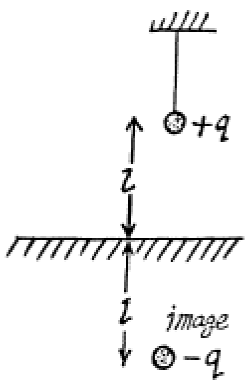
Conductors & Dielectrics In An Electric Field (Part - 1)

Q. 54. A small ball is suspended over an infinite horizontal conducting plane by means of an insulating elastic thread of stiffness k. As soon as the ball was charged, it descended by x cm and its separation Horn the plane became equal to l. Find the charge of the ball.

Solution. 54. When the ball is charged, for the equilibrium of ball, electric force on it must counter balance the excess spring force, exerted, on the ball due to the extension in the spring.



Thus $F_{el} = F_{spr}$

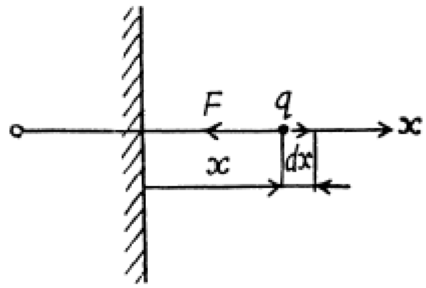
or, $\frac{q^2}{4\pi\varepsilon_0(2l)^2} = \kappa x,$ (The force on the charge q might be considered as arised from attraction by the electrical image)

or, $q = 4l\sqrt{\pi\varepsilon_0 \kappa x},$

Sought charge on the sphere.

Q. 55. A point charge q is located at a distance l from the infinite conducting plane. What amount of work has to be performed in order to slowly remove this charge very far from the plane.

Solution. 55. By definition, the work of this force done upon an elementary displacement dx (Fig.) is given by



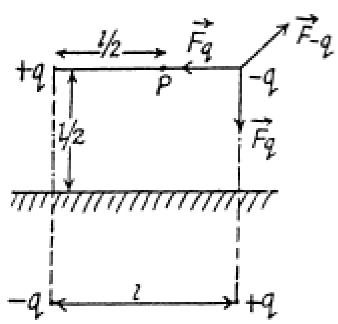
$$dA = F_x\,dx = -\frac{q^2}{4\pi\varepsilon_0 (2x)^2}\,dx,$$

Where the expression for the force is obtained with the help of the image method. Integrating this equation over x between l and ∞, we find

$$A = -\frac{q^2}{16\pi\varepsilon_0}\int_l^{\infty}\frac{dx}{x^2} = -\frac{q^2}{16\pi\varepsilon_0 l}.$$

Q. 56. Two point charges, q and —q, are separated by a distance 1, both being located at a distance l/2 from the infinite conducting plane. Find:
(a) the modulus of the vector of the electric force acting on each charge;
(b) the magnitude of the electric field strength vector at the midpoint between these charges.

Solution. 56. (a) Using the concept of electrical image, it is clear that the magnitude of the force acting on each charge,



$$|\vec{F}| = \sqrt{2}\,\frac{q^2}{4\pi\varepsilon_0 l^2} - \frac{q^2}{4\pi\varepsilon_0 (\sqrt{2}l)^2}$$

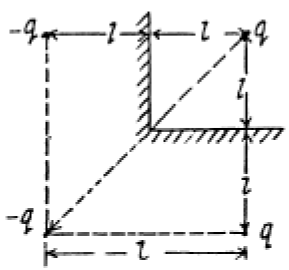
$$= \frac{q^2}{8\pi\varepsilon_0 l^2}(2\sqrt{2}-1)$$

(b) Also, from the figure, magnitude of electrical field strength at P

$$E = 2\left(1 - \frac{1}{5\sqrt{5}}\right)\frac{q}{\pi \varepsilon_0 l^2}$$

Q. 57. A point charge q is located between two mutually perpendicular conducting half-planes. Its distance from each half-plane is equal to l. Find the modulus of the vector of the force acting on the charge.

Solution. 57. 7 Using the concept of electrical image, it is easily seen that the force on the charge q is,

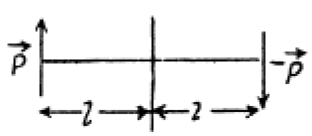


$$F = \frac{\sqrt{2}\, q^2}{4\pi \varepsilon_0 (2l)^2} + \frac{(-q)^2}{4\pi \varepsilon_0 (2\sqrt{2}\, l)^2}$$

$$= \frac{(2\sqrt{2} - 1)\, q^2}{32\pi \varepsilon_0 l^2} \quad \text{(It is attractive)}$$

Q. 58. A point dipole with an electric moment p is located at a distance l from an infinite conducting plane. Find the modulus of the vector of the force acting on the dipole if the vector p is perpendicular to the plane.

Solution. 58. Using the concept of electrical image, force on the dipole $\vec{p}$,



$\vec{F} = p\frac{\partial \vec{E}}{\partial l}$, where $\vec{E}$ is Field at the location of

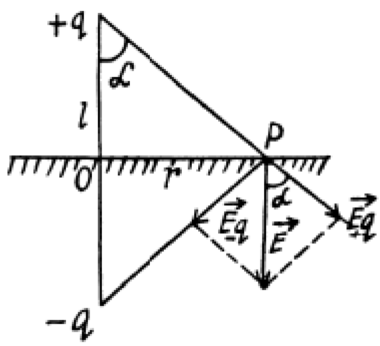
$\vec{p}$ due to $(-\vec{p})$

$$\text{or, } |\vec{F}| = \left|\frac{\partial \vec{E}}{\partial l}\right| p = \frac{3p^2}{32\pi \varepsilon_0 l^4}$$

$$\text{as, } |\vec{E}| = \frac{p}{4\pi \varepsilon_0 (2l)^3}$$

Q. 59. A point charge q is located at a distance l from an infinite conducting plane. Determine the surface density of charges induced on the plane as a function of separation r from the base of the perpendicular drawn to the plane from the charge.

Solution. 59. To find the surface charge density, we must know the electric field at the point P (Fig.) which is at a distance r from the point O.



Using the image mirror method, the field at P,

$$E = 2E\cos\alpha = 2\frac{q}{4\pi\varepsilon_0 x^2}\frac{l}{x} = \frac{q\,l}{2\pi\varepsilon_0(l^2+r^2)^{3/2}}$$

Now from Gauss' theorem the surface charge density on conductor is connected with

the electric field near its surface (in vacuum) through the relation $\sigma = \varepsilon_0 E_n$, where E_n is

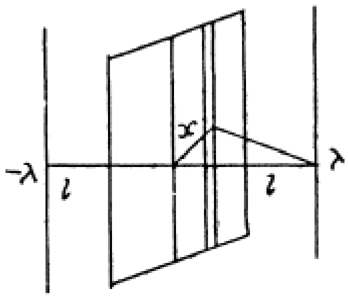
the projection of $\vec{E}$ onto the outward normal $\vec{n}$ (with respect to the conductor).

As our field strength $\vec{E}\uparrow\downarrow\vec{n}$, so

$$\sigma = -\varepsilon_0 E = -\frac{ql}{2\pi(l^2+r^2)^{3/2}}$$

Q. 60. A thin infinitely long thread carrying a charge λ per unit length is oriented parallel to the infinite conducting plane. The distance between the thread and the plane is equal to l. Find:
(a) the modulus of the vector of the force acting on a unit length of the thread;
(b) the distribution of surface charge density σ (x) over the plane, where x is the distance from the plane perpendicular to the conducting surface and passing through the thread.

Solution. 60. (a) The force F_1 on unit length of the thread is given by
$F_1 = \lambda E_1$



Where E_1 is the field at the thread due to image charge:

$$E_1 = \frac{-\lambda}{2\pi\varepsilon_0 (2l)}$$

Thus $$F_1 = \frac{-\lambda^2}{4\pi\varepsilon_0 l}$$

Minus sign means that the force is one of attraction.

(b) There is an image thread with charge density- λ behind the conducting plane. We calculate the electric field on the conductor. It is

$$E(x) = E_n(x) = \frac{\lambda l}{\pi \varepsilon_0 (x^2 + l^2)}$$

On considering the thread and its image.

