\ 4x+y+1=u$

(5) $\dfrac{dy}{dx}=\dfrac{6x-4y+3}{3x-2y+1},\ 3x-2y=u.$

Solution :

(1) $1+\dfrac{dy}{dx}=\operatorname{cosec}(x+y)$ ... (1)

Put $x+y=v$

$\therefore\ 1+\dfrac{dy}{dx}=\dfrac{dv}{dx}$

$\therefore$ (1) becomes, $\dfrac{dv}{dx}=\operatorname{cosec} v$

$\therefore\ \dfrac{1}{\operatorname{cosec} v}\,dv=dx$

Integrating both sides, we get

$\int \sin v\,dv=\int dx$

$\therefore\ -\cos v=x+c$

$\therefore\ x+\cos(x+y)+c=0$

This is the general solution.

---

(2) $x+y\dfrac{dy}{dx}=\sec(x^2+y^2)$ ... (1)

Put $x^2+y^2=u$ $\quad\therefore\ 2x+2y\dfrac{dy}{dx}=\dfrac{du}{dx}$

$\therefore\ x+y\dfrac{dy}{dx}=\dfrac{1}{2}\cdot\dfrac{du}{dx}$

$\therefore$ (1) becomes, $\dfrac{1}{2}\cdot\dfrac{du}{dx}=\sec u \qquad \therefore\ \dfrac{1}{\sec u}du=2\cdot dx$

Integrating both sides, we get

$\int \cos u\, du=2\int dx$

$\therefore\ \sin u = 2x+c \qquad \therefore\ \sin(x^2+y^2)=2x+c$

This is the general solution.

---

(3) $\left(x\dfrac{dy}{dx}-y\right)e^{\frac{y}{x}}=x^2\cos x$ ... (1)

Put $y=vx \qquad \therefore\ \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$ and $\dfrac{y}{x}=v$

$\therefore$(1) becomes, $\left[x\left(v+x\dfrac{dv}{dx}\right)-vx\right]e^{v}=x^2\cos x$

$\therefore\ \left(vx+x^2\dfrac{dv}{dx}-vx\right)e^{v}=x^2\cos x$

$\therefore\ x^2\dfrac{dv}{dx}\times e^{v}=x^2\cos x$

$\therefore\ e^{v}\dfrac{dv}{dx}=\cos x$

$\therefore\ e^{v}\,dv=\cos x\,dx$

Integrating both sides, we get

$\int e^{v}\,dv=\int \cos x\,dx$

$\therefore\ e^{v}=\sin x+c$

$\therefore\ e^{\frac{y}{x}}=\sin x+c$

This is the general solution.

---

(4) $\dfrac{dy}{dx}=(4x+y+1)^2$ ... (1)

Put $4x+y+1=u \qquad \therefore\ 4+\dfrac{dy}{dx}=\dfrac{du}{dx}$

$\therefore\ \dfrac{dy}{dx}=\dfrac{du}{dx}-4 \qquad \therefore$ (1) becomes, $\dfrac{du}{dx}-4=u^2$

$\therefore\ \dfrac{du}{dx}=u^2+4 \qquad \therefore\ \dfrac{1}{u^2+4}du=dx$

Integrating, we get

$$\int \frac{1}{u^2 + 2^2} du = \int dx$$

$$\therefore \ \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) = x + c_1$$

$$\therefore \ \tan^{-1}\left(\frac{4x + y + 1}{2}\right) = 2x + 2c_1$$

$$\therefore \ \tan^{-1}\left(\frac{4x + y + 1}{2}\right) = 2x + c, \text{ where } c = 2c_1$$

This is the general solution.

---

(5) $\dfrac{dy}{dx} = \dfrac{6x - 4y + 3}{3x - 2y + 1} = \dfrac{2(3x - 2y) + 3}{(3x - 2y) + 1}$ ... (1)

Put $3x - 2y = u$

$$\therefore \ 3 - 2\frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore \ \frac{dy}{dx} = \frac{1}{2}\left(3 - \frac{du}{dx}\right)$$

$$\therefore \ \text{(1) becomes, } \frac{1}{2}\left(3 - \frac{du}{dx}\right) = \frac{2u + 3}{u + 1}$$

$$\therefore \ 3 - \frac{du}{dx} = \frac{4u + 6}{u + 1}$$

$$\therefore \ \frac{du}{dx} = 3 - \frac{4u + 6}{u + 1} = \frac{3u + 3 - 4u - 6}{u + 1}$$

$$\therefore \ \frac{du}{dx} = \frac{-u - 3}{u + 1} = -\left(\frac{u + 3}{u + 1}\right)$$

$$\therefore \ \frac{u + 1}{u + 3} du = -dx$$

Integrating, we get

$$\int \frac{u + 1}{u + 3} du = -\int dx$$

$$\therefore \ \int \frac{(u + 3) - 2}{u + 3} du = -\int dx$$

$$\therefore \ \int \left(1 - \frac{2}{u + 3}\right) du = -\int dx$$

$$\therefore \ \int 1\, du - 2\int \frac{1}{u + 3} du = -\int dx$$

$\therefore\ u - 2\log|u+3| = -x + c$

$\therefore\ 3x - 2y - 2\log|3x - 2y + 3| = -x + c$

$\therefore\ 4x - 2y - 2\log|3x - 2y + 3| = c$

This is the general solution.

Examples for Practice	3 marks each

Solve the following differential equations :

1. Using the substitution $x + y = u$:

(1) $(x+y)\,dy = a^2 dx$ (2) $(x+y)^2\dfrac{dy}{dx} = a^2$

(3) $\dfrac{dy}{dx} = \cos(x+y)$.

2. Using the substitution $x - y = u$:

(1) $\cos^2(x-y)\dfrac{dy}{dx} = 1$ (2) $(x-y)\left(1 - \dfrac{dy}{dx}\right) = e^x$.

3. Using the substitution $y = vx$:

(1) $\left(x\dfrac{dy}{dx} - y\right)\sin\left(\dfrac{y}{x}\right) = x^2 e^x$ (2) $\dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 - y^2}}{x}$.

