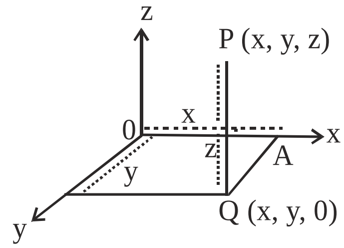
Three Dimensional Geometry

THREE DIMENSIONAL GEOMETRY

1. Coordinates of a point in Space :

Let O be a fixed point, known as origin and let OX, OY and OZ be three mutually perpendicular lines, taken as x-axis, y-axis and z-axis respectively, in such a way that they form a right handed system.

The planes XOY, YOZ and ZOX are known as xy-plane, yz-plane and zx-plane respectively.



Let P be a point in space and distances of P from y-z, z-x and x-y planes be x, y, z respectively (with proper signs) then we say that coordinates of P are (x, y, z).

Also OA = x, OB = y, OC = z

2. Distance Formula :

The distance between two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by

$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Distance from Origin :

Let O be the origin and P (x, y, z) be any point, then

$$OP=\sqrt{(x^2+y^2+z^2)}$$

3. Section Formulae :

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let R (x, y, z) divide PQ in the ratio $m_1 : m_2$, then R is

$$(x,y,z)\equiv\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2},\frac{m_1z_2+m_2z_1}{m_1+m_2}\right)$$

If R be the external point, then $R\equiv\left(\frac{m_1x_2-m_2x_1}{m_1-m_2},\frac{m_1y_2-m_2y_1}{m_1-m_2},\frac{m_1z_2-m_2z_1}{m_1-m_2}\right)$

Mid-Point : Mid point of PQ is given by $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$

4. Centroid of a triangle :

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of a triangle ABC. Then its centroid G is given by $G\equiv\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$

5. To find the ratio in which a given point divide a given line-segment :

Let P (α, β, γ) and A(a_1, b_1, c_1) and B(a_2, b_2, c_2) be three points and we have to find the ratio in which P divides AB. Let k : 1 be the ratio. Then

$$\frac{ka_2 + a_1}{k+1} = \alpha, \frac{kb_2 + b_1}{k+1} = \beta, \frac{kc_2 + c_1}{k+1} = \gamma$$

Solving these, find k. Also if the system is inconsistent, P does not divide AB in any ratio and so P does not lie on AB i.e. P, A, B are non-collinear.

For collinearity of three points it is the best method.

6. Direction Cosines & Direction Ratios :

Direction Cosines :

Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively then cos α, cos β, cos γ are called the direction cosines of the line.

The direction cosines are usually denoted by < l, m, n >. Then l = cos α, m = cos β, n = cos γ.

Direction ratios :

Let a, b, c be proportional to the d.c.'s l, m, n then a, b, c are called the direction ratios.

Relation between D.R.'s and D.C.'s of a line:

Let a, b, c be d.r.'s and l, m, n the d.c.'s of a line, then

$$\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = r \text{ (say)}$$

$$\therefore \quad a^2 + b^2 + c^2 = r^2 (l^2 + m^2 + n^2) = r^2 \Rightarrow r = \sqrt{[a^2 + b^2 + c^2]}$$

Direction cosines l, m, n are given by

$$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r} \qquad \text{where} \qquad r = \sqrt{[(a^2 + b^2 + c^2)]}$$

D.C.'s of Axes :

Since the positive x-axis makes angles 0º, 90º, 90º with axes of x, y and z respectively,

$\therefore$ D.C.'s of x axes are 1, 0, 0.

$\therefore$ D.C.'s of x-axis are 1, 0, 0

D.C.'s of y-axis are 0, 1, 0

D.C.'s of z-axis are 0, 0, 1

Direction ratios of the line joining two points :

Let A(x_1, y_1, z_1) and B(x_2, y_2, z_2) be two points, then d.r.'s of AB are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ and the d.c.'s of AB are $\frac{1}{r}(x_2 - x_1)$, $\frac{1}{r}(y_2 - y_1)$, $\frac{1}{r}(z_2 - z_1)$ where $r = \sqrt{[\Sigma(x_2 - x_1)^2]}$

7. Projection of a line on another line :

Let PQ be a line segment with P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) and let L be a straight line whose d.c.'s are l, m, n. Then the length of projection of PQ on the line L is = $| l (x_2 - x_1) + m (y_2 - y_1) + n (z_2 - z_1) |$

8. Angle between two lines :

Let θ be the angle between the lines with d.c.'s l_1, m_1, n_1 and l_2, m_2, n_2, then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