Thus

$$\sigma(x) = \varepsilon_0 E_n = \frac{\lambda l}{\pi (x^2 + l^2)}$$

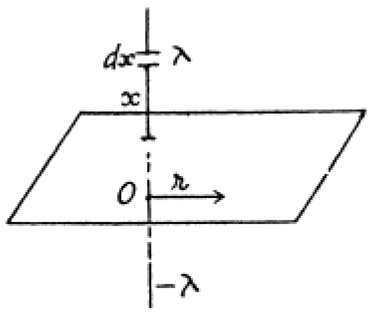
Q. 61. A very long straight thread is oriented at right angles to an infinite conducting plane; its end is separated from the plane by a distance l. The thread carries a uniform charge of linear density λ. Suppose the point 0 is the trace of the thread on the plane. Find the surface density of the induced charge on the plane
(a) at the point 0;
(b) as a function of a distance r from the point 0.

Solution. 61.

(a) At O,

$$E_n(O) = 2\int_l^\infty \frac{\lambda\,dx}{4\pi\varepsilon_0 x^2} = \frac{\lambda}{2\pi\varepsilon_0 l}$$

So $\quad \sigma(O) = \varepsilon_0 E_n = \frac{\lambda}{2\pi l}$



(b) $$E_n(r) = 2\int_l^\infty \frac{\lambda\,dx}{4\pi\varepsilon_0 (x^2+r^2)}\frac{x}{(x^2+r^2)^{1/2}} = \frac{\lambda}{2\pi\varepsilon_0}\int_l^\infty \frac{x\,dx}{(x^2+r^2)^{3/2}}$$

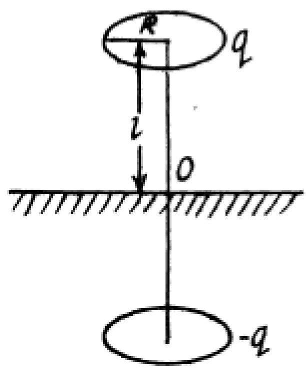
$$= \frac{\lambda}{4\pi\varepsilon_0}\int_{l^2+r^2}^\infty \frac{dy}{y^{3/2}}, \text{ on putting } y = x^2+r^2,$$

$$= \frac{\lambda}{2\pi\varepsilon_0\sqrt{l^2+r^2}}$$

Hence $\sigma(r) = \varepsilon_0 E_n = \frac{\lambda}{2\pi\sqrt{l^2+r^2}}$

Q. 62. A thin wire ring of radius R carries a charge q. The ring is oriented parallel to an infinite conducting plane and is separated by a distance l from it. Find:
(a) the surface charge density at the point of the plane symmetrical with respect to the ring;
(b) the strength and the potential of the electric field at the centre of the ring.

Solution. 62. It can be easily seen that in accordance with the image method, a charge -q must be located on a similar ring but on the other side of the conducting plane. (Fig.) at the same perpendicular distance. From the solution of 3.9 net electric field at O,



$$\vec{E} = 2 \frac{ql}{4 \pi \varepsilon_0 (R^2 + l^2)^{3/2}} (-\vec{n}) \text{ where } \vec{n} \text{ is}$$

Outward normal with respect to the conducting plane.

Now $\quad E_n = \frac{\sigma}{\varepsilon_0}$

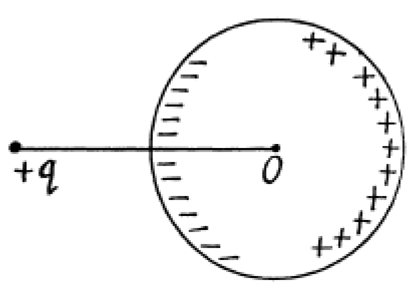
Hence $\quad \sigma = \frac{-ql}{2 \pi (R^2 + l^2)^{3/2}}$

Where minus sign indicates that the induced charge is opposite in sign to that of chaige q > 0.

Q. 63. Find the potential φ of an uncharged conducting sphere outside of which a point charge q is located at a distance l from the sphere's centre.

Solution. 63. Potential φ is the same for all the points of the sphere. Thus we calculate its value at the centre O of the sphere. Thus we can calculate its value at the centre O of the sphere, because only for this point, it can be calculated in the most simple way.

$$\varphi = \frac{1}{4 \pi \varepsilon_0} \frac{q}{l} + \varphi' \qquad (1)$$

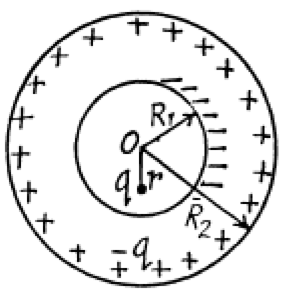


where the first term is the potential of the charge q, while the second is the potential due to the charges induced on the surface of the sphere. But since all induced charges are at the same distance equal to the radius of the circle from the point C and the total induced charge is equal to zero, φ' = 0, as well. Thus equation (1) is reduced to the

form, $\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}$

Q. 64. A point charge q is located at a distance r from the centre O of an uncharged conducting spherical layer whose inside and outside radii are equal to R_1 and R_2respectively. Find the potential at the point O if r < R_1.

Solution. 64. As the sphere has conducting layers, charge -q is induced on the inner surface of the sphere q and consequently chaige + q is induced on the outer layer as the sphere as a whole is uncharged.



Hence, the potential at O is given by,

$$\varphi_0 = \frac{q}{4\pi\varepsilon_0 r} + \frac{(-q)}{4\pi\varepsilon_0 R_1} + \frac{q}{4\pi\varepsilon_0 R_2}$$

It should be noticed that the potential can be found in such a sim ple way only at O, since all the induced charges are at the same distance from this point, and their distribution, (which is unknown to us), does not play any role.

Q. 65. A system consists of two concentric conducting spheres, with the inside sphere of radius a carrying a positive charge q_1. What charge q_2 has to be deposited on the outside sphere of radius b to reduce the potential of the inside sphere to zero? How does the potential φ depend in this case on a distance r from the centre of the system? Draw the approximate plot of this dependence.

Solution. 65. Potential at the inside sphere,

$$\varphi_a = \frac{q_1}{4\pi\varepsilon_0 a} + \frac{q_2}{4\pi\varepsilon_0 b}$$

Obviously $\varphi_a = 0 \text{ for } q_2 = -\frac{b}{a} q_1$ (1)

When $r \geq b$,

$$\varphi_r = \frac{q_1}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0 r} = \frac{q_1}{4\pi\varepsilon_0}\left(1 - \frac{b}{a}\right) \Big/ r, \text{ using Eq. (1).}$$

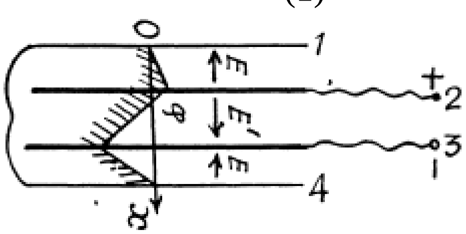
And when $r \leq b$ $\varphi_r = \frac{q_1}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0 b} = \frac{q_1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{a}\right)$

Q. 66. Four large metal plates are located at a small distance d from one another as shown in Fig. 3.8. The extreme plates are inter-connected by means of a conductor while a potential difference Δφ is applied to internal plates. Find:
(a) the values of the electric field strength between neighbouring plates;
(b) the total charge per unit area of each plate.

Solution. 66. (a) As the metallic plates 1 and 4 are isolated and connected by means of a conductor, $\varphi_1 = \varphi_4$. Plates 2 and 3 have the same amount of positive and negative charges and due to induction, plates 1 and 4 are respectively negatively and positively charged and in addition to it all the four plates are located a small but at equal distance d relative to each other, the magnitude of electric field strength between 1 - 2 and 3 - 4 are both equal in magnitude and direction (say $\vec{E}$). Let $\vec{E'}$ be the field strength between the plates 2 and 3 which is directed form 2 to 3. Hence $\vec{E'} \uparrow\downarrow \vec{E}$ (Fig.).