4. Using the substitution shown against them :

(1) $\cos(x - 2y) + 2\dfrac{dy}{dx} = 0,\ x - 2y = u$

(2) $\left(y + x\dfrac{dy}{dx}\right)\sin xy = \cos x,\ xy = u$

(3) $\dfrac{dy}{dx} = (9x + y + 2)^2,\ 9x + y + 2 = u$

(4) $(2x - 2y + 5)\dfrac{dy}{dx} = x - y + 3,\ x - y = u$.

ANSWERS

1. (1) $x + y + a^2 = c \cdot e^{\frac{y}{a^2}}$ (2) $y - a\tan^{-1}\left(\dfrac{x+y}{a}\right) = c$

(3) $\tan\left(\dfrac{x+y}{2}\right) = x + c$.

2. (1) $y = \cot(x-y) + c$ (2) $(x-y)^2 = 2e^x + c$.

3. (1) $e^x + \cos\left(\frac{y}{x}\right) + c = 0$ (2) $\sin^{-1}\left(\frac{y}{x}\right) = \log|x| + c.$

4. (1) $\tan\left(\frac{x-2y}{2}\right) = x + c$ (2) $\sin x + \cos xy + c = 0$

(3) $\tan^{-1}\left(\frac{9x+y+2}{3}\right) = 3x + c$ (4) $x - 2y + \log|x - y + 2| = c.$

3. Homogeneous Differential Equations

A homogeneous D.E. of the first order is of the type $\frac{dy}{dx} = -\frac{f_1(x, y)}{f_2(x, y)}$ or $f_1(x, y)\,dx + f_2(x, y)\,dy = 0$, where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of the same degree in x and y. Such an equation can be reduced to the variables separable form by the substitution $y = vx$.

Solved Examples	3 or 4 marks each

Ex. 9. Solve the following differential equations :

(1) $\left(1 + 2e^{\frac{x}{y}}\right) + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\frac{dy}{dx} = 0.$

(2) $(x^2 + y^2)\,dx - 2xy\,dy = 0.$ ***(Sept. '21)***

Solution :

(1) $\left(1 + 2e^{\frac{x}{y}}\right) + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\frac{dy}{dx} = 0$

$\therefore \left(1 + 2e^{\frac{x}{y}}\right) + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\cdot\frac{1}{\left(\frac{dx}{dy}\right)} = 0$

$\therefore \left(1 + 2e^{\frac{x}{y}}\right)\frac{dx}{dy} + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) = 0$ ... (1)

Put $\frac{x}{y} = u$ $\therefore x = uy$

$\therefore \frac{dx}{dy} = u + y\frac{du}{dy}$

$\therefore$ (1) becomes, $(1 + 2e^u)\left(u + y\frac{du}{dy}\right) + 2e^u(1 - u) = 0$

$\therefore u + 2ue^u + y\left(1 + 2e^u\right)\frac{du}{dy} + 2e^u - 2ue^u = 0$

$$\therefore\ (u+2e^u)+y(1+2e^u)\frac{du}{dy}=0$$

$$\therefore\ \frac{dy}{y}+\frac{1+2e^u}{u+2e^u}\,du=0$$

Integrating both sides, we get

$$\int\frac{1}{y}\,dy+\int\frac{1+2e^u}{u+2e^u}\,du=c_1$$

$$\therefore\ \log|y|+\log|u+2e^u|=\log c,\text{ where } c_1=\log c$$

$$\dots\left[\because\ \frac{d}{du}(u+2e^u)=1+2e^u \text{ and } \int\frac{f'(u)}{f(u)}\,du=\log|f(u)|+c\right]$$

$$\therefore\ \log|y(u+2e^u)|=\log c$$

$$\therefore\ y(u+2e^u)=c$$

$$\therefore\ y\left(\frac{x}{y}+2e^{\frac{x}{y}}\right)=\text{c}$$

$$\therefore\ x+2ye^{\frac{x}{y}}=c$$

This is the general solution.

---

(2) $(x^2+y^2)dx-2xy\,dy=0$

$$\therefore\ 2xy\,dy=(x^2+y^2)\,dx$$

$$\therefore\ \frac{dy}{dx}=\frac{x^2+y^2}{2xy} \quad \dots (1)$$

Put $y=vx$ $\quad\therefore\ \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore\ (1)\text{ becomes, } v+x\frac{dv}{dx}=\frac{x^2+v^2x^2}{2x(vx)}$$

$$\therefore\ v+x\frac{dv}{dx}=\frac{1+v^2}{2v}$$

$$\therefore\ x\frac{dv}{dx}=\frac{1+v^2}{2v}-v=\frac{1+v^2-2v^2}{2v}$$

$$\therefore\ x\frac{dv}{dx}=\frac{1-v^2}{2v}$$

$$\therefore\ \frac{2v}{1-v^2}\,dv=\frac{1}{x}\,dx$$

Integrating both sides, we get

$$\int \frac{2v}{1-v^2}\, dv = \int \frac{1}{x}\, dx$$

$$-\int \frac{-2v}{1-v^2}\, dv = \int \frac{1}{x}\, dx$$

$$\therefore\ -\log|1-v^2| = \log|x| + \log c_1$$

$$\ldots \left[\because \frac{d}{dv}(1-v^2) = -2v \text{ and } \int \frac{f'(v)}{f(v)}\, dv = \log|f(v)| + c\right]$$

$$\therefore\ \log|(1-v^2)^{-1}| = \log|c_1 x|$$

$$\therefore\ \frac{1}{1-v^2} = c_1 x$$

$$\therefore\ \frac{1}{1-\left(\frac{y^2}{x^2}\right)} = c_1 x$$

$$\therefore\ \frac{x^2}{x^2-y^2} = c_1 x$$

$$\therefore\ x^2 - y^2 = cx, \qquad \text{where } c = \frac{1}{c_1}$$

This is the general solution.