Also, $\sin^2\theta = \sum (l_1 m_2 - l_2 m_1)^2$

9. Condition for Perpendicularity and parallelism :

Let the two lines have their d.c.'s as l_1, m_1, n_1 and l_2, m_2, n_2 respectively then they are perpendicular if $\theta = 90^o$ i.e. $\cos\theta = 0$, i.e. $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

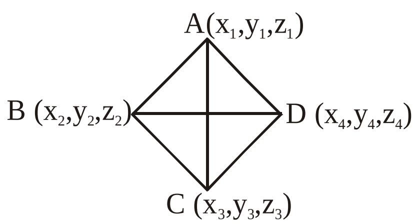
Also the two lines are parallel if $\theta = 0^o$ or 180^o and then sin $\theta = = 0$, i.e. $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

If instead of d.c.'s, d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are given, then the lines are perpendicular if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$ and parallel if $a_1/a_2 = b_1/b_2 = c_1/c_2$.

If θ is the angle between the lines then $\cos\theta = \frac{(a_1 a_2 + b_1 b_2 + c_1 c_2)}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$

10. Centroid of tetrahedron

Centroid $$G \equiv \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$



THE STRAIGHT LINE

1. Straight Line

Definition :

A straight line in space is characterised by the intersection of two planes which are not parallel and, therefore, the equation of a straight line is present as a solution of the system constituted by the equations of the two planes :

$a_1 x + b_1 y + c_1 z + d_1 = 0$; $a_2 x + b_2 y + c_2 z + d_2 = 0$

This form is also known as unsymmetrical form.

Some particular straight lines :

	Straight lines	Equation
(i)	Through the origin	$y = mx, z = nx$
(ii)	x-axis	$y = 0, z = 0$
(iii)	y-axis	$x = 0, z = 0$
(iv)	z-axis	$x = 0, y = 0$
(v)	| | to x-axis	$y = p, z = q$
(vi)	| | to y-axis	$x = h, z = q$
(vii)	| | to z-axis	$x = h, y = p$

2. Equation of a straight line in symmetrical form :

(i) One point form : Let $A(x_1, y_1, z_1)$ be a given point on the straight line and l, m, n the d.c's of the line, then its equation is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \qquad \text{(say)}$$

It should be noted that $P(x_1 + lr, y_1 + mr, z_1 + nr)$ is a general point on this line at a distance r from the point $A(x_1, y_1, z_1)$ i.e. AP = r. One should note that for AP = r, l, m, n must be d.c.'s not d.r.'s. If a, b, c are direction ratios of the line, then equation of the line is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r \qquad \text{but here } AP \neq r$$

(ii) Two point form : Equation of the line through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

3. Reduction of non-symmetrical form to symmetrical form :

Let equation of the line in non-symmetrical form be

$$a_1 x + b_1 y + c_1 z + d_1 = 0; \qquad a_2 x + b_2 y + c_2 z + d_2 = 0$$

To find the equation of the line in symmetrical form, we must know (i) its direction ratios (ii) coordinates of any point on it.

(i) Direction Ratios : Let l, m, n be the direction ratios of the line. Since the line lies in both the planes, it must be perpendicular to normals of both planes. So

$$a_1 l + b_1 m + c_1 n = 0; \qquad a_2 l + b_2 m + c_2 n = 0$$

From these equations, proportional values of l, m, n can be found by cross-multiplication as

$$\frac{l}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$$

(ii) Point on the line : Note that as l, m, n can not be zero simultaneously, so at least one must be non-zero. Let $a_1b_2 - a_2b_1 \neq 0$. Then the line cannot be parallel to x-y plane so it intersect it. Let it intersect x-y plane in $(x_1, y_1, 0)$. Then $a_1 x_1 + b_1 y_1 + d_1 = 0$ and $a_2 x_1 + b_2 y_1 + d_2 = 0$

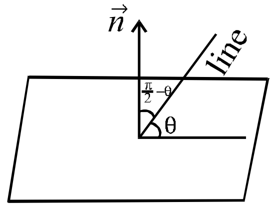
Solving these, we get a point on the line.

Then its equations becomes $\dfrac{x - x_1}{b_1c_2 - b_2c_1} = \dfrac{y - y_1}{c_1a_2 - c_2a_1} = \dfrac{z - 0}{a_1b_2 - a_2b_1}$

If $l \neq 0$, take a point on y-z plane as $(0, y_1, z_1)$ and if $m \neq 0$, take a point on x-z plane as $(x_1, 0, z_1)$.