According to the problem

$E' d = \Delta\varphi = \varphi_2 - \varphi_3$ (1)



In addition to

$$\varphi_1 - \varphi_4 = 0 = (\varphi_1 - \varphi_2) + (\varphi_2 - \varphi_3) + (\varphi_3 - \varphi_4)$$

or, $$0 = -Ed + \Delta\varphi - Ed$$

or, $$\Delta\varphi = 2Ed \text{ or } E = \frac{\Delta\varphi}{2d}$$

Hence $$E = \frac{E'}{2} = \frac{\Delta\varphi}{2d} \quad (2)$$

(b) Since E α σ, we can state that according to equation (2) for part (a) the charge on the p4te 2 is divided into two parts; such that 1/3 rd of it lies on the upper side and 2/3 rd on its lower face.

Thus charge density of upper face of plate 2 or of plate 1 or plate 4 and lower face of

$$3\sigma = \varepsilon_0 E = \frac{\varepsilon_0 \Delta\varphi}{2d}$$ and charge density of lower face of 2 or upper face of 3

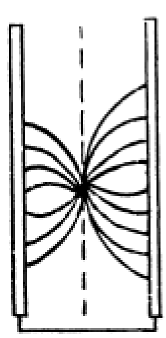
$$\sigma' = \varepsilon_0 E' = \varepsilon_0 \frac{\Delta\varphi}{d}$$

Hence the net charge density of plate 2 or 3 becomes $\sigma + \sigma' = \frac{3\varepsilon_0 \Delta\varphi}{2d}$, which is obvious from the argument.

Q. 67. Two infinite conducting plates 1 and 2 are separated by a distance l. A point charge q is located between the plates at a distance x from plate 1. Find the charges induced on each plate.

Solution. 67. The problem of point charge between two conducting planes is more easily tackled (if we want only the total charge induced on the planes) if we replace the point charge by a uniformly charged plane sheet.

Let σ be the charge density on this sheet and E_1, E_2 outward electric field on the two sides of this sheet.



Then $E_1 + E_2 = \frac{\sigma}{\varepsilon_0}$

The conducting planes will be assumed to be grounded. Then $E_1 x = E_2 (l - x)$.

Hence $E_1 = \frac{\sigma}{l \varepsilon_0}(l - x),\ E_2 = \frac{\sigma}{l \varepsilon_0} x$

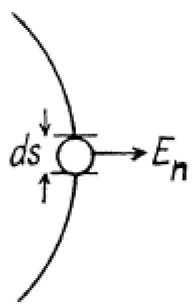
This means that the induced charge density on the plane conductors are

$\sigma_1 = -\frac{\sigma}{l}(l - x),\ \sigma_2 = -\frac{\sigma}{l} x$

Hence $q_1 = -\frac{q}{l}(l - x),\ q_2 = -\frac{q}{l} x$

Q. 68. Find the electric force experienced by a charge reduced to a unit area of an arbitrary conductor if the surface density of the charge equals σ.

Solution. 68. Near the conductor $E = E_n = \frac{\sigma}{\varepsilon_0}$



This field can be written as the sum of two parts E_1 and E_2 E_1 is the electric field due to an infinitesimal area dS.

Very near it $E_1 = \pm \frac{\sigma}{2 \varepsilon_0}$

The remaining part contributes $E_2 = \frac{\sigma}{2 \varepsilon_0}$ both sides. In calculating the force on the

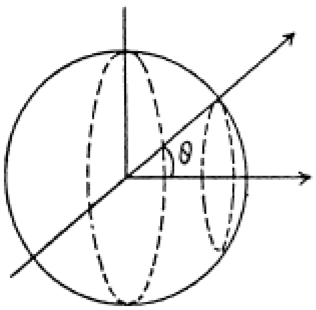
element dS we drop E_1 (because it is a self-force.) Thus

$\frac{dF}{dS} = \sigma \cdot \frac{\sigma}{2 \varepsilon_0} = \frac{\sigma^2}{2 \varepsilon_0}$

Q. 69. A metal ball of radius R = 1.5 cm has a charge q = 10 μC. Find the modulus of the vector of the resultant force acting on a charge located on one half of the

ball.

Solution. 69. The total force on the hemisphere is



$$F = \int_0^{\pi/2} \frac{\sigma^2}{2\varepsilon_0} \cdot \cos\theta \cdot 2\pi R \sin\theta\, R\, d\theta$$

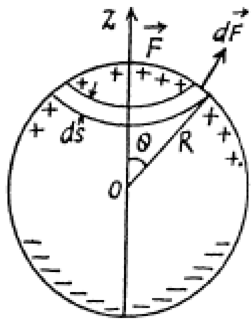
$$= \frac{2\pi R^2 \sigma^2}{2\varepsilon_0} \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta$$

$$= \frac{2\pi R^2}{2\varepsilon_0} \times \frac{1}{2} \times \left(\frac{q}{4\pi R^2}\right)^2 = \frac{q^2}{32\pi\varepsilon_0 R}.$$

Q. 70. When an uncharged conducting ball of radius R is placed in an external uniform electric field, a surface charge density $\sigma = \sigma_0 \cos\theta$ is induced on the ball's surface (here ao is a constant, is a polar angle). Find the magnitude of the resultant electric force acting on an induced charge of the same sign.

Solution. 70. We know that the force acting on the area element dS of a conductor is,

$$d\vec{F} = \frac{1}{2}\sigma \vec{E}\, dS \quad (1)$$



It follows from symmetry considerations that $\vec{F}$ is directed along the z-axis, and hence it can be represented as the sum (integral) of the projection of elementary forces (1) onto the z-axis :

$dF_z = dF \cos\theta$ (2)

For simplicity let us consider an element area $dS = 2\pi R \sin\theta \, R \, d\theta$ (Fig.) Now

considering that $E = \sigma/\varepsilon_0$, Equation (2) takes the from

$$dF_z = \frac{\pi \sigma^2 R^2}{\varepsilon_0} \sin\theta \cos\theta \, d\theta$$

$$= -\left(\frac{\pi \sigma_0^2 R^2}{\varepsilon_0}\right) \cos^3\theta \, d\cos\theta$$

Integrating this expression over the half sphere (i.e. with respect to cos 0 between 1 and 0), we obtain

$$F = F_z = \frac{\pi \sigma_0^2 R^2}{4\varepsilon_0}$$

Conductors & Dielectrics In An Electric Field (Part - 2)

Q. 71. An electric field of strength E = 1.0 kV/cm produces polarization in water equivalent to the correct orientation of only one out of N molecules. Find N. The electric moment of a water molecule equals p = 0.62-10^{-29} C•m.

Solution. 71. The total polarization is $P = (\varepsilon - 1)\varepsilon_0 E$. This must equals $\frac{n_0 p}{N}$ here n_0 is the concertation of water molecules. Thus

$$N = \frac{n_0 p}{(\varepsilon - 1)\varepsilon_0 E} = 2{\cdot}93 \times 10^3 \text{ on putting the values}$$

Q. 72. A non-polar molecule with polarizability β is located at a great distance l from a polar molecule with electric moment p. Find the magnitude of the interaction force between the molecules if the vector p is oriented along a straight line passing through both molecules.

Solution. 72. From the general formula

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{3\vec{p}\cdot\vec{r}\vec{r} - \vec{p}r^2}{r^5}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\vec{p}}{l^3}, \text{ where } r = l \text{ and } \vec{r} \uparrow\uparrow \vec{p}$$

This will cause the induction of a dipole moment.

$$\vec{p}_{ind} = \beta \frac{1}{4\pi\varepsilon_0}\frac{2\vec{p}}{l^3} \times \varepsilon_0$$

Thus the force,

$$\vec{F} = \frac{\beta}{4\pi}\frac{2p}{l^3}\frac{\partial}{\partial l}\frac{1}{4\pi\varepsilon_0}\frac{2p}{l^3} = \frac{3\beta p^2}{4\pi^2\varepsilon_0 l^7}$$

Q. 73. A non-polar molecule is located at the axis of a thin uniformly charged ring of radius R. At what distance x from the ring's centre is the magnitude of the force F acting on the given molecule (a) equal to zero; (b) maximum?
Draw the approximate plot F_x (x).