Examples for Practice	**3 or 4 marks each**

Solve the following differential equations :

1. $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

2. $x^2 y\, dx - (x^3 + y^3)\, dy = 0$

3. $y^2 - x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$

4. $(x^2 + 3xy + y^2)dx - x^2 dy = 0$

5. $x \frac{dy}{dx} = x \tan\left(\frac{y}{x}\right) + y$

6. $y^2\, dx + (xy + x^2)\, dy = 0.$

ANSWERS

1. $\tan^{-1}\left(\frac{y}{x}\right) = \log|x| + c$

2. $\log|y| - \frac{x^3}{3y^3} = c$

3. $\frac{y}{x} + \log|y| = c$

4. $\log|x| + \frac{x}{x+y} = c$

5. $\sin\left(\frac{y}{x}\right) = cx$

6. $xy^2 = c^2\,(x + 2y).$

4. Linear Differential Equations

The general form of a linear differential equation of the first order is $\frac{dy}{dx}+P\cdot y=Q$, where P and Q are the functions of x only or constants. The solution of the linear differential equation is given by $y\,(\text{I.F.})=\int Q\,(\text{I.F.})\,dx+c$, where $\text{I.F.}=e^{\int P dx}$.

Solved Examples	3 or 4 marks each

Ex. 10. Solve the following differential equations :

(1) $x\frac{dy}{dx}+2y=x^2\cdot\log x$ **(2)** $(x+2y^3)\frac{dy}{dx}=y.$

Solution :

(1) $x\frac{dy}{dx}+2y=x^2\cdot\log x$

$\therefore \quad \frac{dy}{dx}+\left(\frac{2}{x}\right)\cdot y=x\cdot\log x$ ... (1)

This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where $P=\frac{2}{x}$ and $Q=x\cdot\log x$

$\therefore \quad \text{I.F.}=e^{\int P\,dx}=e^{\int \frac{2}{x}dx}=e^{2\int \frac{1}{x}dx}$

$=e^{2\log x}=e^{\log x^2}=x^2$

$\therefore$ the solution of (1) is given by

$y\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$

$\therefore \quad y\cdot x^2=\int(x\log x)\cdot x^2\,dx+c$

$\therefore \quad x^2\cdot y=\int x^3\cdot\log x\,dx+c$

$=(\log x)\int x^3\,dx-\int\left[\frac{d}{dx}(\log x)\int x^3\,dx\right]dx+c$

$=(\log x)\cdot\frac{x^4}{4}-\int\frac{1}{x}\cdot\frac{x^4}{4}dx+c$

$=\frac{1}{4}x^4\log x-\frac{1}{4}\int x^3dx+c$

$\therefore \quad x^2\cdot y=\frac{1}{4}x^4\log x-\frac{1}{4}\cdot\frac{x^4}{4}+c$

$$\therefore\ x^2y=\frac{x^4\log x}{4}-\frac{x^4}{16}+c$$

This is the general solution.

---

(2) $(x+2y^3)\dfrac{dy}{dx}=y$

$$\therefore\ \frac{x+2y^3}{y}=\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$\therefore\ \frac{x}{y}+2y^2=\frac{dx}{dy}$$

$$\therefore\ \frac{dx}{dy}-\frac{1}{y}\cdot x=2y^2 \qquad \text{... (1)}$$

This is the linear differential equation of the form $\dfrac{dx}{dy}+P\cdot x=Q$, where

$P=-\dfrac{1}{y}$ and $Q=2y^2$

$$\therefore\ \text{I.F.}=e^{\int P dy}=e^{\int -\frac{1}{y} dy}$$

$$=e^{-\log y}=e^{\log\left(\frac{1}{y}\right)}=\frac{1}{y}$$

$\therefore$ the solution of (1) is given by

$$x\cdot(\text{I.F.})=\int Q\,(\text{I.F.})\,dy+c$$

$$\therefore\ x\left(\frac{1}{y}\right)=\int 2y^2\times\frac{1}{y}\,dy+c$$

$$\therefore\ \frac{x}{y}=2\int y\,dx+c$$

$$\therefore\ \frac{x}{y}=2\cdot\frac{y^2}{2}+c \qquad \therefore\ x=y(c+y^2)$$

This is the general solution.

Ex. 11. The curve passes through the point (0, 2). The sum of the coordinates of any point on the curve exceeds the slope of the tangent to the curve at any point by 5. Find the equation of the curve.

Solution : Let A(x, y) be any point on the curve. Then slope of the tangent to the curve at the point A is $\dfrac{dy}{dx}$.

According to the given condition

$$x+y=\frac{dy}{dx}+5$$

$$\therefore \ \frac{dy}{dx}-y=x-5 \qquad \text{... (1)}$$

This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where $P=-1$ and $Q=x-5$

$\therefore$ I.F. $=e^{\int P\,dx}=e^{\int -1\,dx}=e^{-x}$

$\therefore$ the solution of (1) is given by

$$y\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$$

$$\therefore \ y\cdot e^{-x}=\int(x-5)\,e^{-x}\,dx+c$$

$$\therefore \ e^{-x}\cdot y=(x-5)\int e^{-x}\,dx-\int\left[\frac{d}{dx}(x-5)\int e^{-x}\,dx\right]dx+c$$

$$\therefore \ e^{-x}\cdot y=(x-5)\cdot\frac{e^{-x}}{-1}-\int 1\cdot\frac{e^{-x}}{-1}\,dx+c$$

$$\therefore \ e^{-x}\cdot y=-(x-5)\cdot e^{-x}+\int e^{-x}\,dx+c$$

$$\therefore \ e^{-x}\cdot y=-(x-5)\,e^{-x}+\frac{e^{-x}}{-1}+c$$

$$\therefore \ y=-(x-5)-1+ce^{x}$$

$$\therefore \ y=-x+5-1+ce^{x}$$

$$\therefore \ y=4-x+ce^{x} \qquad \text{... (2)}$$

This is the general equation of the curve.

But the required curve is passing through the point (0, 2).

$\therefore$ by putting $x=0$, $y=2$ in (2), we get

$2=4-0+c \qquad \therefore \ c=-2$

$\therefore$ from (2), the equation of the required curve is $y=4-x-2e^{x}$.

Examples for Practice	3 or 4 marks each

1. Solve the following differential equations :

(1) $\frac{dy}{dx}+\frac{y}{x}=x^2-3$

(2) $\frac{dy}{dx}+y\sec x=\tan x$

(3) $\frac{dy}{dx}+y\cot x=x^2\cot x+2x$

(4) $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy$

(5) $y\,dx+(x-y^2)\,dy=0$

(6) $y\log y=(\log y^2-x)\frac{dy}{dx}$.