4. Angle between a line and a plane :

Let equations of the line and plane be $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ and $ax + by + cz + d = 0$ respectively and θ be the angle which line makes with the plane.Then $(\pi/2 - \theta)$ is the angle between the line and the normal to the plane. So $\sin\theta = \dfrac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(l^2 + m^2 + n^2)}}$



Line is parallel to plane if θ = 0 i.e. if al + bm + cn = 0.

Line is $\perp$ to the plane if line is parallel to the normal of the plane i.e. if $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$.

5. Condition in order that the line may lie on the given plane :

The line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ will lie on the plane Ax + By + Cz + D = 0 if

(i) Al + Bm + Cn = 0 and (ii) $Ax_1 + By_1 + Cz_1 + D = 0$

6. Foot, length and equation of perpendicular from a point to a line :

Let equation of the line be

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}=r \quad \text{(say)} \qquad \text{..........} (1)$$

and A (α,β,γ) be the point. Let l, m, n denote the actual d.c.'s of the line. Any point on the line (1) is

$$P(lr + a, mr + b, nr + c) \qquad \text{..........} (2)$$

If it is the foot of the perpendicular, from A on the line, then AP is $\perp$ to the line, so

$$l(lr + a - \alpha) + m(mr + b - \beta) + n(nr + c - \gamma) = 0$$

i.e. $r = (\alpha - a)\,l + (\beta - b)\,m + (\gamma - c)\,n$

[since $l^2 + m^2 + n^2 = 1$]

Putting this value of r in (2), we get the foot of perpendicular from point A to the line.

Length : Since foot of perpendicular P is known, length of perpendicular,

$$AP=\sqrt{[(lr+a-\alpha)^2+(mr+b-\beta)^2+(nr+c-\gamma)^2]}$$

Equation of perpendicular is given by

$$\frac{x-\alpha}{lr+a-\alpha}=\frac{y-\beta}{mr+b-\beta}=\frac{z-\gamma}{nr+c-\gamma}$$

7. Equation of any plane through a given line :

Equation of any plane through the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, where Al + Bm + Cn = 0.

There can be infinitely many planes through a given line. Particular plane can be found if extra condition is given.

8. Condition that two given lines should intersect i.e. to be coplanar i.e. testing of skewness or coplanarity of two given lines. Equation of plane containing two intersecting lines :

Let the two lines be

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \qquad \text{.......... (1)}$$

and $$\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2} \qquad \text{.......... (2)}$$

These lines will be coplanar if

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

the plane containing the two lines is

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} x-\alpha_2 & y-\beta_2 & z-\gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Condition of coplanarity if both lines are in general from :

Let the lines be $ax + by + cz + d = 0 = a'x + b'y + c'z + d'$

and $\alpha x+\beta y+\gamma z+\delta=0=\alpha' x+\beta' y+\gamma' z+\delta'$

These are coplanar if

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix}=0$$

Skew lines :

The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines.

9. Shortest distance between two skew straight lines :

Shortest distance between two skew lines is perpendicular to both.

(i) If the equations are in cartesian form :

Suppose the equation of the lines are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{.......... (1)}$$

and $$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'} \qquad \text{.......... (2)}$$

Then shortest distance between them is given by

$$\text{S.D.}=\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \sqrt{\{\Sigma(mn'-m'n)^2\}}$$

(ii) If the lines intersect, then S.D.between them is zero. Therefore

$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0$$

(iii) The equations of shortest distance between lines (1) and (2)

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} x-\alpha_2 & y-\beta_2 & z-\gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

(iv) If the equations are in vector form :

Suppose the equation of the lines are

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \quad \text{and} \quad \vec{r} = \vec{a}_2 + \lambda \vec{b}_2$$

Then shortest distance between them is given by

$$\text{S.D.} = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

10. Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$, where λ is a scalar.

11. The vector equation of a line passing through two given points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$.