Solution. 73. The electric field E at distance x from the centre of the ring is,

$$E(x) = \frac{qx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}.$$

The induced dipole moment is $p = \beta\varepsilon_0 E = \frac{q\beta x}{4\pi(R^2 + x^2)^{3/2}}$

The force on this molecule is

$$F = p\frac{\partial}{\partial x}E = \frac{q\beta x}{4\pi(R^2 + x^2)^{3/2}}\frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\frac{x}{(R^2 + x^2)^{3/2}} = \frac{q^2\beta}{16\pi^2\varepsilon_0}\frac{x(R^2 - 2x^2)}{(R^2 + x^2)^4}$$

This vanishes for $x = \frac{\pm R}{\sqrt{2}}$ (apart from $x = 0,\ x = \infty$)

It is maximum when

$$\frac{\partial}{\partial x}\frac{x(R^2 - x^2 \times 2)}{(R^2 + x^2)^4} = 0$$

or, $(R^2 - 2x^2)(R^2 + x^2) - 4x^2(R^2 + x^2) - 8x^2(R^2 - 2x^2) = 0$

or, $R^4 - 13x^2R^2 + 10x^4 = 0$ or, $x^2 = \frac{R^2}{20}\left(13 \pm \sqrt{129}\right)$

or, $x = \frac{R}{\sqrt{20}}\sqrt{13 \pm \sqrt{129}}$ (On either side), Plot of $F_x(x)$ is as shown in the answer sheet.

Q. 74. A point charge q is located at the centre of a ball made of uniform isotropic dielectric with permittivity ε. Find the polarization P as a function of the radius vector r relative to the centre of the system, as well as the charge q' inside a sphere whose radius is less than the radius of the ball.

Solution. 74. inside the ball

$$\vec{D}(\vec{r}) = \frac{q}{4\pi}\frac{\vec{r}}{r^3} = \varepsilon\varepsilon_0\vec{E}.$$

Also $\varepsilon_0\vec{E} + \vec{P} = \vec{D}$ or $\vec{P} = \frac{\varepsilon - 1}{\varepsilon}\vec{D} = \frac{\varepsilon - 1}{\varepsilon}\frac{q}{4\pi}\frac{\vec{r}}{r^3}$

$$q' = -\oint \vec{P} \cdot d\vec{S} = -\frac{\varepsilon - 1}{\varepsilon} \frac{q}{4\pi} \int d\Omega = -\frac{\varepsilon - 1}{\varepsilon} q$$

Also,

Q. 75. Demonstrate that at a dielectric-conductor interface the surface density of the dielectric's bound charge σ' = -σ (ε — 1)/ε, where a is the permittivity, a is the surface density of the charge on the conductor.

Solution. 75.

$$D_{diel} = \varepsilon \varepsilon_0 E_{diel} = D_{conductor} = \sigma \quad \text{or,} \quad E_{diel} = \frac{\sigma}{\varepsilon \varepsilon_0}$$

$$P_n = (\varepsilon - 1) \varepsilon_0 E_{diel} = \frac{\varepsilon - 1}{\varepsilon} \sigma$$

$$\sigma' = -P_n = -\frac{\varepsilon - 1}{\varepsilon} \sigma$$

This is the surface density of bound charges.

Q. 76. A conductor of arbitrary shape, carrying a charge q, is surrounded with uniform dielectric of permittivity 8 (Fig. 3.9). Find the total bound charges at the inner and outer surfaces of the dielectric.

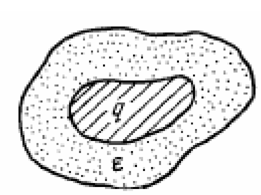


Fig. 3.9.

Solution. 76. From the solution of the previous problem q'in = charge on the interior

surface of the conductor

$$= -(\varepsilon - 1)/\varepsilon \int \sigma \, dS = -\frac{\varepsilon - 1}{\varepsilon} q$$

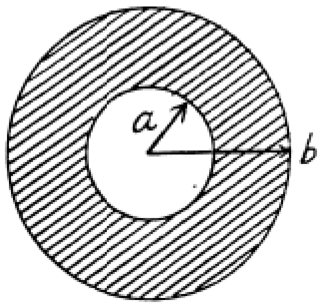
Since the dielectric as a whole is neutral there must be a total charge equal to

$q'_{outer} = +\frac{\varepsilon - 1}{\varepsilon} q$ On the outer surface of the dielectric.

Q. 77. A uniform isotropic dielectric is shaped as a spherical layer with radii a and b. Draw the approximate plots of the electric field strength E and the potential φ vs the distance r from the centre of the layer if the dielectric has a certain positive

extraneous charge distributed uniformly:
(a) over the internal surface of the layer;
(b) over the volume of the layer.

Solution. 77. (a) Positive extraneous charge is distributed uniformly over the internal surface layer. Let σ_0 be the surface density of the charge.



Clearly, $E = 0$, for $r < a$

For $a < r$

$$\varepsilon_0 E \times 4\pi r^2 = 4\pi a^2 \sigma_0 \text{ by Gauss theorem.}$$

or, $$E = \frac{\sigma_0}{\varepsilon_0 \varepsilon}\left(\frac{a}{r}\right)^2, \quad a < r < b$$

For $r > b$, similarly

$$E = \frac{\sigma_0}{\varepsilon_0}\left(\frac{a}{r}\right)^2, \quad r > b$$

Now, $$E = -\frac{\partial \varphi}{\partial r}.$$

So by integration from infinity where $\varphi(\infty) = 0$,

$$\varphi = \frac{\sigma_0 a^2}{\varepsilon_0 r} \quad r > b$$

$$a < r < b \quad \varphi = \frac{\sigma_0 a^2}{\varepsilon \varepsilon r} + B, \; B \text{ is a constant}$$

or by continuity, $\varphi = \frac{\sigma_0 a^2}{\varepsilon_0 \varepsilon}\left(\frac{1}{r} - \frac{1}{b}\right) + \frac{\sigma_0 a^2}{\varepsilon_0 b}, a < r < b$

For $r < a$. $\varphi = A =$ Constant

By continuity, $\varphi = \frac{\sigma_0 a^2}{\varepsilon_0 \varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{\sigma_0 a^2}{\varepsilon_0 b}$

(b) Positive extraneous charge is distributed uniformly over the internal volume of the dielectric

Let p_0 = Volume density of the charge in the dielectric, for a < r < b.

$E = 0, \ r < a$

$$\varepsilon_0 \varepsilon 4 \pi r^2 E = \frac{4\pi}{3}(r^3 - a^3)\rho_0, (a < r < b)$$

Or, $E = \frac{\rho_0}{3 \varepsilon_0 \varepsilon}\left(r - \frac{a^3}{r^2}\right)$

$$E = \frac{4\pi}{3}(b^3 - a^3)\rho_0 \Big/ \varepsilon_0 4 \pi r^2, \ r > b$$

Or, $E = \frac{(b^3 - a^3)\rho_0}{3 \varepsilon_0 r^2}$ for $r > b$

By integration,

$$\varphi = \frac{(b^3 - a^3)\rho_0}{3 \varepsilon_0 r} \quad \text{for } r > b$$

Or, $\varphi = B - \frac{\rho_0}{3 \varepsilon_0 \varepsilon}\left(\frac{r^2}{2} + \frac{a^3}{r}\right), \ a < r < b$

By continuity $\frac{b^3 - a^3}{3 \varepsilon_0 b}\rho_0 = B - \frac{\rho_0}{3 \varepsilon_0 \varepsilon}\left(\frac{b^2}{2} + \frac{a^3}{b}\right)$

Or, $B = \frac{\rho_0}{3 \varepsilon_0 \varepsilon}\left\{\frac{\varepsilon (b^3 - a^3)}{b} + \left(\frac{b^2}{2} + \frac{a^3}{b}\right)\right\}$

$$\varphi = B - \frac{\rho_0}{3\,\varepsilon_0\,\varepsilon}\left(\frac{a^2}{2} + a^2\right) = B - \frac{\rho_0\, a^2}{2\,\varepsilon_0\,\varepsilon},\ r < a$$

On the basis of obtained expressions E (r) and (φ) (r) can be plotted as shown in the answer-sheet.

Q. 78. Near the point A (Fig. 3.10) lying on the boundary between glass and vacuum the electric field strength in vacuum is equal to E_0 = 10.0 V/m, the angle between the vector E_0 and the normal n of the boundary line being equal to α_0 = 30°. Find the field strength E in glass near the point A, the angle α between the vector E and n, as well as the surface density of the bound charges at the point A.

