Chapter 4

Principle of Mathematical Induction

Exercise 4.1

Question 1: Prove the following by using the principle of mathematical induction for all $n \in N$:

$1 + 3 + 3^2 + \ldots + 3^{n-1} = \frac{(3^n - 1)}{2}$

Answer 1:

Let the given statement be P(n), i.e.,

P(n): $1 + 3 + 3^2 + \ldots + 3^{n-1} = \frac{(3^n - 1)}{2}$

For n = 1, we have

P (1): $= \frac{(3^1 - 1)}{2} = \frac{3-1}{2} = \frac{2}{2} = 1$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$1 + 3 + 3^2 + \ldots + 3^{k-1} = \frac{(3^k - 1)}{2}$ … (1)

We shall now prove that P (k + 1) is true.

Consider

$1 + 3 + 3^2 + \ldots + 3k–1 + 3(k+1) – 1$

$= (1 + 3 + 3^2 + \ldots + 3k–1) + 3k$

$= \frac{(3^k - 1)}{2} + 3k$

$= \frac{(3^k - 1) + 2.3^k}{2}$ [using (1)]

$= \frac{(1+2)3^k - 1}{2}$

$= \frac{3.3^k - 1}{2}$

$= \frac{3^{k-1} - 1}{2}$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 2: Prove the following by using the principle of mathematical induction for all n ∈ N:

$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Answer 2:

Let the given statement be P(n), i.e.,

P(n): $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

For n = 1, we have

P (1): $1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{1.2}{2}\right)^2 = 1^2 = 1$which is true.

Let P(k) be true for some positive integer k, i.e.,

$1^3 + 2^3 + 3^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 \ldots (1)$

We shall now prove that P (k + 1) is true.

Consider

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3$

$= (1^3 + 2^3 + 3^3 + \ldots + k^3) + (k + 1)^3$

$= \left\{\frac{k(k+1)}{2}\right\}^2 + (k+1)^3$ [using (1)]

$= \frac{k^2 (k+1)^2}{4} + (k+1)^3$

$= \frac{k^3 (k+1)^3 + 4(k+1)^3}{4}$

$= \frac{(k+1)^2 \{k^2 + 4(k+1)\}}{4}$

$= \frac{(k+1)^2 + (k^2 + 4k + 4)}{4}$

$= \frac{(k+1)^2 + (k+1)^2}{4}$

$= \frac{(k+1)^2 (k+1+1)^2}{4}$

$= \left\{\frac{(k+1)(k+1+1)}{2}\right\}^2$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 3: Prove the following by using the principle of mathematical induction for all n ∈ N:

$1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \ldots + \frac{1}{(1+2+3+\cdots n)} = \frac{2n}{(n+1)}$

Answer 3:

Let the given statement be P(n), i.e.,

P(n): $1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \ldots + \frac{1}{(1+2+3+..+n)} = \frac{2n}{(n+1)}$

For n = 1, we have

P (1): $1 = \frac{2.1}{1+1} = \frac{2}{2} = 1$ which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+..k)}=\frac{2k}{k+1} \ldots. (1)$$

We shall now prove that P (k + 1) is true.

Consider

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+..+k}+\frac{1}{1+2+3+\cdots+k+(k+1)}$$

$$=\left\{1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+k}\right\}+\frac{1}{1+2+3+\cdots+k+(k+1)}$$

$$=\frac{2k}{k+1}+\frac{1}{1+2+3+..+k+(k+1)} \text{ [using (1)]}$$

$$=\frac{2k}{k+1}+\frac{1}{\left(\frac{(k+1)(k+1+1)}{2}\right)}\left[1+2+3+..+n=\frac{n(n+1)}{2}\right]$$

$$=\frac{2k}{k+1}+\frac{2}{(k+1)(k+2)}$$

$$=\frac{2}{k+1}\left\{k+\frac{1}{k+2}\right\}$$

$$=\frac{2}{k+1}\left\{\frac{k^2+2k+1}{k+2}\right\}$$

$$=\frac{2}{k+1}\left[\frac{(k+1)^2}{k+2}\right]$$

$$=\frac{2(k+1)}{k+2}$$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 4: Prove the following by using the principle of mathematical induction for all n ∈ N: 1.2.3 + 2.3.4 + ... + n (n + 1) (n + 2) = $\frac{n(n+1)(n+2)(n+3)}{4}$

Answer 4:

Let the given statement be P(n), i.e.,

P(n): 1.2.3 + 2.3.4 + ... + n (n + 1) (n + 2) = $\frac{n(n+1)(n+2)(n+3)}{4}$

For n = 1, we have

P (1): 1.2.3 = 6 = $\frac{1(1+1)(1+2)(1+3)}{4} = \frac{1.2.3.4}{4} = 6$ which is true.