2. Find the equation of the curve which passes through the origin and has the slope $x+3y-1$ at any point (x, y) on it.

3. If the slope of the tangent to the curve at each of its point is equal to the sum of abscissa and the product of the abscissa and ordinate of the point. Also, the curve passes through the point (0, 1). Find the equation of the curve.

ANSWERS

1. (1) $\frac{x^5}{5}-\frac{3x^2}{2}-xy=c$

(2) $y(\sec x+\tan x)=\sec x+\tan x-x+c$

(3) $y=x^2+c \operatorname{cosec} x$ (4) $x+1-\tan^{-1}y=ce^{-\tan^{-1}y}$

(5) $\frac{y^3}{3}=xy+c$ (6) $x \log y=(\log y)^2+c$

2. $3(x+3y)=2-2e^{3x}$ **3.** $1+y=2e^{\frac{x^2}{2}}$.

5. Particular solution

Solved Examples	2 marks each

Ex. 12. Find the particular solutions of the following differential equations :

(1) $\frac{dx}{x+2}+\frac{dy}{y+2}=0$, **when** $x=1, y=2$

(2) $\sec^2 y \tan x\, dy+\sec^2 x \tan y\, dx=0$, **when** $x=y=\frac{\pi}{4}$

(3) $\frac{dy}{dx}=e^{2y}\cos x$, **when** $x=\frac{\pi}{6}, y=0$.

Solution :

(1) The given differential equation is

$$\frac{dx}{x+2}+\frac{dy}{y+2}=0$$

Integrating, we get

$$\int\frac{dx}{x+2}+\int\frac{dy}{y+2}=c_1$$

$\therefore \log|x+2|+\log|y+2|=\log c$, where $c_1=\log c$

$\therefore \log|(x+2)(y+2)|=\log c$

$\therefore (x+2)(y+2)=c$

$\therefore\ xy + 2x + 2y + 4 = c$

$\therefore\ xy + 2\ (x + y) + 4 = c$

This is the general solution.

When $x = 1$, $y = 2$, we get

$1\ (2) + 2\ (1 + 2) + 4 = c$

$\therefore\ 2 + 6 + 4 = c \qquad \therefore\ c = 12$

$\therefore$ The particular solution is

$xy + 2\ (x + y) + 4 = 12$

i.e. $xy + 2\ (x + y) = 8$.

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(2) The given D.E. is $\sec^2 y \tan x\, dy + \sec^2 x \tan y\, dx = 0$

$$\therefore\ \frac{\sec^2 y}{\tan y} dy + \frac{\sec^2 x}{\tan x} dx = 0$$

$$\therefore\ \int \frac{\sec^2 y}{\tan y} dy + \int \frac{\sec^2 x}{\tan x} dx = c_1$$

Each integral is of the type

$$\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$$

$\therefore\ \log|\tan y| + \log|\tan x| = \log c$, where $c_1 = \log c$

$\therefore\ \log|\tan x \tan y| = \log c$

$\therefore\ \tan x \tan y = c$

This is the general solution.

When $x = y = \dfrac{\pi}{4}$, we get

$$\tan\frac{\pi}{4} \cdot \tan\frac{\pi}{4} = c \qquad \therefore\ c = 1$$

$\therefore$ the particular solution is $\tan x \tan y = 1$.

---

(3) The given D.E. is $\dfrac{dy}{dx} = e^{2y} \cos x$

$$\therefore\ \frac{1}{e^{2y}} dy = \cos x\ dx$$

Integrating, we get

$$\int e^{-2y}\, dy = \int \cos x\ dx$$

$$\therefore\ \frac{e^{-2y}}{-2} = \sin x + c_1$$

$\therefore\ e^{-2y} = -2\sin x - 2c_1$

$\therefore\ e^{-2y} + 2\sin x = c$, where $c = -2c_1$

This is the general solution.

When $x = \frac{\pi}{6}$, $y = 0$, we have

$$e^0 + 2\sin\frac{\pi}{6} = c$$

$\therefore\ 1 + 2\left(\frac{1}{2}\right) = c \qquad \therefore\ c = 2$

$\therefore$ the particular solution is $e^{-2y} + 2\sin x = 2$.

Examples for Practice	**2 marks each**

Find the particular solutions of the following differential equations :

1. $\sin x\dfrac{dy}{dx} = y\log y$, when $x = \frac{\pi}{2}$, $y = 1$.

2. $\dfrac{dy}{dx} = 3^{x+y}$, when $x = y = 0$

3. $\cos\left(\dfrac{dy}{dx}\right) = a$, $a \in R$, $y(0) = 2$

4. $\dfrac{dy}{dx} + xy = xy^2$, when $x = 1$, $y = 4$

5. $y - x\dfrac{dy}{dx} = 0$, when $x = 2$, $y = 3$.

ANSWERS

1. $y = 1$

2. $3^x + 3^{-y} = 2$

3. $\cos\left(\dfrac{y-2}{x}\right) = a$

4. $\dfrac{x^2}{2} + \log\left|\dfrac{y}{y-1}\right| = \dfrac{1}{2} + \log\left(\dfrac{4}{3}\right)$

5. $3x - 2y = 0$.

Solved Examples	**3 or 4 marks each**

Ex. 13. Find the particular solutions of the following differential equations :

(1) $y(1 + \log x)\dfrac{dx}{dy} - x\log x = 0$, **when** $x = e$, $y = e^2$

(2) $\cos(x + y)\,dy = dx$, **when** $x = 0$ **and** $y = 0$.

(3) $xy\dfrac{dy}{dx} = x^2 + 2y^2$, $y(1) = 0$.

(4) $\dfrac{dy}{dx} - 3y\cot x = \sin 2x$, **when** $y\left(\dfrac{\pi}{2}\right) = 2$.

Solution :

(1) $y(1+\log x)\dfrac{dx}{dy}-x\log x=0$

$\therefore\ \dfrac{1+\log x}{x\log x}\,dx-\dfrac{dy}{y}=0$

$\therefore\ \displaystyle\int\frac{1+\log x}{x\log x}\,dx-\int\frac{dy}{y}=c_1$ ... (1)

Put $x\log x=t$.