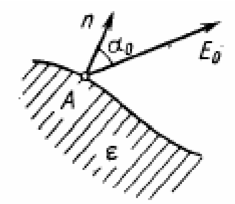
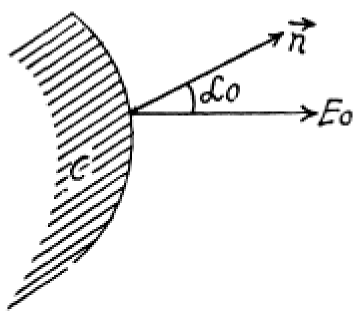


Fig. 3.10.

Solution. 78. Let the field in the dielectric be $\vec{E}$ making an angle α with $\vec{n}$. Then we have the boundary conditions,



$$E_0 \cos \alpha_0 = \varepsilon E \cos \alpha \ \text{ and } \ E_0 \sin c'_0 = E \sin \alpha$$

$$\text{So } \ E = E_0 \sqrt{\sin^2 \alpha_0 + \frac{1}{\varepsilon^2} \cos^2 \alpha_0} \ \text{ and } \ \tan \alpha = \varepsilon \tan \alpha_0$$

In the dielectric the normal component of the induction vector is

$$D_n = \varepsilon_0\, \varepsilon\, E_n = \varepsilon_0\, \varepsilon\, E \cos \alpha = \varepsilon_0 E_0 \cos \alpha_0$$

$$\sigma' = P_n = D_n - \varepsilon_0 E_n = \left(1 - \frac{1}{\varepsilon}\right) \varepsilon_0 E_0 \cos \alpha_0$$

$$\text{or,} \qquad \sigma' = \frac{\varepsilon - 1}{\varepsilon}\, \varepsilon_0 E_0 \cos \alpha_0$$

Q. 79. Near the plane surface of a uniform isotropic dielectric with permittivity ε the electric field strength in vacuum is equal to E_0, the vector E_0 forming an angle θ with the normal of the dielectric's surface (Fig. 3.11). Assuming the field to be uniform both inside and outside the dielectric, find:
(a) the flux of the vector E through a sphere of radius R with centre located at the surface of the dielectric;
(b) the circulation of the vector D around the closed path Γ of length I (see Fig. 3.11) whose plane is perpendicular to the surface of the dielectric and parallel to the vector E_0.

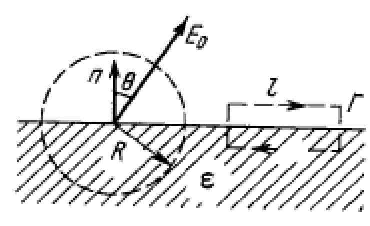
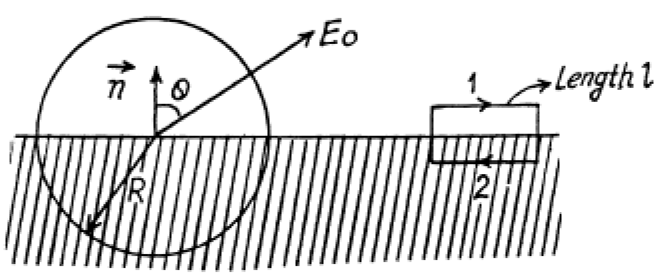


Fig. 3.11.

Solution. 79. From the previous problem, $\sigma' = \varepsilon_0 \frac{\varepsilon - 1}{\varepsilon} E_0 \cos\theta$



(a) Then $\oint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} Q = \pi R^2 E_0 \cos\theta \frac{\varepsilon - 1}{\varepsilon}$

(b) $\oint \vec{D} \cdot d\vec{l} = (D_{1t} - D_{2t})\, l = (\varepsilon_0 E_0 \sin\theta - \varepsilon \varepsilon_0 E_0 \sin\theta) = -(\varepsilon - 1)\, \varepsilon_0 E_0\, l \sin\theta$

Q. 80. An infinite plane of uniform dielectric with permittivity ε is uniformly charged with extraneous charge of space density p. The thickness of the plate is equal to 2d. Find:
(a) the magnitude of the electric field strength and the potential as functions of distance l from the middle point of the plane (where the potential is assumed to be equal to zero); having chosen the x coordinate axis perpendicular to the plate,

draw the approximate plots of the projection $E_x(x)$ of the vector E and the potential φ(x);
(b) the surface and space densities of the bound charge.

Solution. 80. (a) $\text{div}\vec{D} = \frac{\partial D_x}{\partial x} = \rho$ and $D = \rho\, l$

$E_x = \frac{\rho\, l}{\varepsilon\, \varepsilon_0}$, $l < d$ and $E_x = \frac{\rho\, d}{\varepsilon_0}$ constant for $l > d$

$\varphi(x) = -\frac{\rho\, l^2}{2\varepsilon\, \varepsilon_0}$, $l < d$ and $\varphi(x) = A - \frac{\rho\, ld}{\varepsilon_0}$, $l > d$ then $\varphi(x) = \frac{\rho\, d}{\varepsilon_0}\left(d - \frac{d}{2\varepsilon} - l\right)$,

by continuity.

On the basis of obtained expressions Ex (x) and qp (x) can be plotted as shown in the

figure of answer sheet.

(b) $\rho' = -\text{div}\ \vec{P} = -\text{div}\ (\varepsilon - 1)\,\varepsilon_0 \vec{E} = -\rho\frac{(\varepsilon - 1)}{\varepsilon}$

$\sigma' = P_{1n} - P_{2n}$, where n is the normal from 1 to 2.

$= P_{1n}$, ($\vec{P}_2 = 0$ as 2 is vacuum.)

$= (\rho d - \rho\, d/\varepsilon) = \rho\, d\frac{\varepsilon - 1}{\varepsilon}$

Q. 81. Extraneous charges are uniformly distributed with space density p > 0 over a ball of radius R made of uniform isotropic dielectric with permittivity a. Find:
(a) the magnitude of the electric field strength as a function of distance r from the centre of the ball; draw the approximate plots E (r) and φ (r);
(b) the space and surface densities of the bound charges.

Solution. 81.

$\text{div}\vec{D} = \frac{l}{r^2}\frac{\partial}{\partial r} r^2 D_r = \rho$

$r^2 D_r = \rho\frac{r^3}{3} + A$ $D_r = \frac{1}{3}\rho\, r + \frac{A}{r^2}$, $r < R$

$A = 0$ as $D_r \neq \infty$ at $r = 0$,Thus, $E_r = \frac{\rho r}{3\,\varepsilon\,\varepsilon_0}$

For $r > R,\ D_r = \frac{B}{r^2}$

By continuity of D_r at $r = R$; $B = \frac{\rho R^3}{3}$

so, $E_r = \frac{\rho R^3}{3\,\varepsilon_0\,r^2},\ r > R$

$$\varphi = \frac{\rho R^3}{3\,\varepsilon_0\,r},\ r > R \text{ and } \varphi = -\frac{\rho r^2}{6\,\varepsilon\,\varepsilon_0} + C,\ r < R$$

$$C = +\frac{\rho R_s^2}{3\,\varepsilon_0} + \frac{\rho R^2}{6\,\varepsilon\,\varepsilon_0}, \text{ by continuity of } \varphi.$$

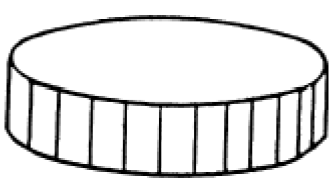
See answer sheet for graphs of E (r) and φ (r)

$$\text{(b)}\quad \rho' = \text{div}\,\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left\{\frac{r^3}{3}\rho\left(1 - \frac{1}{\varepsilon}\right)\right\} = -\frac{\rho\,(\varepsilon - 1)}{\varepsilon}$$

$$\sigma' = P_{1r} - P_{2r} = P_{1r} = \frac{1}{3}\rho R\left(1 - \frac{1}{\varepsilon}\right)$$

Q. 82. A round dielectric disc of radius R and thickness d is statically polarized so that it gains the uniform polarization P, with the vector P lying in the plane of the disc. Find the strength E of the electric field at the centre of the disc if d ≪ R.

Solution. 82. Because there is a discontinuity in polarization at the boundary of the dielectric disc, a bound surface charge appears, which is the source of the electric Held inside and outside the disc.