Let P(k) be true for some positive integer k, i.e.,

1.2.3 + 2.3.4 + ... + k (k + 1) (k + 2) = $\frac{k(k+1)(k+2)(k+3)}{4}$

We shall now prove that P (k + 1) is true.

Consider

1.2.3 + 2.3.4 + ... + k (k + 1) (k + 2) + (k + 1) (k + 2) (k + 3)

= {1.2.3 + 2.3.4 + ... + k (k + 1) (k + 2)} + (k + 1) (k + 2) (k + 3)

= $\frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$ [using (1)]

= $(k+1)(k+20(k+3)\left(\frac{k}{4}+1\right)$

= $\frac{(k+1)(k+2)(k+3)(k+4)}{4}$

= $\frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 5: Prove the following by using the principle of mathematical induction for all n ∈ N:

$1.3 + 2.3^2 + 3.3^2 + \ldots + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}$

Answer 5:

Let the given statement be P(n), i.e.,

P(n): $1.3 + 2.3^2 + 3.3^2 + \ldots + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}$

For n = 1, we have

P (1): $1.3 = 3\ \frac{(2n-1)3^{1+1}+3}{4} = \frac{3^2+3}{4} = \frac{12}{4} = 3$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$1.3 + 2.3^2 + 3.3^2 + k3^k = \frac{(2k-1)3^{k+1}+3}{4}$ … (1)

We shall now prove that P (k + 1) is true.

Consider

$1.3 + 2.3^2 + 3.3^3 + \ldots + k.3^k + (k + 1).3k+1$

$= (1.3 + 2.3^2 + 3.3^3 + \ldots + k.3^k) + (k + 1).3k+1$

Thus, P (k + 1) is true whenever P(k) is true

$= \frac{(2k-1)3^{k+1}+3}{4} + (k+1)3^{k+1}$ [using (1)]

$= \frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$

$= \frac{3^{k+1}\{2k-1+4(k+1)\}+3}{4}$

$= \frac{3^{k+1}(6k+3)+3}{4}$

$= \frac{3^{k+1}.3(2k+1)+3}{4}$

$= \frac{3^{(k+1)+1}(2k+1)+3}{4}$

$= \frac{(2(k+1))3^{(k+1)+1}+3}{4}$

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 6: Prove the following by using the principle of mathematical induction for all n ∈ N:

1.2 + 2.3 + 3.4 + … + n (n + 1) = $\left[\frac{n(n+1)(n+2)}{3}\right]$

Answer 6:

Let the given statement be P(n), i.e.,

P(n): 1.2 + 2.3 + 3.4 + … + n (n+1) = $\left[\frac{n(n+1)(n+2)}{3}\right]$

For n = 1, we have

P (1): 1.2 = 2 = $\frac{1(1+1)(1+2)}{3} = \frac{1.2.3}{3} = 2$which is true.

Let P(k) be true for some positive integer k, i.e.,

We shall now prove that P (k + 1) is true.

Consider1.2 + 2.3 + 3.4 + ... + k. (k + 1) + (k + 1). (k + 2)

= [1.2 + 2.3 + 3.4 + ... + k (k + 1)] + (k + 1) (k + 2)

$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$ [using (1)]

= (k + 1) (k + 2)$\left(\frac{k}{3} + 1\right)$

$= \frac{(k+1)(k+2)(k+3)}{3}$

$= \frac{(k+1)(k+1+1)(k+1+2)}{3}$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 7: Prove the following by using the principle of mathematical induction for all n ∈ N:

$1.3 + 3.5 + 5.7 + \ldots + (2n - 1)(2n + 1) = \frac{n(4n^2+6n-1)}{3}$

Answer 7:

Let the given statement be P(n), i.e.,

P(n): $1.3 + 3.5 + 5.7 + \ldots + (2n - 1)(2n + 1) = \frac{n(4n^2+6n-1)}{3}$

For n = 1, we have

P (1):$1.3 = 3 = \frac{1(4.1^2+6.1-1)}{3} = \frac{4+6-1}{3} = \frac{9}{3} = 3$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$1.3 + 3.5 + 5.7 + \ldots + (2k - 1)(2k + 1) = \frac{k\ (4k^2+6k-1)}{3} \ldots (1)$

We shall now prove that P (k + 1) is true.

Consider

$(1.3 + 3.5 + 5.7 + \ldots + (2k - 1)(2k + 1) + \{2(k + 1) - 1\}\{2(k + 1) + 1\}$

$= \frac{k(4k^2+6k-1)}{3} + (2k + 2 - 1)(2k + 2 + 1)$ [using (1)]

$= \frac{k(4k^2+6k-1)}{3} + (2k + 1)(2k + 3)$

$= \frac{k(4k^2+6k-1)}{3} + (4k^2 + 8k + 3)$

$= \frac{k(4k^2+6k-1)+3(4k^2+8k+3)}{3}$

$= \frac{4k^3+6k^2-k+12k^2+24k+9}{3}$

$= \frac{4k^3+18k^3+23k+9}{3}$

$= \frac{4k^3+14k^3+9k+4k^3+14k+9}{3}$

$= \frac{k\left(4k^2+14k+9\right)+1(4k^2+14k+9)}{3}$

$= \frac{(k+1)\left(4k^2+14k+9\right)}{3}$

$= \frac{(k+1)\left\{4k^2+8k+4+6k+6-1\right\}}{3}$

$= \frac{(k+1)\left\{4\left(k^2+2k+1\right)+6(k+1)-1\right\}}{3}$

$= \frac{(k+1)\left\{4(k+1)^2+6(k+1)-1\right\}}{3}$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 8: Prove the following by using the principle of mathematical induction for all n ∈ N: $1.2 + 2.2^2 + 3.2^2 + ... + n.2^n = (n – 1)\ 2^{n+1} + 2$

Answer 8:

Let the given statement be P(n), i.e.,

P(n): $1.2 + 2.2^2 + 3.2^2 + ... + n.2^n = (n – 1)\ 2^{n+1} + 2$

For n = 1, we have

P (1): $1.2 = 2 = (1 – 1)\ 2^{1+1} + 2 = 0 + 2 = 2$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$1.2 + 2.2^2 + 3.2^2 + ... + k.2^k = (k – 1)\ 2^{k+1} + 2$ ... (i)

We shall now prove that P (k + 1) is true.

Consider

$\{1.2 + 2.2^2 + 3.2^3 + \cdots + k.2^k\} + (k+1).2^{k+1}$

$= (k+1)2^{k+1} + 2 + (k+1)2^{k+1}$

$= 2^{k+1}\{(k-1) + (k+1)\} + 2$

$= 2^{k+1}.2k + 2$

$= \text{k}.2^{(k+1)+1} + 2$

$= \{(k+1) - 1\}2^{(k+1)+1} + 2$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 9: Prove the following by using the principle of mathematical induction for all n ∈ N:

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

Answer 9:

Let the given statement be P(n), i.e.,

P(n): $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

For n = 1, we have

P (1): $\frac{1}{2} = 1 - \frac{1}{2^1} = \frac{1}{2}$which is true.

Let P(k) be true for some positive integer k, i.e.,

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$ … (1)

We shall now prove that P (k + 1) is true.

Consider

$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$

$=\left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$ [using (1)]

$=1-\frac{1}{2^k}+\frac{1}{2.2^k}$

$=1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)$

$=1-\frac{1}{2^k}\left(\frac{1}{2}\right)$

$=1-\frac{1}{2^{k+1}}$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 10: Prove the following by using the principle of mathematical induction for all n ∈ N:

$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$

Answer 10:

Let the given statement be P(n), i.e.,

P(n): $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$

For n = 1, we have

P (1): $\frac{1}{2.5}=\frac{1}{10}=\frac{1}{6.1+4}=\frac{1}{10}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4}$ … (1)

We shall now prove that P (k + 1) is true.

