CHAPTER ELEVEN

Applications of Derivatives

THE DERIVATIVE AS A RATE OF CHANGE

In case of a linear function $y = mx + b$ the graph is a straight line and the slope m measures the steepness of the line by giving the rate of climb of the line, the rate of change of y with respect to x.

As x changes from x_0 to x_1, y changes m times as much:

$$y_1 - y_0 = m(x_1 - x_0)$$

Thus the slope $m = (y_1 - y_0)/(x_1 - x_0)$ gives the change in y per unit change in x.

In more general case of a differentiable function $y = f(x)$, the difference quotient

$$\frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h},\ h \neq 0$$

gives the average rate of change of y (or f) with respect to x. The limit as h approaches zero is the derivative $dy/dx = f'(x)$, which can be interpreted as the *instantaneous rate of change of* y with respect to x. Since the graph is a curve, the rate of change of y can vary from point to point.

Illustration 1

Find the rate of change of volume of a sphere with respect to its radius when $r = 4$ cm

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dr} = 4\pi r^2$$

when $r = 4$, $\quad \frac{dV}{dr} = 64\pi$

Illustration 2

A point is in motion along a curve $12y = x^3$. Which of its coordinate change faster?

Differentiating both the coordinates with respect to t we have

$$12\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$

$$\Rightarrow \qquad \frac{y'_t}{x'_t} = \frac{x^2}{4}$$

Hence, if

(i) $-2 < x < 2$, then $\frac{y'_t}{x'_t} < 1$, i.e. the rate of change of the ordinate is less than that of the abscissa.

(ii) For $x > 2$ or $x < -2$, $\frac{y'_t}{x'_t} > 1$ i.e. the rate of change of the ordinate is greater than that of the abscissa.

(iii) For $x = \pm 2$, the rate of change of ordinate is equal to that of the abscissa.

Velocity and Acceleration

Suppose that an object is moving along a straight line and that, for each time t during a certain time interval, the object has (coordinate) $x(t)$. Then at time $t + h$, the position of the object is $x(t + h)$, and $x(t + h) - x(t)$ is the change in position that the object experienced during the time period t to $t + h$. The ratio

$$\frac{x(t+h) - x(t)}{t+h-t} = \frac{x(t+h) - x(t)}{h}$$

gives the average velocity of the object during this time period. If

$$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = x'(t)$$

exists, then $x'(t)$ gives the (instantaneous) rate of change of position with respect to time. This rate of change of position is called the velocity of the object. If the velocity function is itself differentiable, then its rate of change with respect t time is called the acceleration, in symbols

$$a(t) = v'(t) = x''(t)$$

the speed is by definition the absolute value of the velocity:

$$\text{speed at time } t = |v(t)|$$

If the velocity and acceleration have the same sign then the object is speeding up, but if the velocity and acceleration have opposite signs then the object is slowing down.

A sudden change in acceleration is called a *jerk*. Thus, *Jerk* is the derivative of acceleration. If a body's position at the time t is $x(t)$, the body's jerk at time t is

$$j = \frac{da}{dt} = \frac{d^3x}{dt^3}$$

Illustration 3

A point moves in a straight line so that its distance from the start in t sec is equal to $s = \frac{1}{4}t^4 - 4t^3 + 16t^2$. What will be acceleration and at what times is its velocity equal to zero?

$$v = \frac{ds}{dt} = t^3 - 12t^2 + 32t$$

$$a = \frac{d^2s}{dt^2} = 3t^2 - 24t + 32$$

$$v = 0 \Rightarrow t^3 - 12t^2 + 32t = 0$$

$$\Rightarrow t(t^2 - 12t + 32) = 0$$

$$\Rightarrow t(t-4)(t-8) = 0$$

at $t = 0$, $t = 4$, $t = 8$ the velocity will be zero and the corresponding acceleration will be 32, – 16, 32 respectively.

Illustration 4

A body whose mass is 3 kg performs rectilinear motion according to the formula $s = 1 + t + t^2$, where s is measured in cm and t in secs. Determine the kinetic energy $\left(\frac{mv^2}{2}\right)$ of the body in $5s$ after the start.

$$\frac{ds}{dt} = 1 + 2t,\ v = \left.\frac{ds}{dt}\right|_{t=5} = 11 \text{ cm/s}$$

$$\text{K.E.} = \frac{mv^2}{2} = \frac{3000 \times 121}{2} = 181500 \text{ gm cm}^2/\text{s}^2$$

$$= 181.5 \times 10^3 \text{ erg.}$$

Differentials Let $y = f(x)$ be a differentiable function. Let $h \neq 0$. The difference $f(x + h) - f(x)$ is called the increment of f from x to $x + h$, and is denoted by Δf.

$$\Delta f = f(x + h) - f(x)$$

The product $f'(x)\,h$ is called the differential of f at x with increment h, and is denoted by df

$$df = f'(x)\,h$$

The change in f from x to $x + h$ can be approximated by $f'(x)\,h$;

$$f(x + h) - f(x) \simeq f'(x)\,h.$$

Illustration 5

Find the increment and differential of the function

$$y = 3x^3 + x - 1 \quad \text{at } x = 1, \Delta x = 0.1.$$

$$dy = (9x^2 + 1)\,\Delta x$$

At $\quad x = 1, \Delta x = 0.1, dy = (9 + 1)(0.1)$

$$= 1$$

$$\Delta y = [3(x + \Delta x)^3 + (x + \Delta x) - 1] - (3x^3 + x - 1)$$

$$= 9x^2\,\Delta x + 9x\,\Delta x^2 + 3\Delta x^3 + \Delta x$$

$$\Delta y - dy = 9x\,\Delta x^2 + 3\Delta x^3$$

At $\quad x = 1, \Delta x = 0.1$

$$\Delta y - dy = 0.09 + 0.003 = 0.093.$$

TANGENT AND NORMAL

Let $y = f(x)$ be the equation of a curve, and let $P(x_0, y_0)$ be a point on it. Let PT be the tangent, PN the normal and PM the perpendicular to the x-axis (Fig. 11.1).

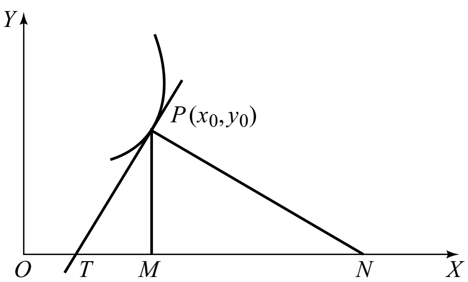


Fig. 11.1

The slope of the tangent to the curve $y = f(x)$ at P is given by $f'(x_0)$ Thus the equation of the tangent to the curve $y = f(x)$ at (x_0, y_0) is

$$y - y_0 = f'(x_0)(x - x_0)$$

Since PN is perpendicular to PT, it follows that, if $f'(x_0) \neq 0$, the slope of PN is $-1/f'(x_0)$. Hence the equation of the normal to the curve $y = f(x)$ at (x_0, y_0) is

$$f'(x_0)(y - y_0) + (x - x_0) = 0$$

An equation of the normal parallel to the x-axis is $y = y_0$. The *length of the tangent* at (x_0, y_0) is PT, and it is equal to

$$|y_0|\sqrt{1 + \frac{1}{(f'(x_0))^2}}.$$

The *length of the normal* is PN and it is equal to

$$|y_0|\sqrt{1 + (f'(x_0))^2}.$$

TM is called subtangent and the *length of the subtangent* is equal to

$$\left|\frac{1}{f'(x_0)}y_0\right|$$

MN is called subnormal and the *length of the subnormal* is equal to $|y_0 f'(x_0)|$.

Illustration 6

Derive the equation of tangent and normal at (x_0, y_0) of the curve $y = \log x$.

$$\frac{dy}{dx} = \frac{1}{x} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{x_0}$$

Hence equation of tangent at (x_0, y_0) is

$$y - \log x_0 = \frac{1}{x_0}(X - x_0)$$

Equation of normal is

$$Y - \log x_0 = -x_0(X - x_0)$$

Length of tangent = $|\log x_0|\sqrt{1+x_0^2}$

Length of normal = $|\log x_0|\sqrt{1+\frac{1}{x_0^2}}$

Length of subtangent = $|x_0 \log x_0|$

Length of subnormal = $|y_0|x_0|$

ANGLE BETWEEN TWO CURVES

An angle of intersection of two curves is defined as the angle between the tangents to the two curves at their point of intersection. Let $y = f(x)$ and $y = g(x)$ be two curves, and let $P(x_0, y_0)$ be their point of intersection. Also, let ψ and φ be the angles of inclination of the two tangents at P with the x-axis, and let θ be the angle between the two tangents. Then

$$\pm\tan\theta = \frac{\tan\varphi - \tan\psi}{1+\tan\varphi\tan\psi} = \frac{g'(x_0) - f'(x_0)}{1+f'(x_0)\,g'(x_0)}$$

Orthogonal Curves

If the angle of intersection between two curves is a right angle then the two curves are said to be intersecting orthogonally. Two curves $y = f(x)$ and $y = g(x)$ cut orthogonally if $f'(x)$ $g'(x) = -1$.

THE ROLLE'S AND LAGRANGE'S THEOREMS

Rolle's theorem Let $f(x)$ be a function defined on a closed interval $[a, b]$, such that (i) $f(x)$ is continuous on $[a, b]$, (ii) $f(x)$ is derivable on $]a, b[$, and (iii) $f(a) = f(b)$. Then there exists a $c \in\ ]a, b[$ such that $f'(c) = 0$.

Rolle's theorem for polynomials If $\phi(x)$ is any polynomial, then between any pair of roots of $\phi(x) = 0$ lies a root of $\phi'(x) = 0$.

If a polynomial equation $P(x) = 0$ has at least n real roots, then $P'(x) = 0$ has at least $(n-1)$ real roots, $P''(x) = 0$ has at least $(n-2)$ real roots and so on.

Lagrange's Mean Value theorem Let $f(x)$ be a function defined on $[a, b]$, such that (i) $f(x)$ is continuous on $[a, b]$, and (ii) $f(x)$ is derivable on $]a, b[$. Then there exists a $c \in\ ]a, b[$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Illustration 7

Check the validity of Rolle's theorem for $y = 4^{\sin x}$ in the interval $[0, \pi]$.

Being exponential function, y is a continuous function x and $y' = (4^{\sin x})$ (log 4) cos x. Also $y(0) = 4° = 1$ and $y(\pi) = 4^{\sin\pi} = 4° = 1$. Hence all the hypothesis of Rolle's theorem are satisfied. For conclusion,

$$y'(\pi/2) = 0 \quad \text{and} \quad \pi/2 \in [0, \pi]$$

Illustration 8

Show that the equation $x^3 - 3x + C = 0$ cannot have two different roots in (0, 1).

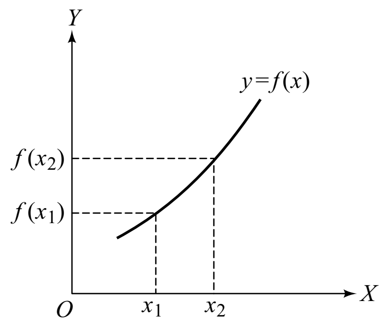
Consider $f(x) = x^3 - 3x + C$. Suppose f has two distinct roots say x_1 and x_2 in (0,1). f is a polynomial so continuous on $[x_1, x_2]$ and differentiable in (x_1, x_2) and $f(x_1) = f(x_2) = 0$ so by Rolle's theorem there is $\alpha \in (x_1, x_2) \subseteq (0, 1)$ such that

$$f'(\alpha) = 0 \quad \Rightarrow \quad 3\alpha^2 - 3 \neq 0 \quad \Rightarrow \quad \alpha = \pm 1 \notin (0, 1).$$

So $x^3 - 3x + c = 0$ cannot have two different roots in (0, 1).

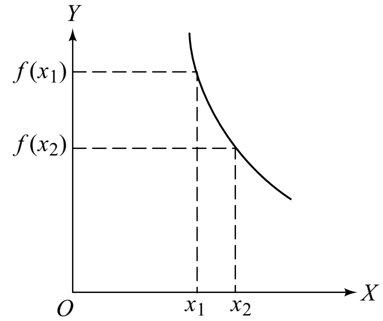
MONOTONICITY

A function $f(x)$ defined on a set D is said to be non-decreasing, increasing, non-increasing and decreasing respectively, if for any $x_1, x_2 \in D$ and $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$, $f(x_1) < f(x_2)$, $f(x_1) \geq f(x_2)$ and $f(x_1) > f(x_2)$, respectively (Figs 11.2 and 11.3). The function $f(x)$ is said to be monotonic if it possesses any of these properties.



An increasing function

Fig. 11.2



A decreasing function

Fig. 11.3

As an immediate consequence of Lagrange's Mean theorem, we have

Testing monotonicity Let $f(x)$ be continuous on $[a, b]$ and differentiable on $]a, b[$. Then

(i) for $f(x)$ to be non-decreasing (non-increasing) on $[a, b]$ it is necessary and sufficient that $f'(x) \geq 0$ ($f'(x) \leq 0$) for all $x \in]a, b[$.

(ii) for $f(x)$ to be increasing (decreasing) on $[a, b]$ it is necessary and sufficient that $f'(x) > 0$ ($f'(x) < 0$) for all $x \in]a, b[$.

Illustration 9

Show that the function $y = \sqrt{2x - x^2}$ increases in the interval (0, 1) and decreases in (1, 2)

$$y'(x) = \frac{1}{2}\frac{2-2x}{\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{(2-x)x}} \quad x \neq 0, 2.$$

y' is not defined at $x = 0$, $x = 2$.

$y'(x) > 0$ if $1 - x > 0$ so y increases on (0, 1) and $y'(x) < 0$ for $x \in (1, 2)$, so y decreases on (1, 2).

Illustration 10

Find the interval of monotonicity of $y = \dfrac{1-x+x^2}{1+x+x^2}$

$$y'(x) = \frac{(1+x+x^2)(-1+2x) - (1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$

$$= \frac{-2+2x^2}{(1+x+x^2)^2} = \frac{2(x^2-1)}{(1+x+x^2)^2}$$

Since the denominator is always positive so the sign of $y'(x)$ according to the sign of $x^2 - 1$.

Thus $\quad y'(x) > 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow x > 1$ or $x < -1$

Hence y increases on $(1, \infty) \cup (-\infty, -1)$ and decreases on $(-1, 1)$.

MAXIMA AND MINIMA

A function has a local *maximum* at the point x_0 if the value of the function $f(x)$ at that point is greater than its values at all points other than x_0 of a certain interval containing the point x_0. In other words, a function $f(x)$ has a maximum at x_0 if it is possible to find an interval (α, β) containing x_0, i.e., with $\alpha < x_0 < \beta$, such that for all points different from x_0 in (α, β), we have $f(x) < f(x_0)$.

A function $f(x)$ has a local minimum at x_0 if there exists an interval (α, β) containing x_0 such that $f(x) > f(x_0)$ for $x \in (\alpha, \beta)$ and $x \neq x_0$.

One should not confuse the local maximum and local minimum of a function with its largest and smallest values over a given interval. The local maximum of a function is the largest value only in comparison to the values it has at all points sufficiently close to the point of local maximum. Similarly, the local minimum is the smallest value only in comparison to the values of the function at all points sufficiently close to the local minimum point.

The general term for the maximum and minimum of a function is *extremum*, or the extreme values of the function. A necessary condition for the existence of an extremum at the point x_0 of the function $f(x)$ is that where $f'(x_0) = 0$, or $f'(x_0)$ does not exist. The points at which $f'(x) = 0$, or where $f'(x)$ does not exist, are called *critical points* of the function f.

First derivative test

(i) If $f'(x)$ changes sign from positive to negative at x_0, i.e., $f'(x) > 0$ for $x < x_0$, and $f'(x) < 0$ for $x > x_0$, then the function attains a local maximum at x_0.

(ii) If $f'(x)$ changes sign from negative to positive at x_0, i.e., $f'(x) < 0$ for $x < x_0$, and $f'(x) > 0$ for $x > x_0$, then the function attains a local minimum at x_0.

(iii) If the derivative does not change sign in moving through the point x_0, there is no extremum at that point.

Second derivative test Let f be twice differentiable, and let c be a root of the equation $f'(x) = 0$. Then

(i) c is a local maximum point if $f''(c) < 0$

(ii) c is a local minimum point if $f''(c) > 0$.

However, if $f''(c) = 0$, then the following result is applicable. Let $f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$ (where f^r denotes the rth derivative), but $f^{(n)}(c) \neq 0$.

(i) If *n is even* and $f^{(n)}(c) < 0$, there is a local maximum at c, while if $f^{(n)}(c) > 0$, there is a local minimum at c.

(ii) If n is *odd*, there is no extremum at the point c.

The *greatest* (*least*) value of continuous function $f(x)$ on the interval $[a, b]$ is attained either at the critical points or at the end-points of the interval. To find the greatest (least) value of the function, we have to compute its values at all the critical points on the interval $]a, b[$, and the values $f(a)$,

$f(b)$ of the function at the end-points of the interval, and choose the greatest (least) out of the numbers so obtained.

Illustration 11

Find the extrema of $y = 2x^3 - 3x^2$. y is differentiable function and $y' = 6x^2 - 6x = 6x(x - 1)$.

So, points extremum are 0 and 1 $y'' = 6(2x - 1)$ $y''(1) = 6 > 0$, $y''(0) = -6 < 0$.

Hence $y_{max} = y(0) = 0$

and $y_{min} = y(1) = 2 - 3 = -1$

Illustration 12

Find the greatest and least value of $y = x^3 - 3x^2 + 6x - 2$ on $[-1, 1]$. y is a differentiable function of x and $y'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 2) = 3((x - 1)^2 + 1) > 0$.

Hence y increases on $[-1, 1]$. Thus the greatest value $= y(1) = 1 - 3 + 6 - 2 = 2$, least value $= y(-1) = -1 - 3 - 6 - 2 = -12$.

Point of Inflection

A point $x = c$ is said to be point of inflection for a curve $y = f(x)$ if $f''(c) = 0$ or is not defined and $f'''(c) \neq 0$

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: The approximate value of cos 31° is

(a) 0.52 (b) 0.851

(c) 0.641 (d) 0.681

Ans. (b)

Solution: Known value near to cos 31° is cos 30°

$$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Let $y = \cos x$ $y'(x) = -\sin x$

$\cos 31 = \cos\left(\frac{\pi}{6} + \frac{\pi}{180}\right)$. So putting $x = \frac{\pi}{6}$ and $\Delta x = \frac{\pi}{180}$,

we have $y'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$

$\cos(x + \Delta x) \simeq \cos x + f'(x)\,\Delta x$. Putting $x = \frac{\pi}{6}$ and $\Delta x = \frac{\pi}{180}$, we have

$$\cos 31 \simeq \frac{\sqrt{3}}{2} - \frac{1}{2}\frac{\pi}{180} = \frac{1.732}{2} - \frac{1}{2} \times \frac{3.14}{180}$$

$$= 0.851$$

Example 2: The tangent line at (2, 4) to the curve $y = x^3 - 3x + 2$ meets the x-axis at

(a) (2, 0) (b) $\left(\frac{7}{2}, 0\right)$

(c) $\left(\frac{11}{9}, 0\right)$ (d) $\left(\frac{14}{9}, 0\right)$

Ans. (d)

Solution: $\frac{dy}{dx} = 3x^2 - 3 = 3(x^2 - 1)$

$$\left.\frac{dy}{dx}\right|_{x=2} = 3(4 - 1) = 9$$

Equation of tangent at (2, 4) is

$$Y - 4 = 9(X - 2)$$

This meets x-axis if $Y = 0$

$$\Rightarrow \quad \frac{-4}{9} = X - 2 \quad \Rightarrow \quad X = 2 - \frac{4}{9} = \frac{14}{9}.$$

Example 3: The slope of the tangent to the curve

$x = t^2 + 3t - 8$

$y = 2t^2 - 2t - 5$

at the point (2, –1) is

(a) 2/3 (b) 6/7

(c) 4/5 (d) 3/2

Ans. (b)

Solution: We need to determine the value of t corresponding to the given point, so

$$\left.\begin{array}{l} 2 = t^2 + 3t - 8 \\ -1 = 2t^2 - 2t - 5 \end{array}\right] \begin{array}{l} \Rightarrow \quad t^2 + 3t - 10 = 0 \\ \Rightarrow \quad t^2 - t - 2 = 0 \end{array} \left.\begin{array}{l} \\ \end{array}\right] \begin{array}{l} \Rightarrow \quad t = 2, -5 \\ \Rightarrow \quad t = 2, -1 \end{array}$$

Hence common value is 2

$$y'|_{x=2} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)_{t=2} = \left(\frac{4t - 2}{2t + 3}\right)_{t=2} = \frac{6}{7}.$$

Example 4: The interval in which $y = \frac{1}{4x^3 - 9x^2 + 6x}$ is increasing is

(a) $(-\infty, \infty)$ (b) (0, 1/2)

(c) (1/2, 1) (d) $(1, \infty)$

Ans. (c)

Solution: $y'(x) = -\frac{12x^2 - 18x + 6}{(4x^3 - 9x^2 + 6x)^2}$

$$= -\frac{6(2x - 1)(x - 1)}{(4x^3 - 9x^2 + 6x)^2}$$

$y'(x) > 0 \Leftrightarrow (2x-1)(x-1) < 0$

$\Leftrightarrow$ (i) $2x - 1 > 0, \quad x - 1 < 0$

(ii) $2x - 1 < 0, \quad x - 1 > 0$

(i) $\Rightarrow x \in (1/2, 1)$ case (ii) is not possible. Hence y increases on (1/2, 1)

Example 5: Let $y = x - \log(1 + x)$, the minimum value of y is

(a) 1 (b) 0

(c) –1 (d) $\frac{1}{2}$

Ans. (b)

Solution: $y'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$. The domain y is $x > -1$. so $y'(x) = 0 \Rightarrow x = 0$

$y''(x) = \frac{1}{(1+x)^2} > 0$. Hence y is min at $x = 0$ and $y_{min} = 0 - \log 1 = 0$

Example 6: A covered box of volume 72 cm^3 and the base sides in a ratio of 1:2 is to be made. The length all sides so that the total surface area is the least possible is

(a) 2, 4, 9 (b) 8, 3, 3

(c) 6, 6, 2 (d) 6, 3, 4

Ans. (d)

Solution: Let l, b, h be the dimensions $l = 2b$, so $V = lbh = 2b^2h$

$\Rightarrow \quad 72 = 2b^2h \quad \Rightarrow \quad h = \frac{36}{b^2}$

The surface area $S = 2(lb + bh + lh)$

$$= 2\left(2b^2 + \frac{3b}{b} + 2b \times \frac{36}{b^2}\right)$$

$$= 2\left(2b^2 + \frac{108}{b}\right)$$

$$= 4\left(b^2 + \frac{54}{b}\right)$$

$\frac{dS}{db} = 4\left(2b - \frac{54}{b^2}\right)$, $\frac{dS}{db}$ is zero if $b = 3$ and

$\frac{d^2S}{db^2} = 4\left(2 + \frac{108}{b^3}\right) > 0$

Hence S is minimum when $b = 3$. So the dimensions are

6, 3, $\frac{36}{9}$ = 6, 3, 4

Example 7: A point on the curve $y = x^3 - 3x + 5$ at which the tangent line is parallel to $y = -2x$ is

(a) (1, 3) (b) (0, 5)

(c) $\left(\frac{1}{\sqrt{3}}, 5 - \frac{8\sqrt{3}}{9}\right)$ (d) $\left(\frac{1}{\sqrt{2}}, 0\right)$

Ans. (c)

Solution: Since $y'(x) = 3x^2 - 3$, so by the condition of parallelism

$3x^2 - 3 = -2$

$x = \pm\frac{1}{\sqrt{3}}$

If $x = \frac{1}{\sqrt{3}}$, then $y = \frac{1}{3\sqrt{3}} - \frac{3}{\sqrt{3}} + 5$

$$= 5 - \frac{8}{3\sqrt{3}} = 5 - \frac{8\sqrt{3}}{9}.$$

Example 8: The greatest value of $y = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is

(a) $\frac{\pi}{2}$ (b) 1

(c) 2 (d) $-\frac{\pi}{2}$

Ans. (a)

Solution $y' = 2\cos 2x - 1 \Rightarrow y'(x) = 0 \Leftrightarrow \cos^2 x = \frac{1}{2}$

So, $y'(x) = 0$ if $x = \frac{\pi}{6}$. Thus the critical points of y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ are $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}$. Now $y\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$, $y\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$ and $y\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{2}$. The greatest value is $\frac{\pi}{2}$.

Example 9: The point of inflection of $y = x^3 - 5x^2 + 3x - 5$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$

(c) $\frac{7}{4}$ (d) $\frac{5}{4}$

Ans. (d)

Solution: Since y is twice differentiable function so for inflection points $y''(x) = 0$, $y'''(x) \neq 0$

$y'(x) = 3x^2 - 10x + 3$

$y'' = 6x - 10$. So, $y''(x) = 0$ if $x = \frac{5}{4}$

$y'''(x) = 6 \neq 0$. So, $x = \frac{5}{4}$ is a point of inflection.

Example 10: The minimal rate of change of the function $f(x) = 3x^5 - 5x^3 + 5x - 7$ is

(a) $\frac{3}{4}$ (b) $\frac{5}{4}$

(c) $\frac{2}{3}$ (d) $\frac{3}{2}$

Ans. (b)

Solution Rate of change of $f(x)$ is $f'(x)$

$$= 15x^4 - 15x^2 + 5 = 15\left(x^4 - x^2 + \frac{1}{3}\right)$$

$$= 15\left(\left(x^2 - \frac{1}{2}\right)^2 + \frac{1}{12}\right) \geq \frac{15}{12} = \frac{5}{4}$$

The minimum vale of f' is attained at $x = +\frac{1}{\sqrt{2}}$ and equals $\frac{5}{4}$.

LEVEL 1

Straight Objective Type Questions

Example 11: A spherical balloon is expanding. If the radius is increasing at the rate of 5 inch per minute, the rate at which the volume increases (in cubic inches per minute) when the radius is 10 inch is

(a) 100π (b) 1000π
(c) 2000π (d) 500π

Ans. (c)

Solution: The volume $V = \frac{4}{3}\pi r^3$, r being the radius. It is given that $\frac{dr}{dt}$ = 5 inch/min and we have to find $\frac{dV}{dt}$ when r = 5 inches.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \quad \left.\frac{dV}{dt}\right|_{r=10} = 4\pi(100)5 = 2000\pi.$$

Example 12: An object is moving in the clockwise direction around the unit circle $x^2 + y^2 = 1$. As it passes through the point $(1/2, \sqrt{3}/2)$, its y-coordinate is decreasing at the rate of 3 units per second. The rate at which the x-coordinate changes at this point is (in units per second)

(a) 2 (b) $3\sqrt{3}$
(c) $\sqrt{3}$ (d) $2\sqrt{3}$

Ans. (b)

Solution: We find $\frac{dx}{dt}$ when $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$ given that $\frac{dy}{dt} = -3$ units/s and $x^2 + y^2 = 1$.

Differentiating $x^2 + y^2 = 1$, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Putting $x = 1/2$, $y = \sqrt{3}/2$ and $dy/dt = -3$, we have

$$\frac{1}{2}\frac{dx}{dt} + \frac{\sqrt{3}}{2}(-3) = 0 \quad \Rightarrow \quad \frac{dx}{dt} = 3\sqrt{3}.$$

Example 13: An approximate value of cos 40° is

(a) 0.7688 (b) 0.7071
(c) 0.7117 (d) 0.7

Ans. (a)

Solution:

Let $f(x) = \cos x$. 40° = 45° – 5°

$$= \frac{\pi}{4} - \frac{\pi}{180} \times 5 = \frac{\pi}{4} - \frac{\pi}{36} \text{ radians}$$

We use a differential to estimate the change in $\cos x$ when x decreases from $\pi/4$ to $\pi/4 - (\pi/36)$

$$f'(x) = -\sin x \text{ and } df = f'(x)\,h = -h\sin x$$

With $x = \pi/4$ and $h = -\pi/36$, df is given by

$$df = -f'(x)h = -\left(-\frac{\pi}{36}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{36}\cdot\frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{72} = 0.0617$$

$$\cos 40 \cong \cos 45 + 0.0617 \cong 0.7071 + 0.0617$$
$$= 0.7688.$$

Example 14: The values of x for which the tangents to the curves $y = x\cos x$, $y = (\sin x)/x$ are parallel to the axis of x are roots of (respectively)

(a) $\sin x = x$, $\tan x = x$ (b) $\cot x = x$, $\sec x = x$
(c) $\cot x = x$, $\tan x = x$ (d) $\tan x = x$, $\cot x = x$

Ans. (c)

Solution: Let $y = f(x) = x\cos x$ and $y = g(x) = (\sin x)/x$. Now $f'(x) = -x\sin x + \cos x$ and $g'(x) = (x\cos x - \sin x)/x^2$. Since the tangents are parallel to x-axis so $f'(x) = 0$ and $g'(x) = 0$, which is turn give $\cot x = x$ and $\tan x = x$ respectively.

Example 15: The length of the subtangent to the ellipse $x = a\cos t$, $y = b\sin t$ at $t = \pi/4$ is

(a) a (b) b
(c) $b/\sqrt{2}$ (d) $a/\sqrt{2}$

Ans. (d)

◎ **Solution:** $\frac{dx}{dt} = -a \sin t$ and $\frac{dy}{dt} = b \cos t$, therefore $\left.\frac{dy}{dx}\right|_{t=\pi/4} = -\frac{b}{a} \cot(\pi/4) = -\frac{b}{a}$. The length of the subtangent = $\left| y_0 \frac{dx}{dy} \right|_{t=\pi/4} = \left| b\sin\frac{\pi}{4} \times -\frac{a}{b} \right| = \frac{a}{\sqrt{2}}$.