We have for the electric field at the origin.

$$\vec{E} = -\int \frac{\sigma'\,dS}{4\pi\,\varepsilon_0\,r^3}\vec{r},$$

where $\vec{r} =$ radius vector to the origin from the element dS.

$\sigma' = P_n = P \cos\theta$ on the curved surface

($P_n = 0$ on the flat surface.)

Here θ = angle between $\vec{r}$ and $\vec{P}$

By symmetry, $\vec{E}$ will be parallel to $\vec{P}$. Thus

$$E = -\int_0^{2\pi} \frac{P \cos\theta R \, d\theta \cdot \cos\theta}{4 \pi \varepsilon_0 R^2} \cdot d$$

where, $r = R$ if $d << R$.

So, $E = -\frac{Pd}{4 \varepsilon_0 R}$ and $\vec{E} = -\frac{\vec{P} d}{4 \varepsilon_0 R}$

Q. 83. Under certain conditions the polarization of an infinite uncharged dielectric plate takes the form P = P_0 (1 — x^2/d^2), where P_0 is a vector perpendicular to the plate, x is the distance from the middle of the plate, d is its half-thickness. Find the strength E of the electric field inside the plate and the potential difference between its surfaces.

Solution. 83. Since there are no free extraneous charges anywhere

$\text{div}\, \vec{D} = \frac{\partial D_x}{\partial x} = 0$ or, D_x = Constant

But $D_x = 0$ at ∞, so, $D_x = 0$, every where.

Thus, $\vec{E} = -\frac{\vec{P}_0}{\varepsilon_0}\left(1 - \frac{x^2}{d^2}\right)$ or, $E_x = -\frac{P_0}{\varepsilon_0}\left(1 - \frac{x^2}{d^2}\right)$

So, $\varphi = \frac{P_0 x}{\varepsilon_0} - \frac{P_0 x^3}{3 \varepsilon_0 d^2}$ + constant

Hence,

$$\varphi(+d) - \varphi(-d) = \frac{2 P_0 d}{\varepsilon_0} - \frac{2 P_0 d^3}{3 d^2 \varepsilon_0} = \frac{4 P_0 d}{3 \varepsilon_0}$$

Q. 84. Initially the space between the plates of the capacitor is filled with air, and

the field strength in the gap is equal to E_0. Then half the gap is filled with uniform isotropic dielectric with permittivity a as shown in Fig. 3.12. Find the moduli of the vectors E and D in both parts of the gap (1 and 2) if the introduction of the dielectric
(a) does not change the voltage across the plates;
(b) leaves the charges at the plates constant.


Fig. 3.12.

Solution. 84.

(a) We have $D_1 = D_2$, or, $\varepsilon E_2 = E_1$

Also, $$E_1 \frac{d}{2} + E_2 \frac{d}{2} = E_0 d \text{ or, } E_1 + E_2 = 2E_0$$

Hence, $$E_2 = \frac{2E_0}{\varepsilon + 1} \text{ and } E_1 = \frac{2\varepsilon E_0}{\varepsilon + 1} \text{ and } D_1 = D_2 = \frac{2\varepsilon\varepsilon_0 E_0}{\varepsilon + 1}$$

(b) $D_1 = D_2$, or, $\varepsilon E_2 = E_1 = \frac{\sigma}{\varepsilon_0} = E_0$

Thus, $$E_1 = E_0,\ E_2 = \frac{E_0}{\varepsilon} \text{ and } D_1 = D_2 = \varepsilon_0 E_0$$

Q. 85. Solve the foregoing problem for the case when half the gap is filled with the dielectric in the way shown in Fig. 3.13.

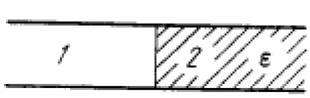


Fig. 3.13.

Solution. 85. (a) Constant voltage across the plates;
$E_1 = E_2 = E_0$, $D_1 = \varepsilon_0 E_0$, $D_2 = \varepsilon_0 \varepsilon E_0$.
(b) Constant charge across the plates;
$E_1 = E_2$, $D_1 = \varepsilon_0 E_1$, $D_2 = \varepsilon\varepsilon_0 E_2 = \varepsilon D_1$

$$E_1(1 + \varepsilon) = 2E_0 \quad \text{or} \quad E_1 = E_2 = \frac{2E_0}{\varepsilon + 1}$$

Conductors & Dielectrics In An Electric Field (Part - 3)

Q. 86. Half the space between two concentric electrodes of a spherical capacitor is filled, as shown in Fig. 3.14, with uniform isotropic dielectric with permittivity ε. The charge of the capacitor is q. Find the magnitude of the electric field strength between the electrodes as a function of distance r from the curvature centre of the electrodes.

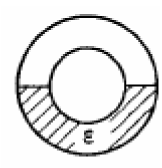
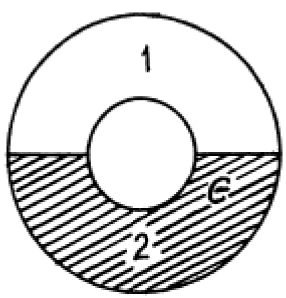


Fig. 3.14.

Solution. 86. At the interface of the dielectric and vacuum,



$$E_{1t} = E_{2t}$$

The electric field must be radial and

$$E_1 = E_2 = \frac{A}{\varepsilon_0 \varepsilon r^2}, \quad a < r < b$$

$$\text{Now,} \quad q = \frac{A}{R^2}(2\pi R^2) + \frac{A}{\varepsilon R^2}(2\pi R^2)$$

$$= A\left(1 + \frac{1}{\varepsilon}\right) 2\pi$$

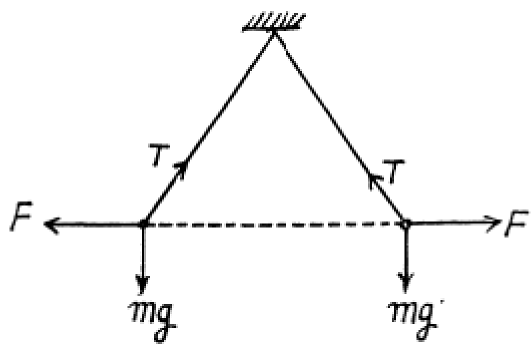
$$\text{or,} \quad E_1 = E_2 = \frac{q}{2\pi \varepsilon_0 r^2 (1 + \varepsilon)}$$

Q. 87. Two small identical balls carrying the charges of the same sign are suspended from the same point by insulating threads of equal length. When the

surrounding space was filled with kerosene the divergence angle between the threads remained constant. What is the density of the material of which the balls are made?

Solution. 87. In air the forces are as shown. In K-oil,

$F \to F' = F/\varepsilon$ and $mg \to mg\left(1 - \frac{\rho_0}{\rho}\right)$.



Since the inclinations do not change

$$\frac{1}{\varepsilon} = 1 - \frac{\rho_0}{\rho}$$

Or, $\frac{\rho_0}{\rho} = 1 - \frac{1}{\varepsilon} = \frac{\varepsilon - 1}{\varepsilon}$

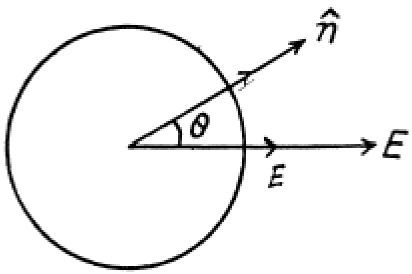
Or, $\rho = \rho_0 \frac{\varepsilon}{\varepsilon - 1}$

Where p_0 is the density of K-oil and p that of the material of which the balls are made.

Q. 88. A uniform electric field of strength E = 100 V/m is generated inside a ball made of uniform isotropic dielectric with permittivity ε = 5.00. The radius of the ball is R = 3.0 cm. Find the maximum surface density of the bound charges and the total bound charge of one sign.