12. Angle between lines

$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is

$$\cos\theta = \frac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$$

13.Angle between lines

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \text{ is}$$

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

If the lines are perpendicular, then

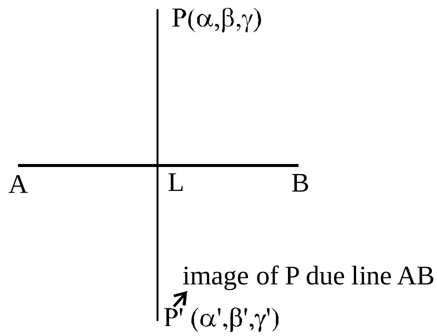
$$\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$$

If the lines are parallel, then

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$$

14. Reflection or Image of a point in a straight line

Let $P(\alpha, \beta, \gamma)$ be the given point and let $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ is the given line. Lete L be the foot of the perpendicular from P in the given line and let P' be the image of P in the given line. Let the coordinates of L be $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, then direction ratios of PL are



$x_1 + a\lambda - \alpha, y_1 + b\lambda - \beta, z_1 + c\lambda - \gamma$

Since PL is perpendicular to the given line, whose direction ratios are a, b, c

$\therefore \quad (x_1 + a\lambda - \alpha)a + (y_1 + b\lambda - \beta)b + (z_1 + c\lambda - \gamma)c = 0$

$$\Rightarrow \quad \lambda = -\frac{[a(x_1 - \alpha) + b(y_1 - \beta) + c(z_1 - \gamma)]}{a^2 + b^2 + c^2}$$

Substituting the value of λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, we obtain coordinates of L.

Let $\quad Q = (\alpha', b', \gamma')$

Since L is the mid point of PQ

$$\therefore \quad \frac{\alpha + \alpha'}{2} = x_1 + a\lambda, \ \frac{\beta + \beta'}{2} = y_1 + b\lambda, \ \frac{\gamma + \gamma'}{2} = z_1 + c\lambda$$

$$\Rightarrow \quad \alpha' = 2(x_1 + a\lambda) - \alpha, \ \beta' = 2(y_1 + b\lambda) - \beta, \ \gamma' = 2(z_1 + c\lambda) - \gamma$$

THE PLANE

1. Plane

Definition :

A geometrical locus is a plane if it is such that if P and Q are any two points on the locus, then every point on the line PQ is also a point on the locus.

2. Equations of a Plane :

The equation of every plane is of the first degree i.e. of the form ax + by + cz + d = 0, in which a, b, c are constants, where $a^2 + b^2 + c^2 \neq 0$ (i.e. a, b, c $\neq$ 0 simultaneously). Here a,b,c are direction ratios of the normal to the plane.

Plane Parallel to the Coordinate Planes :

(i) Equation of y-z plane is x = 0.

(ii) Equation of z-x plane is y = 0.

(iii) Equation of x-y plane is z = 0.

(iv) Equation of the plane parallel to x-y plane at a distance c (if c > 0, towards positive z-axis) is z = c. Similarly, planes parallel to y-z plane and z-x plane are respectively x = c and y = c.

3. Equations of Planes Parallel to the Axes :

If a = 0, the plane is parallel to x-axis i.e. equation of the plane parallel to x-axis is by + cz + d = 0

Similarly, equations of planes || to y-axis and || to z-axis are $ax + cz + d = 0$ and $ax + by + d = 0$, respectively.

4. Equation of a Plane through Origin :

Equation of plane passing through origin is $ax + by + cz = 0$

5. Equation of a Plane through a given Point :

Let the plane $ax + by + cz + d = 0$ passes through the point (x_1, y_1, z_1) then $ax_1 + by_1 + cz_1 + d = 0$.

$\therefore$ Subtracting, we get $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ which is the required equation.

6. Equation of a Plane in Intercept Form :

Equation of the plane which cuts off intercepts a, b, c from the axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

7. Equation of a Plane in Normal Form :

If the length of the perpendicular distance of the plane from the origin is p and direction cosines of this perpendicular are (l, m, n), then the equation of the plane is $lx + my + nz = p$.

- In solving problems of plane, first consider its normal. In the equation $ax + by + cz + d = 0$, a, b, c are the direction ratios of the normal of the plane.

8. Reduciton of equation of plane ax+by+cz+d=0 into

(i) Intercept form:

$$ax + by + cz = -d$$

$$\Rightarrow \frac{x}{(-d/a)} + \frac{y}{(-d/b)} + \frac{z}{(-d/c)} = 1$$

which is required intercept form.

(ii) Normal form:

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = \frac{-d}{\sqrt{a^2+b^2+c^2}}$$

If $d < 0$, then above is the required normal form.