◉ **Example 16:** Tangent of the angle at which the curve $y = a^x$ and $y = b^x$ $(a \neq b > 0)$ intersect is given by

(a) $\frac{\log ab}{1+\log ab}$ (b) $\frac{\log a/b}{1+(\log a)(\log b)}$

(c) $\frac{\log ab}{1+(\log a)(\log b)}$ (d) none of these

Ans. (b)

◎ **Solution:** Intersection of the two curves is given by $a^x = b^x$ which implies $x = 0$. If α is the angle at which the two curves intersect then

$$\tan\alpha = \frac{m_1 - m_2}{1+m_1 m_2} = \left.\frac{a^x \log a - b^x \log b}{1+a^x b^x (\log a)(\log b)}\right|_{x=0}$$

$$= \frac{(\log a/b)}{1+(\log a)(\log b)}. \qquad \text{(Putting } x = 0)$$

◉ **Example 17:** For the parabola $y^2 = 16x$, the ratio of the length of the subtangent to the abscissa is

(a) 2 : 1 (b) 1 : 1

(c) $x : y$ (d) $x^2 : y$

Ans. (a)

◎ **Solution:** Differentiating, $2y\frac{dy}{dx} = 16$ so $\frac{dy}{dx} = \frac{8}{y}$. Thus the length of the subtangent is $y.\frac{dx}{dy} = \frac{y^2}{8} = \frac{16x}{8} = 2x$.

Hence length of the subtangent: abscissa = $2x : x = 2 : 1$.

◉ **Example 18:** If the tangent to the curve $x^3 - y^2 = 0$ at $(m^2, -m^3)$ is parallel to $y = -\frac{1}{m}x - 2m^3$, then the value of m^2 is

(a) 1/3 (b) 1/6

(c) 2/3 (d) – 2/3

Ans. (c)

◎ **Solution:** Differentiating $x^3 - y^2 = 0$, we have

$$\frac{dy}{dx} = \frac{3x^2}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{(m^2 - m^3)} = -\frac{3m^4}{2m^3} = -\frac{3m}{2}.$$

According to the given condition, $-\frac{3}{2}m = -\frac{1}{m}$

$$\Rightarrow m^2 = \frac{2}{3}$$

◉ **Example 19** If the function $f(x) = \frac{cx+d}{(x-1)(x-4)}$ has a turning point at the point (2, – 1) then

(a) $c = 2, d = 0$ (b) $c = 1, d = 0$

(c) $c = 1, d = -1$ (d) $c = 1, d = 1$

Ans. (b)

◎ **Solution:** $f'(x) = \frac{c(x-1)(x-4)-(cx+d)(2x-5)}{(x-1)^2(x-4)^2}$

So $0 = f'(2) = \frac{-2c+(2c+d)}{4} = \frac{d}{4} \Rightarrow d = 0$

Also $-1 = f(2) = \frac{2c+d}{-2} = -c \Rightarrow c = 1.$

◉ **Example 20:** The distance between the origin and the normal to the curve $y = e^{2x} + x^2$ at the point whose abscissa is 0 is

(a) $1/\sqrt{5}$ (b) $2/\sqrt{5}$

(c) $3/\sqrt{5}$ (d) $2/\sqrt{3}$

Ans. (b)

◎ **Solution:** The point on the curve corresponding to $x = 0$ is (0,1)

$$\frac{dy}{dx} = 2e^{2x} + 2x \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 2$$

Hence the equation of the normal at the point (0, 1) is

$$y - 1 = (-1/2)(x-0) \Rightarrow 2y + x - 2 = 0$$

Therefore, the distance of the point (0, 0) from this line is $2/\sqrt{5}$.

◉ **Example 21:** The function $f(x) = 1 + x(\sin x)[\cos x]$, $0 < x \leq \pi/2$

(a) is continuous on $(0, \pi/2)$

(b) is strictly decreasing in $(0, \pi/2)$

(c) is strictly increasing in $(0, \pi/2)$

(d) has global maximum value 2.

Ans. (a)

◎ **Solution:** For $0 < x \leq \pi/2$, $[\cos x] = 0$. Hence $f(x) = 1$ for all $x \in (0, \pi/2]$. Trivially $f(x)$ is continuous on $(0, \pi/2)$. This function is neither strictly increasing nor strictly decreasing and its global maximum is 1.

◉ **Example 22:** Let f be an increasing function on $[a, b]$ and g a decreasing function from $[a, b]$ to $[a, b]$, then

(a) f o g is a decreasing function

(b) g o f is an increasing function

(c) f o g is an increasing function

(d) none of these

Ans. (a)

◎ **Solution:** For $x_1 < x_2$, $x_1, x_2 \in [a, b]$, since g is decreasing so $g(x_1) > g(x_2) \Rightarrow f(g(x_1)) > f(g(x_2))$ i.e. f o g is decreasing function on $[a, b]$. Similarly g o f is also decreasing function.

◉ **Example 23:** If $a < 0$ and $f(x) = e^{ax} + e^{-ax}$. Suppose that $S = \{x : f(x) \text{ is monotonically decreasing}\}$ then

(a) $S = \{x : x > 0\}$ (b) $S = \{x : x < 0\}$
(c) $S = \{x : x > 1\}$ (d) $S = \{x : x < 1\}$

Ans. (b)

◎ **Solution:** $f'(x) = a(e^{ax} - e^{-ax})$. So f is monotonically decreasing if and only if $e^{ax} - e^{-ax} > 0$, i.e. $e^{2ax} > 1$ which is true if and only if $2ax > 0$. Since $a < 0$, we must have $x < 0$.

Example 24: The equation of the horizontal tangent to the graph of the function $y = e^x + e^{-x}$ is

(a) $y = -2$ (b) $y = -1$
(c) $y = 2$ (d) none

Ans. (c)

◎ **Solution:** The slope of the horizontal tangent is 0, so $0 = dy/dx = e^x - e^{-x} \Rightarrow e^{2x} = 1 \Rightarrow x = 0$. If $x = 0$ then $y = 2$. Hence the equation of horizontal tangent at (0, 2) is $y = 2$.

Example 25: Let $f(x)$ and $g(x)$ be defined and differentiable for $x \geq x_0$ and $f(x_0) = g(x_0)$, $f'(x) > g'(x)$ for $x > x_0$ then

(a) $f(x) < g(x)$ for some $x > x_0$
(b) $f(x) = g(x)$ for some $x > x_0$
(c) $f(x) > g(x)$ for all $x > x_0$
(d) none of these

Ans. (c)

◎ **Solution:** Consider the function $\phi(x) = f(x) - g(x)$ on the interval $[x_0, x]$. By Lagrange's theorem we have $\phi(x) - \phi(x_0) = \phi'(z)(x - x_0)$ for some $z \in (x_0, x)$. Since $\phi(x_0) = 0$, $\phi'(z) = f'(z) - g'(z) > 0$. So $\phi(x) - \phi(x_0) = \phi(x) = (f'(z) - g'(z))(x - x_0) > 0 \Rightarrow \phi(x) > 0$

Thus $f(x) > g(x)$ for all $x > x_0$.

Example 26: If $y = 2x + \cot^{-1} x + \log\left(\sqrt{1+x^2} - x\right)$, then y

(a) decreases on $(-\infty, \infty)$
(b) decreases on $[0, \infty)$
(c) neither decreases nor increases on $[0, \infty)$
(d) increases on $(-\infty, \infty)$

Ans. (d)

Solution:

$$y'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2} - x}\frac{d}{dx}\left(\sqrt{1+x^2} - x\right)$$

$$= 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2} - x} \times \left(\frac{x}{\sqrt{1+x^2}} - 1\right)$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} \geq 0$$

since $1/(1+x^2)$ and $1/\sqrt{1+x^2}$ are less than or equal to 1 for all x. So $f(x)$ increases on $(-\infty, \infty)$.

Example 27 The equation $x^4 - 7x + 2 = 0$ has

(a) exactly two real and distinct solutions
(b) has four real roots
(c) no real root
(d) all the four roots lie between 0 and 2

Ans. (a)

◎ **Solution:** Let $f(x) = x^4 - 7x + 2$. Since $f(0) = 2$, $f(1) = -4$ and $f(2) = 4$

so by Intermediate value theorem there is $x_1 \in (0, 1)$ and $x_2 \in (1, 2)$ such that $f(x_1) = f(x_2) = 0$. Also $f'(x) = 4x^3 - 7$, so $f(x)$ decreases for $x < (7/4)^{1/3}$ and increases for $x > (7/4)^{1/3}$ and $1 < (7/4)^{1/3} < 2$. Thus $f(x)$ cannot be zero at any other point.

Example 28: The maximum value of $x^{1/x}$ is

(a) $(1/e)^e$ (b) $e^{1/e}$
(c) e (d) $1/e$

Ans. (b)

◎ **Solution:** Let $f(x) = x^{1/x} \Rightarrow \log f(x) = (1/x)\log x$. Differentiating both the sides, we have $f'(x) = f(x)\left[\frac{1 - \log x}{x^2}\right]$.

So $f'(x) = 0 \Leftrightarrow x = e$. Also $f'(x) > 0$ for $0 < x < e$ and $f'(x) < 0$ for $e < x < \infty$. Thus, $f(x)$ has a maximum at $x = e$ and max $f(x) = e^{1/e}$

Example 29: Let $P(x) = a_0 + a_1 x^2 + a_2 x^4 + \ldots + a_n x^{2n}$ be a polynomial in a real variable x with $0 < a_0 < a_1 \ldots < a_n$. The function $P(x)$ has

(a) neither a maximum nor a minimum
(b) only one maximum
(c) only one minimum
(d) none of these

Ans. (c)

◎ **Solution:** Since $P(x)$ is a polynomial, it is differentiable, and its extremum values are given by $P'(x) = 0$

$$\Rightarrow \quad 2a_1x + 4a_2x^3 + \ldots + 2na_n x^{2n-1} = 0$$

$$\Rightarrow \quad 2x(a_1 + 2a_2x^2 + \ldots + na_n x^{2n-2}) = 0$$

But, $a_1 > 0$ and other terms of $2a_2 x^2 + \ldots + n a_n x^{2n-2}$ are non negative so we have $x = 0$. Also $P''(x) = 2a_1 + 12a_2 x^2 + \ldots + 2n(2n-1) a_n x^{2n-2}$ so that $P''(0) = 2a_1 > 0$. Therefore, $P(x)$ has only one minimum at $x = 0$.

Example 30: If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval

(a) $f(x)$ and $g(x)$ are increasing functions
(b) both $f(x)$ and $g(x)$ are decreasing functions
(c) $f(x)$ is an increasing function
(d) $g(x)$ is an increasing function

Ans. (c)

◎ **Solution:** Let $f(x) = \frac{x}{\sin x} \Rightarrow f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$

Let $u(x) = \sin x - x\cos x$ so $u'(x) = \cos x - \cos x + x\sin x = x\sin x > 0$ for $0 < x \le 1$. Hence u is an increasing so $u(x) > u(0) = 0$. Thus $f'(x) > 0$ for $0 < x \le 1$. i.e. f is an increasing function. Now

$$g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}.$$

Let $v(x) = \tan x - x\sec^2 x$. Since $v'(x) = \sec^2 x - \sec^2 x - 2x\sec^2 x\tan x = -2x\sec^2 x\tan x < 0$ for $0 < x \le 1$. Hence $v(x) < v(0) = 0$ so $g'(x) < 0$ for $0 < x \le 1$. Thus g is a decreasing function.

Example 31: If $f(x) = x^{2/3}$ then

(a) (0, 0) is a point of maximum
(b) (0, 0) is not a point of minimum
(c) (0, 0) is a critical point
(d) There is no critical point

Ans. (c)

Solution: $\frac{dy}{dx} = \frac{2}{3}x^{-1/3}$. This derivative is never zero, but there is no derivative for $x = 0$. So (0, 0) is a critical point. If $x < 0$ then $\frac{dy}{dx} < 0$ and if $x > 0$ then $\frac{dy}{dx} > 0$. Thus (0, 0) is a point of minimum.

Example 32: Let $f(x) = (ax + b)/(cx + d)$ $(da - cb \ne 0, c \ne 0)$ then $f(x)$ has

(a) a critical point (b) no point of inflection
(c) a maximum (d) a minimum

Ans. (b)

Solution: $f'(x) = \frac{da - cb}{(cx+d)^2} \ne 0$ for any $x \in \mathbf{R}$ so f has no critical point in particular no minimum and no maximum. Now

$f''(x) = \frac{-2(da-cb)c}{(cx+d)^3} \ne 0$ for any $(x, f(x))$. Hence f has no point of inflection.

Example 33: Let $f(x) = (x - a)^m (x - b)^n$, where $m, n \in \mathbf{I}$ and $m, n > 1$. Then

(a) $(a, 0)$, $(b, 0)$ are the only critical points of f
(b) there are $m + n$ critical points of f
(c) there are exactly three critical points of f
(d) none of these

Ans. (c)

Solution: $f'(x) = m(x - a)^{m-1}(x - b)^n + n(x - a)^m (x - b)^{n-1}$

$= (x - a)^{m-1}(x - b)^{n-1}[(m + n)x - (mb + na)]$. Thus $f'(x) = 0 \Rightarrow x = a, x = b, x = \frac{mb + na}{m + n}$. So f has exactly three critical points.

Example 34: A ball is dropped from a platform 19.6 m high. Its position function is

(a) $x = -4.9t^2 + 19.6 (0 \le t \le 1)$
(b) $x = -4.9t^2 + 19.6 (0 \le t \le 2)$
(c) $x = -9.8t^2 + 19.6 (0 \le t \le 2)$
(d) $x = -4.9t + 19.6 (0 \le t \le 2)$

Ans. (b)

Solution: We have $a = \frac{d^2x}{dt^2} = -9.8$. The initial conditions are $x(0) = 19.6$ and $v(0) = 0$.

So $\quad v = \frac{dx}{dt} = -9.8t + v(0) = -9.8t$

$\therefore \quad x = -4.9t^2 + x(0) = -4.9t^2 + 19.6$

Now, the domain of the function is restricted since the ball hits the ground after a certain time. To find this time we set $x = 0$ and solve for t.

$$0 = -4.9t^2 + 19.6 \quad \Rightarrow \quad t = 2$$

Thus $\quad x = -4.9t^2 + 19.6 \quad (0 \le t \le 2)$

Example 35: Let $f(n) = 20n - n^2$ $(n = 1, 2, 3\ldots)$, then

(a) $f(n) \to \infty$ as $n \to \infty$
(b) $f(n)$ has no maximum
(c) The maximum value of $f(n)$ is greater than 200.
(d) None of these

Ans. (d)

Solution: Consider $f(x) = 20x - x^2$, defined for all real number x. $f'(x) = 20 - 2x$ and $f''(x) = -2$. Hence $x = 10$ is a point of maximum. Thus the maximum value of $f(n)$ is $200 - 100 = 100$.

Example 36: The point of the curve $y = x^2$ that is closest to $(4, -1/2)$ is

(a) (1, 1) (b) (2, 4)
(c) (2/3, 4/9) (d) (4/3, 16/9)

Ans. (a)

Solution: Let the required point be (x, y) on the curve so $d = \sqrt{(x-4)^2 + (y+1/2)^2}$ should be minimum. It is enough to consider

$$D = (x - 4)^2 + (y + 1/2)^2 = (x - 4)^2 + (x^2 + 1/2)^2.$$

$D' = 4x^3 + 4x - 8$. Now for critical points

$D' = 0$ so $x^3 + x - 2 = 0 \quad \Rightarrow \quad x = 1$. Clearly D'' at $x = 1$ is $16 > 0$. Thus D is minimum when $x = 1$. So the required point is (1, 1)

Example 37: The smallest value of M such that $|x^2 - 3x + 2| \le M$ for all x in the interval $[1, 5/2]$ is

(a) 1/4 (b) 3/4
(c) 5/4 (d) 5/16

Ans. (b)

Solution: Consider $f(x) = x^2 - 3x + 2$ on $[1, 5/2]$. Now $f'(x) = 2x - 3$ so the only critical point is 3/2. Since $f(1) = 0$, $f(3/2) = -5/4$ and $f(5/2) = 3/4$. min $\{f(x): x \in [1,5/2]\} = 0$, hence the

$\max\{|f(x)|: x \in [1, 5/2]\} = \max\{f(x): x \in [1, 5/2]\} = 3/4$. Thus $M = 3/4$.

Example 38: The number of solutions of the equation $a^{f(x)} + g(x) = 0$, where $a > 0$, $g(x) \neq 0$ and $g(x)$ has minimum value 1/2, is

(a) one (b) two
(c) infinitely many (d) zero

Ans. (d)

Solution: $a^{f(x)} = -g(x) \le -1/2$, since $g(x) \ge 1/2$ for all x. But this is not possible as $a^{f(x)} > 0$ for all x. Thus the number of solution is zero.

Example 39: The minimum value of $f(x) = |3 - x| + |2 + x| + |5 - x|$ is

(a) 0 (b) 7
(c) 8 (d) 10

Solution:

$$f(x) = \begin{cases} 6 - 3x, & x < -2 \\ 10 - x, & -2 \le x < 3 \\ x + 4, & 3 \le x < 5 \\ 3x - 6, & x \ge 5 \end{cases}$$

The function f is decreasing for $x \in (-\infty, 3)$ and increases on $(3, \infty)$. Hence $x = 3$ is a point of minimum and $\min f(x) = 7$.

Example 40: If $f(x) = x(x - 2)(x - 4)$, $1 \le x \le 4$, then a number satisfying the conclusion of the mean value theorem is

(a) 1 (b) 2
(c) 5/2 (d) 7/2

Ans. (a)

Solution: $f'(x) = (x - 2)(x - 4) + x(x - 4) + x(x - 2) = 3x^2 - 12x + 8$. Also $f(4) = 0$ and $f(1) = 3$. Thus $\dfrac{f(4) - f(1)}{4 - 1} = -1$. We must have $-1 = f'(x)$

$\Rightarrow \quad 3x^2 - 12x + 9 = 0$

$\Rightarrow \quad x^2 - 4x + 3 = 0$

$\Rightarrow \quad x = 1 \quad \text{or} \quad x = 3.$

Example 41: The sum of the intercepts of a tangent to $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $a > 0$ upon the coordinate axes is

(a) $2a$ (b) a
(c) $a/2$ (d) $\sqrt{a}$

Ans. (b)

Solution: $\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}$.

Equation of tangent at any point (x, y) of the curve is $Y - y = -\dfrac{\sqrt{y}}{\sqrt{x}}(X - x)$. So intercepts of X-axis and Y-axis are $x + \sqrt{xy}$ and $y + \sqrt{xy}$.

Therefore, the sum of intercepts $= x + y + 2\sqrt{xy} = \left(\sqrt{x} + \sqrt{y}\right)^2 = a$.

Example 42: Let x and y be two real numbers such that $x > 0$ and $xy = 1$. The minimum value of $x + y$ is

(a) 1 (b) 1/2
(c) 2 (d) 1/4

Ans. (c)

Solution: If $xy = 1$ then $S = x + y = x + 1/x$ and $S'(x) = 1 - 1/x^2 \Rightarrow S'(x) = 0 \Leftrightarrow x = 1$ (as $x > 0$). Also $S''(x) = 2/x^3 > 0$ for $x = 1$. Hence minimum value of $x + y$ is 2.

Alternatively $S = x + \dfrac{1}{x} = \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2 + 2 \ge 2$ and $S = 2$ for $x = 1$. Hence the minimum value of S is 2.

Example 43: The function $\dfrac{\sin(x + \alpha)}{\sin(x + \beta)}$ has no maximum or minimum if (k an integer)

(a) $\beta - \alpha = k\pi$ (b) $\beta - \alpha \neq k\pi$
(c) $\beta - \alpha = 2k\pi$ (d) none of the above

Ans. (b)

Solution: Let $f(x) = \dfrac{\sin(x + \alpha)}{\sin(x + \beta)}$, so that

$$f'(x) = \frac{\sin(x + \beta)\cos(x + \alpha) - \cos(x + \beta)\sin(x + \alpha)}{\sin^2(x + \beta)} = \frac{\sin(\alpha - \beta)}{\sin^2(x + \beta)}$$

Thus, $f(x)$ has no maximum or minimum if $f'(x) \neq 0$. i.e. if $\sin(\beta - \alpha) \neq 0$ equivalently $\beta - \alpha \neq k\pi$, $k \in \mathbf{I}$.

Example 44: The tangent to the curve $y = x^3 - 6x^2 + 9x + 4$, $0 \le x \le 5$ has maximum slope at x which is equal to

(a) 2 (b) 3
(c) 4 (d) none of these

Ans. (d)

Solution: $u = \dfrac{dy}{dx} = 3x^2 - 12x + 9$. For u to be maximum or minimum $\dfrac{du}{dx} = 0$ which is true if and only if $6x - 12 = 0 \Leftrightarrow x = 2$. Now $u(0) = 9$, $u(2) = -3$ and $u(5) = 24$. Thus the maximum of $u(x)$, $0 \le x \le 5$ is $u(5)$ so $x = 5$.

Example 45: The set of all values of the parameters a for which the points of minimum of the function $y = 1 + a^2x - x^3$ satisfy the inequality $\dfrac{x^2 + x + 2}{x^2 + 5x + 6} \le 0$ is

(a) an empty set

(b) $\left(-3\sqrt{3}, -2\sqrt{3}\right)$

(c) $\left(2\sqrt{3}, 3\sqrt{3}\right)$

(d) $\left(-3\sqrt{3},-2\sqrt{3}\right) \cup \left(2\sqrt{3},3\sqrt{3}\right)$

Ans. (c)

◎ **Solution:** $\frac{dy}{dx} = a^2 - 3x^2 = 0 \Leftrightarrow x = \pm a/\sqrt{3}$. Since $\frac{d^2y}{dx^2} = -6x$ so y is minimum for $x = -a/\sqrt{3}$.

Since $x^2 + x + 2 > 0$ for all x so for $\frac{x^2+x+2}{x^2+5x+6} \le 0$, we must have $x^2 + 5x + 6 < 0$. If $x = -a/\sqrt{3}$, we have $a^2/3 - 5a/\sqrt{3} + 6 < 0$ i.e. $a^2 - 5\sqrt{3}\ a + 18 < 0 \Leftrightarrow (a - 2\sqrt{3})(a - 3\sqrt{3}) < 0$ i.e. $a \in (2\sqrt{3}, 3\sqrt{3})$.

◉ **Example 46:** The point of intersection of the tangents drawn to the curve $x^2y = 1 - y$ at the points where it is met by the curve $xy = 1 - y$ is given by

(a) (0, – 1) (b) (1, 1)
(c) (0, 1) (d) none of these

Ans. (c)

◎ **Solution:** The intersection of $x^2y = 1 - y$ and $xy = 1 - y$ are given by $x^2 y = xy$. Since $y \ne 0$ on any of two curves so we must have $x = 0, 1$. Thus the curves intersects at (0, 1); (1, 1/2).

Now, differentiating $x^2 y = 1 - y$, we have $\frac{dy}{dx} = -\frac{2xy}{1+x^2}$.

Thus $\left.\frac{dy}{dx}\right|_{(0,1)} = 0$ and $\left.\frac{dy}{dx}\right|_{(1,1/2)} = -\frac{1}{2}$. Equations of tangents are $Y = 1$ and $Y - 1/2 = (-1/2)(X - 1)$. Their intersection is given by (0, 1).

◉ **Example 47:** The equation of the tangent to the curve $y = (2x - 1)\,e^{2(1-x)}$ at the point of its maximum is

(a) $y = 1$ (b) $x = 1$
(c) $x + y = 1$ (d) $x - y = -1$

Ans. (a)

◎ **Solution:** $y'(x) = -2(2x-1)\,e^{2(1-x)} + 2e^{2(1-x)} = 2e^{2(1-x)}(-2x+2)$

Thus $y'(x) = 0 \Rightarrow x = 1$. Since, $y''(1) < 0$ so (1, 1) is the point of maximum and an equation of tangent to the curve is $y - 1 = 0\,(x - 1)$, i.e. $y = 1$.

◉ **Example 48:** If the function $f(x) = x^2 + \alpha/x$ has a local minimum at $x = 2$, then the value of α is

(a) 8 (b) 18
(c) 16 (d) none of these

Ans. (c)

◎ **Solution:** $f'(x) = 2x - \alpha/x^2$, so $f'(x) = 0 \Leftrightarrow 2x = \alpha/x^2 \Leftrightarrow x = (\alpha/2)^{1/3}$. Clearly for $x = (\alpha/2)^{1/3}$, we have $f''(x) > 0$. So $2 = (\alpha/2)^{1/3} \Rightarrow \alpha = 16$.

◉ **Example 49:** Three normals are drawn to the parabola $y^2 = 4x$ from the point $(c, 0)$. These normals are real and distinct when

(a) $c = 0$ (b) $c = 1$
(c) $c = 2$ (d) $c = 3$

Ans. (d)

◎ **Solution:** Any point on $y^2 = 4x$ is $(t^2, 2t)$. Since $\frac{dy}{dx} = \frac{2}{y}$ so $\left.\frac{dy}{dx}\right|_{(t^2,2t)} = \frac{2}{2t} = \frac{1}{t}$. Hence equation of normal at $(t^2, 2t)$ is

$$Y - 2t = -t(X - t^2)$$

This passes through $(c, 0)$ if $-2t = -t(c - t^2)$

$$\Rightarrow \quad t = 0,\ t^2 = c - 2$$

Thus the roots are real and distinct if $c > 2$ so $c = 3$ is the correct choice.

◉ **Example 50:** The function $f(x) = (\log(x-1))^2 (x-1)^2$ has

(a) local extremum at $x = 1$
(b) point of inflection at $x = 1$
(c) local extremum at $x = 2$
(d) point of inflection at $x = 2$

Ans. (c)

◎ **Solution:** $f'(x) = (x-1)^2\, 2\log(x-1) \cdot 1/(x-1) + 2(\log(x-1))^2(x-1)$

$= 2(x-1)[\log(x-1)][1 + \log(x-1)]$

$f'(x) = 0 \Rightarrow x = 2$ or $1 + e^{-1}$, since the domain of f is $\{x : x > 1\}$.

$$f''(x) = 2\log(x-1)(1 + \log(x-1)) + 2(x-1) \times \left[\frac{1}{x-1} + \frac{2}{x-1}\log(x-1)\right]$$

$f''(2) = 2 > 0$. Hence $x = 2$ is a point of extremum.

◉ **Example 51:** If $f(x) = \log x$ satisfies Lagrange's theorem on $[1, e]$ then value of $c \in (1, e)$ such that the tangent at c is parallel to line joining $(1, f(1))$ and $(e, f(e))$ is

(a) $e - \frac{3}{2}$ (b) $\frac{1+e}{2}$

(c) $e - 1$ (d) $e - \frac{1}{2}$

Ans. (c)

◎ **Solution:** $f'(c) = \frac{1}{c}$

So $\frac{f(e) - f(1)}{e-1} = f'(c) = \frac{1}{c}$

$\Rightarrow \frac{\log e - \log 1}{e - 1} = \frac{1}{c}$

$\Rightarrow \frac{1}{e-1} = \frac{1}{c} \quad \Rightarrow \quad c = e - 1$

Example 52: The value of c for which the conclusion of Lagrange's theorem holds for the function $f(x) = \sqrt{a^2 - x^2}, a > 1$ on the interval [1, a] is

(a) $\frac{a(a+1)}{2}$ (b) $\frac{1+a}{2}$

(c) $\frac{\sqrt{a(a+1)}}{2}$ (d) $\frac{a(a-1)}{2}$

Ans. (c)

Solution: $\frac{f(a)-f(1)}{a-1} = f'(c) = \frac{-c}{\sqrt{a^2-c^2}}$

$$\frac{\sqrt{a^2-a^2}-\sqrt{a^2-1}}{a-1} = \frac{-c}{\sqrt{a^2-c^2}}$$

$$\Rightarrow \quad -\sqrt{\frac{a+1}{a-1}} = -\frac{c}{\sqrt{a^2-c^2}}$$

$$\Rightarrow \quad \frac{a+1}{a-1} = \frac{c^2}{a^2-c^2}$$

$$\Rightarrow \quad (a+1)(a^2-c^2) = c^2(a-1)$$

$$\Rightarrow \quad c^2(-a-1+1-a) = -(a+1)a^2$$

$$\Rightarrow \quad c^2(2a) = (a+1)a^2$$

$$\Rightarrow \quad c^2 = \frac{(a+1)a}{2}$$

Since $c \in (1, a)$ so $c = \sqrt{\frac{(a+1)a}{2}}$

Example 53: Let $f(x) = \begin{cases} |x-2|+a, & \text{if } x \le 2 \\ 4x^2+3x+1, & \text{if } x > 2 \end{cases}$

If $f(x)$ has a local minimum at $x = 2$, then

(a) $a > 21$ (b) $a \le 21$

(c) $a > 30$ (d) $a > 24$

Ans. (b)

Solution: Since f has a local minimum at $x = 2$

So $f(2) \le f(x)$ for x in an open interval around 2.

$$\Rightarrow \quad f(2) \le \lim_{x\to 2+} f(x)$$

Thus, $a = f(2) \le \lim_{x\to 2} 4x^2+3x+1 = 21$

Example 54 If $y = mx + 2$ is parallel to a tangent to curve $e^{4y} = 1 + 16x^2$ then

(a) $|m| \le 1$ (b) $|m| < 1$

(c) $|m| > 1$ (d) $|m| \ge 1$

Ans. (a)

Solution Differentiating, we have

$$4e^{4y}\frac{dy}{dx} = 32x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{8x}{1+16x^2}$$

According to the given condition

$$m = \frac{8x}{1+16x^2}$$

$$|m| = \left|\frac{8x}{1+16x^2}\right| \le 1 \quad (\text{since } 16x^2 - 8|x| + 1 = (4|x|-1)^2 \ge 0)$$

Example 55: Given the function $f(x) = x^2 e^{-2x}$, $x > 0$. Then $f(x)$ has the maximum value equal to

(a) e^{-2} (b) $(2e)^{-1}$

(c) e^{-1} (d) none of these.

Ans. (a)

Solution: $f'(x) = 2xe^{-2x} - 2x^2e^{-2x} = 2(1-x)x\,e^{-2x}$. Now, $f'(x) = 0$
$\Leftrightarrow \quad x = 1, 0$. Also $f''(x) = 2(1-x)e^{-2x} - 2xe^{-2x}$
$-4(1-x)xe^{-2x}$, so $f''(1)$
$= -2e^{-2} < 0$ and $f''(0) > 0$. Thus max $f(x) = f(1) = e^{-2}$.