Then $\left[x\cdot\dfrac{d}{dx}(\log x)+(\log x)\cdot\dfrac{d}{dx}(x)\right]dx=dt$

$\therefore\ \left[\dfrac{x}{x}+(\log x)(1)\right]dx=dt \quad \therefore\ (1+\log x)\,dx=dt$

$\therefore\ \displaystyle\int\frac{1+\log x}{x\log x}\,dx=\int\frac{dt}{t}=\log|t|=\log|x\log x|$

$\therefore$ from (1), the general solution is

$\log|x\log x|-\log|y|=\log c$, where $c_1=\log c$

$\therefore\ \log\left|\dfrac{x\log x}{y}\right|=\log c \quad \therefore\ \dfrac{x\log x}{y}=c$

$\therefore\ x\log x=cy$

Now, $y=e^2$, when $x=e$,

$\therefore\ e\log e=c\cdot e^2 \qquad 1=c\cdot e$ ... $[\because\ \log e=1]$

$\therefore\ c=\dfrac{1}{e}$

$\therefore$ the particular solution is $x\log x=\left(\dfrac{1}{e}\right)y$

$\therefore\ y=ex\log x.$

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(2) $\cos(x+y)\,dy=dx$

$\therefore\ \dfrac{dy}{dx}=\dfrac{1}{\cos(x+y)}$ ... (1)

Put $x+y=v$. Then $1+\dfrac{dy}{dx}=\dfrac{dv}{dx}$

$\therefore\ \dfrac{dy}{dx}=\dfrac{dv}{dx}-1$

$\therefore$ (1) becomes, $\dfrac{dv}{dx}-1=\dfrac{1}{\cos v} \qquad \therefore\ \dfrac{dv}{dx}=\dfrac{1}{\cos v}+1=\dfrac{1+\cos v}{\cos v}$

$$\therefore \frac{\cos v}{1+\cos v}\, dv = dx$$

Integrating, we get

$$\int \frac{\cos v}{1+\cos v}\, dv = \int dx$$

$$\therefore \int \frac{1+\cos v-1}{1+\cos v}\, dv = \int dx$$

$$\therefore \int \left(1-\frac{1}{1+\cos v}\right) dv = \int dx$$

$$\therefore \int \left[1-\frac{1}{2\cos^2\left(\frac{v}{2}\right)}\right] dv = \int dx$$

$$\int 1\, dv - \frac{1}{2}\int \sec^2\left(\frac{v}{2}\right) dv = \int dx$$

$$\therefore v - \frac{1}{2}\cdot\frac{\tan\left(\frac{v}{2}\right)}{\left(\frac{1}{2}\right)} = x + c$$

$$\therefore x+y-\tan\left(\frac{x+y}{2}\right) = x+c$$

$$\therefore y = \tan\left(\frac{x+y}{2}\right) + c$$

This is the general solution.

When $x = 0$, $y = 0$, we get

$0 = \tan 0 + c \qquad \therefore c = 0$

$\therefore$ the particular solution is $y = \tan\left(\frac{x+y}{2}\right)$.

---

(3) $xy\frac{dy}{dx} = x^2 + 2y^2$

$$\therefore \frac{dy}{dx} = \frac{x^2+2y^2}{xy} \qquad \text{... (1)}$$

Put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \text{(1) becomes, } v + x\frac{dv}{dx} = \frac{x^2+2v^2x^2}{x\cdot vx} = \frac{1+2v^2}{v}$$

$\therefore \; x\dfrac{dv}{dx} = \dfrac{1+2v^2}{v} - v = \dfrac{1+2v^2-v^2}{v}$

$\therefore \; x\dfrac{dv}{dx} = \dfrac{1+v^2}{v}$

$\therefore \; \dfrac{v}{1+v^2}dv = \dfrac{1}{x}dx$

Integrating, we get

$\int \dfrac{v}{1+v^2}dv = \int \dfrac{1}{x}dx$

$\therefore \; \dfrac{1}{2}\int \dfrac{2v}{1+v^2}dv = \int \dfrac{1}{x}dx$

$\therefore \; \dfrac{1}{2}\log|1+v^2| = \log|x| + \log c_1$

$\therefore \; \log|1+v^2| = 2\log|x| + 2\log c_1$

$\therefore \; \log|1+v^2| = \log|x^2| + \log {c_1}^2$

$\therefore \; \log|1+v^2| = \log|cx^2|$, where $c = {c_1}^2$

$\therefore \; 1+v^2 = cx^2$

$\therefore \; 1+\dfrac{y^2}{x^2} = cx^2$

$\therefore \; \dfrac{x^2+y^2}{x^2} = cx^2$

$\therefore \; x^2+y^2 = cx^4$

This is the general solution.

Now, $y(1) = 0$, i.e. when $x = 1$, $y = 0$, we get

$1 + 0 = c\,(1) \qquad \therefore \; c = 1$

$\therefore$ the particular solution is $x^2+y^2 = x^4$.

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(4) $\dfrac{dy}{dx} - 3y\cot x = \sin 2x$

$\therefore \; \dfrac{dy}{dx} - (3\cot x)\,y = \sin 2x \qquad \text{... (1)}$

This is the linear differential equation of the form

$\dfrac{dy}{dx} + Py = Q$, where $P = -3\cot x$ and $Q = \sin 2x$

$\therefore$ I.F. $= e^{\int P dy} = e^{\int -3 \cot x \, dx}$

$$= e^{-3 \log \sin x} = e^{\log (\sin x)^{-3}}$$

$$= (\sin x)^{-3} = \frac{1}{\sin^3 x}$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) \, dx + c$$

$$\therefore \; y \times \frac{1}{\sin^3 x} = \int \sin 2x \times \frac{1}{\sin^3 x} \, dx + c$$

$$\therefore \; y \operatorname{cosec}^3 x = \int 2 \sin x \cos x \times \frac{1}{\sin^3 x} \, dx + c$$

$$\therefore \; y \operatorname{cosec}^3 x = 2 \int \frac{\cos x}{\sin^2 x} \, dx + c$$

Put $\sin x = t$ $\quad \therefore \; \cos x \, dx = dt$

$$\therefore \; y \operatorname{cosec}^3 x = 2 \int \frac{1}{t^2} \, dt + c$$

$$\therefore \; y \operatorname{cosec}^3 x = 2 \int t^{-2} dt + c$$

$$\therefore \; y \operatorname{cosec}^3 x = 2 \left[\frac{t^{-1}}{-1} \right] + c$$

$$\therefore \; y \operatorname{cosec}^3 x = \frac{-2}{\sin x} + c$$

$$\therefore \; y \operatorname{cosec}^3 x + 2 \operatorname{cosec} x = c$$

This is the general solution.