If $d > 0$, then the required normal form is

$$\left(-\frac{a}{\sqrt{\sum a^2}}\right)x + \left(-\frac{b}{\sqrt{\sum a^2}}\right)y + \left(-\frac{c}{\sqrt{\sum a^2}}\right)z = \frac{d}{\sqrt{\sum a^2}}, \text{ where } \sum a^2 = a^2 + b^2 + c^2$$

9. Equation of a Plane through three points :

The equation of the plane through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

10. Angle between two planes :

Consider two planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Angle between these planes is the angle between their normals.

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\,\sqrt{a'^2 + b'^2 + c'^2}}$$

$\therefore$ Planes are perpendicular if aa' + bb' + cc' = 0 and they are parallel if a/a' = b/b' = c/c'.

11. Planes parallel to a given Plane :

Equation of a plane parallel to the plane ax + by + cz + d = 0 is ax + by + cz + d' = 0. d' is to be found by other given function.

12. A plane through the line of intersection of two given planes :

Consider two planes u = ax + by + cz + d = 0 and v = a' x + b' y + c' z + d' = 0.

The equation u + λv = 0, l a real parameter, represents the plane passing through the line of intersection of given planes and if planes are parallel, this represents a plane parallel to them.

13. Perpendicular distance of a point from the plane :

Perpendicular distance p, of the point $A(x_1, y_1, z_1)$ from the plane ax + by + cz + d = 0 is given by

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{(a^2 + b^2 + c^2)}}$$

14. Bisectors of angles between two planes :

Let the equations of the two planes be ax + by + cz + d = 0 and $a_1x + b_1y + c_1z + d_1 = 0$.

Then equations of bisectors of angles between them are given by

$$\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}} \quad (*)$$

(i) Equation of bisector of the angle containing origin : First make both constant terms positive. Then +ve sign in (*) give the bisector of the angle which contains the origin.

(ii) Bisector of acute/obtuse angle : First making both constant terms positive,

$aa_1 + bb_1 + cc_1 > 0 \Rightarrow$ origin lies in obtuse angle

$< 0 \Rightarrow$ origin lies in acute angle

15. Area of a triangle :

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of a triangle ABC. Form two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then area is given by

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\begin{vmatrix} i & j & k \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

16. Volume of a Tetrahedron :

Volume of the tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ is given by

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

17. Vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to the plane as vector $\vec{n}$ is $(\vec{r}-\vec{a}).\vec{n}=0$

18. Vector equation of a plane in normal form is $\vec{r}.\hat{n}=d$, where $\hat{n}$ is the unit normal vector to the plane and d is its distance from the origin.

19. The angle θ between the planes

$\vec{r}.\vec{n_1}=d_1$ and $\vec{r}.\vec{n_2}=d_2$ is given by

$$\cos\theta=\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

If planes are perpendicular then $\vec{n_1}.\vec{n_2}=0$

If planes are parallel then $\vec{n_1}=\lambda\vec{n_2}$

20. Parametric form:

The equation of the plane passing through a point having vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$, where λ and μ are scalers.

21. Non parametric form:

The equation of the plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is $(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$

22. The plane parallel to the plane

$\vec{r}.\vec{n}=d_1$ is $\vec{r}.\vec{n}=d_2$, where d_2 is a constant.

23. Equation of plane parallel to the intersection of two planes $\vec{r_1}.\vec{n_1}=d_1$ and $\vec{r_2}.\vec{n_2}=d_2$ is $\vec{r}.(\vec{n_1}+\lambda\vec{n_2})=d_1+\lambda d_2$, where λ is an arbitary constant.

24. The length of the perpendicular from a point having positive vector $\vec{a}$ to the plane $\vec{r}.\vec{n}=d$

$$p=\frac{|\vec{a}.\vec{n}-d|}{|\vec{n}|}$$

25. Distance between the parallel planes

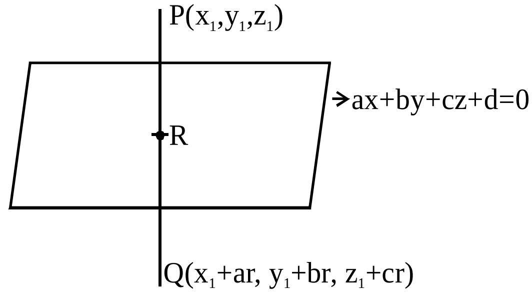
$ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is $\dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

26. If the line $\vec{r}=\vec{a}+\lambda\vec{b}$ lies in the plane $\vec{r}.\vec{n}=d$, then (i) $\vec{b}.\vec{n}=0$ and (ii) $\vec{a}.\vec{n}=d$