Example 56: Let $f(x) = (x-4)(x-5)(x-6)(x-7)$ then

(a) $f'(x) = 0$ has four real roots

(b) three roots of $f'(x) = 0$ lie in $(4, 5) \cup (5, 6) \cup (6, 7)$

(c) the equation $f'(x) =$ has only two roots

(d) three roots of $f'(x) = 0$ lie in $(3, 4) \cup (4, 5) \cup (5, 6)$

Ans. (b)

Solution: Since $f(4) = f(5) = f(6) = f(7) = 0$, so by Rolle's theorem applied to the intervals [4, 5], [5, 6], [6, 7] there exist $x_1 \in (4, 5)$, $x_2 \in (5, 6)$, $x_3 \in (6,7)$ such that $f'(x_1) = f'(x_2) = f'(x_3) = 0$. Since f' is a polynomial of degree 3 so cannot have four roots.

Example 57: If $f(x) = \frac{x^2-1}{x^2+1}$ for every real number x, then the minimum value of f

(a) does not exist because f is unbounded

(b) is not attained even though f is bounded

(c) is equal to 1

(d) is equal to -1

Ans. (d)

Solution: We have $f(x) = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$

$f(x)$ will attain its minimum value when $2/(x^2+1)$ is maximum i.e. when x^2+1 is minimum i.e. at $x = 0$. Thus min $f(x) = f(0) = -1$. Since $-1 \le f(x) < 1$ so f is bounded.

Example 58: For all $x \in (0, 1)$

(a) $e^x < 1 + x$ (b) $\log_e(1 + x) < x$

(c) $\sin x > x$ (d) $\log_e x > x$

Ans. (b)

Solution: Since $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} > 1 + x$ for $x > 0$ so (a) is not true. Also log x is an increasing function so $x > \log(1 + x)$. Again $e^x > x$ for $x > 0$ so $x > \log_e x$. We know that

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots = x - \left(\frac{x^3}{3!} - \frac{x^5}{5!} + \ldots\right)$ The terms inside the brackets are positive since $x \in (0, 1)$ so $\sin x < x$.

Example 59: Let f be a differentiable function on $\mathbf{R}$ and $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for all $x \in \mathbf{R}$. Then

(a) h increases whenever f decreases
(b) h decreases whenever f increases
(c) h increases or decreases accordingly as f increases or decreases
(d) nothing can be claimed in general

Ans. (c)

Solution: $h'(x) = f'(x) - 2f(x)f'(x) + 3(f(x))^2 f'(x)$

$$= f'(x)[1 - 2f(x) + 3(f(x))^2]$$

Since $1 - 2f(x) + 3(f(x))^2 = 3\left[\left(f(x) - \frac{1}{3}\right)^2 + \frac{2}{9}\right] > 0$ for all x

so $h'(x) > 0$ or < 0 accordingly as $f'(x) > 0$ or < 0. Hence h increases or decreases according as f increases or decreases.

Example 60: Let $f(x) = ax^3 + bx^2 + cx + d$, $b^2 - 3ac > 0$, $a > 0$, $c < 0$. Then $f(x)$ has

(a) local maximum at some $x \in \mathbf{R}^+$
(b) a local maximum at some $x \in \mathbf{R}^-$
(c) a local minima at $x = 0$
(d) local minima at some $x \in \mathbf{R}^-$, $\mathbf{R}^+ = (0, \infty)$. $\mathbf{R}^- = (-\infty, 0)$

Ans. (b)

Solution: $f'(x) = 3ax^2 + 2bx + C$. The discriminant $D = 4b^2 - 12ac = 4(b^2 - 3ac) > 0$

So $f'(x) = 0$ has exactly two real roots say α, β so $f'(x) = 3a(x - \alpha)(x - \beta)$. But $\frac{c}{a} < 0 \Rightarrow \alpha\beta < 0$. So we have $\alpha < 0$, $\beta > 0$ or $\alpha > 0$, $\beta > 0$. If $\alpha < 0$, $\beta > 0$, for $x < \alpha$, $f'(x) > 0$ and for $x > \alpha$ but near to α, $f'(x) > 0$. Also for $x < \beta$, but close to β, $f'(x) < 0$ and for $x > \beta$, $f'(x) > 0$. Thus f has local maxima at $x = \alpha$ which is $\mathbf{R}^-$ and f has local minima at $x = \beta \in \mathbf{R}^+$
In case $\alpha > 0$, $\beta < 0$, our conclusion is same *i.e.*, local minima at $x = \alpha$ and local maxima at $x = \beta$.

Example 61: If $f(x) = \begin{cases} 3 - x^2, & x \le 2 \\ \sqrt{a + 14} - |x - 48|, & x > 2 \end{cases}$ and $f(x)$ has a local maxima at $x = 2$, then

(a) a cannot be determined
(b) least value of a is 2011
(c) greatest value of a is 2011
(d) $a \ge 3010$

Ans. (c)

Solution: Since f has local maxima at $x = 2$, so $\lim_{h \to 0} f(2 + h) \le f(2)$

Since $\lim_{h \to 0-} f(2 + h) = \lim_{h \to 0-} 3 - (2 + h)^2 = f(2)$

So we must have $\lim_{h \to 0+} f(2 + h) \le f(2)$

$\Rightarrow \quad \lim_{h \to 0+} \sqrt{a + 14} - |2 + h - 48| \le -1$

$\Rightarrow \quad \sqrt{a + 14} \le 46 - 1 = 45$

$\Rightarrow \quad a \le 2011.$

The greatest value of a is 2011.

Example 62: The total number of local maxima and local minima of the function

$$f(x) = \begin{cases} (2 + x)^3, & -3 < x \le -1 \\ x^{2/3}, & - < x < 2 \end{cases}$$

(a) 0 (b) 1
(c) 2 (d) 3

Ans. (c)

Solution: Since $\lim_{x \to -1-} f(x) = \lim_{x \to -1+} f(x) = 1$

So, f is continuous on $(-3, 2)$. Also

$$f'(x) = \begin{cases} 3(x + 2)^2, & -3 < x < -1 \\ (2/3)\, x^{-1/3}, & -1 < x < 0, 0 < x < 2 \end{cases}$$

$f'(0)$ does not exist

$$\mathrm{L}f'(-1) = \lim_{x \to -1-} \frac{f(x) - f(1)}{x + 1} = \lim_{x \to -1-} \frac{(x + 2)^3 - 1}{x + 1}$$

$$= \lim_{x \to -1-} \frac{(x + 1)\left[(x + 2)^2 + (x + 2) + 1\right]}{x + 1} = 3.$$

$$\mathrm{R}f'(-1) = \lim_{x \to -1+} \frac{x^{2/3} - 1}{x + 1}$$

$$= \lim_{x \to 1+} \frac{\left(x^{1/3} - 1\right)\left(x^{1/3} + 1\right)}{\left(x^{1/3} + 1\right)\left(x^{2/3} + x^{1/3} + 1\right)} = -2$$

So f is not differentiable at $x = -1$
For $x \ne 0, -1$, $f'(x) \ne 0$. Critical points of f are -1, 0. At $x = -1$, f has a local maximum and at $x = 0$, f has a local minimum.

Example 63: Let $g : \mathbf{R} \to (-\pi/2, \pi/2)$ be given by $g(u) = 2\tan^{-1}(e^u) - \pi/2$, then g is

(a) even and is strictly increasing in $(0, \infty)$
(b) odd and is strictly decreasing in $(-\infty, \infty)$
(c) odd and is strictly increasing in $(-\infty, \infty)$
(d) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$.

Ans. (c)

Solution: $g(-u) = 2\tan^{-1}\left(\frac{1}{e^u}\right) - \pi/2$

$$= 2\cot^{-1} e^u - \pi/2$$

$$= 2\left[\pi/2 - \tan^{-1} e^u\right] - \pi/2$$

$$= \pi/2 - \tan^{-1} e^u = -g(u)$$

Also, $\qquad g'(u) = \dfrac{2\,e^u}{1+e^{2u}} > 0 \;\forall\; u \in \mathbf{R}$

Thus g is an odd function which is strictly increasing on **R**.

Example 64: Let $f(x) = \begin{cases} x^\alpha \log x & , \text{ if } x > 0 \\ 0 & , \text{ if } x = 0 \end{cases}$

If Rolle's theorem can be applied to f on [0, 1] then value of α can be

(a) – 1 (b) – 1/2
(c) 0 (d) 1/2

Ans. (d)

Solution: For the Rolle's theorem to be applicable on [0, 1], we must have that f is continuous on [0, 1] and in particular at $x = 0$ from the right.

So $\qquad \lim\limits_{x\to 0+} x^\alpha \log x = f(0) = 0$

$$\Rightarrow \qquad \lim_{x\to 0+} \frac{\log x}{x^{-\alpha}} = 0$$

$$\Rightarrow \quad \lim_{x\to 0+} \frac{(1/x)}{(-\alpha)x^{-\alpha-1}} = 0 \quad \Rightarrow \quad \lim_{x\to 0+} \frac{x^\alpha}{-\alpha} = 0$$

This is possible only if $\alpha > 0$.

Example 65: Suppose that $f(x) = x^3 - px + q$ has three distinct real roots where $p, q > 0$. Then which one of the following hold?

(a) $f(x)$ has minima at $\sqrt{\dfrac{p}{3}}$ and maxima at $-\sqrt{\dfrac{p}{3}}$.

(b) $f(x)$ has minima at $-\sqrt{\dfrac{p}{3}}$ and maxima at $\sqrt{\dfrac{p}{3}}$.

(c) $f(x)$ has minima at both $\sqrt{\dfrac{p}{3}}$ and $-\sqrt{\dfrac{p}{3}}$.

(d) $f(x)$ has maxima at both $\sqrt{\dfrac{p}{3}}$ and $-\sqrt{\dfrac{p}{3}}$.

Ans. (a)

Solution: As $f(x) = 0$ has three real and distinct zero so $f'(x) = 3x^2 - p = 0$ has two real and distinct zeros by Rolle's Theorem.

So $\qquad x = \pm\sqrt{\dfrac{p}{3}}.$

$$f'(x) > 0 \quad \text{if} \quad x < -\sqrt{\frac{p}{3}}\,;\; f'(x) < 0 \quad \text{if} \quad -\sqrt{\frac{p}{3}} < x < \sqrt{\frac{p}{3}}\,;$$

$$f'(x) > 0 \quad \text{if} \quad x > \sqrt{\frac{p}{3}}.$$

Thus $f(x)$ has a local maximum at $x = -\sqrt{\dfrac{p}{3}}$ and a local minimum at $x = \sqrt{\dfrac{p}{3}}$.

Example 66: Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$.
If $P(-1) < P(1)$, then in the interval [– 1, 1]

(a) $P(-1)$ is the minimum but $P(1)$ is not the maximum of P
(b) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P
(c) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
(d) $P(-1)$ is not minimum but $P(1)$ is the maximum of P

Ans. (d)

Solution: $P'(x) = 4x^3 + 3ax^2 + 2\,bx + c$

As $\qquad P'(0) = 0 \quad$ so $\quad c = 0$

$\therefore \qquad P'(x) = x(4x^2 + 3ax + 2b)$

As $x = 0$ is the only real root of $P'(x) = 0$, roots of $4x^2 + 3ax + 2b$ must have imaginary roots,
therefore $4x^2 + 3ax + 2\,b > 0 \;\forall\; x \in \mathbb{R}$

Thus $\qquad P'(x) < 0$ for $x < 0$
$\qquad\qquad > 0$ for $x > 0$

Therefore, $x = 0$ is a point of local minimum at $x = 0$ As $P(-1) < P(1)$, we get $P(1)$ is maximum but $P(-1)$ is not minimum of P on [– 1, 1].

Example 67: Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} k - 2x & , \text{ if } x \le -1 \\ 2x + 3 & , \text{ if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k is

(a) –1/2 (b) – 1
(c) 1 (d) 0

Ans. (b)

Solution: If f has a local minimum at $x = -1$ then $\lim\limits_{h\to 0} f(-1+h) \ge f(-1) = k + 2$

$$\Rightarrow \qquad \lim_{h\to 0+} f(-1+h) \ge k + 2$$

$$\Rightarrow \qquad \lim_{h\to 0+} 2(-1+h) + 3 \ge k + 2$$

$$\Rightarrow \quad 1 \ge k + 2 \Rightarrow k \le -1.$$

Example 68: The value of a in order that $f(x) = \sin x - \cos x - ax + b$ decreases for all real values of x is given by

(a) $a \ge \sqrt{2}$ (b) $a < \sqrt{2}$
(c) $a \ge 1$ (d) $a < 1$

Ans. (a)

Solution: $f'(x) = \cos x + \sin x - a$

$$= \sqrt{2}\sin(x + \pi/4) - a$$

$f'(x) \le 0$ for all x if

$$a \ge \max_{x \in \mathbf{R}} \sqrt{2}\sin(x + \pi/4) = \sqrt{2}$$

Example 69: The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact, is given by :

(a) $2y - 3x = 0$ (b) $y = \dfrac{6}{x}$

(c) $x^2 + y^2 = 13$ (d) $\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{3}\right)^2 = 2$

Ans. (b)

Solution: The equation of tangent at (x, y) to the curve $y = f(x)$ is given by

$$Y - y = \frac{dy}{dx}(X - x)$$

This tangent meets the axes at $A\left(x - y\dfrac{dx}{dy}, 0\right)$ and $B\left(0, y - x\dfrac{dy}{dx}\right)$. We are given mid point of AB as (x, y) so,

$$\frac{1}{2}\left(x - y\frac{dx}{dy}\right) = x \text{ and } \frac{1}{2}\left(y - \frac{xdy}{dx}\right) = y$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} = 0 \quad \Rightarrow \quad \log|xy| = C$$

$$\Rightarrow \quad xy = \pm e^c = A$$

As it passes through (2, 3) so $A = 6$

$$\therefore \quad xy = 6 \text{ or } y = 6/x$$

Example 70: A spherical balloon is filled with 4500 π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72 π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is :

(a) 7/9 (b) 2/9

(c) 9/2 (d) 9/7

Ans. (b)

Solution: Let r be the radius of the balloon and V be its volume. It is given that

$$\frac{dV}{dt} = -72\pi$$

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -72\pi \Rightarrow \frac{4}{3}\pi r^3 = -72\pi t + C$$

When $t = 0$, $\dfrac{4}{3}\pi r^3 = V = 4500\pi$

$$\therefore \quad \frac{4}{3}\pi r^3 = 4500\pi - 72\pi t$$

When $t = 49$, $\dfrac{4}{3}\pi r^3 = 4500\pi - (72\pi)(49) = 972\pi$

$$\Rightarrow \quad r^3 = 729 \quad \text{so} \quad r = 9$$

Also, $\dfrac{d}{dt}\left(\dfrac{4}{3}\pi r^3\right) = -72\pi \quad \Rightarrow 4\pi r^2 \dfrac{dr}{dt} = -72\pi$

$$\Rightarrow \quad \frac{dr}{dt} = -\frac{72\pi}{4\pi r^2} = -\frac{18}{r^2}. \text{ So } \left.\frac{dr}{dt}\right|_{r=9} = -\frac{2}{9}.$$

Assertion-Reason Type Questions

Example 71: Let $f(x) = 2\sqrt{x}$ and $g(x) = 3 - 1/x$, $x > 1$

Statement-1: $f(x) > g(x)$ $(x > 1)$

Statement-2: $f(x) - g(x)$ increases on $(1, \infty)$

Ans. (a)

Solution: $f'(x) - g'(x) = \dfrac{x^{3/2} - 1}{x^2} > 0$ for $x > 1$

Hence $f(x) - g(x) > f(1) - g(1) = 0$, $x > 1$

Therefore $f(x) > g(x)$, $x > 1$.

Example 72: Let $f(x) = x^4 - 2x^2 + 5$ be defined on $[-2, 2]$

Statement-1: The range of $f(x)$ is [2, 13]

Statement-2: The greatest value of f is attained at $x = 2$

Ans. (d)

Solution: $f'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1)$. The critical points of f are 0, – 1, 1 But $f(0) = 5$, $f(1) = 4$, $f(-1) = 4$, $f(2) = 13$. So the range of f is [4, 13] and grea, test value of f is at $x = 2$.

Example 73: Let $f(x) = 2x^3 + 3x^2 - 12x + 1$

Statement-1: f decreases on $(-2, 1)$

Statement-2: The solution set of $x^2 + x - 2 < 0$ is $(-2, 1)$.

Ans. (a)

Solution: $f'(x) = 6x^2 + 6x - 12 = 6(x - 1)(x + 2)$

So $f'(x) < 0$ if and only if $x \in (-2, 1)$.

Example 74: Let $y = x + \dfrac{a^2}{x}$ $(a > 0)$

Statement-1: $y_{max} = -2a$ (local mix)

Statement-2: $y_{min} = 2a$ (local min)

Ans. (b)

◎ **Solution:** $y'(x) = 1 - \frac{a^2}{x^2} = \frac{(x-a)(x+a)}{x^2}$

⇒ $y'(x) > 0$ for $x \in (a, \infty) \cup (-\infty, -a)$ and $y'(x) < 0$ for $x \in (-a, a)$. Therefore y is local maximum at $x = -a$ and loval minimum at $x = a$.

◉ **Example 75:** Let $f(x) = (x-1)(x-2)(x-3)(x-4)$

Statement-1: $f'(x) = 0$ has three roots lying in $(1, 2) \cup (2, 3) \cup (3, 4)$

Statement-2: Rolle's theorem is valid for f on $[1, 4]$

Ans. (b)

◎ **Solution:** Applying Rolle's theorem on [1, 2], [2, 3], [3, 4], we get three roots of $f'(x) = 0$ on (1, 2) (2, 3), (3, 4).

◉ **Example 76:** Let f be a function defined by

$$f(x) = \begin{cases} \frac{\tan x}{x} & , \text{ if } x \neq 0 \\ 1 & , \text{ if } x = 0 \end{cases}$$

Statement-1: $x = 0$ is point of maxima of f

Statement-2: $f'(0) = 0$

Ans (b)

◎ **Solution:** Let $g(x) = \tan x - x$, $-\pi/2 < x < x/2$

⇒ $g'(x) = \text{see}^2 x - 1 = \tan^2 x > 0$ for $-\pi/2 < x < \pi/2$

⇒ g increases on $(-\pi/2, \pi/2)$

⇒ $\tan x < x$ for $-\pi/2 < x < \pi/2$

Thus $f(x) = \frac{\tan x}{x} < 1$ for $-\pi/2 < x < \pi/2, x \neq 0$

and $f(1) = 1$, ∴ $x = 0$ in point of maxima.

Also $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\tan x - x}{x^2}$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{2x} = \frac{1}{2} \lim \frac{\tan^2 x}{x} = 0$$

Hence $f'(0) = 0$

Thus, both the statements are true. However statement: 2 in not a correct explanation of statement: 1.

◉ **Example 77:** Let $a, b \in \mathbf{R}$ be such that the function f given by $f(x) = \log|x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$

Statement-1: f has local maximum at $x = -1$ and at $x = 2$

Statement-2: $a = 1/2$ and $b = -1/4$.

Ans. (a)

◎ **Solution:** $f'(x) = \frac{1}{x} + 2bx + a \;,\; x \neq 0$

As $x = -1$ and $x = 2$ are extreme value of f, $f'(-1) = 0$ and $f'(2) = 0$

$\Rightarrow -1 - 2b + a = 0$ and $\frac{1}{2} + 4b + a = 0$

Solving these equations, we get $a = 1/2$, $b = -1/4$

Thus $f'(x) = \frac{1}{x} - \frac{1}{2}x + \frac{1}{2} = -\frac{1}{2x}(x+1)(x-2)$.

We have $f'(x) > 0$, if $x < -1$; $f'(x) < 0$, if $-1 < x < 0$; $f'(x) > 0$, if $0 < x < 2$ and $f'(x) < 0$, if $x > 2$.

Therefore, $f(x)$ has a local maximum at $x = -1$ and $x = 2$. Thus statement-1 and statement-2 are True and statement-2 is a correct explanation for statement :1.

◉ **Example 78:** **Statement 1:** $x^y > y^x$, $e < x < y$

Statement 2: $f(x) = \frac{\log x}{x}$ is a decreasing function for $x > e$.

Ans. (a)

◎ **Solution:** Let $f(x) = \frac{\log x}{x}$

$$f'(x) = \frac{1 - \log x}{x^2}$$

Thus $f'(x) > 0$ if $1 - \log x > 0$ i.e. $\log x < 1$ i.e. $x < e$ and $f'(x) < 0$ if $x > e$.

So f increases on $(0, e)$ and decreases on (e, ∞)

$$e < x < y \Rightarrow f(x) > f(y) \Rightarrow \frac{\log x}{x} > \frac{\log y}{y}$$

⇒ $y \log x > x \log y \Rightarrow x^y > y^x$.

◉ **Example 79:** Let $f(x) = |x - 7| + |x - 10| + |x - 12|$

Statement 1: $f(x)$ has a minimum at $x = 12$

Statement 2: f is not differentiable at $x = 12$

Ans. (d)

◎ **Solution:** $f(x) = \begin{cases} 29 - 3x & , \quad x < 7 \\ 15 - x & , \quad 7 \leq x < 10 \\ x - 5 & , \quad 10 \leq x < 12 \\ 3x - 29 & , \quad x \geq 12 \end{cases}$

Since $f'(x)$ changes sign around $x = 10$ from negative to positive f has minimum at $x = 10$.

f is clearly not differentiable at $x = 12$.

◉ **Example 80:** Let b, c be two non-zero real numbers such that $b^2 \leq 3c$

Let $f(x) = x^3 + bx^2 + cx + d$, $x \in \mathbf{R}$.

Statement 1: f is 1–1 function

Statement 2: f is strictly function

Ans. (c)

◎ **Solution:** $f'(x) = 3x^2 + 2bx + c$

The discriminant of f' is $4b^2 - 12c = 4(b^2 - 3c) < 0$.

So, $f'(x) > 0$. Hence f is an increasing function.

If there exists x_1, x_2 s.t. $x_1 \neq x_2$ (say $x_1 < x_2$) and $f(x_1) = f(x_2)$ then by Rolle's theorem there is $c \in (x_1, x_2)$ such that $f'(c) = 0$ but $f'(x) > 0$ for all x. So f is $1 - 1$.

LEVEL 2

Straight Objective Type Questions

Example 81: The point M (x, y) of the graph of the function $y = e^{-|x|}$ so that area bounded by the tangent at M and the coordinate axes is greatest is

(a) $(1, e^{-1})$ (b) $(2, e^{-2})$
(c) $(-2, e^{2})$ (d) $(0, 1)$

Ans. (a)

Solution: For $x \geq 0$, $y = e^{-x}$. The equation of tangent is $Y - y = -e^{-x}(X - x)$. This will intersect coordinate axes at $(x + ye^{x}, 0)$ and $(0, y + xe^{-x})$. Hence the area of the required triangle A is $\frac{1}{2}(y + xe^{-x})(x + ye^{x})$

$$= \frac{1}{2}(1 + x)^2 e^{-x} \qquad [\because y = e^{-x}]$$

Now $$\frac{dA}{dx} = \frac{1}{2}[-(1 + x)^2 e^{-x} + 2(1 + x)e^{-x}]$$

$$= \frac{1}{2}(1 + x)e^{-x}(1 - x)$$

Note that $\frac{dA}{dx} = 0 \Rightarrow \quad x = 1, -1$

Also, $\frac{dA}{dx} > 0$, if $0 \leq x < 1$ and $\frac{dA}{dx} < 0$ if $x < 1$. Hence A is maximum when $x = 1$ so $y = e^{-1}$. Since y is even function other possibility of M is $(-1, e^{-1})$.

Example 82: The abscissa of the point on the curve $9y^2 = x^3$, the normal at which cuts off equal intercepts on the coordinate axes is

(a) 2 (b) 4
(c) – 4 (d) – 2

Ans. (b)

Solution: Differentiating $9y^2 = x^3$ we have $\frac{dy}{dx} = \frac{x^2}{6y}$.

Any point on the curve is of the form $(t^2, t^3/3)$ and so $\frac{dy}{dx}$ at this point is $t/2$. Thus an equation of normal is

$$y - t^3/3 = (-2/t)(X - t^2)$$

This will intersect coordinate axes at $(0, 2t + t^3/3)$ and $\left(\frac{t^4}{6} + t^2, 0\right)$. Hence we must have

$$2t + \frac{t^3}{3} = t^2 + \frac{t^4}{6} \Rightarrow 2 + \frac{t^2}{3} = t + \frac{t^3}{6}$$

Clearly $t = 2$ satisfies, the last equation. Hence the abscissa of the required point is 4. (For $t = 0$, the normal meets both the axes only at origin.)

Example 83: A curve passes through the point (2, 0) and the slope of the tangent at any point (x, y) is $x^2 - 2x$ for all value of x. The point of maximum ordinate on the curve is

(a) (0, 4/3) (b) (0, 2/3)
(c) (1, 2/3) (d) (2, 4/3)

Ans. (a)

Solution: $$\frac{dy}{dx} = x^2 - 2x = \frac{d}{dx}\left(\frac{x^3}{3} - x^2 + c\right)$$

$$y = \frac{x^3}{3} - x^2 + c$$

Since the curve passes through (2, 0), we get $0 = 8/3 - 4 + c$, i.e., $c = 4/3$

Hence the equation of the curve is

$$y = \frac{x^3}{3} - x^2 + \frac{4}{3}$$

Now, from $dy/dx = 0$, we get $x = 0$ or 2. Also

$$\frac{d^2y}{dx^2} = 2x - 2 \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=0} = -2 \text{ and } \left.\frac{d^2y}{dx^2}\right|_{x=2} = 2$$

Hence at $x = 0$, y has a maximum. Thus the required point is (0, 4/3).

Example 84: If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a, a) cuts off intercepts α and β on the coordinate axes, where $\alpha^2 + \beta^2 = 61$ then the value of $|a|$ is

(a) 16 (b) 28
(c) 30 (d) 31

Ans. (c)

Solution: The slope of the tangent is

$$\frac{dy}{dx} = \frac{2ax + 3x^2}{6y^2}$$

and the value of this slope at (a, a) is 5/6. Therefore, the equation

$$y - a = \frac{5}{6}(x - a) \Rightarrow \frac{x}{-a/5} + \frac{y}{a/6} = 1,$$

represents the tangent. Thus the x-intercept α is $-a/5$, and the y-intercept β is $a/6$. From $\alpha^2 + \beta^2 = 61$, we now get

$$\frac{a^2}{25} + \frac{a^2}{36} = 61 \Rightarrow a^2 = 25 \times 36 \Rightarrow |a| = 30.$$

Example 85: The coordinates of the point on the parabola $y^2 = 8x$, which is at minimum distance from the circle $x^2 + (y + 6)^2 = 1$ are

(a) (2, – 4) (b) (18, – 12)
(c) (2, 4) (d) none of these

Ans. (a)

◎ **Solution:** A point on the parabola is at a minimum distance from the circle if and only if it is at a minimum distance from the centre of the circle. Any point on the parabola $y^2 = 8x$ is of the form $P(2t^2, 4t)$. The centre of the circle $x^2 + (y + 6)^2 = 1$ is $O(0, -6)$

$$OP^2 = 4t^4 + (-6 - 4t)^2 = 4(t^4 + 4t^2 + 12t + 9)$$

Let $A = t^4 + 4t^2 + 12t + 9$

$$\frac{dA}{dt} = 4t^3 + 8t + 12 = 4(t^3 + 2t + 3) = 4(t + 1)(t^2 - t + 3)$$

So $\frac{dA}{dt} = 0$ if $t = -1$. Moreover,

$$\left.\frac{d^2A}{dt^2}\right|_{t=-1} = 4(3(-1)^2 + 2) > 0.$$

Hence required point is $P(2, -4)$.

◉ **Example 86:** The equation $e^{x-8} + 2x - 17 = 0$ has

(a) two real roots (b) one real root
(c) eight real roots (d) four real roots

Ans. (b)

◎ **Solution:** Clearly $x = 8$ satisfies the given equation. Assume that $f(x) = e^{x-8} + 2x - 17 = 0$ has a real root α other than $x = 8$. We may suppose that $\alpha > 8$ (the case for $\alpha < 8$ is exactly similar). Applying Rolle's theorem on $[8, \alpha]$, we get $\beta \in (8, \alpha)$, such that $f'(\beta) = 0$. But $f'(\beta) = e^{\beta-8} + 2$, so that $e^{\beta-8} = -2$ which is not possible. Hence there is no real root other than 8.

◉ **Example 87:** The maximum and minimum value of $f(x) = ab\sin x + b\sqrt{1-a^2}\cos x + c$ lie in the interval (assuming $|a| < 1, b > 0$)

(a) $[b - c, b + c]$ (b) $(b - c, b + c)$
(c) $[c - b, b + c]$ (d) none of these

Ans. (c)

◎ **Solution:** $f'(x) = ab\cos x - b\sqrt{1-a^2}\sin x = b\cos(x + \phi)$ where $\phi = \cos^{-1} a$. So $f'(x) = 0 \Rightarrow x = (2m + 1)(\pi/2 - \phi)$

Also $f''(x) = -b\sin(x + \phi)$. Thus $f''(\pi/2 - \phi) = -b < 0$ and $f''(3\pi/2 - \phi) = b > 0$. Hence f has maximum at $\pi/2 - \phi$ and minimum at $3\pi/2 - \phi$. Moreover, max $f = ab\sin(\pi/2 - \phi) + b\sqrt{1-a^2}\cos(\pi/2 - \phi) + c = a^2b + b(1 - a^2) + c = b + c$ and min $f = c - b$.

Alternatively $f(x) = b\sin(x + \phi) + c$, $\phi = \cos^{-1} a$ so max $f(x) = b.1 + c$ and min $f(x) = -b + c = c - b$

◉ **Example 88:** The maximum area of the rectangle whose sides pass through the angular points of a given the rectangle is of sides a and b is

(a) $(1/2)(ab)^2$ (b) $(1/2)(a + b)$
(c) $(1/2)(a + b)^2$ (d) none of these

Ans. (c)

◎ **Solution:** Let $ABCD$ be the given rectangle of sides a and b and $EFGH$ be any rectangle, whose sides pass through A, B, C, D.