Now, $y\left(\frac{\pi}{2}\right) = 2$, i.e. $y = 2$, when $x = \frac{\pi}{2}$

$$\therefore \; 2 \operatorname{cosec}^3 \frac{\pi}{2} + 2 \operatorname{cosec} \frac{\pi}{2} = c$$

$$\therefore \; 2(1)^3 + 2(1) = c \qquad \therefore \; c = 4$$

$\therefore$ the particular solution is

$$y \operatorname{cosec}^3 x + 2 \operatorname{cosec} x = 4$$

$$\therefore \; y \operatorname{cosec}^2 x + 2 = 4 \sin x.$$

Examples for Practice	3 or 4 marks each

Find the particular solutions of the following differential equations :

1. $(x - y^2x)\ dx - (y + x^2y)\ dy = 0$, when $x = 2$, $y = 0$

2. $(x + 1)\dfrac{dy}{dx} - 1 = 2e^{-y}$, $y = 0$, when $x = 1$

3. $\left(y + x\dfrac{dy}{dx}\right)\sin xy = \cos x$, when $x = 0$, $y = 0$

4. $x + y\dfrac{dy}{dx} = \sec(x^2 + y^2)$, when $x = y = 0$

5. $(x + y)\,dy + (x - y)\,dx = 0$, when $x = 1 = y$

6. $(2x - 2y + 3)\,dx - (x - y + 1)\,dy = 0$, when $x = 0$, $y = 1$

7. $(x + 2y^2)\dfrac{dy}{dx} = y$, when $x = 2$, $y = 1$.

ANSWERS

1. $(1 + x^2)(1 - y^2) = 5$
2. $2(2 + e^y) = 3(x + 1)$
3. $\sin x + \cos xy = 1$
4. $\sin(x^2 + y^2) = 2x$
5. $\tan^{-1}\left(\dfrac{y}{x}\right) + \log\sqrt{x^2 + y^2} = \dfrac{\pi}{4} + \log\sqrt{2}$
6. $(2x - y) - \log|x - y + 2| + 1 = 0$
7. $x = 2y^2$.

13.3 APPLICATIONS OF DIFFERENTIAL EQUATION

1. If x denotes the amount or quantity (which grows or decay) at time t, then rate of change of x w.r.t. time t is $\dfrac{dx}{dt}$.

   In case of growth, x increases as t increases.

   $\therefore\ \dfrac{dx}{dt}$ is positive. Hence, $\dfrac{dx}{dt} = kx$, where $k > 0$.

   In case of decay, x decreases as t increases.

   $\therefore\ \dfrac{dx}{dt}$ is negative. Hence, $\dfrac{dx}{dt} = -kx$, where $k > 0$.

2. Newton's law of cooling states that the rate of change of temperature of a body at any time is proportional to the difference between the temperature of the body and that of its surrounding medium.

Let θ be the temperature of the body at time t and θ_0 be the temperature of the surrounding medium.

Then $\frac{d\theta}{dt}$ is the rate of change of temperature with respect to time t.

According to the Newton's law of cooling

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\therefore \frac{d\theta}{dt} = -k(\theta - \theta_0), \text{ where } k \text{ is a constant and } k > 0.$$

Solved Examples	**3 or 4 marks each**

Ex. 14. If the population of a country doubles in 60 years; in how many years will it be triple (treble) under the assumption that the rate of increase is proportional to the number of inhabitants ?

(Given : $\log 2 = 0.6912$, $\log 3 = 1.0986$)

Solution : Let P be the population at time t years.

Then $\frac{dP}{dt}$, the rate of increase of population is proportional to P.

$$\therefore \frac{dP}{dt} \propto P$$

$$\therefore \frac{dP}{dt} = kP, \text{ where } k \text{ is a constant}$$

$$\therefore \frac{dP}{P} = k\,dt$$

On integrating, we get

$$\int \frac{dP}{P} = k \int dt$$

$$\therefore \log P = kt + c$$

Initially, i.e. when $t = 0$, let $P = P_0$

$\therefore \log P_0 = k \times 0 + c \qquad \therefore c = \log P_0$

$\therefore \log P = kt + \log P_0 \qquad \therefore \log P - \log P_0 = kt$

$$\therefore \log\left(\frac{P}{P_0}\right) = kt \qquad \text{... (1)}$$

Since the population doubles in 60 years, i.e. when $t = 60$, $P = 2P_0$

$\therefore \log\left(\frac{2P_0}{P_0}\right) = 60\,k \qquad \therefore k = \frac{1}{60}\log 2$

$\therefore$ (1) becomes, $\log\left(\dfrac{P}{P_0}\right)=\dfrac{t}{60}\log 2$

When population becomes triple, i.e. when $P = 3P_0$, we get

$$\log\left(\frac{3P_0}{P_0}\right)=\frac{t}{60}\log 2$$

$$\therefore\ \log 3 = \frac{t}{60}\log 2$$

$$\therefore\ t = 60\left(\frac{\log 3}{\log 2}\right)$$

$$= 60\left(\frac{1\cdot 0986}{0\cdot 6912}\right)$$

$$= 60 \times 1\cdot 5894 = 95\cdot 364 \approx 95.4 \text{ years}$$

$\therefore$ the population becomes triple in 95.4 years (approximately).

Ex. 15. The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number doubles in 1 hour, find the number of bacteria after $2\frac{1}{2}$ hours. [Take $\sqrt{2}=1.414$]

Solution : Let x be the number of bacteria at time t.