Consider

$$= \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \cdots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$

$$= \frac{k}{6k+4} + \frac{1}{(3k+3-1)(3k+3+2)} \text{ [using (1)]}$$

$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{1}{3k+2}\left(\frac{k}{2} + \frac{1}{3k+5}\right)$$

$$= \frac{1}{3k+2}\left(\frac{k(3k+5)+2}{2(3k+5)}\right)$$

$$= \frac{1}{3k+2}\left(\frac{3k^2+5k+2}{2(3k+5)}\right)$$

$$= \frac{1}{3k+2}\left(\frac{(3k+2)(k+1)}{2(3k+5)}\right)$$

$$= \frac{k+1}{2(3k+5)}$$

$$= \frac{k+1}{6k+10}$$

$$= \frac{k+1}{6(k+1)+4}$$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 11: Prove the following by using the principle of mathematical induction for all n ∈ N:

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Answer 11:

Let the given statement be P(n), i.e.,

P(n): $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

For n = 1, we have, which is true.

P (1): $\frac{1}{1.2.3}=\frac{1.(1+3)}{4(1+1)(1+2)}=\frac{1.4}{4.2.3}=\frac{1}{1.2.3}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)} \ldots (1)$$

We shall now prove that P (k + 1) is true.

Consider

$$=\left[\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots+\frac{1}{k(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \text{ [using (1)]}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)}{4}+\frac{1}{k+3}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)^2+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\left(k^2+6k+9\right)+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+6k^2+9k+6}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+2k^3+k+4k^3+8k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\left(k^2+2k+1\right)+4(k^2+2k+1)}{4(k+3)}\right\}$$

$= \frac{1}{(k+1)(k+2)}\left\{\frac{k(k+1)^2+4(k+1)^2}{4(k+3)}\right\}$

$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$

$= \frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 12:

Prove the following by using the principle of mathematical induction for all n ∈ N:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$$

Answer 12:

Let the given statement be P(n), i.e.,

P (n): $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$

For n = 1, we have

P (1): $a = \frac{a(r^1-1)}{r-1} = a$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$= a + ar + ar^2 + \cdots + ar^{k-1} = \frac{a(r^k-1)}{r-1}$ … (1)

We shall now prove that P (k + 1) is true.

Consider

$\{a + ar + ar^2 + \cdots + ar^{k-1}\} + ar^{(k+1)-1}$

$= \frac{a(r^k-1)}{r-1} + ar^k$ [using (1)]

$= \frac{a(r^k-1)+ar^k(r-1)}{r-1}$

$= \frac{a(r^k-1)+ar^{k-1}-ar^k}{r-1}$

$= \frac{ar^k-a+ar^{k+1}-ar^k}{r-1}$

$= \frac{ar^{k+1}-a}{r-1}$

$= \frac{a(r^{k+1}-1)}{r-1}$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 13: Prove the following by using the principle of mathematical induction for all n ∈ N:

$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2$

Answer 13:

Let the given statement be P(n), i.e.,

P (n): $= \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2$

For n = 1, we have

P (1): $\left(1+\frac{3}{1}\right)+4 = (1+1)^2 = 2^2 = 4$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$= \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)....\left(1+\frac{(2k+1)}{k^2}\right) = (k+1)^2 \ldots$ (1)

We shall now prove that P (k + 1) is true.

Consider

$$\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)\right]\left\{1+\frac{(2(k+1)+1)}{(k+1)^2}\right\}$$

$= (k+1)^2\left(1+\frac{2(k+1)+1}{(k+1)^2}\right)$ [using (1)]

$= (k+1)^2\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$

= (k + 1) + 2 (k + 1) + 1

= {(k+1) + 1} + 2

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 14: Prove the following by using the principle of mathematical induction for all n ∈ N:

Answer 14:

$= \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots.\left(1+\frac{1}{n}\right) = (n+1)$

Let the given statement be P(n), i.e.,

P (n): $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right) = (n+1)$

For n = 1, we have

P (1): $\left(1+\frac{1}{1}\right) = 2 = (1+1)$, which is true.