A = Area $EFGH = (b\sin\theta + a\cos\theta)(a\sin\theta + b\cos\theta)$
$= ab + (a^2 + b^2)\sin\theta\cos\theta.$

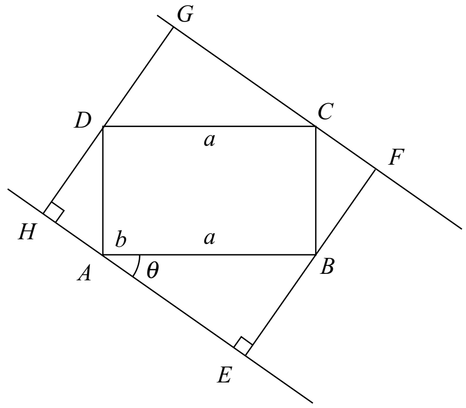


Fig. 11.4

$dA/d\theta = (a^2 + b^2)\cos 2\theta$ so $dA/d\theta = 0 \Rightarrow \theta = \pi/4$

$$\Rightarrow \frac{d^2A}{d\theta^2} = -2(a^2 + b^2)\sin 2\theta, \text{ so } \left.\frac{d^2A}{d\theta^2}\right|_{\theta=\pi/4} < 0$$

Hence $A_{max} = (1/2)(a + b)^2$.

◉ **Example 89:** The image of the interval $[-1, 3]$ under $f(x) = 4x^3 - 12x$ is

(a) $[-2, 0]$ (b) $[-8, 72]$
(c) $[-8, 0]$ (d) $[8, 72]$

Ans. (b)

◎ **Solution:** To find the image of the given interval, we must find the set of values of $f(x)$ for $x \in [-1, 3]$. By virtue of the continuity of $f(x)$, the image is the interval

$$\left[\min_{x\in[-1,3]} f(x), \max_{x\in[-1,3]} f(x)\right]$$

The critical points of $f(x)$ are given by $f'(x) = 12x^2 - 12 = 12(x^2 - 1) = 0$. That is, $x = \pm 1$, so that $f(1) = -8$, $f(-1) = 8$ and $f(3) = 72$.

$$\therefore \max_{x\in[-1,3]} f(x) = f(3) = 72 \text{ and } \min_{x\in[-1,3]} f(x) = f(1) = -8$$

Hence the image of $[-1, 3]$ under the mapping $f(x)$ is $[-8, 72]$.

◉ **Example 90:** The difference between the greatest and least values of the function $f(x) = \cos x + (1/2)\cos 2x - (1/3)\cos 3x$ is

(a) 3/8 (b) 2/3
(c) 8/7 (d) 9/4

Ans. (d)

◎ **Solution:** The given function is periodic, with period 2π. So the difference between the greatest and least values of the function is the difference between these values on the interval $[0, 2\pi]$. We have

$$f'(x) = -(\sin x + \sin 2x - \sin 3x)$$
$$= -4 \sin x \sin (3x/2) \sin (x/2).$$

Hence $x = 0, 2\pi/3, \pi$ and 2π are the critical points. Also, $f(0) = 1 + 1/2 - 1/3 = 7/6$, $f(2\pi/3) = -13/12$, $f(\pi) = -1/6$ and $f(2\pi) = 7/6$. Hence the greatest value is 7/6 and the least value is $-13/12$. Thus the difference is

$$7/6 - (-13/12) = 27/12 = 9/4.$$

◉ **Example 91:** The longest distance of the point $(a, 0)$ from the curve $2x^2 + y^2 - 2x = 0$ is

(a) $\sqrt{1-2a+2a^2}$ (b) $\sqrt{1-2a+a^2}$

(c) $\sqrt{1+2a+2a^2}$ (d) $\sqrt{1+a+a^2}$

Ans. (a)

◎ **Solution** Let (x, y) be any point on the curve $2x^2 + y^2 - 2x = 0$. Its distance S from the point $(a, 0)$ is given by

$$T = S^2 = (x-a)^2 + y^2 = (x-a)^2 + (2x - 2x^2)$$
$$= -x^2 + 2x(1-a) + a^2 \quad (1)$$

For S to be maximum, we have $dT/dx = 0$
$\Rightarrow -2x + 2(1-a) = 0 \Rightarrow x = 1 - a$. Since $d^2T/dx^2 < 0$, so S^2 is maximum when $x = 1 - a$ and this maximum value is given by $-(1-a)^2 + 2(1-a)^2 + a^2 = 1 - 2a + 2a^2$. Hence $S_{max} = (1 - 2a + 2a^2)^{1/2}$.

◉ **Example 92:** The sides of the rectangle of the greatest area, that can be inscribed in the ellipse $x^2 + 2y^2 = 8$, are given by

(a) $4\sqrt{2}, 4$ (b) $4, 2\sqrt{2}$

(c) $2, \sqrt{2}$ (d) $2\sqrt{2}, 2$

Ans. (b)

◎ **Solution:** Any point on the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$ is $(2\sqrt{2}\cos\theta, 2\sin\theta)$. [see Fig. 11.5]

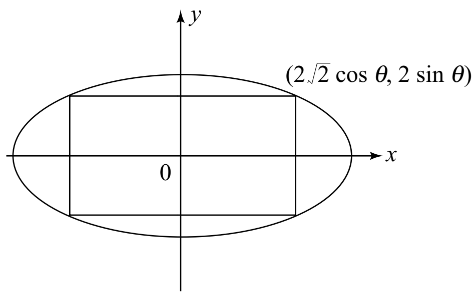


Fig. 11.5

$$A = \text{Area of the inscribed rectangle}$$
$$= 4(2\sqrt{2}\cos\theta)(2\sin\theta)$$
$$= 8\sqrt{2}\sin 2\theta$$
$$\frac{dA}{d\theta} = 16\sqrt{2}\cos 2\theta = 0 \Rightarrow \theta = \pi/4$$

Also $\frac{d^2A}{d\theta^2} = -32\sqrt{2}\sin 2\theta < 0$ for $\theta = \pi/4$.

Hence, the inscribed rectangle is of largest area if the sides are $4\sqrt{2}\cos\pi/4$ and $4\sin(\pi/4)$ i.e. 4 and $2\sqrt{2}$.

◉ **Example 93:** An equation of the circle that is tangent to $y = x^3$ at (1, 1) and has the same second derivative there is

(a) $x^2 + y^2 + 24x - 28y + 2 = 0$

(b) $2(x^2 + y^2) + 12x - 8y - 8 = 0$

(c) $3(x^2 + y^2) - 24x + 10y + 8 = 0$

(d) none of these

Ans. (d)

◎ **Solution:** $\frac{dy}{dx} = 3x^2$ so $\left.\frac{dy}{dx}\right|_{(1,1)} = 3$ and $\left.\frac{d^2y}{dx^2}\right|_{(1,1)} = 6$

Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.
Since this should pass through (1, 1) so we have

$$2 + 2g + 2f + c = 0 \quad (1)$$

Also $2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$

Putting $x = 1$, $y = 1$ and $\frac{dy}{dx} = 3$, we have

$1 + 3 + g + 3f = 0$

Again differentiating (1), we have

$$1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} + f\frac{d^2y}{dx^2} = 0$$

Putting $y = 1$, $\frac{dy}{dx} = 3$ and $\frac{d^2y}{dx^2} = 6$, we have

$$16 + 6f = 0 \Rightarrow f = -8/3$$

so $g = 4$ and $c = -14/3$. Thus required circle is

$$3(x^2 + y^2) + 24x - 16y - 14 = 0.$$

◉ **Example 94:** Let $f(x) = 6x^{4/3} - 3x^{1/3}$, $x \in [-1, 1]$. Then

(a) The maximum value of $f(x)$ on $[-1, 1]$ is 3

(b) The maximum value of $f(x)$ on $[-1, 1]$ is 9

(c) The minimum value of $f(x)$ on $[-1, 1]$ is 0

(d) none of these

Ans. (b)

◎ **Solution:** $f'(x) = \frac{8x-1}{x^{2/3}}$. Thus $f'(x) = 0$ when $x = 1/8$ and $f'(x)$ does not exist when $x = 0$. Now $f(-1) = 9$, $f(0) = 0$, $f(1/8) = -9/8$ and $f(1) = 3$.
The maximum value of $f(x)$ is 9 and the minimum value is $-9/8$ on $[-1, 1]$.

Example 95: Let $g(x) = (\log(1+x))^{-1} - x^{-1}, x > 0$ then

(a) $1 < g(x) < 2$ (b) $-1 < g(x) < 0$

(c) $0 < g(x) < 1$ (d) none of these

Ans. (c)

Solution: For $x > 0$, consider the function $f(x) = \log(1 + x)$ on the interval $[0, x]$. Since the function is differentiable on $(0, x)$, by Lagrange's theorem, there is *a* $\xi \in (0, x)$ such that

$$\frac{\log(1+x) - \log(1+0)}{x - 0} = f'(\xi) = \frac{1}{1+\xi}$$

$$\Rightarrow \quad \frac{\log(1+x)}{x} = \frac{1}{1+\xi} < 1$$

$$\Rightarrow \quad \log(1+x) < x \quad \Rightarrow \quad g(x) > 0$$

Also $\quad \xi \in (0, x) \quad \Rightarrow \quad \xi < x \quad \Rightarrow \quad \dfrac{1}{1+\xi} > \dfrac{1}{1+x}$

$$\Rightarrow \quad \frac{\log(1+x)}{x} > \frac{1}{1+x}$$

$$\Rightarrow \quad \log(1+x) > \frac{x}{1+x} \Rightarrow g(x) < 1.$$

Example 96: The maximum value of $\dfrac{x^2 - x + 1}{x^2 + x + 1}$ for all real values of x is

(a) 1/2 (b) 1

(c) 2 (d) 3

Ans. (d)

Solution: Let $S = \dfrac{x^2 - x + 1}{x^2 + x + 1} = 1 - \dfrac{2x}{x^2 + x + 1}$

$$S'(x) = -2\,\frac{(x^2 + x + 1) - x(2x + 1)}{(x^2 + x + 1)^2}$$

$$= -2\,\frac{-x^2 + 1}{(x^2 + x + 1)^2}$$

$S'(x) = 0 \Leftrightarrow x = \pm 1$. Since $S'(x) > 0$ for $x < -1$ and $S'(x) < 0$ for $-1 < x < 1$, also $S'(x) > 0$ for $x > 1$. So S is maximum when $x = -1$. Hence $S_{max} = 1 + 2/1 = 3$.

Example 97: If the tangent at (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P, then P is

(a) (4, 4) (b) (– 1, 2)

(c) (9/4, 3/8) (d) none of these

Ans. (c)

Solution: $2y\dfrac{dy}{dx} = (2 - x)^2 - 2x(2 - x)$

$$= 3x^2 - 8x + 4. \text{ So } \left.\frac{dy}{dx}\right|_{(1,1)} = -1/2.$$

An equation of tangent at (1, 1) is $Y - 1 = (-1/2)(X - 1)$ i.e. $Y = (-1/2)x + 3/2$. The intersection of this line with the given curve is given by $(-x/2 + 3/2)^2 = x(2 - x)^2$

$$\Rightarrow \quad x^2 - 6x + 9 = 16x + 4x^3 - 16x^2. \text{ So,}$$

$$4x^3 - 17x^2 + 22x - 9 = 0$$

$$\Rightarrow \quad (x - 1)(4x - 9)(x + 1) = 0.$$

Thus $x = 1, 9/4, -1$. But $x = -1$ cannot lie on the given curve so required point is (9/4, 3/8).

Example 98: If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$, cut each other at right angles then the value of b is

(a) 2 (b) 4

(c) 9/2 (d) none of these

Ans. (c)

Solution: The intersection of the two curves is given by

$$9x^2 + 6bx - x = x \qquad \text{(i)}$$

Differentiating $y^2 = 6x$, we have $\dfrac{dy}{dx} = \dfrac{3}{y}$.

Differentiating $9x^2 + 6y^2 = 16$, we have $\dfrac{dy}{dx} = -\dfrac{9x}{by}$.

For curves to intersect at right angles, we must have at the points of intersection $\dfrac{3}{y}\left(\dfrac{-9x}{by}\right) = -1 \quad \Rightarrow \quad 27x = by^2$. Thus we must have

$$9x^2 + 9y^2 = 16 \quad \Rightarrow \quad 9x^2 + 27x - 16 = 0 \qquad \text{(ii)}$$

(i) and (ii) must be identical so $27 = 6b \Rightarrow b = 9/2$.

Example 99: The distance of that point on $y = x^4 + 3x^2 + 2x$ which is nearest to the line $y = 2x - 1$ is

(a) $4/\sqrt{5}$ (b) $3/\sqrt{5}$

(c) $2/\sqrt{5}$ (d) $1/\sqrt{5}$

Ans. (d)

Solution: Distance of any point (x, y) from $y = 2x - 1$ is: $\left|\dfrac{y - 2x + 1}{\sqrt{5}}\right|$. If (x, y) is on $y = x^4 + 3x^2 + 2x$ then this distance is $S = \dfrac{x^4 + 3x^2 + 1}{\sqrt{5}}$

$$\frac{dS}{dx} = \frac{4x3 + 6x}{\sqrt{5}} \quad \Rightarrow \quad \frac{dS}{dx} = 0 \quad \Rightarrow \quad x = 0.$$

Also, $S'(x) < 0$ for $x < 0$ and $S'(x) > 0$ for $x > 0$.

Thus S is minimum when $x = 0$, and min. S is $1/\sqrt{5}$.

Example 100: A given right circular cone has a volume p, and the largest right circular cylinder that can be inscribed in the cone has a volume q. Then $p : q$ is

(a) 9 : 4 (b) 8 : 3

(c) 7 : 2 (d) none of these

Ans. (a)

◎ **Solution:** Let H be the height of the cone and α be its semi vertical angle. Suppose that x is the radius of the inscribed cylinder and h be its height $h = QL = OL - OQ = H - x \cot \alpha$

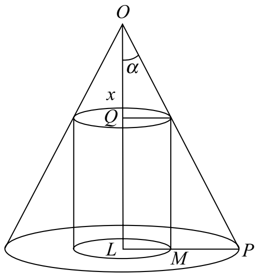


Fig. 11.6

$$V = \text{volume of the cylinder} = \pi x^2 (H - x \cot \alpha)$$

Also $p = \dfrac{1}{3} \pi (H \tan \alpha)^2 H$ (i)

$$\frac{dV}{dx} = \pi (2Hx - 3x^2 \cot \alpha)$$

so $\dfrac{dV}{dx} = 0 \Leftrightarrow x = 0,$

$$x = \frac{2}{3} H \tan \alpha, \left.\frac{d^2V}{dx^2}\right|_{x=\frac{2}{3}H\tan\alpha} = -2\pi H < 0. \text{ so}$$

V is maximum when $x = \dfrac{2}{3} H \tan \alpha$ and $q = V_{max} = \pi \dfrac{4}{9} H^2 \tan^2 \alpha \dfrac{1}{3} H = \dfrac{4}{9} p$. [using (i)]

Hence $p : q = 9 : 4$.

◉ **Example 101:** The set of all values of a for which $f(x) = (a^2 - 3a + 2)(\cos^2 (x/4) - \sin^2 (x/4)) + (a - 1)x + \sin 1$ doesn't possess critical points is

(a) $[1, \infty)$ (b) $(-2, 4)$
(c) $(1, 3) \cup (3, 5)$ (d) $(0, 1) \cup (1, 4)$

Ans. (d)

◎ **Solution:** The given function can be written as

$$f(x) = (a - 1)(a - 2) \cos (x/2) + (a - 1) x + \sin 1.$$

It is clearly differentiable, so its critical points are given by

$$f'(x) = (-1/2)(a - 1)(a - 2) \sin (x/2) + (a - 1) = 0$$

If $a = 1$, $f'(x) = 0$ for all x, while for values of a other than 1, $f'(x)$ will be zero if

$$(a - 2) \sin (x/2) = 2$$

In order not to have critical point, a must therefore satisfy $a \neq 1$ and $\left|\dfrac{2}{a-2}\right| > 1$ which is the same as saying $a \in (0, 1) \cup (1, 4)$.

◉ **Example 102:** Let $x, p \in \mathbf{R}$, $x + 1 > 0$, $p \neq 0, 1$. Then

(a) $(1 + x)^p > 1 + px$ for $p > 0$
(b) $(1 + x)^p > 1 + px$ for $p \in (-\infty, 0) \cup (1, \infty)$
(c) $(1 + x)^p > 1 + px$ for $0 < p < 1$
(d) $(1 + x)^p < 1 + px$ for $p < 1$

Ans. (b)

◎ **Solution:** Let $f(x) = (1 + x)^p - (1 + px)$ so $f(0) = 0$. Now $f'(x) = p(1 + x)^{p-1} - p$ and $f''(x) = p(p - 1)(1 + x)^{p-2}$. Let $p \in (-\infty, 0) \cup (1, \infty)$ then $f''(x) > 0$ so f' is increasing. This implies $f' < 0$ on $(-1, 0)$ and $f' > 0$ on $(0, \infty)$ as $f'(0) = 0$. Therefore f decreases on $(-1, 0)$ and increases on $(0, \infty)$ but $f(0) = 0$ so $f(x) > 0$ if $x > -1$ ($x \neq 0$). Thus $(1 + x)^p > 1 + px$ for $p \in (-\infty, 0) \cup (1, \infty)$. Similarly it follows that $(1 + x)^p < 1 + px$, if $0 < p < 1$.

◉ **Example 103:** If $f(x) = \dfrac{a \sin x + b \cos x}{c \sin x + d \cos x}$ decreases for all x if

(a) $ad - bc < 0$ (b) $ad - bc > 0$
(c) $ab - cd > 0$ (d) $ab - cd < 0$

Ans. (a)

Solution:
$$f'(x) = \frac{(a \cos x - b \sin x)(c \sin x + d \cos x) - (a \sin x + b \cos x)(c \cos x - d \sin x)}{(c \sin x + d \cos x)^2}$$

$$= \frac{ad(\cos^2 x + \sin^2 x) - bc(\cos^2 x + \sin^2 x)}{(c \sin x + d \cos x)^2}$$

$$= \frac{ad - bc}{(c \sin x + d \cos x)^2}$$

f decreases for all x if and only if $f'(x) < 0$ for all x i.e. $ad - bc < 0$.

◉ **Example 104:** On the interval [0, 1] the function $x^{25}(1 - x)^{75}$ takes its maximum value at the point

(a) 0 (b) 1/3
(c) 1/2 (d) 1/4

Ans. (d)

◎ **Solution:** Let $f(x) = x^{25}(1 - x)^{75}$. The critical points of f are given by $f'(x) = 0$. But

$$f'(x) = x^{24}(1 - x)^{74}[25 - 25x - 75x]$$
$$= 25x^{24}(1 - x)^{74}(1 - 4x)$$

Thus the critical points are 0, 1, 1/4

Since $f(0) = 0$, $f(1) = 0$ and $f(1/4) = (1/4)^{25}(3/4)^{75}$

so f takes its maximum value at $x = 1/4$.

◉ **Example 105:** For $a > 0$, the value of a for which the equation $ax^2 = \log x$ possess a single root is

(a) 1/2 (b) $1/2e$
(c) $1/e$ (d) $2e^{-1}$

Ans. (b)

◎ **Solution:** For $a > 0$, the curves $y_1 = ax^2$ and $y_2 = \log x$ can have only one point in common if they touch each other. At the point of tangency $y'_1(x) = y'_2(x) \Rightarrow 2ax = 1/x \Rightarrow x = 1/\sqrt{2a}$ (clearly x cannot be negative). Putting this value in $ax^2 = \log x$, we have $(1/2) = \log (2a)^{-1/2} \Rightarrow \log 2a = -1 \Rightarrow a = 1/2e$.

EXERCISE

Concept-based Straight Objective Type Questions

1. The length of the tangent to the curve $x = a \sin^3 t$, $y = a \cos^3 t$ $(a > 0)$ at an arbitrary is
(a) $a \cos^2 t$ (b) $a \sin^2 t$
(c) $\frac{a \sin^2 t}{\cos t}$ (d) $\frac{a \cos^2 t}{\sin t}$

2. Equation of normal to $x = 2e^t$, $y = e^{-t}$ at $t = 0$ is
(a) $x + y - 4 = 0$
(b) $x + 2y - 4 = 0$
(c) $2x - y - 3 = 0$
(d) $x - 2y - 3 = 0$

3. A point moves according $s = \frac{2}{9}\sin\frac{\pi}{2}t + s_0$. The acceleration at the end of first second is
(a) $-\frac{\pi}{18}$ (b) $-\frac{\pi^2}{18}$
(c) $\frac{\pi}{18}$ (d) $\frac{\pi^2}{18}$

4. Let $f(x) = x \log x - x + 1$ then the set $\{x : f(x) > 0\}$ is equal to
(a) $(1, \infty)$ (b) $(1/e, \infty)$
(c) $[e, \infty)$ (d) $(0, 1) \cup (1, \infty)$

5. On the curve $y = x^3$, the point at which the tangent line in parallel to the chord through the point $(-1, -1)$ and $(2, 8)$ is
(a) $(1, 1)$ (b) $\left(\frac{1}{2}, \frac{1}{8}\right)$
(c) $\left(\frac{1}{3}, \frac{1}{27}\right)$ (d) $\left(-\frac{1}{2}, -\frac{1}{8}\right)$

6. Let $f(x) = 2x^2 - \log x$, then
(a) f increases on $(0, \infty)$
(b) f decrease on $\left(\frac{1}{2}, \infty\right)$
(c) f increases on $\left(\frac{1}{2}, \infty\right)$
(d) f decreases on $(0, 1)$

7. Let $f(x) = \frac{3}{4}x^4 - x^3 - 9x^2 + 7$, then the number of critical points in $[-1, 4]$ is
(a) 4 (b) 3
(c) 2 (d) 1

8. On the curve $x^3 = 12y$, the values of x for which the abscissa changes at a faster rate than the ordinate is
(a) $(-2, 2) \sim \{0\}$ (b) $(-3, 3) \sim \{0\}$
(c) $(1, 4)$ (d) $(2, 4)$

9. The value of $k > 0$ for which the curves $\frac{x^2}{k^2} + \frac{y^2}{4} = 1$ and $y^2 = 16x$ cut each other orthogonally is
(a) 1 (b) $\frac{2\sqrt{3}}{3}$
(c) $3\sqrt{3}$ (d) $5\sqrt{5}$

10. The least value of $g(t) = 8t - t^4$ on $[-2, 1]$ is
(a) -16 (b) -20
(c) -32 (d) 7

LEVEL 1

Straight Objective Type Questions

11. Let $f(x) = \tan^{-1} x$ and $g(x) = \frac{x}{1+x^2}, x > 0$ then
(a) $f(x) < g(x)$, on $(0, \infty)$
(b) $f(x) \le g(x)$ on $[1, \infty)$
(c) $g(x) < f(x)$ on $(0, \infty)$
(d) none of these

12. Let $f(x) = (x - 2)(x - 3)(x - 4)(x - 5)(x - 6)$ then
(a) $f'(x) = 0$ has five real roots
(b) four roots of $f'(x) = 0$ lie in $(2, 3) \cup (3, 4) \cup (4, 5) \cup (5, 6)$
(c) the equation $f'(x)$ has only three roots
(d) four roots of $f'(x) = 0$ lie in $(1, 2) \cup (2, 3) \cup (3, 4) \cup (4, 5)$

13. Let $f(x) = (x - 3)^5 (x + 1)^4$ then
(a) $x = -1$ is point of minima
(b) $x = -1$ is point of maxima
(c) $x = 7/9$ is a point of maxima
(d) $x = -1$ is neither a point of maxima and minima.

14. The normal to the curve represented parametrically by $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$ at any point θ, is such that it
 (a) makes a constant angle with the x-axis
 (b) is at a constant distance from the origin
 (c) does not touch a fixed circle
 (d) passes through the origin.

15. The value of a for which the equation $x^3 - 3x + a = 0$ has two distinct roots in [0, 1], is given by
 (a) – 1 (b) 1
 (c) 3 (d) none of these

16. If the sum of the squares of the intercepts on the axes cut off by the tangent to the curve $x^{1/3} + y^{1/3} = a^{1/3}$ $(a > 0)$ at $(a/8, a/8)$ is 2, then a has the value
 (a) 1 (b) 2
 (c) 4 (d) 8

17. The value of m for which the area of the triangle included between the axes and any tangent to the curve $x^m y = b^m$ is constant, is
 (a) 1/2 (b) 1
 (c) 3/2 (d) 2

18. If the tangent at any point on the curve $x^4 + y^4 = a^4$ cuts off intercepts p and q on the coordinate axes, the value of $p^{-4/3} + q^{-4/3}$ is
 (a) $a^{-4/3}$ (b) $a^{-1/2}$
 (c) $a^{1/2}$ (d) none of these

19. The interval of increase of the function $y = x - 2\sin x$ if $0 \le x \le 2\pi$, is
 (a) $(0, \pi)$ (b) $(0, \pi)$
 (c) $(\pi/2, \pi)$ (d) $(\pi/3, 5\pi/3)$

20. The greatest value of $y = (x + 1)^{1/3} - (x - 1)^{1/3}$ on [0, 1] is
 (a) 1 (b) 2
 (c) 3 (d) $2^{1/3}$

21. Let f be a function defined by $f(x) = 2x^2 - \log|x|$, $x \ne 0$ then
 (a) f increase on $[-1/2, 0] \cup [1/2, \infty)$
 (b) f decreases on $(-\infty, 0)$
 (c) f increases on $(-\infty, -1/2)$
 (d) f decreases on $[1/2, \infty]$

22. The shortest distance of (0, 0) from the curve $y = \dfrac{e^x + e^{-x}}{2}$ is
 (a) 1/2 (b) 1/3
 (c) 2 (d) none of these

23. The normal to the circle $x^2 + y^2 - 2x - 2y = 0$ passing through (2, 2) is
 (a) $x = y$ (b) $2x + y - 6 = 0$
 (c) $x + 2y - 6 = 0$ (d) $x + y - 4 = 0$

24. If $f(0) = 0$ and $f''(x)$ exists and > 0, for all $x > 0$ then $f(x)/x$
 (a) decreases on $(0, \infty)$
 (b) increases on $(0, \infty)$
 (c) decreases on $(1, \infty)$
 (d) neither increases nor decreases on $(0, \infty)$

25. The value of k so that the equation $x^3 - 12x + k = 0$ has distinct roots in [0, 2] is
 (a) 4 (b) 2
 (c) –2 (d) none of these

26. Let $f(x) = 6x^{4/3} - 3x^{1/3}$ defined on $[-1, 1]$ then
 (a) maximum value of f is 7
 (b) maximum value of f is 5
 (c) maximum value of f is 9
 (d) minimum value of f is – 3/2

27. An equation of tangent line at an inflection point of $f(x) = x^4 - 6x^3 + 12x^2 - 8x + 3$ is
 (a) $y = 3x + 4$ (b) $y = 4$
 (c) $y = 3x + 2$ (d) none of these

28. The number of real roots of the equation $2x^3 - 3x^2 + 6x + 6 = 0$ is
 (a) 1 (b) 2
 (c) 3 (d) none of these

29. Let $f(x) = (x - 2)(x^4 - 4x^3 + 6x^2 - 4x + 1)$ then value of local minimum of f is
 (a) – 2/3 (b) $-(4/5)^4$
 (c) $-4^4/5^5$ (d) $-(4/5)^5$

30. Let $f(x) = x^2 - 2|x| + 2$, $x \in [-1/2, 3/2]$ then
 (a) $\min\limits_{x \in [-1/2, 3/2]} f(x) = 1/2$ (b) $\min\limits_{x \in [-1/2, 3/2]} f(x) = 1$
 (c) $\max\limits_{x \in [-1/2, 3/2]} f(x) = 3/2$ (d) none of these

31. A critical point of the function $f(x) = \dfrac{|x-1|}{x^2}$ is
 (a) – 1 (b) 3
 (c) 2 (d) 1/2

32. The function $f(x) = x^x$ decreases on the interval
 (a) $(0, e)$ (b) (0, 1)
 (c) $(0, 1/e)$ (d) none of these

33. The interval of increase of the function $y = x - e^x + \tan(\pi/7)$ is
 (a) $(-\infty, 1)$ (b) $(0, \infty)$
 (c) $(-\infty, 0)$ (d) $(1, \infty)$

34. Let $f(x) = x^2 + px + q$. The value of (p, q) so that $f(1) = 3$ is an extreme value of f on [0, 2] is
 (a) (– 2, 2) (b) (1, 4)
 (c) (– 2, 4) (d) (– 2, 3)

35. The number of inflection points of a function given by a third degree polynomial is exactly

(a) 2 (b) 1
(c) 3 (d) 0

36. Let $f(x) = 2\tan^{-1} x + \sin^{-1} \dfrac{2x}{1+x^2}$ then
(a) $\max f(x) = \pi/2$ (b) $\min f(x) = \pi/4$
(c) $\max f(x) = \pi$ (d) none of these

37. If the normal to the curve $x^3 = y^2$ at the point $(m^2, -m^3)$ is $y = mx - 2m^3$, then the value of m^2 is
(a) 1 (b) 1/2
(c) 1/3 (d) 2/3

38. Let $f(x) = 2\sin x + \cos 2x$ $(0 \le x \le 2\pi)$ and $g(x) = x + \cos x$ then
(a) g is a decreasing function
(b) f increases on $(0, \pi/2)$
(c) f increases on $(0, \pi/6) \cup (\pi/2, 5\pi/6)$
(d) f decreases on $(0, \pi/2)$

39. The minimum value of the function $f(x) = \tan x + \cot x$ in the interval $(0, \pi/2)$ is
(a) 1 (b) 0
(c) 2 (d) 1/2

40. The number of points of extremum of the function $f(x) = 3x^4 - 4x^3 + 6x^2 + b$ for any value of b is
(a) 4 (b) 3
(c) 1 (d) 2

41. The shortest distance of the line $y - x - 1 = 0$ from $x = y^2$ is
(a) 3/8 (b) $3\sqrt{2}/4$
(c) 3/4 (d) $3\sqrt{2}/8$

42. The value of a for which the extremum of the function $f(x) = x^3 - 3ax^2 + 3(a^2 - 1)x + 1$ lie in the interval $(-2, 4)$ lie in
(a) $(-1, 0)$ (b) $(-2, 4)$
(c) $(-1, 5)$ (d) $(-1, 3)$

43. If $A > 0$, $B > 0$ and $A + B = \pi/3$ then the maximum value of $\tan A \tan B$ is
(a) 1/3 (b) 1/2
(c) $1/\sqrt{2}$ (d) $\sqrt{3}/2$

44. The maximum value of $|x \log x|$ for $0 < x \le 1$ is
(a) 0 (b) $1/e$
(c) $2e^{-1}$ (d) none of these

45. The greatest value of the function $\log_x 1/9 - \log_3 x^2$ $(x > 1)$ is
(a) 2 (b) 0
(c) -4 (d) -2

46. Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \ge 2$ for $x \in [1, 6]$ then
(a) $f(6) < 8$ (b) $f(6) \ge 8$
(c) $f(6) \ge 10$ (d) $f(6) \le 5$

47. An extremum value of the function
$f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$ $(-1 < x < 1)$ is
(a) $7\pi^3/8$ (b) $\pi^3/8$
(c) $\pi^3/32$ (d) $\pi^3/16$

48. Let $f(x) = x \log x + 3x$. Then
(a) f increases on (e^{-4}, ∞)
(b) f increases on $(0, \infty)$
(c) f decreases on $(0, \infty)$
(d) f decreases on $(0, e^{-2})$

49. Let $f(x) = x^2 e^{-x}$ then
(a) $\max f(x) = e^{-1}$ (b) $\max f(x) = 4e^{-2}$
(c) $\min f(x) = e^{-1}$ (d) $\min f(x) > 0$

50. The minimum value of $f(x) = |3 - x| + |2 + x| + |5 - x|$ is
(a) 0 (b) 7
(c) 8 (d) 10

51. Let $f(x) = 2 + 2x - 3x^{2/3}$ on $[-1, 10/3]$. Then f has
(a) Absolute maximum at an end point
(b) Absolute minimum at an interior point
(c) Absolute minimum is $f(10/3)$
(d) Absolute minimum is $f(-1)$

52. If f and g are defined on $[0, \infty)$ by

$$f(x) = \lim_{n\to\infty} \frac{x^n - 1}{x^n + 1} \text{ and } g(x) = \int_0^x f(t)\,dt.$$

Then
(a) g has local maximum at $x = 1$
(b) g has local minimum at $x = 1$
(c) g is an increasing function on $(0, \infty)$
(d) g is a decreasing function on $(0, \infty)$.