Then the rate of increase is $\dfrac{dx}{dt}$ which is proportional to x.

$$\therefore\ \frac{dx}{dt} \propto x$$

$$\therefore\ \frac{dx}{dt} = kx, \text{ where } k \text{ is a constant}$$

$$\therefore\ \frac{dx}{x} = k\,dt$$

On integrating, we get

$$\int \frac{dx}{x} = k\int dt$$

$$\therefore\ \log x = kt + c$$

Initially, i.e. when $t = 0$, $x = 1000$

$\therefore\ \log 1000 = k \times 0 + c \qquad \therefore\ c = \log 1000$

$$\therefore\ \log x = kt + \log 1000$$

$$\therefore\ \log x - \log 1000 = kt$$

$$\therefore\ \log\left(\frac{x}{1000}\right) = kt \qquad \text{... (1)}$$

Now, when $t=1$, $x=2\times 1000=2000$

$\therefore\ \log\left(\frac{2000}{1000}\right)=k \qquad \therefore\ k=\log 2$

$\therefore$ (1) becomes, $\log\left(\frac{x}{1000}\right)=t\log 2$

If $t=2\frac{1}{2}=\frac{5}{2}$, then

$$\log\left(\frac{x}{1000}\right)=\frac{5}{2}\log 2=\log (2)^{\frac{5}{2}}$$

$$\therefore\ \left(\frac{x}{1000}\right)=(2)^{\frac{5}{2}}=4\sqrt{2}=4\times 1.414=5.656$$

$\therefore\ x=5.656\times 1000=5656$

Hence, number of bacteria after $2\frac{1}{2}$ hours $=5656$.

Ex. 16. The rate of decay of certain substance is directly proportional to the amount present at that instant. Initially, there are 25 gm of certain substance and two hours later it is found that 9 gm are left. Find the amount left after one more hour.

Solution : Let x gm be the amount of the substance left at time t.

Then the rate of decay is $\frac{dx}{dt}$, which is proportional to x.

$\therefore\ \frac{dx}{dt}\propto x$

$\therefore\ \frac{dx}{dt}=-kx$, where $k>0$

$\therefore\ \frac{1}{x}dx=-k\,dt$

On integrating, we get

$$\int\frac{1}{x}dx=-k\int dt$$

$\therefore\ \log x=-kt+c$

Initially, i.e. when $t=0$, $x=25$

$\therefore\ \log 25=-k\times 0+c \qquad \therefore\ c=\log 25$

$\therefore\ \log x=-kt+\log 25$

$\therefore\ \log x-\log 25=-kt$

$$\therefore \ \log\left(\frac{x}{25}\right) = -kt \qquad \text{... (1)}$$

Now, when $t = 2$, $x = 9$

$$\therefore \ \log\left(\frac{9}{25}\right) = -2k$$

$$\therefore \ -2k = \log\left(\frac{3}{5}\right)^2 = 2\log\left(\frac{3}{5}\right)$$

$$\therefore \ k = -\log\left(\frac{3}{5}\right)$$

$$\therefore \ \text{(1) becomes, } \log\left(\frac{x}{25}\right) = t\log\left(\frac{3}{5}\right)$$

When $t = 3$, then

$$\log\left(\frac{x}{25}\right) = 3\log\left(\frac{3}{5}\right) = \log\left(\frac{3}{5}\right)^3$$

$$\therefore \ \frac{x}{25} = \frac{27}{125} \qquad \therefore \ x = \frac{27}{5}$$

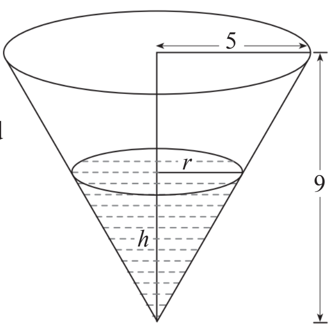
Hence, the amount left after 3 hours $= \frac{27}{5}$ gm.

Ex. 17. A right circular cone has height 9 cm and radius of the base 5 cm. It is inverted and water is poured into it. If at any instant the water level rises at the rate of $\left(\frac{\pi}{A}\right)$ cm/sec, where A is the area of the water surface at that instant, show that the vessel will be full in 75 seconds.

Solution : Let r be the radius of the water surface and h be the height of the water at time t.

$\therefore$ area of the water surface

$A = \pi r^2$ sq cm.



Since height of the right circular cone is 9 cm and radius of the base is 5 cm,

$$\frac{r}{h} = \frac{5}{9} \qquad \therefore \ r = \frac{5}{9}h$$

$\therefore$ area of water surface, i.e. $A = \pi\left(\frac{5}{9}h\right)^2$

$\therefore \ A = \dfrac{25\pi h^2}{81}$ ... (1)

The water level, i.e. the rate of change of h is $\dfrac{dh}{dt}$ rises at the rate of $\left(\dfrac{\pi}{A}\right)$ cm/sec.

$\therefore \ \dfrac{dh}{dt} = \dfrac{\pi}{A} = \dfrac{\pi \times 81}{25\pi h^2}$ ... [By (1)]

$\therefore \ \dfrac{dh}{dt} = \dfrac{81}{25h^2} \qquad \therefore \ h^2 dh = \dfrac{81}{25} dt$

On integrating, we get

$\int h^2\, dh = \dfrac{81}{25} \int dt \qquad \therefore \ \dfrac{h^3}{3} = \dfrac{81}{25} \cdot t + c$

Initially, i.e. when $t = 0$, $h = 0$ $\qquad \therefore \ 0 = 0 + c \qquad \therefore \ c = 0$

$\therefore \ \dfrac{h^3}{3} = \dfrac{81}{25} t$

When the vessel will be full, $h = 9$ $\qquad \therefore \ \dfrac{(9)^3}{3} = \dfrac{81}{25} \times t$

$\therefore \ t = \dfrac{81 \times 9 \times 25}{3 \times 81} = 75$

Hence, the vessel will be full in 75 seconds.

Ex. 18. A body cools according to Newton's law from 100 °C to 60 °C in 20 minutes. The temperature of the surrounding being 20 °C. How long will it take to cool down to 30 °C ?

Solution : Let θ °C be the temperature of the body at time t. The temperature of the surrounding is given to be 20 °C.