Let P(k) be true for some positive integer k, i.e.,

P (k): $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right) = (k+1)$ … (1)

We shall now prove that P (k + 1) is true.

Consider

$$\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

$= (k+1)\left[1+\frac{1}{k+1}\right]$ [using (1)]

$= (k+1)\left\{\frac{(k+1)+1}{(k+1)}\right\}$

$= (k+1)+1$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 15: Prove the following by using the principle of mathematical induction for all n ∈ N:

$= 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Answer 15:

Let the given statement be P(n), i.e.,

P (n): $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

For n = 1 we have

P (1): $1^2 = 1 = \frac{1(2.1-1)(2.1+1)}{3} = \frac{1.1.3}{3} = 1$, which is true

Let P(k) be true for some positive integer k, i.e.,

P (k): $1^2 + 3^2 + 5^2 + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{3}$ … (1)

We shall now prove that P (k + 1) is true.

Consider

$\{1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2\} + \{2(k+1)-1\}^2$

$= \frac{k(2k-1)(2k+1)}{3} + (2k + 2 - 1)^2$ [using (1)]

$= \frac{k(2k-1)(2k+1)}{3} + (2k + 1)^2$

$= \frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$

$= \frac{(2k+1)\{k(2k-1)+3(2k+1)\}}{3}$

$= \frac{(2k+1)\{2k^2-k+6k+3\}}{3}$

$= \frac{(2k+1)\{2k^2+5k+3\}}{3}$

$= \frac{(2k+1)(2k^2+2k+3k+3)}{3}$

$= \frac{(2k+1)\{2k(k+1)+3(k+1)\}}{3}$

$= \frac{(2k+1)(k+1)(2k+3)}{3}$

$= \frac{(k+1)\{2(k+1)-1\}\{2(k+10+1\}}{3}$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 16: Prove the following by using the principle of mathematical induction for all n ∈ N:

$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$

Answer 16:

Let the given statement be P(n), i.e.,

P (n): $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$

For n = (1) we have

P (1) = $\frac{1}{1.4} = \frac{1}{3.1+4} = \frac{1}{4} = \frac{1}{1.4}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

P (k): = $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{(3k+1)}$ … (1)

We shall now prove that P (k + 1) is true.

Consider

$= \left\{\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3k-2)(3k+1)}\right\} + \frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$

$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$ [using (1)]

$= \frac{1}{(3k+1)}\left\{k + \frac{1}{(3k+4)}\right\}$

$= \frac{1}{(3k+1)}\left\{\frac{k(3k+4)+1}{(3k+4)}\right\}$

$= \frac{1}{(3k+1)}\left\{\frac{3k^2+4k+1}{(3k+4)}\right\}$

$= \frac{1}{3k+1}\left\{\frac{3k^2+3k+k+1}{(3k+4)}\right\}$

$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$

$= \frac{(k+1)}{3(k+1)+1}$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 17: Prove the following by using the principle of mathematical induction for all n ∈ N:

$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+..+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

Answer 17:

Let the given statement be P(n), i.e.,

P (n): $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

For n = 1, we have

P (1): $\frac{1}{3.5}=\frac{1}{3(2.1+3)}=\frac{1}{3.5}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

P (k): $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)}$ … (1)

We shall now prove that P (k + 1) is true.

Consider

$=\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{\{2(k+1)+1\}\{2(k+1))+3\}}$

$=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)}$ [using (1)]

$=\frac{1}{(2k+3)}\left[\frac{k}{3}+\frac{1}{(2k+5)}\right]$

$=\frac{1}{(2k+3)}\left[\frac{k(2k+5)+3}{3(2k+5)}\right]$

$=\frac{1}{(2k+3)}\left[\frac{2k^2+5k+3}{3(2k+5)}\right]$

$=\frac{1}{(2k+3)}\left[\frac{2k^2+2k+3k+3}{3(2k+5)}\right]$

$=\frac{1}{2k+3}\left[\frac{2k(k+1)+3(k+1)}{3(2k+5)}\right]$

$=\frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$

$= \frac{(k+1)}{3\{2(k+1)+3\}}$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 18: Prove the following by using the principle of mathematical induction for all n ∈ N:

$1 + 2 + 3 + \ldots + n < \frac{1}{8}(2n+1)^2$

Answer 18:

Let the given statement be P(n), i.e.,

P(n): $1 + 2 + 3 + \ldots + n < \frac{1}{8}(2n+1)^2$

It can be noted that P(n) is true for n = 1 since

$1 < \frac{1}{8}(2.1+1)^2 = \frac{9}{8}$

Let P(k) be true for some positive integer k, i.e.,

P (k) = $1 + 2 + 3 + \ldots + k < \frac{1}{8}(2k+1)^2$

We shall now prove that P (k + 1) is true whenever P(k) is true.

Consider

$(1+2+..+k)+(k+1) < \frac{1}{8}(2k+1)^2+(k+1)$ [using (1)]

$< \frac{1}{8}\{(2k+1)^2+8(k+1)\}$

$< \frac{1}{8}\{4k^2+4k+1+8k+8\}$

$< \frac{1}{8}\{4k^2+12k+9\}$

$< \frac{1}{8}(2k+3)^2$

$< \frac{1}{8}\{2(k+1)+1\}^2$

Hence, $(1+2+3+\cdots+k)+(k+1) < \frac{1}{8}(2k+1)^2+(k+1)$

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 19: Prove the following by using the principle of mathematical induction for all n ∈ N:

n (n + 1) (n + 5) is a multiple of 3.

Answer 19:

Let the given statement be P(n), i.e.,

P(n): n (n + 1) (n + 5), which is a multiple of 3.

It can be noted that P(n) is true for n = 1 since 1 (1 + 1) (1 + 5) = 12,

which is a multiple of 3.

Hence,

Let P(k) be true for some positive integer k, i.e.,

k (k + 1) (k + 5) is a multiple of 3.

∴ k (k + 1) (k + 5) = 3m, where m ∈ N ... (1)

We shall now prove that P (k + 1) is true whenever P(k) is true.

Consider

(k + 1) {(k + 1) + 1} {(k + 1) + 5}

= (k + 1) (k + 2) {(k + 5) + 1}

= (k + 1) (k + 2) (k + 5) + (k + 1) (k + 2)

$= \{k (k + 1) (k + 5) + 2 (k + 1) (k + 5)\} + (k + 1) (k + 2)$

$= 3m + (k + 1) \{2 (k + 5) + (k + 2)\}$

$= 3m + (k + 1) \{2k + 10 + k + 2\}$

$= 3m + (k + 1) \{3k + 12\}$

$= 3m + 3 (k + 1) (k + 4)$

$= 3 \{m + (k + 1) (k + 4)\} = 3 \times q$, where $q = [m + (k + 1) (k + 4)\}$ is some natural number.

Therefore, $(k + 1) \{(k + 1) + 1\} \{(k + 1) + 5\}$ is a multiple of 3.

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

Question 20: Prove the following by using the principle of mathematical induction for all $n \in N$:

$10^{2n-1} + 1$ is divisible by 11.

Answer 20:

Let the given statement be P(n), i.e.,

P(n): $10^{2n-1} + 1$ is divisible by 11.

It can be observed that P(n) is true for n = 1

since $P(1) = 10^{2.1-1} + 1 = 11$, which is divisible by 11.

Let P(k) be true for some positive integer k,

i.e., $10^{2k-1} + 1$ is divisible by 11.

$\therefore 10^{2k-1} + 1 = 11m$, where $m \in N$ ... (1)

We shall now prove that P (k + 1) is true whenever P(k) is true.

Consider

$10^{2(k+1)-1} + 1$

$= 10^{2k+2-1} + 1$

$= 10^{2k+1} + 1$

$10^2 (10^{2k-1} + 1 – 1) + 1$

$= 10^2 (10^{2k-1} + 1) – 102 + 1$

$= 10^2 .11m – 100 + 1$ [using (1)]

$= 100 \times 11m – 99$

$= 11 (100m – 9)$

$= 11r$, where $r = (100m - 9)$ is some natural number

Therefore, $10^{2(k+1)-1} + 1$ is divisible by 11.

Thus, P (k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.