53. Let $f(x) = \sin x + \cos x$ then
(a) $x = 17\pi/4$ is a point of minima
(b) $x = 13\pi/4$ is a point of maxima
(c) $x = 21\pi/4$ is a point of minima
(d) $x = 29\pi/4$ is a point of maxima

54. If $f(x) = x\, e^{x(1-x)}$, then $f(x)$ is
(a) increasing on $[-1/2, 1]$
(b) decreasing on **R**
(c) increasing on **R**
(d) decreasing on $[-1/2, 1]$

55. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$
(a) on the left of $x = c$ (b) on the right of $x = c$
(c) at no point (d) at all points

Assertion-Reason Type Questions

56. **Statement-1:** $e^{\pi} > \pi^{e}$
Statement-2: The function $x^{1/x}$ $(x > 0)$ has local maximum at $x = e$.

57. **Statement-1:** $|\cot x - \cot y| \le |x - y|$ for all $x, y \in (-\pi/2, \pi/2)$
Statement-2: If f is differentiable on an open interval and $|f'(x)| \le M$ then $|f(x) - f(y)| \le M |x - y|$.

58. **Statement-1:** The function $f(x) = 2\sin x + \cos 2x$ $(0 \le x \le 2\pi)$ has minimum at $x = \pi/3$ and maximum at $5\pi/3$.
Statement-2: The function $f(x)$ above decreases on $(0, \pi/3)$, increases on $(\pi/3, 5\pi/3)$ and decreases on $(5\pi/2, 2\pi)$

59. Let $f(x) = \tan^{-1}\dfrac{1-x}{1+x}$
Statement-1: The difference between the greatest and smallest value of $f(x)$ on $[0, 1]$ is $\pi/4$
Statement-2: If a function g decreases on $[a, b]$ then the greatest value of $g = g(a)$ and least value of g is $g(b)$.

60. Let $f(x) = \dfrac{x}{\log x}$
Statement-1: The minimum value of $f(x)$ is e
Statement-2: $\log x > 1$ for $x > e$ and < 1 for $x < e$.

LEVEL 2

Straight Objective Type Questions

61. The critical points of the function $f(x) = (x-2)^{2/3}(2x+1)$ are
(a) -1 and 2 (b) 1
(c) 1 and $-1/2$ (d) 1 and $1/2$

62. The function $f(x) = (1/4)x^3 - \sin \pi x + 3$ on $[-2, 2]$ takes the value
(a) 1 (b) 16/3
(c) 6 (d) 8

63. The greatest value of $f(x) = \tan^{-1} x - \dfrac{1}{2}\log x$ on $[1/\sqrt{3}, \sqrt{3}]$ is
(a) $\pi/2 + (1/2)\log 3$ (b) $\pi/6 + (1/4)\log 3$
(c) $\pi/6 + (1/2)\log 3$ (d) $\pi/4 - (1/4)\log 3$

64. Equations of those tangents to $4x^2 - 9y^2 = 36$ which are perpendicular to the straight line $2y + 5x = 10$; are
(a) $5(y-3) = \left(x - \sqrt{117/4}\right)$
(b) $5(y-2) = 2(x - \sqrt{18})$
(c) $5(y+2) = 2(x - \sqrt{18})$
(d) none of these

65. If $a, b, c, \in \mathbf{R}$, then
$$f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} \text{ decrease on}$$
(a) $(-(2/3)(a^2+b^2+c^2), 0)$
(b) $(0, (2/3)(a^2+b^2+c^2))$
(c) $((1/3)(a^2+b^2+c^2), 0)$
(d) none of these

66. A channel 27 m wide falls at a right angle into another channel 64 m wide. The greatest length of the log that can be floated along this system of channels is
(a) 120 (b) 125
(c) 100 (d) 110

67. For $a \in [\pi, 2\pi]$, the function
$$f(x) = \frac{1}{3}\sin a \tan^3 x + (\sin a - 1)\tan x + \frac{\sqrt{a-2}}{\sqrt{8-a}}$$
has
(a) $x = n\pi$ ($n \in \mathbf{I}$) as critical points
(b) no critical points
(c) $x = 2n\pi$ ($n \in \mathbf{I}$) as critical points
(d) $x = (2n+1)\pi$ ($n \in \mathbf{I}$) as critical points.

68. The value of a for which the function
$f(x) = (4a-3)(x + \log 5) + 2(a-7)\cot(x/2)\sin^2(x/2)$ does not possess critical point is
(a) $(-\infty, -4/3]$ (b) $(-\infty, -1)$
(c) $[1, \infty)$ (d) $(0, \infty)$

69. The interval to which b may belong so that the function
$$f(x) = \left(1 - \frac{\sqrt{21-4b-b^2}}{b+1}\right)x^3 + 5x + \sqrt{6}$$
is increasing at every point of its domain is
(a) $[-7, 0]$ (b) $[-6, 0]$
(c) $[1, 4]$ (d) $[2, 3]$

70. A tangent is drawn at a variable point on an ellipse $x^2/a^2 + y^2/b^2 = 1$, then minimum area of the triangle formed by the tangent and the coordinate axes is
(a) ab (b) $(a^2 + b^2)/2$
(c) $(a + b)^2/4$ (d) $2\,ab$

71. The set of all x for which $\log(1 + x) \le x$ is
(a) $(1, \infty)$ (b) $(0, \infty)$
(c) $(-1, \infty)$ (d) none of these

72. The minimum value of $2^{(x^2-3)^3} + 27$ is
(a) 2^{27} (b) 2
(c) 1 (d) none of these

73. Let $f(x) = \begin{cases} |x| & \text{for } 0 < |x| \le 2 \\ 1 & \text{for } x = 0 \end{cases}$

Then at $x = 0$, f has
(a) a local maximum (b) no local maximum
(c) a local minimum (d) no extremum

74. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is
(a) increasing on $[-1/2, 1]$
(b) decreasing on **R**
(c) increasing on **R**
(d) decreasing on $[-1/2, 1]$

75. Let $f(x) = \begin{cases} x^\alpha \log x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

Rolle's theorem can be applied to f on [0, 1] then value of α can be
(a) -1 (b) $-1/2$
(c) 0 (d) 1/2

76. A cone is made from a circular sheet of radius $\sqrt{3}$ by cutting out a sector and gluing the cut edges of the remaining piece together. The maximum volume attainable for the cone is
(a) $\pi/3$ (b) $\pi/6$
(c) $2\pi/3$ (d) $3\sqrt{3}\pi$

77. The dimensions of the rectangle of maximum area that can be inscribed in the ellipse $(x/4)^2 + (y/3)^2 = 1$ are
(a) $\sqrt{8}, \sqrt{2}$ (b) 4, 3
(c) $2\sqrt{8}, 3\sqrt{2}$ (d) none of these

78. The condition for $y = ax^4 + bx^3 + cx^2 + dx + e$ to have points of inflection is
(a) $b^2 - 4ac > 0$ (b) $3b^2 - 8ac = 0$
(c) $3b^2 - 8ac > 0$ (d) $3b^2 - 8ac < 0$

79. The largest value of m such that $|x^2 - 3x + 2| \ge m$ for all x in the interval [3/2, 7/4] is
(a) 3/4 (b) 3/8
(c) 3/16 (d) 7/4

80. The point in the interval $[0, \pi]$ for which the curve $y = (1/2)x$ and $y = \sin x$ are farthest apart is
(a) $\pi/2$ (b) $\pi/4$
(c) $\pi/6$ (d) π

81. The points at which the tangents to the curve $ax^2 + 2hxy + by^2 = 1$ is parallel to y-axis is
(a) (0, 0)
(b) where $hx + by = 0$ meets it
(c) where $ax + hy$ meets it
(d) none of these

82. If the point on $y = x\tan\alpha - \dfrac{ax^2}{2u^2\cos^2\alpha}$ $(0 < \alpha < \pi/2)$ where the tangent is parallel to $y = x$ has an ordinate $u^2/4a$ then the value of α is
(a) $\pi/2$ (b) $\pi/6$
(c) $\pi/3$ (d) none of these

83. Let $f(x) = \begin{cases} |x-1| + a, & x \le 1 \\ 2x + 3, & x > 1 \end{cases}$

If $f(x)$ has local minimum at $x = 1$ and $a \ge 5$ then the value of a is
(a) 5 (b) 6
(c) 11/2 (d) 15/2

84. Let $g(x) = \int_0^x f(t)\,dt$ and $f(x)$ satisfies the equation $f(x+y) = f(x) + f(y) + 2xy - 1$ for all $x, y \in \mathbf{R}$ and $f'(0) = 2$ then
(a) g increases on $(0, \infty)$ and decreases on $(-\infty, -0)$
(b) g increases on $(0, \infty)$
(c) g decreases on $(0, \infty)$ and increases $(-\infty, 0)$
(d) g decreases on $(-\infty, \infty)$

85. The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is
(a) $2\sqrt{3}$ (b) $\sqrt{3}$
(c) $4\sqrt{3}$ (d) 3

86. The interval into which the function $y = \dfrac{x-1}{x^2 - 2x + 3}$ transforms the entire real line is
(a) [1/3, 2] (b) [– 1/3, 2]
(c) [–1/3, 1] (d) none of these

87. The angle at which $x^2 + y^2 = 16$ can be seen from the point (8, 0) is
(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/2$ (d) $\pi/3$

88. The critical points of the function $f(x) = (x + 2)^{2/3}(2x - 1)$ are
(a) – 1 and 2 (b) 1
(c) 1 and – 1/2 (d) –1 and –2

89. The function $f(x) = \dfrac{\log(\pi + x)}{\log(e + x)}$ is
   (a) increasing on $[0, \infty)$
   (b) decreasing on $[0, \infty)$
   (c) increasing on $[0, \pi/e)$ and decreasing on $[\pi/e, \infty)$
   (d) decreasing on $[0, \pi/e)$ and increasing on $[\pi/e, \infty)$

90. A rectangle with perimeter 32 cm has greatest area if its length is
   (a) 12 (b) 10
   (c) 8 (d) 14

91. The greatest value of the function
   $f(x) = \cot^{-1} x + (1/2) \log x$ on $[1, \sqrt{3}]$ is
   (a) $(\pi/6) + 0.25 \log 3$ (b) $(\pi/3) - 0.25 \log 3$
   (c) $\pi/4$ (d) $\tan^{-1} e - 1/2$

92. A particle is moving along the parabola $y^2 = 4(x + 2)$. As it passes through the point (7, 6) its y-coordinate is increasing at the rate of 3 units per second. The rate at which x-coordinate change at this instant is (in units/sec)
   (a) 4 (b) 6
   (c) 8 (d) 9

93. The perimeter of a rectangle is fixed at 24 cm. If the length l of the rectangle is increasing at the rate of 1 cm per second, the value of l for which the area of rectangle start to decrease is
   (a) 2 cm (b) 6 cm
   (c) 4 cm (d) 8 cm

94. The rate at which fluid level inside vertical cylindrical tank of radius r drop if we pump fluid out at the rate of 3cm^3 l/min is
   (a) $-\dfrac{1}{\pi r^2}$ (b) $\dfrac{3}{\pi r^2}$
   (c) $\dfrac{2}{\pi r^2}$ (d) $\dfrac{4}{\pi r}$

95. The length l of a rectangle is decreasing at the rate of 2 cm/sec while the widths w is increasing at the rate of 2 cm/sec. When $l = 12$ and $w = 5$, the rate of change of area is (in cm^2/sec)
   (a) 14 (b) 12
   (c) 8 (d) 4

96. Let f be twice differentiable function such that $f(x) = x^2$, $x = 1, 2, 3$. Then
   (a) $f''(x) = 2\ \forall\ x \in (1, 3)$
   (b) $f''(x) = 2$ for some $x \in (1, 3)$
   (c) $f''(x) = 3\ \forall\ x \in (2, 3)$
   (d) $f''(x) = f'(x)$ for some $x \in (2, 3)$

97. A tangent drawn to the curve $y = f(x)$ at $P(x, y)$ cuts the x-axis and y-axis at A and B respectively such that $BP : AP = 3 : 1$, given that $f(1) = 1$ then
   (a) equation of the curve is $x\dfrac{dy}{dx} - 3y = 0$.
   (b) normal at (1, 1) is $x + 3y = 4$.
   (c) curve passes through (2, 1/8)
   (d) equation of the curve is $x\dfrac{dy}{dx} + 4y = 0$.

98. If $0 < b^2 < c$ then $f(x) = x^3 + bx^2 + cx + d$
   (a) has no local minima
   (b) has no local maxima
   (c) is strictly increasing on **R**
   (d) is strictly decreasing on **R**

Previous Years' AIEEE/JEE Main Questions

1. If $2a + 3b + 6c = 0 (a, b, c \in \mathbf{R})$ then the quadratic equation $ax^2 + bx + c = 0$ has
   (a) at least one root in [0, 1]
   (b) at least one root in [2, 3]
   (c) at least one root.
   (d) none of these **[2002, 2004]**

2. The maximum distance from origin of a point on the curve $x = a \sin t - b \sin\left(\dfrac{at}{b}\right)$, $y = a \cos t - b \cos\left(\dfrac{at}{b}\right)$, both $a, b > 0$ is
   (a) $a - b$ (b) $a + b$
   (c) $\sqrt{a^2 + b^2}$ (d) $\sqrt{a^2 - b^2}$ **[2002]**

3. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals
   (a) 1 (b) 2
   (c) 1/2 (d) 3 **[2003]**

4. If $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$

then the difference between the maximum and minimum values of u^2 is given by

(a) $(a + b)^2$ (b) $2\sqrt{a^2 + b^2}$

(c) $2(a^2 + b^2)$ (d) $(a - b)^2$ **[2004]**

5. A function $y = f(a)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is

(a) $(x + 1)^3$ (b) $(x - 1)^3$

(c) $(x - 1)^2$ (d) $(x + 1)^2$ **[2004]**

6. The normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at θ always passes through the fixed point

(a) (0, 0) (b) $(0, a)$

(c) $(a, 0)$ (d) (a, a) **[2004]**

7. A function is matched below against an interval where it is supported to be increasing which is of the following pairs in incorrectly matched?

	Interval	Function
(a)	$(-\infty, 1/3)$	$3x^2 - 2x + 1$
(b)	$(-\infty, -4)$	$x^3 + 6x^2 + 6$
(c)	$(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(d)	$(2, \infty)$	$2x^3 - 3x^2 - 12x + 6$

[2005]

8. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point Q is such that

(a) it passes through $\left(\dfrac{a\pi}{2}, -a\right)$

(b) it is at constant distance from origin

(c) it passes through origin

(d) it makes angle $\dfrac{\pi}{2} + \theta$ with the x-axis **[2005]**

9. Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \geq 2$ for $x \in [1, 6]$ then

(a) $f(6) < 5$ (b) $f(6) = 5$

(c) $f(6) \geq 8$ (d) $f(6) < 8$ **[2005]**

10. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases, is

(a) $1/54\pi$ cm/min (b) $5/6\pi$ cm/min

(c) $1/36\pi$ cm/min (d) $1/8\pi$ cm/min **[2005]**

11. The function $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$ has a local minimum at

(a) $x = 1$ (b) $x = 2$

(c) $x = -2$ (d) $x = 0$ **[2006]**

12. A value of c for which the conclusion of mean value theorem holds for the function $f(x) = \log x$ on the interval [1, 3] is

(a) $2\log_3 e$ (b) (1/2)log 3

(c) $\log_3 c$ (d) log 3. **[2007]**

13. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function is

(a) $(\pi/4, \pi/2)$ (b) $(-\pi/2, \pi/4)$

(c) $(0, \pi/2)$ (d) $(-\pi/2, \pi/2)$ **[2007]**

14. Suppose the cube $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds?

(a) The cubic has minima at $\sqrt{\dfrac{p}{3}}$ and maxima at $-\sqrt{\dfrac{p}{3}}$

(b) The cubic has minima at $-\sqrt{\dfrac{p}{3}}$ and maxima at $\sqrt{\dfrac{p}{3}}$

(c) The cubic has minima at both $\sqrt{\dfrac{p}{3}}$ and $-\sqrt{\dfrac{p}{3}}$

(d) The cubic has maxima at both $\sqrt{\dfrac{p}{3}}$ and $-\sqrt{\dfrac{p}{3}}$

[2008]

15. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$.
If $P(-1) < P(1)$, then in the interval $[-1, 1]$

(a) $P(-1)$ is the minimum but $P(1)$ is not the maximum of P

(b) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

(c) $P(-1)$ is the minimum and $P(1)$ is the maximum of P

(d) $P(-1)$ is not minimum but $P(1)$ is the maximum of P. **[2009]**

16. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} k - 2x, & \text{if } x \leq -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k is

(a) $-1/2$ (b) -1

(c) 1 (d) 0 **[2010]**

17. The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact, is given by:

(a) $2y - 3x = 0$ (b) $y = 6/x$

(c) $x^2 + y^2 = 13$ (d) $\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{3}\right)^2 = 2$

[2011]

18. Let f be a function defined by

$$f(x) = \begin{cases} \dfrac{\tan x}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

Statment-1: $x = 0$ is point of maxima of f **[2011]**

Statement-2: $f'(0) = 0$.

19. Let $a, b \in \mathbf{R}$ be such that the function f given by $f(x) = \log |x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$

Statement-1: f has local maximum at $x = -1$ and at $x = 2$

Statement-2: $a = 1/2$ and $b = -1/4$. **[2012]**

20. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is

(a) 7/9 (b) 2/9
(c) 9/2 (d) 9/7 **[2012]**

21. The cost of running a bus from A to B is Rs. $\left(av + \dfrac{b}{v}\right)$, where v km/h is the average speed of bus. When the bus travels at 30 km/h, the cost comes out to be Rs. 75 while at 40 km/h, it is Rs. 65. Then the most economical speed in km/h of the bus is

(a) 45 (b) 50
(c) 60 (d) 40 **[2013, online]**

22. If the surface area of a sphere of radius r is increasing uniformly at the rate of 8 cm^2/s then the rate of change of its volume is

(a) constant (b) proportional to $\sqrt{r}$
(c) proportional to r^2 (d) proportional to r **[2013, online]**

23. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]

(a) lies between 2 and 3
(b) lies between –1 and 0
(c) does not exist
(d) lies between 1 and 2 **[2013]**

24. The maximum area of a right angled triangle with hypotenuse h is

(d) $\dfrac{h^2}{2\sqrt{2}}$ (b) $\dfrac{h^2}{2}$
(c) $\dfrac{h^2}{\sqrt{2}}$ (d) $\dfrac{h^2}{4}$ **[2013, online]**

25. **Statement-1:** The equation $x \log x = 2 - x$ is satisfied by at least one value of x lying between 1 and 2

Statement-2: The function $f(x) = x \log x$ is an increasing function in [1, 2] and $g(x) = 2 - x$ is a decreasing function in [1, 2] and the graphs represented by these functions intersect at a point in [1, 2].**[2013, online]**

26. **Statement-1:** The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$

Statement 2: The function $x^2 e^x$ and $x^2 e^{-x}$ are increasing for all $x > 0$ and sum of two increasing functions in any interval (a, b) is an increasing function in (a, b) **[2013, online]**

27. If f and g are differentiable function in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$ then for some $c \in (0, 1)$

(a) $2f'(c) = g'(c)$ (b) $2f'(c) = 3g'(c)$
(c) $f'(c) = g'(c)$ (d) $f'(c) = 2g'(c)$ **[2014]**

28. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log |x| + \beta x^2 + x$ then:

(a) $\alpha = -6, \beta = \dfrac{1}{2}$ (b) $\alpha = -6, \beta = -\dfrac{1}{2}$
(c) $\alpha = 2, \beta = -\dfrac{1}{2}$ (d) $\alpha = 2, \beta = \dfrac{1}{2}$ **[2014]**

29. If the volume of a spherical ball is increasing at the rate of 4π cc/sec, then the rate of increase of its radius (in cm/sec), when the volume is 288π cc is

(a) $\dfrac{1}{9}$ (b) $\dfrac{1}{6}$
(c) $\dfrac{1}{36}$ (d) $\dfrac{1}{24}$ **[2014, online]**

30. If non-zero real number b and c are such that $\min f(x) > \max g(x)$ where $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ ($x \in \mathbf{R}$) then $\left|\dfrac{c}{b}\right|$ lies in the interval:

(a) $\left[\dfrac{1}{\sqrt{2}}, \sqrt{2}\right]$ (b) $\left[0, \dfrac{1}{2}\right]$
(c) $\left[\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}\right]$ (d) $\left[\sqrt{2}, \infty\right]$ **[2014, online]**

31. Let f and g be two differentiable functions on **R** such that $f'(x) > 0$ and $g'(x) < 0$ for all $x \in \mathbf{R}$. Then for all x

(a) $g(f(x)) > g(f(x-1))$
(b) $f(g(x)) > f(g(x+1)$
(c) $f(g(x)) > f(g(x-1))$
(d) $g(f(x)) < g(f(x+1))$ **[2014 online]**

32. If Rolle's theorem holds for the function $f(x) = 2x^3 + ax^2 + bx$ in the interval [–1, 1] for the point $c = \frac{1}{2}$, then the value of $2a + b$ is

(a) 1 (b) – 1
(c) 2 (d) – 2
[2014, 2015 online]

33. Let $f(x)$ be a polynomial of degree four having extreme value at $x = 1$ and $x = 2$. If $\lim_{x \to 0}\left[1+\frac{f(x)}{x^2}\right] = 3$, then $f(2)$ is equal to

(a) –8 (b) – 4
(c) 0 (d) 4 **[2015]**

34. The equation of a normal to the curve $\sin y = x\sin\left(\frac{\pi}{3}+y\right)$ at $x = 0$ is

(a) $2x + \sqrt{3}y = 0$ (b) $2y - \sqrt{3}x = 0$
(c) $2y + \sqrt{3}x = 0$ (d) $2x - \sqrt{3}y = 0$
[2015, online]

35. Let k and K be the minimum and maximum values of the function $f(x) = \frac{(1+x)^{0.6}}{1+x^{0.6}}$ in [0, 1] respectively, then the order pair (k, K) is equal to

(a) $(1, 2^{0.6})$ (b) $(2^{-0.4}, 2^{0.6})$
(c) $(2^{-0.6}, 1)$ (d) $(2^{-0.4}, 1)$ **[2015, online]**

36. From the top of a 64 metres high tower, a stone is thrown upward vertically with the velocity of 48m/s. The greatest height (in metres) attained by stone, assuming the value of the gravitational acceleration $g = 32$ m/s^2, is:

(a) 100 (b) 88
(c) 128 (d) 112 **[2015, online]**

37. The distance, from the origin, of the normal to the curve, $x = 2\cos t + 2t\sin t$, $y = 2\sin t - 2t\cos t$ at $t = \frac{\pi}{4}$, is

(a) 4 (b) $2\sqrt{2}$
(c) 2 (d) $\sqrt{2}$ **[2015, online]**

38. Let the tangents drawn to the circle, $x^2 + y^2 = 16$ from the point $P(0, h)$ meet the x-axis at points A and B. If the area of ΔAPB is minimum, then h is equal to:

(a) $4\sqrt{3}$ (b) $3\sqrt{3}$
(c) $3\sqrt{2}$ (d) $4\sqrt{2}$ **[2015]**

39. Consider

$$f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right),\ x \in \left(0, \frac{\pi}{2}\right).$$

A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point:

(a) (0, 0) (b) $\left(0, \frac{2\pi}{3}\right)$
(c) $\left(\frac{\pi}{6}, 0\right)$ (d) $\left(\frac{\pi}{4}, 0\right)$ **[2016]**

40. If m and M are are the minimum and the maximum values of $4+\frac{1}{2}\sin^2 2x - 2\cos^4 x$, $x \in \mathbf{R}$, then $M - m$ is equal to

(a) $\frac{9}{4}$ (b) $\frac{15}{4}$
(c) $\frac{7}{4}$ (d) $\frac{1}{4}$ **[2016, online]**

41. If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 - 1$, $t \in \mathbf{R}$ meets the curve again at a point Q, then the coordinates of Q are:

(a) $(16t^2 + 3, -64t^3 - 1)$ (b) $(4t^2 + 3, -8t^3 - 1)$
(c) $(t^2 + 3, t^3 - 1)$ (d) $(t^2 + 3, -t^3 - 1)$
[2016, online]

42. Let $f(x) = \sin^4 x + \cos^4 x$. Then f is an increasing function in the interval

(a) $\left]\frac{5\pi}{8}, \frac{3\pi}{4}\right[$ (b) $\left]\frac{\pi}{2}, \frac{5\pi}{8}\right[$
(c) $\left]\frac{\pi}{4}, \frac{\pi}{2}\right[$ (d) $\left]0, \frac{\pi}{4}\right[$ **[2016, online]**

43. Let C be a curve given by $y(x) = 1+\sqrt{4x-3}$, $x > \frac{3}{4}$. If P is a point on C, such that the tangent at P has slope $\frac{2}{3}$, then a point through which the normal at P passes, is

(a) (1, 7) (b) (3, –4)
(c) (4, –3) (d) (2, 3) **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. The slope of the normal to curve $y = x^3 - 4x^2$ at (2, –1) is
   (a) $\frac{1}{4}$ (b) $\frac{1}{2}$
   (c) 4 (d) –4 **[2006]**

2. For the curve $x = t^2 - 1$, $y = t^2 - t$, the tangent line is perpendicular to the x-axis when
   (a) $t = 0$ (b) $t = 1$
   (c) $t = \frac{1}{\sqrt{3}}$ (d) $t = \frac{1}{2}$ **[2006]**

3. If $f(x) = 4^{\sin x}$ satisfies the Rolle's theorem on $[0, \pi]$, then the value of $c \in (0, \pi)$ for which $f'(c) = 0$ is
   (a) $c = \frac{\pi}{6}$ (b) $c = \frac{\pi}{4}$
   (c) $c = \frac{\pi}{2}$ (d) $c = \frac{\pi}{3}$ **[2006]**

4. The value of c for which the conclusion of Lagrange's mean value theorem holds for the function $f(x) = \sqrt{25 - x^2}$ on the interval [1, 5] is
   (a) $\sqrt{3}$ (b) $\sqrt{5}$
   (c) $\sqrt{15}$ (d) 2 **[2007]**

5. Let $f(x) = \begin{cases} |x-1| + a & \text{if } x \le 1 \\ 2x + 3 & \text{if } x > 1 \end{cases}$
   If $f(x)$ has a local minimum at $x = 1$, then
   (a) $a > 5$ (b) $0 < a \le 5$
   (c) $a \le 5$ (d) $a = 5$ **[2007]**

6. If m is the slope of a tangent to the curve $e^{2y} = 1 + 4x^2$ then
   (a) $|m| \le 1$ (b) $|m| > 1$
   (c) $|m| \ge 1$ (d) $|m| < 1$ **[2007]**

7. Let $f : (-\infty, \infty) \to (-\infty, \infty)$ be a continuous and differentiable function and let $f'(\cdot)$ denote the derivative of $f(\cdot)$. If $f(0) = -2$ and $f'(x) \le 3$ for each $x \in [0, 2]$, then the largest possible value of $f(2)$ is
   (a) 1 (b) 2
   (c) 3 (d) 4 **[2008]**

8. Let f: $[-1, 2] \to (-\infty, \infty)$ be given by
   $$f(x) = \frac{x^4 + 3x^2 + 1}{x^2 + 1}$$
   Then the maximum possible value of $f(\cdot)$ on [–1, 2] is
   (a) 1 (b) $\frac{29}{5}$
   (c) $\frac{21}{5}$ (d) $\frac{28}{5}$ **[2008]**

9. Let $y = f(x)$ be a curve which passes through (3, 1) and is such that normal at any point on it passes through (1, 1). Then $y = f(x)$ describes
   (a) a circle of area π
   (b) an ellipse of area 2π
   (c) an ellipse of area 3π
   (d) a circle of area 4π **[2008]**