According to Newton's law of cooling

$$\frac{d\theta}{dt} \propto \theta - 20$$

$\therefore \ \dfrac{d\theta}{dt} = -k(\theta - 20)$, where $k > 0$

$\therefore \ \dfrac{d\theta}{\theta - 20} = -k\, dt$

On integrating, we get

$$\int \frac{1}{\theta - 20}\, d\theta = -k \int dt$$

$\therefore \ \log(\theta - 20) = -kt + c$

Initially, i.e. when $t=0$, $\theta=100$

$\therefore \log(100-20)=-k\times 0+c \qquad \therefore c=\log 80$

$\therefore \log(\theta-20)=-kt+\log 80$

$\therefore \log(\theta-20)-\log 80=-kt$

$$\therefore \log\left(\frac{\theta-20}{80}\right)=-kt \qquad \text{... (1)}$$

Now, when $t=20$, $\theta=60$

$$\therefore \log\left(\frac{60-20}{80}\right)=-k\times 20$$

$$\therefore \log\left(\frac{40}{80}\right)=-20k \qquad \therefore k=-\frac{1}{20}\log\left(\frac{1}{2}\right)$$

$$\therefore \text{(1) becomes, } \log\left(\frac{\theta-20}{80}\right)=\frac{t}{20}\log\left(\frac{1}{2}\right)$$

When $\theta=30$, then

$$\log\left(\frac{30-20}{80}\right)=\frac{t}{20}\log\left(\frac{1}{2}\right)$$

$$\therefore \log\left(\frac{1}{8}\right)=\log\left(\frac{1}{2}\right)^{\frac{t}{20}}$$

$$\therefore \left(\frac{1}{2}\right)^{\frac{t}{20}}=\frac{1}{8}=\left(\frac{1}{2}\right)^{3}$$

$$\therefore \frac{t}{20}=3 \qquad \therefore t=60$$

Hence, the body will cool down to 30 °C in 60 minutes, i.e. in 1 hour.

Examples for Practice	3 or 4 marks each

1. The population of a town increasing at a rate proportional to the population at that time. If the population increases from 40 thousands to 60 thousands in 40 years, what will be the population in another 20 years.

$$\left(\text{Given : } \sqrt{\frac{3}{2}}=1.2247\right)$$

2. Bacteria increases at the rate proportional to the number of bacteria present. If the original number N doubles in 3 hours, find in how many hours the number of bacteria will be $4N$?

3. The rate of disintegration of a radioactive element at any time t is proportional to its mass at that time. Find the time during which the original mass of 1.5 gm will disintegrate into its mass of 0.5 gm.

4. Water at 100°C cools in 10 minutes to 88°C in a room temperature of 25°C. Find the temperature of water after 20 minutes.

5. A person's assets start reducing in such a way that the rate of reduction of assets is proportional to the square root of the assets existing at that moment. If the assets at the beginning was ₹ 10 lakhs and they dwindle down to ₹ 10,000 after 2 years, show that the person will be bankrupt in $2\frac{2}{9}$ years from the start.

ANSWERS

1. 73,482 **2.** 6 hours **3.** $\frac{1}{k}\log 3$ **4.** (77.92)°C.

MULTIPLE CHOICE QUESTIONS	2 marks each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. The order and degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{7}{3}}=7\left(\frac{d^2y}{dx^2}\right)$ are respectively

(a) 2, 3 (b) 3, 2 (c) 2, 2 (d) 3, 3

2. The order and degree of the differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}+x=\sqrt{1+\frac{d^3y}{dx^3}}$ are respectively

(a) 2, 3 (b) 3, 2 (c) 1, 3 (d) 3, 1. ***(Sept. '21)***

3. The differential equation $y\frac{dy}{dx}+x=0$ represents family of

(a) circles (b) parabolas (c) ellipses (d) hyperbolas

4. The differential equation of $y=c^2+\frac{c}{x}$ is

(a) $x^4\left(\frac{dy}{dx}\right)^2-x\frac{dy}{dx}=y$ (b) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

(c) $x^3\left(\frac{dy}{dx}\right)^2+x\frac{dy}{dx}=y$ (d) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0$

5. The solution of $\frac{dy}{dx}+y=\cos x-\sin x$ is

(a) $ye^x=\cos x+c$ (b) $ye^x+e^x\cos x=c$

(c) $ye^x=e^x\cos x+c$ (d) $y^2e^x=e^x\cos x+c$

6. The solution of $\frac{dy}{dx}=\frac{y+\sqrt{x^2-y^2}}{x}$ is

(a) $\sin^{-1}\left(\frac{y}{x}\right)=2\log|x|+c$ (b) $\sin^{-1}\left(\frac{y}{x}\right)=\log|x|+c$

(c) $\sin\left(\frac{y}{x}\right)=\log|x|+c$ (d) $\sin\left(\frac{y}{x}\right)=2\log|x|+c$

7. The solution of the differential equation $\frac{dy}{dx}=\sec x-y\tan x$ is

(a) $y\sec x+\tan x=c$ (b) $y\sec x=\tan x+c$

(c) $\sec x+y\tan x=c$ (d) $\sec x=y\tan x+c$

8. The particular solution of $\frac{dy}{dx}=xe^{y-x}$, when $x=y=0$ is

(a) $e^{x-y}=x+1$ (b) $e^{x+y}=x+1$

(c) $e^x+e^y=x+1$ (d) $e^{y-x}=x-1$

9. The integrating factor of linear differential equation $x\frac{dy}{dx}+2y=x^2\log x$ is

(a) $\frac{1}{x}$ (b) x (c) $\frac{1}{x^2}$ (d) x^2

10. If the surrounding air is kept at 20 °C and a body cools from 80 °C to 70 °C in 5 minutes, the temperature of the body after 15 minutes will be

(a) 51.7 °C (b) 54.7 °C (c) 52.7 °C (d) 50.7 °C

ANSWERS

1. (a) 2, 3

2. (d) 3, 1

3. (a) circles

4. (a) $x^4\left(\frac{dy}{dx}\right)^2-x\frac{dy}{dx}=y$

5. (c) $ye^x=e^x\cos x+c$

6. (b) $\sin^{-1}\left(\frac{y}{x}\right)=\log|x|+c$

7. (b) $y\sec x=\tan x+c$

8. (a) $e^{x-y}=x+1$

9. (d) x^2

10. (b) 54.7 °C.