Solution. 88. Within the ball the electric field can be resolved into normal and tangential components.

$$E_n = E\cos\theta, E_t = E\sin\theta$$



Then, $D_n = \varepsilon \varepsilon_0 E \cos\theta$

and $P_n = (\varepsilon - 1)\ \varepsilon_0 E \cos\theta$

or, $\sigma' = (\varepsilon - 1)\varepsilon_0 E \cos\theta$

so, $\sigma_{max} = (\varepsilon - 1)\varepsilon_0 E,$

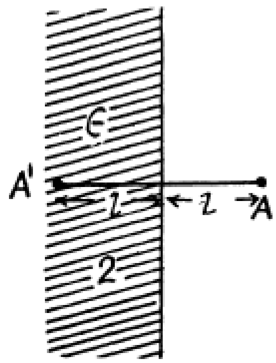
and total charge of one sign,

$$q' = \int_0^1 (\varepsilon - 1)\varepsilon_0 E \cos\theta\, 2\pi R^2\, d(\cos\theta) = \pi R^2 \varepsilon_0 (\varepsilon - 1) E$$

(Since we are interested in the total charge of one sign we must integrate cos θ from 0 to 1 only).

Q. 89. A point charge q is located in vacuum at a distance l from the plane surface of a uniform isotropic dielectric filling up all the half-space. The permittivity of the dielectric equals a. Find:
(a) the surface density of the bound charges as a function of distance r from the point charge q; analyse the obtained result at 1→0;
(b) the total bound charge on the surface of the dielectric.

Solution. 89. The chaige is at A in the medium 1 and has an image point at A’ in the medium 2. The electric field in the medium 1 is due to the actual charge q at A and the image charge q' at A'. The electric field in 2 is due to a corrected charge c [at A. Thus on the boundary between 1 and 2,



$$E_{1n} = \frac{q'}{4\pi\varepsilon_0 r^2}\cos\theta - \frac{q}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$E_{2n} = \frac{-q''}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$E_{1t} = \frac{q'}{4\pi\varepsilon_0 r^2}\sin\theta + \frac{q}{4\pi\varepsilon_0 r^2}\sin\theta$$

$$E_{2t} = \frac{q''}{4\pi\varepsilon_0 r^2}\sin\theta$$

The boundary conditions are

$$D_{1n} = D_{2n} \text{ and } E_{1t} = E_{2t}$$

$$\varepsilon q'' = q - q'$$

$$q'' = q + q'$$

So, $q'' = \frac{2q}{\varepsilon + 1},\ q' = -\frac{\varepsilon - 1}{\varepsilon + 1}q$

(a) The surface density of the bound charge on the surface of the dielectric

$$\sigma' = P_{2n} = D_{2n} - \varepsilon_0 E_{2n} = (\varepsilon - 1)\varepsilon_0 E_{2n}$$

$$= -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{q}{2\pi r^2}\cos\theta = -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{ql}{2\pi r^3}$$

(b) Total bound charge is, $-\frac{\varepsilon - 1}{\varepsilon + 1}q\int_0^\infty \frac{l}{2\pi(l^2 + x^2)^{3/2}}2\pi x\,dx = -\frac{\varepsilon - 1}{\varepsilon + 1}q$

Q. 90. Making use of the formulation and the solution of the foregoing problem, find the magnitude of the force exerted by the charges bound on the surface of the dielectric on the point charge q.

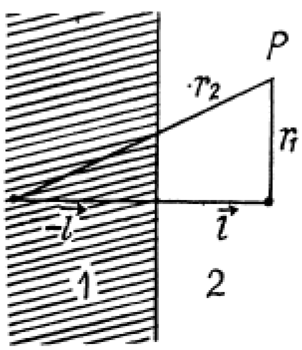
Solution. 90. The force on the point chaige q is due to the bound charges. This can be calculated from the field at this chaige after extracting out the self-field. This image field is

$$E_{image} = \frac{\varepsilon - 1}{\varepsilon + 1} \frac{q}{4 \pi \varepsilon_0 (2l)^2}$$

Thus, $$F = \frac{\varepsilon - 1}{\varepsilon + 1} \frac{q^2}{16 \pi \varepsilon_0 l^2}$$

Q. 91. A point charge q is located on the plane dividing vacuum and infinite uniform isotropic dielectric with permittivity ε. Find the moduli of the vectors D and E as well as the potential q as functions of distance r from the charge q.

Solution. 91.



$$E_p = \frac{q \vec{r_1}}{4 \pi \varepsilon_0 r_1^3} + \frac{q' \vec{r_2}}{4 \pi r_2^3 \varepsilon_0}, \; P \text{ in } 1$$

$$E_p = \frac{q'' \vec{r_1}}{4 \pi \varepsilon_0 r_1^3}, \; P \text{ in } 2$$

where $q'' = \dfrac{2q}{\varepsilon + 1}$, $q' = q'' - q$

In the limit $\vec{l} \to 0$

Thus,

$$E_p = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)r^2}$$

$$\varphi = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)r}$$

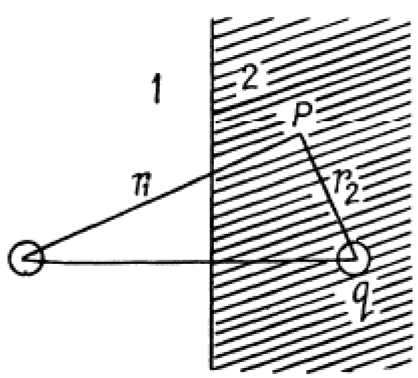
$$D = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)r^2} \times \begin{cases} 1 \text{ in vacuum} \\ \varepsilon \text{ in dielectric} \end{cases}$$

Q. 92. A small conducting ball carrying a charge q is located in a uniform isotropic dielectric with permittivity ε at a distance l from an infinite boundary plane between the dielectric and vacuum. Find the surface density of the bound charges on the boundary plane as a function of distance r from the ball. Analyse the obtained result for l→ 0.

Solution. 92.

$$\vec{E}_p = \frac{q\vec{r_2}}{4\pi\varepsilon_0\varepsilon r_2^3} + \frac{q'\vec{r_1}}{4\pi\varepsilon_0 r_1^3};\ P \text{ in } 2$$

$$\vec{E}_p = \frac{q''\vec{r_2}}{4\pi\varepsilon_0 r_2^3};\ P \text{ in } 1$$



Using the boundary conditions,

$$E_{1n} = \varepsilon E_{2n},\ E_{1t} = E_{2t}$$

This implies

$$q - \varepsilon q' = q'' \text{ and } q + \varepsilon q' = \varepsilon q''$$

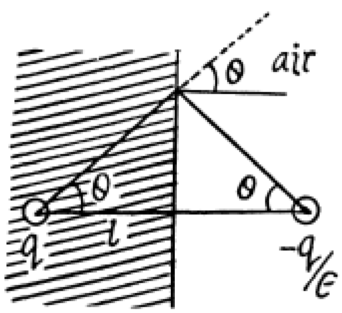
$$\text{So,} \quad q'' = \frac{2q}{\varepsilon + 1}, \; q' = \frac{\varepsilon - 1}{\varepsilon + 1} \frac{q}{\varepsilon}$$

Then, as earlier,

$$\sigma' = \frac{ql}{2 \pi r^3} \cdot \left(\frac{\varepsilon - 1}{\varepsilon + 1}\right) \cdot \frac{1}{\varepsilon}$$

Q. 93. A half-space filled with uniform isotropic dielectric with permittivity a has the conducting boundary plane. Inside the dielectric, at a distance l from this plane, there is a small metal ball possessing a charge q. Find the surface density of the bound charges at the boundary plane as a function of distance r from the ball.

Solution. 93. To calculate the electric field, first we note that an image charge will be needed to ensure that the electric field on the metal boundary is normal to the surface.



The image chaige must have magnitude $-\frac{q}{\varepsilon}$

$$E_n = \frac{1}{4 \pi \varepsilon_0} \left(\frac{q}{\varepsilon r^2}\right) 2 \cos \theta = \frac{ql}{2 \pi \varepsilon_0 \varepsilon r^3}$$

$$\text{Then } P_n = D_n - \varepsilon_0 E_n = \frac{(\varepsilon - 1) ql}{2 \pi \varepsilon r^3} = \sigma'$$

This is the density of bound charge on the surface.

Q. 94. A plate of thickness h made of uniform statically polarized dielectric is placed inside a capacitor whose parallel plates are interconnected by a conductor. The polarization of the dielectric is equal to P (Fig. 3.15). The separation between the capacitor plates is d. Find the strength and induction vectors for the electric field both inside and outside the plates.