10. Let $f(x) = \begin{cases} x \sin\frac{\pi}{x} & , \ 0 < x \le 1 \\ 0 & , \ x = 0 \end{cases}$
    then $f'(x) = 0$ for
    (a) exactly two values of x
    (b) no value of x
    (c) infinitely many values of x
    (d) exactly one value of x **[2009]**

11. Let $f(x) = [1 - x^2]$, $x \in \mathbf{R}$, where [ ] is the greatest integer function. Then
    (a) f is increasing
    (b) $x = 0$ is the point of maxima of f
    (c) f is continuous at $x = 0$
    (d) f is decreasing **[2009]**

12. A particle is constrained to move along the curve $y = \sqrt{x}$ starting at the origin at time t = 0. The point on the curve where the abscissa and the ordinate are changing at the same rate is:
    (a) $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(\frac{1}{8}, \frac{1}{2\sqrt{2}}\right)$
    (c) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (d) (1, 1) **[2009]**

13. If the tangent and normal to the hyperbola $x^2 - y^2 = 4$ at a point cut off intercepts a_1 and a_2 respectively on x-axis and b_1 and b_2 respectively on y-axis then the value of $a_1 a_2 + b_1 b_2$ is
    (a) –1 (b) 0
    (c) 4 (d) 1 **[2010]**

14. Let f be a differentiable function defined on **R** such that $f(0) = -3$. If $f'(x) \le 5$ for all x then

(a) $f(2) > 7$ (b) $f(2) \leq 7$
(c) $f(2) > 8$ (d) $f(2) = 8$ **[2010]**

15. Let f be a function defined on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by $f(x) = 3\cos^4 x - 6\cos^3 x - 6\cos^2 x - 3$. Then the range of $f(x)$ is
(a) $[-12, -3]$ (b) $[-6, -3]$
(c) $[-6, 3]$ (d) $(-12, 3]$ **[2010]**

16. The function $f(x) = xe^{-x}$ has
(a) neither maximum nor minimum at $x = 1$
(b) a minimum at $x = 1$
(c) a maximum at $x = 1$
(d) a maximum at $x = -1$ **[2011]**

17. Each side of a square is increasing at the uniform rate of 1m/sec. If after sometime the area of the square is increasing at the rate of 8 m^2/sec, then the area of square at that time in sq. meters is
(a) 4 (b) 9
(c) 16 (d) 25 **[2012]**

18. The rate of change of the volume of a sphere with respect to its surface area when the radius is 2 units is:
(a) 4 (b) 3
(c) 2 (d) 1 **[2013]**

19. If m is the slope of a tangent to curve $e^y = 1 + x^2$ at the point (x, y) on the curve then all possible values of m lie in the interval:
(a) $[0, 1]$ (b) $(1, \infty)$
(c) $(-\infty, -1)$ (d) $[-1, 1]$ **[2013]**

20. $f(x) = |x \log x|$, $x > 0$ is monotonically decreasing in:
(a) $\left(0, \frac{1}{e}\right)$ (b) $\left[\frac{1}{e}, 1\right]$
(c) $(1, e)$ (d) (e, ∞) **[2014]**

21. Let $f(x) = |x - x_1| + |x - x_2|$, where x_1 and x_2 are distinct real numbers. Then the number of points at which $f(x)$ is minimum
(a) 1 (b) 2
(c) 3 (d) more than 3 **[2014]**

22. The maximum value of $f(x) = 2\sin x + \sin 2x$, in the interval $\left[0, \frac{3}{2}\pi\right]$ is
(a) $\sqrt{2}+1$ (b) $2\sqrt{3}$
(c) $\frac{3\sqrt{3}}{2}$ (d) $\sqrt{3}$ **[2015]**

23. The abscissa of a point, tangent at which to the curve $y = e^x \sin x$, $x \in [0, \pi]$, has maximum slope, is:
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) π (d) 0 **[2016]**

24. Let $p(x)$ be a real polynomial of degree 4 having extreme values at $x = 1$ and $x = 2$. If $\lim_{x \to 0} \frac{p(x)}{x^2} = 1$, then $p(4)$ is equal to:
(a) 16 (b) 32
(c) 64 (d) 8 **[2016]**

Reasoning Type

25. Let b, c be two non-zero real numbers such that $b^2 \leq 3c$. Let $f(x) = x^3 + bx^2 + cx + d$, $x \in \mathbf{R}$.
Statement-1: f is a 1 – 1 function.
Statement-2: f is a strictly decreasing function on **R**.
[2009]

26. **Statement-1:** The function f defined on **R** as $f(x) = \min\{x, x^2)$ is not differentiable at $x = 1$.
Statement-2: The smaller angle between the tangents to the curve $y = x$ and $y = x^2$ at $x = 1$ is $\tan^{-1}\frac{1}{3}$.
[2010]

27. **Statement-1:** If $f(x) = e^{(x-1)(x-3)}$ then Rolle's theorem is applicable to $f(x)$ in the interval [1, 3].
Statement-2: Mean value theorem is applicable to $f(x) = e^{(x-1)(x-3)}$ in the interval [1, 4]. **[2011]**

28. Let $f(x) = x^2 - 8x + 12$, $x \in [2, 6]$.
Statement 1: $f'(c) = 0$ for some $c \in (2, 6)$.
Statement 2: f is continuous on [2, 6] and differentiable on (2, 6) with $f(2) = f(6)$. **[2012]**

29. Let $a, b, c \in \mathbf{R}$, $a > 0$ and the function $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = ax^2 + bx + c$
Statement-1: $b^2 < 4ac \Rightarrow f(x) > 0$ for every value of x.
Statement-2: f is strictly decreasing in the interval $\left(-\infty, -\frac{b}{2a}\right)$ and strictly increasing in the interval $\left(-\frac{b}{2a}, \infty\right)$ **[2012]**

Answers

Concept-based

1. (a)	**2.** (c)	**3.** (b)	**4.** (d)
5. (a)	**6.** (c)	**7.** (c)	**8.** (a)
9. (b)	**10.** (c)		

Level 1

11. (c)	**12.** (b)	**13.** (b)	**14.** (b)
15. (d)	**16.** (c)	**17.** (b)	**18.** (a)
19. (d)	**20.** (c)	**21.** (a)	**22.** (d)
23. (a)	**24.** (b)	**25.** (d)	**26.** (c)
27. (d)	**28.** (a)	**29.** (c)	**30.** (b)
31. (c)	**32.** (c)	**33.** (c)	**34.** (c)
35. (b)	**36.** (c)	**37.** (d)	**38.** (c)
39. (c)	**40.** (c)	**41.** (d)	**42.** (d)
43. (a)	**44.** (b)	**45.** (c)	**46.** (b)
47. (c)	**48.** (a)	**49.** (b)	**50.** (b)
51. (d)	**52.** (b)	**53.** (c)	**54.** (a)
55. (a)	**56.** (d)	**57.** (d)	**58.** (a)
59. (a)	**60.** (a)		

Level 2

61. (b)	**62.** (a)	**63.** (b)	**64.** (d)
65. (a)	**66.** (b)	**67.** (b)	**68.** (a)
69. (d)	**70.** (a)	**71.** (c)	**72.** (d)
73. (a)	**74.** (a)	**75.** (d)	**76.** (c)
77. (c)	**78.** (c)	**79.** (c)	**80.** (d)
81. (b)	**82.** (d)	**83.** (a)	**84.** (b)
85. (a)	**86.** (d)	**87.** (d)	**88.** (d)
89. (b)	**90.** (c)	**91.** (a)	**92.** (d)
93. (b)	**94.** (b)	**95.** (a)	**96.** (b)
97. (c)	**98.** (c)		

Previous Years' AIEEE/JEE Main Questions

1. (a)	**2.** (b)	**3.** (b)	**4.** (a)
5. (b)	**6.** (c)	**7.** (a)	**8.** (b)
9. (c)	**10.** (d)	**11.** (b)	**12.** (a)
13. (b)	**14.** (a)	**15.** (d)	**16.** (b)
17. (b)	**18.** (b)	**19.** (a)	**20.** (b)
21. (c)	**22.** (d)	**23.** (c)	**24.** (c)
25. (a)	**26.** (c)	**27.** (d)	**28.** (c)
29. (c)	**30.** (d)	**31.** (b)	**32.** (b)
33. (c)	**34.** (a)	**35.** (d)	**36.** (a)
37. (c)	**38.** (d)	**39.** (b)	**40.** (a)
41. (d)	**42.** (c)	**43.** (a)	

Previous Years' B-Architecture Entrance Examination Questions

1. (a)	**2.** (b)	**3.** (c)	**4.** (c)
5. (c)	**6.** (a)	**7.** (d)	**8.** (b)
9. (d)	**10.** (c)	**11.** (d)	**12.** (c)
13. (b)	**14.** (b)	**15.** (a)	**16.** (c)
17. (c)	**18.** (d)	**19.** (d)	**20.** (b)
21. (b)	**22.** (c)	**23.** (b)	**24.** (a)
25. (c)	**26.** (b)	**27.** (b)	**28.** (a)
29. (b)			

Hints and Solutions

Concept-based

1. $$\frac{dy}{dx} = \frac{-a\cos^2 t + \sin t}{a\sin^2 t\cos t} = -\frac{\cos t}{\sin t}$$

Length of the tangent $= |y|\sqrt{1+\frac{\sin^2 t}{\cos^2 t}}$

$$= \frac{a\cos^3 t}{\cos t} = a\cos^2 t$$

2. $\frac{dy}{dx} = -\frac{e^{-t}}{2e^t} = -\frac{1}{2}e^{-2t}$. If $t = 0$ then $X = 2$, $y = 1$

Equation of normal is

$$Y - 1 = 2e^{2t}\Big|_{t=0}(X-2)$$

$\Rightarrow$ $Y - 1 = 2(X - 2) \Rightarrow 2X - Y - 3 = 0.$

3. $\frac{ds}{dt} = \frac{\pi}{9}\cos\frac{\pi}{2}t$ and $a = \frac{d^2s}{dt^2} = \frac{-\pi^2}{18}\sin\frac{\pi}{2}t$

$a(\text{t} = 1) = -\frac{\pi^2}{18}$

4. $f'(x) = 1 + \log x - 1 = \log x > 0$ for $x > 1$. So f is increasing for $x > 1$ $\Rightarrow f(x) > f(1) = 0$ for $x > 1$. $f'(x) < 0$ for $0 < x < 1$ $\Rightarrow f$ is decreasing for $0 < x < 1$ $\Rightarrow f(x) > f(1) = 0$ for $0 < x < 1$.

5. $\frac{dy}{dx} = 3x^2$, we need x such that $3x^2 = \frac{8+1}{2+1} = \frac{9}{3} = 3$ $\Rightarrow x^2 = 1$ so $x = 1$ or -1. Thus two points are $(-1, -1)$ and $(1, 1)$.

6. $$f'(x) = 4x - \frac{1}{x}$$

The function f increases if $4x - \frac{1}{x} > 0$ *i.e.* $x > 1/2$

The function f decreases if $4x-\frac{1}{x}<0$ *i.e.* $0<x<\frac{1}{2}$

7. f is differentiable function, so the critical points are given by $f'(x) = 0$

   $f'(x) = 3x^3 - 3x^2 - 18x = 3x(x+2)(x-3)$

   The critical points of f are 0, –2, 3 but only 0, 3 $\in$ [–1 4].

8. $3x^2\frac{dx}{dy}=12 \Rightarrow \frac{dx}{dy}=\frac{4}{x^2}$. We need $\left|\frac{dx}{dy}\right|>1$

   $\Rightarrow \frac{4}{x^2}>1 \Rightarrow x^2<4, x\neq 0 \Rightarrow x\in(-2,2)\sim\{0\}$.

9. For $\frac{x^2}{k^2}+\frac{y^2}{4}=1$, $\frac{dy}{dx}=\frac{-4x}{k^2y}$ and for $y^3 = 16x$,

   $\frac{dy}{dx}=\frac{16}{3y^2}$. For orthogonality $\left(\frac{-4x}{k^2y}\right)\left(\frac{16}{3y^2}\right)=-1$

   $\Rightarrow \quad 64x = 3k^2y^3 = 48k^2x$

   $\Rightarrow \quad k^2=\frac{4}{3} \Rightarrow k=\frac{2\sqrt{3}}{3}$

10. $g'(t) = 0 \Rightarrow 8 - 4t^3 = 0 \Rightarrow t^3 = 2$ so $t = 2^{1/3}$ but $2^{1/3} \notin [-2, 1]$. Now $g(-2) = -32$, $g(1) = 7$ So –32 in the least value.

Level 1

11. Let $h(x) = \tan^{-1}x - \frac{x}{1+x^2}$, $h'(x) = \frac{1}{1+x^2}-\frac{1}{x^2+1}$ $+\frac{2x^2}{(x^2+1)^2}>0$ for all x. For $x > 0$ $h(x) > h(0) = 0$

    $\Rightarrow f(x) > g(x)$ on $(0, \infty)$

12. $f(2) = f(3) = f(4) = f(5) = f(6) = 0$,

    so, by Rolle's theorem applied on [2, 3], [3, 4], [4, 5], [5, 6] there are $x_1 \in (2, 3)$, $x_2 \in (3,4)$, $x_3 \in (4, 5)$, $x_4 \in (5, 6)$ such that $f'(x_i) = 0$, $i = 1, 2, 3, 4$. Since f' is polynomial of degree 4 so cannot have five roots.

13. $f'(x) = (x-3)^4(x+1)^3(9x-7)$

    For $x < -1$, $f'(x) > 0$ and for $-1 < x < 0$, $f'(x) < 0$. $x = -1$ is a point of maxima.

14. $\frac{dy}{dx}$ = tan θ, so equation the equation of normal is $y - a(\sin\theta - \theta\cos\theta) = -\cot\theta(x - a(\cos\theta + \theta\sin\theta)$ which simplifies to $x\cos\theta + y\sin\theta = a$. The distance of the normal from origin is $|a|$.

15. Apply Rolle's theorem to show that the given equation cannot have two real distinct roots.

16. $\left(\frac{dy}{dx}\right)$ $(a/8, a/8) = -1$. The equation of tangent at $(a/8, a/8)$ is $y + x = a/4$. The x and y intercepts of this line both equal $a/4$, so

    $2 = \frac{a^2}{16}+\frac{a^2}{16} \Rightarrow a = 4.$

17. The slope of tangent to $x^m y = b^m$ is $\frac{dy}{dx}=-\frac{my}{x}$. So the equation of tangent is $Y - y = -\frac{my}{x}(X - x)$ which simplifies to $\frac{X}{x+\frac{x}{m}}+\frac{Y}{y+my}=1$. Thus the base of the triangle is the X-intercept, $x + x/m$ and the height is the y-intercept, $y + my$. So the area of the triangle is

    $$(1/2)\left(x+\frac{x}{m}\right)(y+my)$$

    $$=\frac{xy(1+m)^2}{2m}=\frac{b^m(1+m)^2}{2m\,n^{m-1}}$$

    which is constant if $m = 1$.

18. The slope of the tangent is $-x^3/y^3$ and its equation is $\frac{X}{a^4/x^3}+\frac{Y}{a^4/y^3}=1$.

    Thus the required intercepts are $p = a^4/x^3$ and $q = a^4/y^3$ so that

    $p^{-4/3} + q^{-4/3} = a^{-16/3}(x^4 + y^4) = a^{-16/3}a^4 = a^{-4/3}$

19. $1 - 2\cos x > 0$ if $\cos x < 1/2 \Rightarrow x \in \left(\frac{\pi}{3},\frac{5\pi}{3}\right)$.

20. For $x \neq -1, 1$, $\frac{dy}{dx}=\frac{(x-1)^{2/3}-(x+1)^{2/3}}{3(x^2-1)^{2/3}} = 0$ only at $x = 0$. $y(0) = 2$, $y(-1) = y(1) = 2^{1/3}$, so the greatest value of y is 2.

21. $f'(x) = 4x - \frac{1}{x}=\frac{4x^2-1}{x}=\frac{(2x-1)(2x+1)x}{x^2}$

	$2x-1$	$2x+1$	x	$f'(x)$
$x < -1/2$	–	–	–	–
$-1/2 < x < 0$	–	+	–	+
$0 < x < 1/2$	–	+	+	–
$x > 1/2$	+	+	+	+

Thus f increases on $[-1/2, 0] \cup [1/2, \infty]$.

22. Let (x, y) be any point on $y = \frac{e^x+e^{-x}}{2}$

$U = S^2$ = square of distance of (x, y) from $(0, 0)$

$$= x^2 + \frac{e^{2x} + e^{-2x}}{4} + \frac{1}{2}$$

$$\frac{dU}{dx} = 2x + \frac{e^{2x} - e^{-2x}}{2} = 0 \text{ for } x = 0$$

and $\frac{d^2U}{dx^2} > 0$ at $x = 0$. Hence S^2 is min for $x = 0$ and min $S^2 = \frac{1}{2} + \frac{1}{2} = 1$.

23. $2x + 2y\frac{dy}{dx} - 2 - 2\frac{dy}{dx} = 0 \Rightarrow \left.\frac{dy}{dx}\right|_{(2,2)} = -1$
    Required equation of normal is $y - 2 = (x - 2)$ $\Rightarrow$ $y = x$.
24. Let $g(x) = \frac{f(x)}{x} \cdot g'(x) = \frac{xf'(x) - f(x)}{x^2}$.
    Let $h(x) = x f'(x) - f(x)$, $h'(x) = xf''(x) > 0$
    So $h(x) > h(0) = 0$ for $x > 0'$. Hence $g'(x) > 0$, $x > 0$ g increases on $(0, \infty)$.
25. If α and β are real and distinct roots of $x^3 - 12x + k = 0$. Then $f(x) = x^3 - 12x + k$ satisfies $f(\alpha) = f(\beta) = 0$ then there $\gamma \in [\alpha, \beta] \subseteq [0, 2]$ such that $f'(\gamma) = 0$ i.e., $3v^2 - 12 = 0 \Rightarrow \gamma = \pm 2$. Not possible.
26. $f'(x) = 8x^{1/3} - x^{-2/3} = \frac{8x - 1}{x^{2/3}} = 0$ if $x = 1/8$
    $f(1) = 3$, $f(-1) = 6 + 3 = 9$, $f\left(\frac{1}{8}\right) = -9/8$ thus max $f = 9$.
27. $f'(x) = 4x^3 - 18x^2 + 24x - 8$, $f''(x) = 12x^2 - 36x + 24 = 12(x^2 - 3x + 2) = 12(x - 1)(x - 2)$, $f''(x) = 0 \Rightarrow x = 1, 2$
    If $x = 1$ then $y = 2$. If $x = 2$, $y = 3$. So equation of tangent at $(1, 2)$ is $y - 2 = f'(1)(x - 1) = 2(x - 1)$. Equation of the tangent at $(2, 3)$ is $y - 3 = f'(2)(x - 2) = 0$.
28. The odd degree has at least one real root if the equation has at two real roots then by Rolle's theorem there is $\gamma \in [\alpha, \beta]$ such that $6\gamma^2 - 6\gamma + 6 = 0 \Rightarrow \gamma^2 - \gamma + 1 = 0$ which is not possible for $\gamma \in \mathbf{R}$.
29. $f(x) = (x - 2)(x - 1)^4 \Rightarrow f'(x) = (x - 1)^4 + 4(x - 1)^3(x - 2) = (x - 1)^3(5x - 9)$ so for $1 < x < 9/5$, $f'(x) < 0$ and for $x > 9/5$, $f'(x) > 0$. Thus $x = 9/5$
    f has local minimum $f(9/5) = -\frac{1}{5}\left(\frac{4}{5}\right)^4$.
30. $f(x) = \begin{cases} x^2 + 2x + 2, & -1/2 \le x \le 0 \\ x^2 - 2x + 2, & 0 < x \le 3/2 \end{cases}$
    and $f'(1) = 0$
    $f'(0)$ doesn't exist. Since $f(0) = 2$, $f(-1/2) = 5/4$, $f(3/2) = 5/4$, $f(1) = 1$ so Max. $f(x) = 2$ and min $f(x) = 1$.
31. The function is not differentiable at $x = 0$ and $x = 1$. Now
    $$f(x) = \begin{cases} \frac{x-1}{x^2} & x \ge 1 \\ \frac{1-x}{x^2}, & x < 1, x \ne 0 \end{cases} \Rightarrow f'(x) = \begin{cases} \frac{-x+2}{x^3} & \text{if } x > 1 \\ \frac{x-2}{x^3}, & \text{if } x < 1, x \ne 0 \end{cases}$$
    so we get a third critical point $x = 2$.
32. The derivative of the function, $x^x(\log x + 1)$, is negative when $\log < -1$ i.e., when $x < e^{-1}$.
33. $y'(x) = 1 - e^x$ is positive when $e^x < 1$ i.e., when $x < 0$
34. $3 = f(1) = 1 + p + q$ and $f'(1) = 0 \Rightarrow 2 + p = 0 \Rightarrow p = -2$, $q = 4$.
35. Let $f(x) = ax^3 + bx^2 + cx + d$, $a \ne 0$, $f''(x) = 6ax + 2b$, $f''(x) = 0$
    $\Rightarrow x = \frac{-b}{3a}$ $f'''(x) = 6a \ne 0$. So only one point of inflection.
36. $f(x) = \begin{cases} 4\tan^{-1} x & \text{if } -1 \le x \le 1 \\ 4\tan^{-1} x - \pi & \text{if } x > 1 \\ 4\tan^{-1} x + \pi & \text{if } x < -1 \end{cases}$
    So max $f(x) = 4\frac{\pi}{2} - \pi = \pi$
37. $\left.\frac{dy}{dx}\right|_{(m^2 - m^3)} = -\frac{3m}{2}$ The equation of normal at $(m^2, -m^3)$ is $Y + m^3 = (2/3m)(X - m^2)$. Simplifying we have $Y = (2/3m)X - (2m/3 + m^3)$ which is identical with $y = 2mx - 2m^3$ if $m^2 = 2/3$.
38. $f'(x) = 2\cos x - 2\sin 2x = 2\cos x(1 - 2\sin x) > 0$
    if (i) $\cos x > 0$, $\sin x < 1/2$ (ii) $\cos x < 0$, $\sin x > 1/2$
    i.e., if $x \in (0, \pi/6) \cup (3\pi/2, 2\pi)$ or $x \in (\pi/2, 5\pi/6)$.
    So f increases on $(0, \pi/6) \cup (\pi/2, 5\pi/6)$.
39. $f'(x) = \sec^2 x - \csc^2 x = \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} = \frac{-\cos 2x}{\sin^2 x \cos^2 x}$ $f'(\pi/4) = 0$, $f'(3\pi/4) = 0$. For $0 < x < \pi/4$, $f'(x) < 0$ and for $\pi/4 < x < \pi/2$, $f'(x) > 0$. Min $f(x) = f(\pi/4) = 2$.
40. $f'(x) = 12x(x^2 - x + 1)$, $f'(x) = 0$ only for $x = 0$.
41. Distance of (t^2, t) on $y^2 = x$ from $y - x - 1 = 0$ is $\frac{|t - t^2 - 1|}{\sqrt{2}} = \frac{t^2 - t + 1}{\sqrt{2}} + 1$. This is minimum if $t = 1/2$. So the shortest distance $= 3\sqrt{2}/8$.

42. $f'(x) = 3(x - (a + 1))(x - (a - 1)) = 0 \Rightarrow x = a \pm 1$.
$a - 1 \in (-2, 4) \Rightarrow a \in (-1, 5)$, $a + 1 \in (-2, 4)$
$\Rightarrow a \in (-3, 3)$

43. $y = \tan A \tan(\pi/3 - A) = \dfrac{x(\sqrt{3}-x)}{1+\sqrt{3}x}$, where $x = \tan A$.

$\dfrac{dy}{dx} = \dfrac{-(\sqrt{3}x-1)(x+\sqrt{3})}{(1+\sqrt{3}x)^2}$. As $0 < A < \pi/3$, $x = \tan A > 0$

$\dfrac{dy}{dx} = 0 \Rightarrow x = \dfrac{1}{\sqrt{3}} \cdot \dfrac{dy}{dx} > 0$ for $0 < x < \dfrac{1}{\sqrt{3}}$ and is < 0 for $x > \dfrac{1}{\sqrt{3}}$. So $y_{max} = y = \left(x = 1/\sqrt{3}\right) = 1/3$.

44. $f(x) = -x \log x$, for $0 < x < 1$, and $f'(x) = -(1 + \log x)$. So $f'(x) = 0 \Rightarrow x = e^{-1}$. It is easy to see max $f(x) = f(e^{-1}) = e^{-1}$.

45. $f(x) = \log_x (1/9) - \log_3 x^2 = -\dfrac{\log 9}{\log x} - \dfrac{2\log x}{\log 3}$

$f'(x) = \dfrac{2((\log 3)^2 - (\log x)^2)}{x(\log x)^2 \log 3}$ so $f'(x) = 0 \Rightarrow \log x = \pm \log 3$
$\Rightarrow x = 3$ or $1/3$. One can see by first derivative test that max $f(x) = f(3) = -4$.

46. Applying Lagrange's mean value theorem there is $c \in (1, 6)$ such that $\dfrac{f(6)-+(1)}{5} = f'(c) \geq 2$
$\Rightarrow f(6) \geq f(1) + 10 = 8$.

47. $f'(x) = 3[(\sin^{-1} x)^2 - (\cos^{-1} x)^2]\sqrt{1-x^2}$
$f'(x) = 0 \Rightarrow \sin^{-1} x = \cos^{-1} x$ or $\sin^{-1} x = -\cos^{-1} x$ but
$\sin^{-1} x + \cos^{-1} x = \pi/2$ so $\sin^{-1} x = \cos^{-1} x \Rightarrow \sin^{-1} x = \pi/4$
$\Rightarrow x = 1/\sqrt{2}$. Thus $f(1/\sqrt{2}) = \pi^3/32$.

48. $f'(x) = \log x + 4$ so $f'(x) > 0$ if $\log x > -4$ i.e., $x > e^{-4}$.

49. $f'(x) = (2 - x)xe^{-x}$. $f'(x) = 0 \Rightarrow x = 0, 2$
For $0 < x < 2$, $f'(x) > 0$ and for $x > 2$, $f'(x) < 0$
So max $f(x) = f(2) = 4e^{-2}$.

50. $f(x) =$

$$\begin{cases} 6-3x & , x \le -2 \\ 10-x & , \ 2 < x \le 3 \\ 4+x & , 3 < x \le 5 \\ 3x-6 & , x > 5 \end{cases} \qquad \min f(x) = f(3) = 7.$$

51. f is not differentiable at $x = 0$. For $x \neq 0$, $f'(x) = 2 - 3x^{-1/3}$. So $f'(x) = 0$

$\Rightarrow x = \dfrac{27}{8} > \dfrac{10}{3}$, hence $f'(x)$ is never zero, for $x \neq 0$.

Now $f(-1) = -3$, $f(0) = 2$

$f(10/3) = \dfrac{26}{3} - 3\left(\dfrac{10}{3}\right)^{2/3} > 0$ but is less then 2.

Hence absolute minimum is $f(-1)$ and absolute maximum is $f(0)$.

52. $f(x) = \lim\limits_{n\to\infty} \dfrac{x^n - 1}{x^n + 1} = \begin{cases} -1, & \text{if } \ 0 \le x < 1 \\ 1, & \text{if } \quad x > 1 \\ 0, & \text{if } \quad x = 1 \end{cases}$

$g'(x) = f(x)$. Hence g has local minimum at $x = 1$.

53. $f'(x) = \cos x - \sin x = 0 \Rightarrow x = n\pi + \pi/4$
$f''(x) = -\sin x - \cos x = -(\sin x + \cos x)$
$f''\left(n\pi + \dfrac{\pi}{4}\right) = (-1)^{n+1}\left(\sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4}\right)$
$= (-1)^{n+1}\sqrt{2}$
If n is even then $f(x)$ has maxima and if n is odd then $f(x)$ has minima.

54. Let $f(x) = xe^{x(1-x)}$, we can write
$f'(x) = -(x-1)(2x+1)e^{x(1-x)}$
since $e^{x(1-x)} > 0 \ \forall x \in \mathbf{R}$ so
$f'(x) < 0$ for $x < -1/2$; $f'(x) > 0$ for $-1/2 < x < 1$ and is < 0 for $x > 1$.
Thus $f(x)$ increases on $[-1/2, 1]$

55. Equation tangent at $P(c, e^c)$ is $y - e^c = e^c(x - c)$ (1)
The equation of line joining $(c + 1, e^{c+1})$ and (c^{-1}, e^{c-1}) is

$y - e^{c+1} = \dfrac{1}{2}(e^{c+1} - e^{c-1})(x - c - 1)$ (2)

Subtracting (2) from (1)
$e^c(e - 1) =$

$$e^c\left[(x-c)\left\{1-\frac{1}{2}\left(e-e^{-1}\right)\right\}+\frac{1}{2}\left(e-e^{-1}\right)\right]$$

$\Rightarrow e - 1 = (x-c)\left[1-\dfrac{1}{2}\left(e-e^{-1}\right)\right]+\dfrac{1}{2}\left(e-e^{-1}\right)$

$\Rightarrow c - x = \dfrac{e+e^{-1}-2}{e-e^{-1}-2} > 0 \ \Rightarrow \ x < c$

Thus two lines meet on the left of $x = c$.

56. Consider $f(x) = x^{1/x}$
$f'(x) = x^{1/x}\left(\dfrac{1-\log x}{x^2}\right)$
f has local maximum at $x = e$
Since $\pi > e$ and f decreases on (e, ∞) so $f(\pi) < f(e) \ \Rightarrow \ \pi^e < e^\pi$.

57. Applying Lagrange's mean value on $[x, y]$, we have $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

$| f(y) - f(x) | = | f'(c) | \, | y - x | \le M | y - x |$ (if $| f'(x) | \le M$)
and $| f'(y) - f(x) | \ge M | x - y |$ (if $|f'(x)| \ge M$)
Putting $f(x) = \cot x$, $f'(x) = - \operatorname{cosec}^2 x$, $| f'(x) | \ge 1$
Hence $| \cot x - \cot y | \ge | x - y |$

58. $f'(x) = 2 \cos x - 2 \sin 2x = 2 \cos x (1 - 2 \sin x)$
Now determine the sign of $f'(x)$ on $[0, 2\pi]$.