27. If the lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ are coplanar, then $[\vec{a_1}\ \vec{b_1}\ \vec{c_1}]=[\vec{a_2}\ \vec{b_2}\ \vec{c_2}]$ and the equation of the plane containing them is $[\vec{r}\ \vec{b_1}\ \vec{b_2}]=[\vec{a_1}\ \vec{b_1}\ \vec{b_2}]$ or $[\vec{r}\ \vec{b_1}\ \vec{b_2}]=[\vec{a_2}\ \vec{b_1}\ \vec{b_2}]$

28. Image of A point in a plane:

Equation of PQ is $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}=r,\text{say}$



$\therefore\ Q = (x_1 + ar, y_1 + br, z_1 + cr)$, where Q is the image of the point P due to the plane.

Then find the coordinates of the mid-point R of PQ.

Since R lies on the plane, so putting the coordinates of R in equation of plane, we find value of r.

Then putting value of r we get coordinates of image Q of point P due to the plane.

29. If $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of planes, then the angle between the planes is given by

$$\tan\theta = \frac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{a + b + c}$$

THE SPHERE

1. Definition :

A sphere is the locus of a point in space which remains at a constant distance from a fixed point. The constant distance is called the radius and the fixed point, the centre of the sphere.

2. Equation of a sphere in cartesian form:

(i) Let (a, b, c) be the centre and r the radius of a given sphere. Then the equation of the sphere is $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$

$x^2 + y^2 + z^2 = r^2$ is the equation of a sphere whose centre is (0, 0, 0) and radius r.

(ii) Equation of sphere having centre as $\vec{a}$ and radius R is $|\vec{r} - \vec{a}| = R$.

3. General equation of a sphere :

The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere with centre at (– u, – v, – w) and radius $\sqrt{(u^2 + v^2 + w^2 - d)}$ if $u^2 + v^2 + w^2 - d > 0$.

4. Sphere with a given diameter :

(i) The equation of the sphere described on the segment joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

(ii) The equation of the sphere having extremities of diameter as $\vec{a}$ and $\vec{b}$ is

$$(\vec{r} - \vec{a}).(\vec{r} - \vec{b}) = 0 \Rightarrow |\vec{r}|^2 - \vec{r}.(\vec{a} - \vec{b}) + \vec{a}.\vec{b} = 0$$

5. Equation of sphere through four points :

Equation of the sphere passing through four non-coplanar points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

6. Plane and Sphere :

Let $P = ax + by + cz + d = 0$ be a plane and $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + k = 0$ be a sphere. If p be the length of perpendicular from the centre $(-u, -v, -w)$ of sphere to the plane $P = 0$. $r = \sqrt{(u^2 + v^2 + w^2 - k)}$ be the radius of the sphere. Then

$p < r \Rightarrow$ Plane intersect the sphere

$p = r \Rightarrow$ Plane touches the sphere

$p > r \Rightarrow$ Plane neither intersect nor touches the sphere.

7. Plane section of a sphere :

If plane intersect sphere, its intersection is a circle. The centre of the circle is the foot N of the perpendicular from the centre O of the sphere to the plane and its radius is $\sqrt{(r^2 - ON^2)}$, r being the radius of the sphere.

8. Intersection of two Spheres :

Consider two spheres

$$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

and $$S' = x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$$

Here $C(-u, -v, -w)$ and $C'(-u', -v', -w')$ are their centres and if r, r' be their respective radii, then

$CC' > r + r' \Rightarrow$ Sphere are exterior to each other i.e. neither intersecting nor including.

$|r - r'| < CC' < r + r' \Rightarrow$ Sphere are intersecting.

$CC' = |r + r'| \Rightarrow$ They touch each other externally.

$CC' = |r - r'| \Rightarrow$ They touch internally.

$CC' < |r - r'| \Rightarrow$ One is contained in the other.

9. Common Plane on intersection :

If $S = 0$ and $S' = 0$ intersect, then $S - S' = 0$ represents the equation of common plane of intersection and this intersection yields a circle.

10. Orthogonal Intersection of two spheres

Two spheres $x^2 + y^2 + z^2 + 2u_1x + 2u_1y + 2w_1 + z + dy = 0$ and $x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$ cut each other. Orthogonally if $2(u_1u_2 + v_1v_2 + w_1w_2) = d_1 + d_2$

11. The plane $\vec{r}.\vec{n} = d$ touches the sphere $|\vec{r} - \vec{a}| = R$, if $\dfrac{|\vec{a}.\vec{n} - d|}{|\vec{n}|} = R$