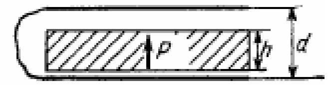


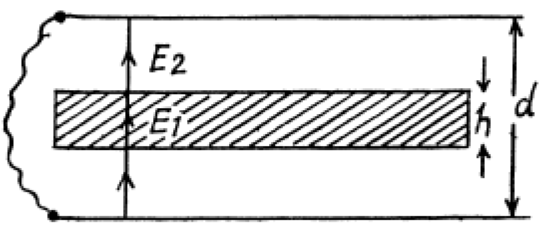
Fig. 3.15.

Solution. 94. Since the condenser plates are connected,

$$E_1 h + E_2 (d - h) = 0$$

and $\quad P + \varepsilon_0 E_1 = \varepsilon_0 E_2$

or, $\quad E_1 + \frac{P}{\varepsilon_0} = E_2$



Thus, $E_2 d - \frac{Ph}{\varepsilon_0} = 0$, or, $E_2 = \frac{Ph}{\varepsilon_0 d}$

$$E_1 = -\frac{P}{\varepsilon_0}\left(1 - \frac{h}{d}\right)$$

Q. 95. A long round dielectric cylinder is polarized so that the vector P = αr, where α is a positive constant and r is the distance from the axis. Find the space density p' of bound charges as a function of distance r from the axis.

Solution. 95. Given $\vec{P} = \alpha \vec{r}$, where $\vec{r} =$ distance from the axis. The space density of

charges is given by, p' = - div $\vec{P} = -2\alpha$

On using.

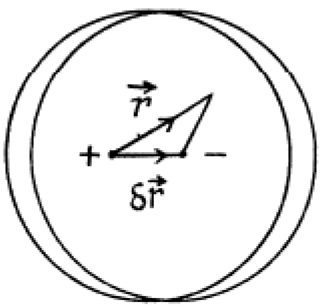
$$\text{div } \vec{r} = \frac{1}{r}\frac{\partial}{\partial r}(\vec{r} \cdot \vec{r}) = 2$$

Q. 96. A dielectric ball is polarized uniformly and statically. Its polarization equals P. Taking into account that a ball polarized in this way may be represented as a result of a small shift of all positive charges of the dielectric relative to all negative charges,
(a) find the electric field strength E inside the ball;

(b) demonstrate that the field outside the ball is that of a dipole located at the centre of the ball, the potential of that field being equal to φ = $p_0 r/4\pi \varepsilon_0$, where P_0 is the electric moment of the ball, and r is the distance from its centre.

Solution. 96. In a uniformly charged sphere,

$$E_r = \frac{\rho_0 r}{3\varepsilon_0} \quad \text{or,} \quad \vec{E} = \frac{\rho_0}{3\varepsilon_0}\vec{r}$$



The total electric field is

$$\vec{E} = \frac{1}{3\varepsilon_0}\rho_0\vec{r} - \frac{1}{3\varepsilon_0}(\vec{r} - \delta\vec{r})\rho_0$$
$$= \frac{1}{3\varepsilon_0}\rho_0\delta\vec{r} = -\frac{\vec{P}}{3\varepsilon_0}$$

Where $\rho\delta\vec{r} = -\vec{P}$ (dipole moment is defined with its direction being from the -vc chaige to +ve charge.)

The potential outside is

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} - \frac{Q}{|\vec{r} - \delta\vec{r}|}\right), \cdot = \frac{\vec{p_0}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}, \; r > R$$

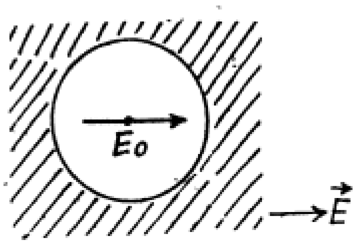
where $\vec{p_0} = -\frac{4\pi}{3}R^3\rho_0\,\partial\vec{r}$ is the total dipole moment.

Q. 97. Utilizing the solution of the foregoing problem, find the electric field strength E_0 in a spherical cavity in an infinite statically polarized uniform dielectric if the dielectric's polarization is P, and far from the cavity the field strength is E.

Solution. 97. The electric field $\vec{E_0}$ in a spherical cavity hi a uniform dielectric of permittivity e is related to the far away field $\vec{E}$, in the following mannenjmagine the cavity to be filled wp with the dielectric. Then there will be a

uniform field $\vec{E}$ everywhere and a polarization $\vec{P}$, given by,

$$\vec{P} = (\varepsilon - 1)\varepsilon_0 \vec{E}$$



Now take out the sphere making the cavity, the electric field inside the sphere will be

$$-\frac{\vec{P}}{3\varepsilon_0}$$

By superposition. $\vec{E_0} - \frac{\vec{P}}{3\varepsilon_0} = \vec{E}$

or, $\vec{E_0} = \vec{E} + \frac{1}{3}(\varepsilon - 1)\vec{E} = \frac{1}{3}(\varepsilon + 2)\vec{E}$

Q. 98. A uniform dielectric ball is placed in a uniform electric field of strength E_0. Under these conditions the dielectric becomes polarized uniformly. Find the electric field strength E inside the ball and the polarization P of the dielectric whose permittivity equals e. Make use of the result obtained in Problem Q.96.

Solution. 98. By superposition the field $\vec{E}$ inside the bll is given by

$$\vec{E} = \vec{E_0} - \frac{\vec{P}}{3\varepsilon_0}$$

On the other hand, if the sphere is not too small, the macroscopic

equation $\vec{P} = (\varepsilon - 1)\varepsilon_0 \vec{E}$ must hold. Thus,

$$\vec{E}\left(1 + \frac{1}{3}(\varepsilon - 1)\right) = \vec{E_0} \quad \text{or,} \quad \vec{E} = \frac{3\vec{E_0}}{\varepsilon + 2}$$

Also $\vec{P} = 3\varepsilon_0 \frac{\varepsilon - 1}{\varepsilon + 2}\vec{E}_0$

Q. 99. An infinitely long round dielectric cylinder is polarized uniformly and statically, the polarization P being perpendicular to the axis of the cylinder. Find the electric field strength E inside the dielectric.

Solution. 99. This is to be handled by the same trick as in Q.96. We have effectively a two dimensional situation. For a uniform cylinder full of charge with chaige density p_0 (charge per unit volume), the electric field E at an inside point is along the (cylindrical) radius vector $\vec{r}$ and equal to,

$$\vec{E} = \frac{1}{2\varepsilon_0}\rho\,\vec{r}$$

$$\left(\text{div}\,\vec{E} = \frac{l}{r}\frac{\partial}{\partial r}(rE_r) = \frac{\rho}{\varepsilon_0}, \quad \text{hence,} \quad E_r = \frac{\rho}{2\varepsilon_0}r\right)$$

Therefore the polarized cylinder can be thought of as two equal and opposite charge distributions displaced with respect to each other

$$\vec{E} = \frac{1}{2\varepsilon_0}\rho\,\vec{r} - \frac{1}{2\varepsilon_0}\rho\,(\vec{r} - \delta\vec{r}) = \frac{1}{2\varepsilon_0}\rho\,\delta\,\vec{r} = -\frac{\vec{P}}{2\varepsilon_0}$$

Since $\vec{P} = -\rho\,\delta\,\vec{r}$. (direction of electric dipole moment vector being from the negative charge to positive charge.)

Q. 100. A long round cylinder made of uniform dielectric is placed in a uniform electric field of strength E_0. The axis of the cylinder is perpendicular to vector E_0. Under these conditions the dielectric becomes polarized uniformly. Making use of the result obtained in the foregoing problem, find the electric field strength E in the cylinder and the polarization P of the dielectric whose permittivity is equal to ε.

Solution. 100. As in Q.98, we write $\vec{E} = \vec{E}_0 - \frac{\vec{P}}{2\varepsilon_0}$

using here the result of the foregoing problem.

Also $\vec{P} = (\varepsilon - 1)\,\varepsilon_0\,\vec{E}$

So, $\vec{E}\left(\frac{\varepsilon + 1}{2}\right) = \vec{E}_0$, or, $\vec{E} = \frac{2\vec{E}_0}{\varepsilon + 1}$ and $\vec{P} = 2\varepsilon_0 \frac{\varepsilon - 1}{\varepsilon + 1}\vec{E}_0$