59. $f(x) = \pi/4 - \tan^{-1} x \Rightarrow f'(x) = -\frac{1}{1+x^2} < 0$ for all x. So the greatest value is $f(0) = \pi/4$ and least value is $f(1) = 0$.

60. $f'(x) = \frac{1 - \log x}{(\log x)^2}$.

Level 2

61. $f'(x) = (2/3)(x - 2)^{-1/3}(2x + 1) + 2(x - 2)^{2/3}$
$= \left(\frac{10}{3}\right)(x - 1)(x - 2)^{-2/3}$

$f'(x) = 0 \Rightarrow x = 1$ also $f'(x)$ does not exist at $x = 2$, hence the critical points are $x = 1$ and

$x = 2$.

62. $f(-2) = 1$

63. $f'(x) = \frac{1}{1+x^2} - \frac{1}{2x} = \frac{-(1-x)^2}{(1+x^2)2x} < 0$

for $x > 0$ i.e. f decreases on $\left[1/\sqrt{3}, \sqrt{3}\right]$
Therefore, the greatest value is $f\left(1/\sqrt{3}\right)$
$= \tan^{-1} \frac{1}{\sqrt{3}} - \frac{1}{2} \log \frac{1}{\sqrt{3}} = \frac{\pi}{6} + \frac{1}{4} \log 3.$

64. $\frac{x^2}{9} - \frac{y^2}{4} = 1 \Rightarrow \frac{2x}{9} - \frac{y}{2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{4x}{9y}$

The tangent will be perpendicular to $2y + 5x = 10$ if $\frac{4x}{9y} \times \left(\frac{-5}{2}\right) = -1 \Rightarrow 10x = 9y$

Substituting the value of x in $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we have $\left(\frac{9}{100} - \frac{1}{4}\right)y^2 = 1$, which is not possible. Thus there is no point on the curve at which the tangent is perpendicular to $2y + 5x = 10$.

65. $$f'(x) = \begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$= (x + b^2)(x + c^2) - b^2c^2 + (x + a^2)(x + c^2) - a^2 c^2 + (x + a^2)(x + b^2) - a^2 b^2$
$= 3x^2 + 2(a^2 + b^2 + c^2) x$
$= x(3x + 2(a^2 + b^2 + c^2))$
f decreases if $x < 0$ and $x > -\frac{2}{3}(a^2 + b^2 + c^2)$
i.e. on $((-2/3)(a^2 + b^2 + c^2), 0)$.

66. Let 'a' be the part of log in the channel of width 27 m and 'b' be the part of log in the channel of width 64 m.

So $l = a + b$
$= 27 \operatorname{cosec} \theta + 64 \sec \theta$

$\frac{dl}{d\theta} = -27 \operatorname{cosec} \theta \cot \theta + 64 \sec \theta \tan \theta$

$\frac{dl}{d\theta} = 0 \Rightarrow \tan \theta = \frac{3}{4}$

One can verify that $\frac{d^2 l}{d\theta^2} < 0$ for $\theta = \tan^{-1} \frac{3}{4}$

max $l = 27 \times \frac{5}{3} + 64 \times \frac{5}{4} = 125.$

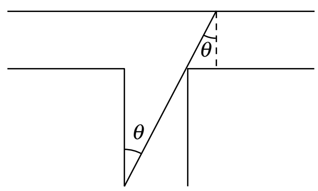


Fig. 11.7

67. $f'(x) = \sin a \tan^2 x \sec^2 x + (\sin a - 1) \sec^2 x$
$= \sec^2 x (\sin a \tan^2 x + (\sin a - 1))$
Since $\sec^2 x \ne 0$ so $\tan^2 x = \frac{1 - \sin a}{\sin a}$
Now $a \in [\pi, 2\pi]$ so $\sin a < 0$. Hence L.H.S. is ≥ 0 but R.H.S. < 0. Thus there is no critical point.

68. The derivative of $f(x)$ is $(4a - 3) + (a - 7) \cos x$. It is
zero when $\cos x = \frac{4a-3}{7-a} \Rightarrow \left|\frac{4a-3}{7-a}\right| \le 1$

$\Rightarrow -1 \le \dfrac{4a-3}{7-a} \le 1 \Rightarrow a - 7 \le 4a - 3 \le 7 - a$

i.e. when $a \ge -4/3$ or $a \le 2$. Hence the interval when f has no critical points are $(-\infty\ -4/3]$ and $[2, \infty)$.

69. $f'(x) = 3\left(1 - \dfrac{\sqrt{21-4b-b^2}}{b+1}\right) x^2 + 5 > 0$ if

$x^2 > \dfrac{-5}{3\left(1-\dfrac{\sqrt{21-4b-b^2}}{b+1}\right)}$ if

$1 - \dfrac{\sqrt{21-4b-b^2}}{b+1} > 0$

which is trivially true if $b + 1 < 0$ i.e. $b < -1$

If $b + 1 > 0$ then

$(b+1)^2 > 21 - 4b - b^2 \Rightarrow (b-2)(b+5) > 0$

i.e. $b > 2$ or $b < -5$

so $b \in [2, 3]$.

70. Any point on the ellipse is given by ($a\cos\theta$, $b\sin\theta$)

Also $\dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y} = \dfrac{-b}{a}\cot\theta$

The equation of tangent at ($a\cos\theta$. $b\sin\theta$) is $Y - b\sin\theta = \dfrac{-b}{a}\cot\theta\,(X - a\cos\theta)$

This tangent cuts the coordinate axes at the points $\left(0, \dfrac{b}{\sin\theta}\right)$ and $\left(\dfrac{a}{\cos\theta}, 0\right)$

The area of the triangle formed by the tangent and the coordinate axes is

$A = \dfrac{1}{2}\dfrac{ab}{\sin\theta\cos\theta} = ab\ \text{cosec}\ 2\theta$

But cosec 2θ is smallest when $\theta = \pi/4$

$A_{\min} = ab.$

71. Let $f(x) = x - \log(1+x)$. The domain of f is $(-1, \infty)$. $f'(x) = 1 - \dfrac{1}{1+x} = \dfrac{1+x-1}{1+x} = \dfrac{x}{1+x}$. Since $1 + x > 0$ for $x \in$ dom f so $f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ for $(-1, 0)$. Thus f increases $(0, \infty)$. Morever, $f(0) = 0$. Hence $f(x) \ge f(0) = 0 \Rightarrow x \ge \log(1+x)$ for $x \in (0, \infty)$.

For $x \in (-1, 0)$, f decreases so

$f(x) \ge f(0)$. Hence $x \ge \log(1+x)$ on $(-1, \infty)$.

72. $2^{(x^2-3)^2} + 27 \ge 28$ and the value 28 is attained at $x = \sqrt{3}$. Hence the minimum value of $2^{(x^2-3)^2} + 27$ is 28.

73. $f(x) = \begin{cases} -x & -2 < x < 0 \\ 1 & x = 0 \\ x & 0 < x < 2 \end{cases}$

f is not continuous at $x = 0$. Morever

$f(0) > f(a)$ for $a \in (-1/2, 1/2) \sim \{0\}$. f has local maximum at $x = 0$.

74. $f'(x) = e^{x(1-x)} + x(1-2x)\,e^{x(1-x)}$

$= (1 + x - 2x^2)\,e^{x(1-x)} = -(x-1)(2x+1)e^{x(1-x)}$

Since $e^{x(1-x)} > 0$ for all x, so $f'(x) > 0$ if and only if $(x-1)(1+2x) < 0$ i.e $-1/2 = \min(1, -1/2) < x < \max(1, -1/2) = 1$.

Thus f increases on $[-1/2, 1]$.

75. Clearly $f(0) = f(1) = 0$. For f to be continuous on $[0, 1]$, we must have $\lim\limits_{x\to 0+} x^\alpha \log x = 0$. This is possible only if $\alpha > 0$. Thus $\alpha = 1/2$ is only such value.

76. Let h be the height of cone, then the radius r of the cone satisfies $r^2 = 3 - h^2$

$V = \dfrac{1}{3}\pi(3-h^2)h$ so $\dfrac{dV}{dh} = \dfrac{1}{3}\pi(-3h+3)$

$\dfrac{dV}{dh} = 0 \Rightarrow h = 1$. Also $\dfrac{d^2V}{dh^2} = -\pi < 0$

so $V_{\max} = \dfrac{1}{3}\pi(3-1) = \dfrac{2\pi}{3}$.

77. Any point on the ellipse is of the form ($4\cos\theta$, $3\sin\theta$), $0 \le \theta \le 2\pi$. Area A of a rectangle inscribed in the ellipse $= 4 \times 4\cos\theta \times 3\sin\theta = 24\sin 2\theta$. This will be maximum when $2\theta = \pi/2$ or $\theta = \pi/4$. Hence the dimensions of the ellipse are

$2\cdot 4\cdot\dfrac{1}{\sqrt{2}}$ and $2\cdot 3\cdot\dfrac{1}{\sqrt{2}}$ i.e. $2\sqrt{8}$ and $3\sqrt{2}$.

78. $y' = 4ax^3 + 3bx^2 + 2cx + d$

$y'' = 12ax^2 + 6bx + 2c$, $y'' = 0$ should have real distinct roots. This is true if $36b^2 - 96ac > 0$

i.e. $3b^2 - 8ac > 0$.

79. $f(x) = |x^2 - 3x + 2|$

$= -(x^2 - 3x + 2)$, $3/2 \le x \le 7/4$

$f'(x) = -(2x - 3)$. So the critical points are

3/2, 7/4 . $f(3/2) = \frac{1}{4}$, $f(7/4) = \frac{3}{16}$. Thus

$f(x) \geq \frac{3}{16}$. So $m = \frac{3}{16}$

80. We are looking for $x \in [0, \pi]$ for which $|(1/2)x - \sin x|$ is maximum.

$$S = \begin{cases} (1/2)x - \sin x & \text{if } (1/2)x \geq \sin x \\ \sin x - (1/2)x & \text{if } \sin x < (1/2)x \end{cases}$$

$\frac{dS}{dx} = 1/2 - \cos x = 0 \Rightarrow \cos x = 1/2 \Rightarrow x = \pi/3$

Thus critical points are $x = 0, \pi/3, \pi$. $S(0) = 0$,

$S(\pi/3) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$, $S(\pi) = \frac{\pi}{2}$. Thus

$S_{\max} = \pi/2$ at $x = \pi$.

81. $2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{ax + hy}{hx + by}$

The tangent will be ∥ to Y-axis if $hx + by = 0$

82. $\frac{dy}{dx} = \tan\alpha - \frac{ax}{u^2\cos^2\alpha}$. The tangent will be parallel to $y = x$ if $\tan\alpha - \frac{ax}{u^2\cos^2\alpha} = 1$

$\Rightarrow x = \frac{(\tan\alpha - 1)u^2\cos^2\alpha}{a}$

$= \frac{u^2}{a}(\sin\alpha - \cos\alpha)\cos\alpha$

The corresponding ordinate is given by

$y = \frac{u^2}{a}(\sin\alpha - \cos\alpha)\cos\alpha\tan\alpha$

$-\frac{a}{2u^2\cos^2\alpha}\frac{u^4}{a^2}(\sin\alpha - \cos\alpha)^2\cos^2\alpha$

$= \frac{u^2}{a}\left[\sin^2\alpha - \sin\alpha\cos\alpha - \frac{1}{2}(1 - 2\sin\alpha\cos\alpha)\right]$

$= \frac{u^2}{a}(\sin^2\alpha - 1/2)$

According to given condition

$\frac{u^2}{a}(\sin^2\alpha - 1/2) = \frac{u^2}{4a}$.

$\Rightarrow \sin^2\alpha = 3/4$

83. $f(x) = \begin{cases} 1 - x + a, & x \leq 1 \\ 2x + 3, & x > 1 \end{cases}$

If $f(x)$ has local minimum at $x = 1$, then $f(1) \leq f(1 + h)$, $h > 0$ i.e. $a \leq 2(1 + h) + 3$

Hence $a \leq 5$. So $a = 5$.

84. $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0}\frac{f(x) + f(h) + 2xh - 1 - f(x)}{h}$

$= \lim_{h\to 0}\frac{f(h) - f(0)}{h} + 2x$

(Putting $x = y = 0$ we obtain $f(0) = 1$)

$= f'(0) + 2x = 2(x + 1)$

$f(x) = (x + 1)^2 + C$. Putting $x = 0$, we obtain $C = 0$. Thus $f(x) = (x + 1)^2$. But $g'(x) = f(x) = (x + 1)^2 > 0$ so g increases on $(-\infty, \infty)$ in particular on $(0, \infty)$.

85. $\frac{dy}{dx} = -\frac{x}{y}$, $\left.\frac{dy}{dx}\right|_{(1,\sqrt{3})} = -\frac{1}{\sqrt{3}}$

Equation of tangent at $(1, \sqrt{3})$ is

$Y - \sqrt{3} = \left(1/\sqrt{3}\right)(X - 1)$

This will intersect X – axis at (4, 0). Equation of the normal at $(1, \sqrt{3})$ is

$Y - \sqrt{3} = \sqrt{3}(X - 1)$

This will intersect X-axis at (0, 0) so the required area of the triangle

$= \frac{1}{2}4\sqrt{3} = 2\sqrt{3}$.

86. $y' = \frac{x^2 - 2x + 3 - 2(x-1)^2}{(x^2 - 2x + 3)^2} = \frac{-(x-1)^2 + 2}{(x^2 - 2x + 3)^2}$

$y' = y\ 0$ for $x = 1 \pm \sqrt{2}$

$y' > 0$ for $1 < x < 1 + \sqrt{2}$ and $y' < 0$ for $x > 1 + \sqrt{2}$

$y_{\max} = y\left(1 + \sqrt{2}\right)$. Similarly $y_{\min} = y\left(1 - \sqrt{2}\right)$

$y\left(1 + \sqrt{2}\right) = \frac{\sqrt{2}}{4}$ and $y\left(1 - \sqrt{2}\right) = -\frac{\sqrt{2}}{4}$.

87. Equation of tangent at any point (x_1, y_1) is

$xx_1 + yy_1 = 16$

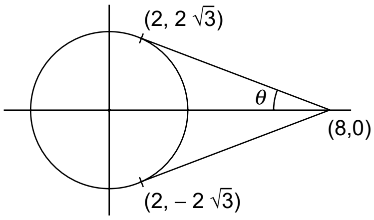


Fig. 11.8

This passes through (8, 0) if $8x_1 = 16$ i.e. $x_1 = 2$ so $y_1 = 2\sqrt{3}$. The slope of tangent is $-\sqrt{3}$.

Therefore $\theta = \pi/6$ and $2\theta = \pi/3$.

88. For $x \neq -2$,

$f'(x) = \frac{2}{3}(x+2)^{-1/3}(2x-1) + (x+2)^{2/32}\,2$

$= \frac{(10/3)(x+1)}{(x+2)^{1/3}}$

$f'(x) = 0 \Rightarrow x = -1$ clearly f is not differentiable at $x = -2$. Thus the critical points are $-1, -2$.

89. $f'(x) = \dfrac{\log(e+x)\times\dfrac{1}{\pi+x} - \log(\pi+x)\times\dfrac{1}{e+x}}{(\log(e+x))^2}$

$= \dfrac{\log(e+x)\times(e+x) - (\pi+x)\log(\pi+x)}{(\pi+x)(e+x)(\log(e+x))^2}$

Since log function is an increasing function and $e < \pi$, $\log(e+x) < \log(\pi+x)$

$\Rightarrow (e+x)\log(e+x) < (e+x)\log(\pi+x) < (\pi+x)\log(\pi+x)$ for all $x > 0$. Thus $f'(x) < 0$ for all $x > 0$.

$\Rightarrow f$ decreases on $[0, \infty)$.

90. $2(1 + b) = 32 \Rightarrow l + b = 16$.

Area of rectangle $A = lb = l(16 - l)$

$\frac{dA}{dl} = 16 - 2l$ so $\frac{dA}{dl} = 0 \Rightarrow l = 8$

$\frac{dA}{dl} = -2 < 0$. Thus A is maximum when $l = 8$.

91. $f'(x) = -\dfrac{1}{1+x^2} + \dfrac{1}{2x} = \dfrac{(1-x)^2}{(1+x^2)2x} > 0$ for $x \in (1, \sqrt{3})$

f increases on $[1, \sqrt{3}]$. The greatest value of f is equal to $f(\sqrt{3}) = \cot^{-1}\sqrt{3} + \frac{1}{2}\log\sqrt{3}$

$= \frac{\pi}{6} + \frac{1}{4}\log 3$.

92. $2y\frac{dy}{dt} = 4\frac{dx}{dt}$. According to the given condition

$\left.\frac{dy}{dt}\right|_{(7,6)} = 3$. So

$2 \times 6 \times 3 = 4\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 9$ unit/s.

93. $2(l + b) = 24 \Rightarrow l + b = 12$

$\frac{dl}{dt} + \frac{db}{dt} = 0$. If $\frac{dl}{dt}$ is 1 cm/s

then $\frac{db}{dt} = -1$. $\frac{dA}{dt} = l\frac{db}{dt} + b\frac{dl}{dt} = b - l$

$\frac{dA}{dt}$ will start decreasing if $b - l \leq 0$ but $b + l = 12$ so $l = 6$.

94. $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2\frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = -\frac{3}{\pi r^2}$.

95. $\frac{dA}{dt} = l\frac{db}{dt} + b\frac{dl}{dt}$

$\left.\frac{dA}{dt}\right|_{l=12, b=5} = 12 \times 2 + 5 \times (-2) = 14$.

96. Let $g(x) = f(x) - x^2$. As f is twice differentiable so g is also twice differentiable.

Also

$g(1) = f(1) - 1^2 = 1 - 1 = 0$

$g(2) = f(2) - 2^2 = 2^2 - 2^2 = 0$

$g(3) = f(3) - 3^2 = 3^2 - 3^2 = 0$

By Rolle's theorem $\exists\ \alpha \in (1, 2)$ and $\beta \in (2, 3)$ such that $g'(\alpha) = 0$, $g'(\beta) = 0$.

Again applying Rolle's theorem $\exists\ \gamma \in (\alpha, \beta)$ such that $g''(\gamma) = 0$

As $\alpha \in (1, 2)$, $\beta \in (2, 3)$, $(\alpha, \beta) \subset (1, 3)$

$g''(\gamma) = f''(\gamma) - 2 = 0 \Rightarrow f''(\gamma) = 2$

So there is some $x \in (1, 3)$ such that $f''(x) = 2$.

97. Equation of tangent at $P(x, y)$

$Y - y = f'(x)(X - x)$

so $A = \left(x - y\dfrac{1}{f'(x)}, 0\right)$ and $B = (0, y - x f'(x))$.

$3:1 = BP:AP$

$= \sqrt{x^2 + x^2(f'(x))^2} : \sqrt{y^2/(f'(x))^2 + y^2}$

$$\frac{9}{1} = \frac{x^2(1+(f'(x))^2)}{y^2(1+(f'(x))^2)} \quad (f'(x))^2 = \frac{x^2(f'(x))^2}{y^2}$$

$$\Rightarrow 9\frac{y^2}{x^2} = (f'(x)^2) \Rightarrow 3\frac{y}{x} = \pm\frac{dy}{dx}$$

$$\Rightarrow \frac{x^3}{y} = \text{Const or } x^3y = C$$

Since $f(1) = 1$, so Const $= 1$

Thus the curve is either $x^3 = y$ or $x^3y = 1$.
The last curve passes through (2, 1/8)

98. f is a differentiable function and

$$f'(x) = 3x^2 + 2bx + c$$

$$= 3\left[\left(x + \frac{b}{3}\right)^2 + \frac{c}{3} - \frac{b^2}{9}\right]$$

$$= 3\left[(x + b/3)^2 + \frac{3c - b^2}{9}\right]$$

Since $b^2 < c < 3c$ so $3c - b^2 > 0$ and hence

$f'(x) > 0$ i.e. f is strictly increasing on **R**.

Previous Years' AIEEE/JEE Questions

1. Let $f(x) = \dfrac{1}{3}ax^3 + \dfrac{1}{2}bx^2 + cx$ ·

Note that f is continuous and differentiable on **R**. We have

$f(0) = 0$ and $f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c = 0$.

∴ By the Rolle's theorem $\exists\ \alpha \in (0, 1)$ such that $f'(\alpha) = 0$.
i.e. $a\alpha^2 + b\alpha + c = 0$.

2. $$\frac{dx}{dt} = a\cos t - b\cdot\frac{a}{b}\cos\left(\frac{at}{b}\right) = a\left[\cos t - \cos\frac{at}{b}\right]$$

$$\frac{dy}{dt} = -a\sin t + b\cdot\frac{a}{b}\sin\left(\frac{at}{b}\right) = a\left[\sin\frac{at}{b} - \sin t\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\frac{at}{b} - \sin t}{\cos t - \cos\frac{at}{b}} = \frac{2\cos\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\sin\left(\left(\frac{a}{b}-1\right)\frac{t}{2}\right)}{-2\sin\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\sin\left(\left(1-\frac{a}{b}\right)\frac{t}{2}\right)}$$

$$= \cot\left(\frac{a}{b}+1\right)\frac{t}{2}$$

Equation tangent at any point is

$$Y - \left(a\cos t - b\cos\frac{at}{b}\right) = \cot\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\left(X - \left(a\sin t - b\sin\frac{a}{b}t\right)\right) \quad \text{(i)}$$

Distance of origin from (i) is

$$S = \left|\frac{\left(a\cos t - b\cos\frac{at}{b}\right) - \cot\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\left(a\sin t - b\sin\frac{a}{b}t\right)}{\sqrt{1+\cot^2\left(\frac{a}{b}+1\right)\frac{t}{2}}}\right|$$

$$= \left|\sin\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\left(a\cos t - b\cos\frac{at}{b}\right) - \cos\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\left(a\sin t - b\sin\frac{a}{b}t\right)\right|$$

$$= \left|a\sin\left(\left(\frac{a}{b}+1\right)\frac{t}{2} - t\right) - b\sin\left(-\frac{a}{b}t + \left(\frac{a}{b}+1\right)\frac{t}{2}\right)\right|$$

$$= \left|a\sin\left(-\frac{t}{2} + \frac{a}{b}\frac{t}{2}\right) + b\sin\left(\frac{t}{2}\frac{a}{b} - \frac{t}{2}\right)\right|$$

$$= \left|(a+b)\sin\left(\frac{a}{b}\frac{t}{2} - \frac{t}{2}\right)\right|$$

So the maximum value of S is $a + b$.

3. $f'(x) = 6x^2 - 18ax + 12a^2 = 6(x^2 - 3ax + 2a^2)$

$f''(x) = 6(2x - 3a)$. According to the given condition $f'(p) = 0\ f'(q)$ and $f''(p) < 0$, $f''(q) > 0$.

$p + q = 3a$, $pq = 2a^2$. Since $p^2 = q$, so $p^3 = 2a^2$ and $p + p^2 = 3a$ and $p < \dfrac{3a}{2}$ and $q > \dfrac{3a}{2}$. $p = \dfrac{-1 \pm \sqrt{1+12a}}{2}$

thus $\left(\dfrac{-1 \pm \sqrt{1+12a}}{2}\right)^3 = 2a^2$. Clearly $a = 2$ satisfy this equation as for $a = 1$, 1/2, 3 the L.H.S. is irrational and R.H.S. is a rational number.

4. We have

$$u = \sqrt{\frac{a^2+b^2}{2} + \frac{a^2-b^2}{2}\cos 2\theta} + \sqrt{\frac{a^2+b^2}{2} - \frac{a^2-b^2}{2}\cos 2\theta}$$

Squaring, we get

$$u^2 = a^2 + b^2 + 2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2 - \left(\frac{a^2-b^2}{2}\right)^2 \cos^2 2\theta}$$

Thus, max $(u^2) = a^2 + b^2 + 2\left(\frac{a^2+b^2}{2}\right)$

$= 2(a^2 + b^2)$

and min $(u^2) = a^2 + b^2 + 2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2 - \left(\frac{a^2-b^2}{2}\right)^2}$

$= a^2 + b^2 + 2ab = (a + b)^2$

$\therefore$ max (u^2) – min $(u^2) = (a - b)^2$.

5. $f''(x) = 6(x - 1) \Rightarrow f'(x) = 3(x - 1)^2 + C_1$

But $f'(x)]_{(2,\,1)} = \left.\frac{dy}{dx}\right]_{(2,1)} = 3$

$\therefore 3 = 3(2 - 1)^2 + C_1 \Rightarrow C_1 = 0$

Thus, $f'(x) = 3(x - 1)^2$

$\Rightarrow f(x) = (x - 1)^3 + C_2$

As $y = f(x)$ passes through (2, 1)

$1 = (2 - 1)^3 + C_2 \Rightarrow C_2 = 0$

$\therefore f(x) = (x - 1)^3$

6. $\frac{dx}{d\theta} = -a \sin \theta$, $\frac{dy}{d\theta} = a \cos \theta$

$\therefore \frac{dy}{dx} = \frac{-a\cos\theta}{a\sin\theta} \Rightarrow$ slope of normal at θ is $\frac{\sin\theta}{\cos\theta}$.

Thus, equation of normal at θ is

$$y - a \sin \theta = \frac{\sin\theta}{\cos\theta}[x - a(1 + \cos \theta)]$$

This clearly passes through $(a, 0)$

7. If $f(x) = 3x^2 - 2x + 1$, then

$f'(x) = 6x - 2 = 6\left(x - \frac{1}{3}\right) < 0$ for $x < \frac{1}{3}$

Thus, $f(x)$ decreases on the interval $(-\infty, 1/3]$.

If $f(x) = x^3 + 6x^2 + 6$,

$f'(x) = 3x^2 + 12x = 3x(x + 4) > 0$ if $x < -4$

$\Rightarrow f(x)$ increases on $(-\infty, -4]$

If $f(x) = x^3 - 3x^2 + 3x + 3$, then

$f'(x) = 3x^2 - 6x + 3 = 3(x - 1)^2 \geq 0 \ \forall \ x$

$f(x)$ increases on $(-\infty, \infty)$

If $f(x) = 2x^3 - 3x^2 - 12x + 6$, then

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

$= 6(x - 2)(x + 1) > 0$ for $x > 2$

$\Rightarrow f(x)$ increases on $[2, \infty)$

8. $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$

$\Rightarrow \frac{dx}{d\theta} = a(-\sin \theta + \sin \theta + \theta \cos \theta) = a\theta \cos \theta$

and $\frac{dy}{d\theta} = a(\cos\theta - \cos \theta + \theta \sin \theta) = a\theta \sin \theta$

$\therefore \frac{dy}{dx} = \frac{\sin\theta}{\cos\theta}$

Equation of normal at θ

$$y - a(\sin \theta - \theta \cos \theta) = -\frac{\cos\theta}{\sin\theta}\{x - a(\cos \theta + \theta \sin \theta)\}$$

$\Rightarrow x \cos \theta + y \sin \theta = a\{\cos^2 \theta + \theta \cos \theta \sin \theta + \sin^2\theta - \theta \cos \theta \sin \theta\}$

$\Rightarrow x \cos \theta + y \sin \theta = a$

This line is at a distance a from the origin.

9. By the Lagrange's mean value theorem,

$\frac{f(6) - f(1)}{6 - 1} = f'(\alpha)$ for some $\alpha \in (1, 6)$

$\Rightarrow f(6) + 2 = 5f'(\alpha) \geq 10$

$\Rightarrow f(6) \geq 8$

10. Let thickness of ice at time t be r cm. Then volume of ice:

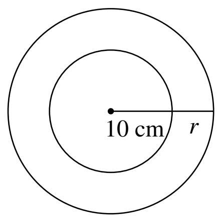


Fig. 11.9

$$V = \frac{4}{3}\pi(10 + r)^3 - \frac{4}{3}\pi(10)^3$$

$$\frac{dV}{dt} = 4\pi(10 + r)^2\frac{dr}{dt}$$

$$\Rightarrow -50 = 4\pi(10 + r)^2\frac{dr}{dt}$$

$\Rightarrow \dfrac{dr}{dt} = \dfrac{-50}{4\pi(10+r)^2}$

$\Rightarrow \left.\dfrac{dr}{dt}\right]_{r=5} = \dfrac{-50}{4\pi(15)^2}$

$= -\dfrac{1}{18\pi}$

$\therefore$ thickness is decreasing at the rate $\dfrac{1}{18\pi}$ cm/min.

11. $f'(x) = \dfrac{1}{2} - \dfrac{2}{x^2} = \dfrac{(x-2)(x+2)}{2x^2}$

For local maximum/minimum, set $f'(x) = 0$

$\Rightarrow x = \pm 2$.

As $f'(x) < 0$ for $0 < x < 2$

and $f'(x) > 0$ for $2 < x < \infty$,

$f(x)$ has a local minimum at $x = 2$.

12. $\dfrac{f(3) - f(1)}{3-1} = f'(c)$

$\Rightarrow \dfrac{\log_e 3}{2} = \dfrac{1}{c}$

$\Rightarrow c = 2 \log_3 e$

13. $f'(x) = \dfrac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}$

$f'(x) > 0$ if and only if $\cos x > \sin x$

This is true if $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$

14. As $f(x) = x^3 - px + q$ has three real and distinct, roots, $f'(x) = 3x^2 - p$ has two distinct zeros by Rolle's theorem.

Now $f'(x) = 0 \Rightarrow x = \pm\sqrt{\dfrac{p}{3}}$

We have $f'(x) > 0$ if $x < -\sqrt{\dfrac{p}{3}}$

< 0 if $-\sqrt{\dfrac{p}{3}} < x < \sqrt{\dfrac{p}{3}}$

> 0 if $x > \sqrt{\dfrac{p}{3}}$

Thus $f(x)$ has a local maximum at $x = -\sqrt{\dfrac{p}{3}}$ and local minimum at $x = \sqrt{\dfrac{p}{3}}$

15. $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$\Rightarrow P'(x) = 4x^3 + 3ax^2 + 2bx + c$

As $P'(0) = 0$, we get $c = 0$

$\therefore P'(x) = x(4x^2 + 3ax + 2b)$

As $x = 0$ is the only real root of $P'(x) = 0$, roots of $4x^2 + 3ax + 2b = 0$ must be imaginary, therefore

$4x^2 + 3ax + 2b > 0 \ \forall\ x \in \mathbf{R}$

Thus, $P'(x) < 0$ for $x < 0$

> 0 for $x > 0$.

$\therefore x = 0$ is a point of local minimum at $x = 0$.

Graph of $y = P(x)$ is given in Fig. 11.10.

As $P(-1) < P(1)$, we get $P(1)$ is maximum but $P(-1)$ is not minimum of P on $[-1, 1]$.

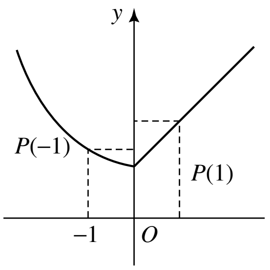


Fig. 11.10

16. f will be continuous at $x = -1$ if

$\Rightarrow \lim\limits_{x \to -1-} f(x) = \lim\limits_{x \to -1+} f(x) = f(-1)$

$\Rightarrow k + 2 = 2(-1) + 3 = k + 2$

$\Rightarrow k = -1$

For this value of k, f is continuous at $x = -1$, $f'(-1)$ does not exist and $f'(x) < 0$ for $x < -1$

and $f'(x) > 0$ for $x > -1$,

therefore, f has a local minimum at $x = -1$.

17. Equation of tangent at (x, y) to the curve $y = f(x)$ is given by

$Y - y = \dfrac{dy}{dx}(X - x)$

This tangent meets the axes at $A\left(x - y\dfrac{dx}{dy}, 0\right)$ and $B\left(0, y - x\dfrac{dy}{dx}\right)$.

We are given mid-point of $AB = (x, y)$

$\Rightarrow \frac{1}{2}\left(x - y\frac{dx}{dy}\right) = x$ and $\frac{1}{2}\left(y - x\frac{dy}{dx}\right) = y$

$\Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$

$\Rightarrow \log|xy| = c \Rightarrow xy = \pm e^c = A.$

As it passes through (2, 3), we get $6 = A$

$\therefore xy = 6$ or $y = 6/x$

18. Let $g(x) = \tan x - x, -\pi/2 < x < \pi/2$

$\Rightarrow g'(x) = \sec^2 x - 1 = \tan^2 x > 0$ for $-\pi/2 < x < \pi/2$.

Thus, $g(x)$ is a strictly increasing function on $(-\pi/2, \pi/2)$.

Therefore, $\tan x < x$ for $-\pi/2 < x < \pi/2$

$\Rightarrow f(x) = \frac{\tan x}{x} < 1$ for $-\pi/2 < x < \pi/2, x \neq 0$.

$\therefore x = 0$ is a point of maxima.

Also,

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x}$$

$$= \lim_{x\to 0}\frac{\frac{\tan x}{x}-1}{x}$$

$$= \lim_{x\to 0}\frac{\tan x - x}{x^2} \quad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x\to 0}\frac{\sec^2 x - 1}{2x}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{(\tan x)^2}{x} = 0$$

Thus, both the statements are true. However, statement-2 is not a correct explanation of statement-1.

19. $f'(x) = \frac{1}{x} + 2bx + a, x \neq 0.$

As $x = -1$ and $x = 2$ are extreme values of f so

$f'(-1) = 0 = f'(2)$

$\Rightarrow -1 - 2b = a = 0$ and $\frac{1}{2} + 4b + a = 0$

Solving we get, $a = \frac{1}{2}$ and $b = -\frac{1}{4}$

So $f'(x) = \frac{1}{x} - \frac{1}{2}x + \frac{1}{2} = -\frac{1}{2x}(x+1)(x-2)$

We have

$$\begin{aligned} f'(x) &> 0 \text{ if } x < -1 \\ &< 0 \text{ if } -1 < x < 0 \\ &> 0 \text{ if } 0 < x < 2 \\ &< 0 \text{ if } x > 2 \end{aligned}$$

Thus $f(x)$ has a local maximum at $x = -1$ and $x = 2$

$\therefore$ statement-1 and statement-2 are both true and statement-2 is a correct explanation for the statement-1.

20. Let r be the radius of balloon and V be its volume.

It is given that $\frac{dV}{dt} = -72\pi$

$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -72\pi$

$\Rightarrow \frac{4}{3}\pi r^3 = -72\pi t + k$

When $t = 0$, $k = \frac{4}{3}\pi r^3 = V = 4500\pi$

$\therefore \frac{4}{3}\pi r^3 = 4500\pi - 72\pi t$

When $t = 49$, $\frac{4}{3}\pi r^3 = 4500\pi - (72\pi)(49) = 972\pi$

$\Rightarrow r^3 = 729 \Rightarrow r = 9$

Also, $\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -72\pi \Rightarrow 4\pi r^2\frac{dr}{dt} = -72\pi$

$\frac{dr}{dt} = \frac{-72\pi}{4\pi r^2} = \frac{-18}{r^2} \Rightarrow \left.\frac{dr}{dt}\right|_{r=9} = -\frac{2}{9}.$

21. $C = av + \frac{b}{v}$. According to the given conditions,

$75 = a.30 + \frac{b}{30}$ and $65 = a.40 + \frac{b}{40}$

Solving $a = \frac{1}{2}$, $b = 1800$. Thus

$C = \frac{1}{2}v + \frac{1800}{v}$

$\frac{dC}{dv} = \frac{1}{2} - \frac{1800}{v^2}$ so $\frac{dc}{dv} = 0 \Rightarrow v = 60$

$\frac{d^2C}{dv^2} = \frac{3600}{v^3} > 0$. Thus C is minimum when $v = 60$.

22. $S = 4\pi r^2$, $8 = \frac{dS}{dt} = 8\pi r\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi r}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4r$ so $\frac{dV}{dt} \alpha r$.

23. Let $f(x) = 2x^3 + 3x + k$

Suppose $\alpha, \beta \in [0, 1]$, $\alpha < \beta$ be such that $f(\alpha) = 0 = f(\beta)$.

As f is differentiable on $[\alpha, \beta]$,

By the Rolle's theorem there exists $r \in (a, b)$ such that $f'(r) = 0 \Rightarrow 6r^2 + 3 = 0$.

But $6r^2 + 3 = 0$ is not true for any real r.

Thus, there is no real value of k, for which $2x^3 + 3x + k = 0$ has two real and distinct roots in [0, 1].

24. $A = \frac{1}{2}pb$, p is the perpendicular and b is the base

$p^2 + b^2 = h^2$ so $A = \frac{1}{2}p\sqrt{h^2 - p^2}$

$\frac{dA}{dp} = \frac{1}{2}\sqrt{h^2 - p^2} + \frac{1}{4}p\frac{(-2p)}{\sqrt{h^2 - p^2}} = \frac{1}{2}\frac{h^2 - p^2 - p^2}{\sqrt{h^2 - p^2}}$

$\frac{dA}{dp} = 0 \Rightarrow p = \frac{h}{\sqrt{2}}$. It is easy to see that A is maximum for $p = \frac{h}{\sqrt{2}}$, so $A_{\max} = \frac{1}{2}\frac{h}{\sqrt{2}}\sqrt{h^2 - \frac{h^2}{2}}$

$= \frac{h^2}{4}$.

25. If $f(x) = x \log x$ then $f'(x) = 1 + \log x > 0$ for $x \in [1, 2]$ so f increases on [1, 2]. For $g(x) = 2 - x$, $g'(x) = -1$ so g decreases on [1, 2]. Therefore $g(x) = f(x)$ for some $x \in [1, 2]$. Hence the statement-1 follows.

26. Let $f(x) = x^2e^x$, $f'(x) = 2xe^x + x^2e^x > 0$ for $x > 0$.

$g(x) = x^2e^{-x}$, $g'(x) = 2xe^{-x} - x^2e^{-x} = (2x - x^2)\, e^{-x}$

$g'(x) > 0$ for $x < 2$ and $g'(x) < 0$ for $x > 2$. Hence g decreases for $x > 2$ and increases on (0, 2). Statement-2 is not true. If $h(x) = x^2(e^x + e^{-x})$ then $h'(x) = 2x(e^x + e^{-x}) + x^2\,(e^x - e^{-x}) > 0$ for $x > 0$. Hence h increases for $x > 0$.

27. Let $F(x) = f(x) - 2g(x)$

$F(0) = f(0) - 2g(0) = 2 - 0 = 2$.

$F(1) = f(1) - 2g(1) = 6 - 4 = 2$

F is continuous on [0, 1] and differentiable on (0, 1) with $F(0) = F(1)$, so by Rolle's theorem there is $c \in (0, 1)$ s. t. $f(c) = 0 \Rightarrow f'(c) = 2g'(c)$.

28. $f'(x) = \frac{\alpha}{x} + 2\beta x + 1$

As $x = -1$ and $x = 2$ are extreme points so $f'(-1) = f'(2) = 0$ i.e.

$-\alpha - 2\beta + 1 = 0$

$\frac{\alpha}{2} + 4\beta + 1 = 0$

Solving $\alpha = 2$, $\beta = -\frac{1}{2}$.

29. $V = \frac{4}{3}\pi r^3$. When $V = 288\pi$ C.C then $288\pi = \frac{4}{3}\pi r^3$

$\Rightarrow r = 6$. Also $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 4\pi = 4\pi 36\frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{36}$.

30. $f(x) = x^2 + 2bx + 2c^2 = (x + b)^2 + 2c^2 - b^2$ so $\min f(x) = 2c^2 - b^2$. Also $g(x) = -x^2 - 2cx + b^2 = b^2 + c^2 - (x + c)^2 \Rightarrow \max g(x) = b^2 + c^2$

Thus $2c^2 - b^2 > b^2 + c^2$

$\Rightarrow \left|\frac{c}{b}\right| > \sqrt{2} \Rightarrow \left|\frac{c}{b}\right| \in \left(\sqrt{2}, \infty\right)$.

31. Note that f is a strictly increasing function and g is a strictly decreasing function.

Now, $x < x + 1 \Rightarrow g(x) > g(x + 1)$

$\Rightarrow f(g(x)) > f(g(x + 1))$

32. $f(-1) = f(1)$

$\Rightarrow -2 + a - b = 2 + a + b$

$\Rightarrow b = -2$

Also, $f'(c) = 0 \Rightarrow 6c^2 + 2ac + b = 0$

$\Rightarrow \frac{3}{2} + a - 2 = 0 \Rightarrow a = \frac{1}{2}$

Thus, $2a + b = 1 - 2 = -1$

33. $\lim_{x \to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$

$\Rightarrow \lim_{x \to 0}\frac{f(x)}{x^2} = 2$

$\therefore f(x)$ must be of the form

$f(x) = ax^4 + bx^3 + 2x^2$

$\Rightarrow f'(x) = 4ax^3 + 3bx^2 + 4x$

As $f(x)$ has extreme values at $x = 1$ and $x = 2$,

$4a + 3b + 4 = 0$

$32a + 12b + 8 = 0$

Solving these we get $a = \frac{1}{2}$, $b = -2$

Thus, $f(2) = \frac{1}{2}(2^4) - 2(2^3) + 2(2^2) = 0.$

34. When $x = 0$, $\sin y = 0 \Rightarrow y = 0$ (since all lines given pass through origin)

$$\cos y \frac{dy}{dx} = \sin\left(\frac{\pi}{3}+y\right)+x\cos\left(\frac{\pi}{3}+y\right)\frac{dy}{dx}$$

When $x = 0$, $y = 0$, therefore

$$\left.\frac{dy}{dx}\right]_{(0,0)} = \sin\left(\frac{\pi}{3}\right)+0 = \frac{\sqrt{3}}{2}$$

Thus, slope of normal at (0, 0) is $-2/\sqrt{3}$ and its equation is

$$y = \left(-2/\sqrt{3}\right)x \text{ or } 2x+\sqrt{3}y = 0$$

35. Let $m = 0.6$,

$$f'(x) = \frac{(1+x^m)(m)(1+x)^{m-1} - mx^{m-1}(1+x)^m}{(1+x^m)^2}$$

$$= \frac{m(1+x)^{m-1}(1+x^m - x^{m-1} - x^m)}{(1+x^m)^2}$$

$$= \frac{m(1+x)^{m-1}(1-x^{m-1})}{(1+x^m)^2}$$

For $0 < x < 1$, $f'(x) < 0$

as $x^{m-1} = x^{-0.4} = 1/x^{0.4} > 1$ for $0 < x < 1$.

Thus, $f'(x) < 0$ for $0 < x < 1$

$\Rightarrow f(x)$ is strictly decreasing on [0, 1]

$\therefore\ k = \max f(x) = f(0) = 1$

and $k = \min f(x) = f(1) = 2^{-0.4}$

Hence, $(k, K) = (2^{-0.4}, 1)$

36. If s is the height at any instant, then

$-2g(s - 64) = v^2 - u^2$

where v is the initial speed and v is the speed at that instant.

At the highest point,

$-2(32)(s - 64) = 0 - 48^2$

$$\Rightarrow s = 64 + \frac{48^2}{64} = 100.$$

37. Let $x_1 = x(\pi/4)$ and $y_1 = y(\pi/4)$.

$$\frac{dx}{dt} = -2 \sin t + 2 \sin t + 2t \cos t = 2t \cos t$$

$$\frac{dy}{dt} = 2 \cos t - 2 \cos t + 2t \sin t = 2t \sin t$$

$$\therefore\ \frac{dy}{dx} = \tan t \Rightarrow \left.\frac{dy}{dx}\right]_{t=\pi/4} = 1$$

$\Rightarrow$ slope of normal at $t = \pi/4$ is -1.

Thus, equation of normal at (x_1, y_1) is

$y - y_1 = -(x - x_1)$ or $x + y - (x_1 + y_1) = 0$

Its distance from the origin is

$$d = \frac{|x_1 + y_1|}{\sqrt{2}}$$

But $x_1 + y_1 = 2\left(\frac{1}{\sqrt{2}}\right)\left(1+\frac{\pi}{4}\right)+2\left(\frac{1}{\sqrt{2}}\right)\left(1-\frac{\pi}{4}\right)$

$= 2\sqrt{2}$

$\therefore\ d = 2$

38. Let coordinates of A be $(k, 0)$ then equation of tangent PA is

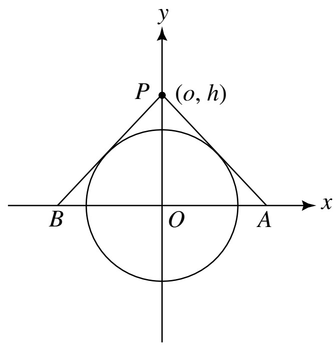


Fig. 11.11

$$\frac{x}{k}+\frac{y}{h} = 1, \text{ where } |h|, |k| > 4.$$

$$\frac{1}{\sqrt{\frac{1}{k^2}+\frac{1}{h^2}}} = 4 \qquad (1)$$

$$\Rightarrow \frac{|h||k|}{\sqrt{h^2+k^2}} = 4$$

Let Δ = Area of ΔPAB = 2(area of ΔOAP)

$$= 2\left(\frac{1}{2}\right)(|h|\ |k|) = 4\sqrt{h^2+k^2}$$

But from (1), $k^2 = \frac{16h^2}{h^2-16}$

$\therefore\ \Delta^2 = 16(h^2 + k^2) = 16\left(h^2 + \dfrac{16h^2}{h^2 - 16}\right)$

$= \dfrac{16h^4}{h^2 - 16}$

But Δ is minimum if and only if Δ^2 is, we therefore minimize, Δ^2.

We have

$\dfrac{d}{dh}(\Delta^2) = 16\left[\dfrac{4h^3(h^2-16) - h^4(2h)}{(h^2-16)^2}\right]$

$= \dfrac{32h^3(h - 4\sqrt{2})(h + 4\sqrt{2})}{(h^2-16)^2}$

As $\dfrac{d}{dh}(\Delta^2) = 0$ for $h = 4\sqrt{2}$ and $\dfrac{d}{dh}(\Delta^2) < 0$ if $4 < h < 4\sqrt{2}$ and is > 0 if $h > 4\sqrt{2}$

Thus Δ^2 is minimum when $h = 4\sqrt{2}$.

39. For $x \in (0, \pi/2)$

$1 + \sin x = 1 - \cos(\pi/2 + x)$

$= 2\sin^2(\pi/4 + x/2)$

and $1 - \sin x = 1 + \cos(\pi/2 + x)$

$= 2\cos^2(\pi/4 + x/2)$

$\therefore f(x) = \tan^{-1}\left[\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)\right] = \dfrac{\pi}{4} + \dfrac{x}{2}$

$\Rightarrow f'(x) = 1/2$

An equation of normal to $y = f(x)$ at $x = \pi/6$ is

$y - \left(\dfrac{\pi}{4} + \dfrac{\pi}{12}\right) = -2\left(x - \dfrac{\pi}{6}\right)$

$\Rightarrow y - \dfrac{\pi}{3} = -2x + \dfrac{\pi}{3}$

$\Rightarrow 2x + y = 2\pi/3$

It passes through $(0, 2\pi/3)$

40. Let $E = 4 + \dfrac{1}{2}\sin^2 2x - 2\cos^4 x$

$= 4 + \dfrac{1}{2}\sin^2 2x - \dfrac{1}{2}(2\cos^2 x)^2$

$= 4 + \dfrac{1}{2}(1 - \cos^2 2x) - \dfrac{1}{2}(1 + \cos x)^2$

$= 4 + \dfrac{1}{2}(1 + \cos 2x)\,[1 - \cos 2x - 1 - \cos 2x]$

$= 4 - \dfrac{1}{2}(1 + \cos 2x)\,(2\cos 2x)$

$= 4 - \cos 2x - \cos^2 2x$

$= \dfrac{9}{4} - \left(\dfrac{1}{2} + \cos 2x\right)^2$

Note that

$m = 0$ when $\cos 2x = 1$

and $M = \dfrac{9}{4}$ when $\cos 2x = -1/2$.

So $M - m = \dfrac{9}{4}$.

41. An equation of tangent at $P(4t^2 + 3, 8t^3 - 1)$ is

$y - (8t^3 - 1) = \dfrac{24t^2}{8t}(x - (4t^3 + 3))$ as $\dfrac{dy}{dx} = \dfrac{24t^2}{8t}$

It will meet the curve again at

$Q(4t_1^2 + 3, 8t_1^3 - 1)$ if

$8t_1^3 - 8t^3 = 3t[4t_1^2 - 4t^2]$

$\Rightarrow 2(t_1 - t)(t_1^2 + t^2 + tt_1) = 3(t_1 - t)(tt_1 + tt^2)$

$\Rightarrow (t_1 - t)(2t_1^2 - tt_1 - t^2) = 0$

$\Rightarrow (t_1 - t)(t_1 - t)(2t_1 + t) = 0 \Rightarrow t_1 = t, -\dfrac{1}{2}t.$

Thus, coordinates of Q are $(t^2 + 3, -t^3 - 1)$

42. $f(x) = \sin^4 x + \cos^4 x$, since f is differentiable and

$f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$

$= -2\sin x\cos x(\cos^2 x - \sin^2 x)$

$= -\sin(4x) > 0$

$\forall x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

$\Rightarrow f(x)$ increases on $\left]\dfrac{\pi}{4}, \dfrac{\pi}{2}\right[$

43. $\dfrac{dy}{dx} = \dfrac{4}{2\sqrt{4x-3}} = \dfrac{2}{\sqrt{4x-3}} = \dfrac{2}{3}$

$\Rightarrow \sqrt{4x-3} = 3 \Rightarrow x = 3$

For $x = 3$, $y = 1 + \sqrt{(4)(3) - 3} = 4$

The equation of normal at (3, 4) is

$y - 4 = -\dfrac{3}{2}(x - 3)$

It passes through (1, 7).

Previous Years' B-Architecture Entrance Examination Questions

1. $\frac{dy}{dx} = 3x^2 - 8x$. $\left.\frac{dy}{dx}\right|_{(2,-1)} = 12 - 16 = -4$. So the slope of the normal is $\frac{1}{4}$.

2. $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 2t - 1 \Rightarrow \frac{dy}{dx} = \frac{2t-1}{2t}$ so tangent line is perpendicular to x-axis if $t = 0$.

3. $f(0) = f(\pi) = 1$. $f'(x) = \cos x\, 4^{\sin x}$ so $f'(c) = 0 \Rightarrow \cos c = 0 \Rightarrow c = \frac{\pi}{2}$.

4. $f(5) = 0$, $f(1) = \sqrt{24}$. $\frac{f(5)-f(1)}{5-1} = \frac{1}{4}\sqrt{24} = \sqrt{\frac{3}{2}}$

   $f'(x) = \frac{-x}{\sqrt{25-x^2}}$ so $f'(c) = \sqrt{\frac{3}{2}} \Rightarrow \frac{c^2}{25-c^2} = \frac{3}{2}$

   $\Rightarrow 2c^2 = 75 - 3c^2 \Rightarrow c^2 = 15$. For $c \in (1, 5)$, we have $c = \sqrt{15}$.

5. $f(x) = \begin{cases} 1-x+a & \text{if } x \le 1 \\ 2x+3 & \text{if } x > 1 \end{cases}$, $f(1) = a$

   So we must have $\lim_{x \to 1+} f(x) \ge a$ i.e. $5 \ge a$ or $a \le 5$

6. $2y = \log(1 + 4x^2) \Rightarrow 2\frac{dy}{dx} = \frac{8x}{1+4x^2}$

   $\Rightarrow \frac{dy}{dx} = \frac{4x}{1+4x^2}$

   Since $4x \le 1 + 4x^2$ so $|m| \le 1$

7. We have $\frac{f(2)-f(0)}{2-0} = f'(c)$ for some $c \in (0, 2)$

   $f(2) = 2f'(c) + f(0) \le 6 - 2 = 4$.

8. $f'(x) = \frac{(x^2+1)(4x^3+6x)-(x^4+3x^2+1)2x}{(x^2+1)^2}$

   $= \frac{2x(x^2(x^2+2)+2)}{(x^2+1)^2}$

   $f'(x) = 0$ for $x = 0$, $f'(x) < 0$, for $x < 0$, $f'(x) > 0$ for $x > 0$

   $f(-1) = \frac{5}{2}$, $f(2) = \frac{29}{5}$, $f(0) = 1$ so the maximum value of f is $\frac{29}{5}$.

9. Equation of normal at any point is $Y - y = -\frac{1}{f'(x)}(X - x)$.

   This passes through (1, 1), so $1 - y = -\frac{1}{f'(x)}(1 - x)$

   $\Rightarrow 1 - y = -\frac{dx}{dy}(1 - x)$

   $\Rightarrow \frac{-(1-y)^2}{2} = \frac{(1-x)^2}{2} + c$

   Since it passes through (3, 1) so

   $0 = 2 + C \Rightarrow C = -2$

   $\Rightarrow (x - 1)^2 + (y - 1)^2 = 4$ which is circle of radius 2 so the area is 4π.

10. $f'(x) = \sin\frac{\pi}{x} - \frac{\pi}{x}\cos\frac{\pi}{x}$, $0 < x \le 1$

    $f'(0)$ does not exist. $f'(x) = 0 \Rightarrow \tan\frac{\pi}{x} = \frac{\pi}{x}$

    Since $0 < x \le 1$ so $\pi \le \frac{\pi}{x} < \infty$. By drawing graph of $\tan x$ and x, we can see that there are infinitely many solutions.

11. Since $1 - x^2$ is a decreasing function and $[x]$ is an increasing function so $f(x) = [1 - x^2]$ is a decreasing function.

12. $\frac{dy}{dt} = \frac{1}{2\sqrt{x}}\frac{dx}{dt}$. If $\frac{dy}{dt} = \frac{dx}{dt}$ then $2\sqrt{x} = 1$

    or $x = \frac{1}{4}$ so $y = \frac{1}{2}$.

13. $2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$.

    Equation of tangent at (x, y)

    $Y - y = \frac{x}{y}(X - x)$

    So $a_1 = -\frac{y^2}{x} + x = \frac{4}{x}$, $b_1 = y - \frac{x^2}{y} = -\frac{4}{y}$

    Equation of normal at (x, y)

    $Y - y = \frac{y}{x}(X - x)$

    So $a_2 = 2x$, $b_2 = 2y$

    $a_1a_2 + b_1b_2 = \frac{4}{x}\times 2x - \frac{4}{y}\times 2y = 0$.

14. $\dfrac{f(2)-f(0)}{2-0} = f'(c)$ for some $c \in (0, 2)$

$f(2) = f(0) + 2f'(c) \le -3 + 2.5 = 7$.

15. $f\left(\dfrac{\pi}{2}\right) = f\left(-\dfrac{\pi}{2}\right) = -3$. Also $f'(x) =$

$6 \cos x \sin x\ (2 \cos x + 1)(\cos x - 2)$

For $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, $2 \cos x + 1 \ne 0$, $\cos x - 2 \ne 0$ so

$f'(x) = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0$.

$f(0) = 3 - 6 - 6 - 3 = -12$. Hence range of $f(x)$ is $[-12, -3]$.

16. $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$, $f'(x) = 0 \Rightarrow x = 1$

$f''(x) = -e^{-x}(1 - x) + e^{-x}(-1)$, $f''(1) = -e^{-1} < 0$

f has maximum at $x = 1$.

17. Area $A(x) = x^2 \Rightarrow \dfrac{dA}{dt} = 2x\dfrac{dx}{dt}$

$\Rightarrow 8 = 2x\dfrac{dx}{dt} = 2x$

$\Rightarrow x = 4 \Rightarrow A = 16$

18. $V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dr} = 4\pi r^2$ and $S = 4\pi r^2 \Rightarrow \dfrac{dS}{dr} = 8\pi r$

$\dfrac{dV}{dS} = \dfrac{4\pi r^2}{8\pi r} = \dfrac{1}{2}r$, $\left.\dfrac{dV}{dS}\right|_{r=2} = 1$.

19. $e^y = 1 + x^2 \Rightarrow y = \log(1 + x^2)$, so $m = \dfrac{dy}{dx} = \dfrac{2x}{1+x^2}$

$|m| \le 1 \Rightarrow m \in [-1, 1]$

20. $f(x) = |x \log x| = x\,|\log x| = \begin{cases} -x\log x, & 0 < x \le 1 \\ x\log x, & x > 1 \end{cases}$

$f'(x) = \begin{cases} -(1+\log x), & 0 < x < 1 \\ 1+\log x, & x > 1 \end{cases}$

$f'(x) < 0$ if $1 + \log x > 0$ for $0 < x < 1 \Rightarrow \log x > -1 \Rightarrow x > e^{-1}$ for $0 < x < 1$

So $x \in \left[\dfrac{1}{e}, 1\right]$.

21. By drawing the graph, we get two points of minimum.

22. $f(x) = 2 \sin x + \sin 2x = 2 \sin x + 2 \sin x \cos x$

$= 2 \sin x\ (1 + \cos x)$

$f'(x) = 2 \cos x\ (1 + \cos x) - 2 \sin x \sin x$

$= 2\ [\cos x + \cos^2 x - (1 - \cos^2 x)]$

$= 2\ [2\cos^2 x + \cos x - 1]$

$= 2\ (\cos x + 1)(2 \cos x - 1)$

Now, $f'(x) = 0 \Rightarrow x = \pi/3$ ($\because\ 0 < x < \pi$)

We have $f(0) = 0$, $f\left(\dfrac{\pi}{3}\right) = 2\left(\dfrac{\sqrt{3}}{2}\right) + \dfrac{\sqrt{3}}{2} = \dfrac{3}{2}\sqrt{3}$,

and $f(\pi) = 0$.

Thus, maximum value is $\dfrac{3}{2}\sqrt{3}$.

23. $m = \dfrac{dy}{dx} = e^x(\sin x + \cos x)$

$\dfrac{dm}{dx} = 2e^x \cos x = 0 \Rightarrow x = \dfrac{\pi}{2}$

$\dfrac{d^2m}{dx^2} = 2e^x\ (\cos x - \sin x)$

$\left.\dfrac{d^2m}{dx^2}\right]_{x=\pi/2} = 2e^{\pi/2}(-1) < 0$

Thus m is maximum when $x = \pi/2$

24. As $\lim\limits_{x\to 0} \dfrac{p(x)}{x^2} = 1$, $p(x)$ must be of the from

$p(x) = x^2 + ax^3 + bx^4$ $a, b \in \mathbf{R}$

$p'(x) = 2x + 3ax^2 + 4bx^3$

As $p'(1) = 0$, $p'(2) = 0$, we get

$2 + 3a + 4b = 0$ and $4 + 12a + 32b = 0$

$\Rightarrow 3a + 4b = -2$, $3a + 8b = -1$

$\Rightarrow a = -1$, $b = \dfrac{1}{4}$

$\therefore\ p(4) = 4^2(1 + 4a + 16b) = 16(1 - 4 + 4) = 16$

25. $f'(x) = 3x^2 + 2bx + c$. Discriminant of $f'(x) = 4(b^2 - 3ac) \le 0$

So $f'(x) > 0$ and hence f increases. Since f is continuous and increasing so f is 1 – 1.

26. $\min(x, x^2) = \begin{cases} x^2 & 0 < x \le 1 \\ x & x > 1 \end{cases}$, $\lim\limits_{h\to 0+} \dfrac{f(1+h)-f(1)}{h} =$

$\lim\limits_{h\to 0+} \dfrac{1+h-1}{h} = 1$

$\lim\limits_{h\to 0-} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0-} \dfrac{(1+h)^2-1}{h} = 2$. So f is not differentiable at $x = 1$.

For $y = x$, $\frac{dy}{dx} = 1$ so $\tan\theta_1 = 1$. For $y = x^2$, $\frac{dy}{dx} = 2x$ so $\left.\frac{dy}{dx}\right]_{x=1} = 2$. $\tan\theta = \frac{2-1}{1+2\times 1} = \frac{1}{3}$

27. f in statement-1 is continuous function on [1, 3] and differentiable on (1, 3) $f(3) = 1 = f(1)$ so Rolle's theorem is applicable. Again the mean value is applicable so statement-2 is true but is not a correct explanation for statement-1.

28. f is continuous on [2, 6] and differentiable on (2, 6) being a polynomial also $f(2) = 0 = f(6)$ so by Rolle's theorem there is $c \in (2, 6)$ such that $f'(c) = 0$.

29. $f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right) > 0$$

$f'(x) = 2ax + b$. So $f'(x) > 0$ if $x > -\frac{b}{2a}$ and

$f'(x) < 0$ if $x < -\frac{b}{2a}$.