Congruency of Triangles

10

What is Congruency?

See the triangles in *Fig.*1. Are they of same measure? If you put one triangle on the other, do they cover each other? In this figure we can see that they are not same, as the sides of the triangle are not same.

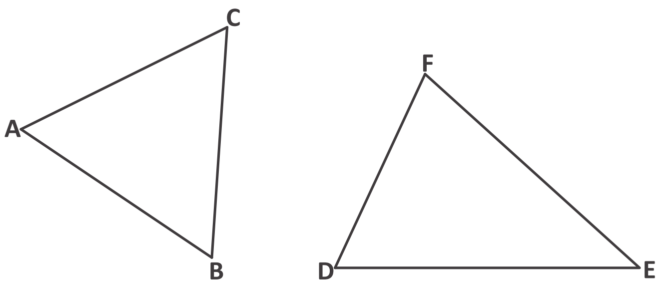


Fig. 1

How can we know whether two shapes cover each other completely or not? In the given triangles it will only happen when point A falls on point D, point B on point E and point C falls on point F and that is only possible if all the sides and angles of one triangle are equal to the other. That means the two triangles are congruent.

Congruence means that all parts are equal. Those shapes, in which all their parts are equal will cover each other completely.

In reference to triangles it means all sides and angles of one triangle are equal to the corresponding side and angle of the other triangle. Similarly we can also check congruency in quadrilaterals and pentagons. But is it necessary to check the equality of all parts when two figures are congruent ? Or is there some special curcumstances when we can see some parts of those figures and comment on the congruency?

Circle, Square and Rectangle

A Square has four sides and four angles and each of its side is equal and every angle is 90°. If the side of two squares are equal, we can say that both are congruent, and they would cover each other completely.

But if one side of a rectangle is equal to the corresponding side of an other rectangle, will they be congruent? Obviously it won't be so. When the two adjacent sides of one rectangle is equal to the corresponding sides of the other rectangle only then will they congruent. Now think when we can say the two circles are congruent? Just by equality of the radii of those circles, we can say that they are congruent.

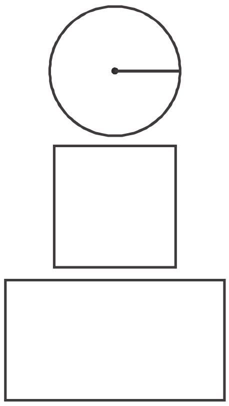


Fig. 2

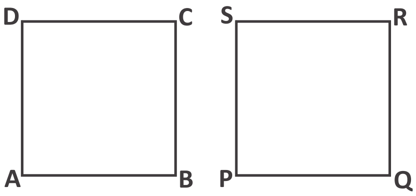


Fig. 3

Precisely we can say-

1. Two squares ABCD and PQRS are congruent if AB = PQ.

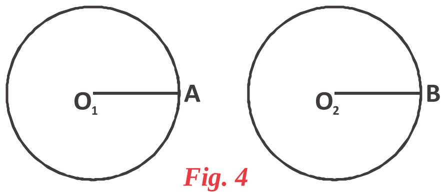


Fig. 4

2. Two circles are congruent if their radii are equal that is $OA_1 = O_2B$.

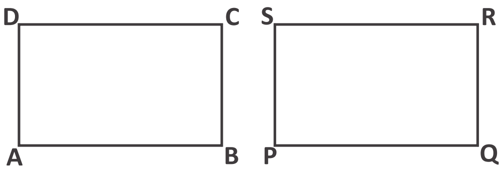


Fig. 5

3. In this way two rectangles are congruent. If there corresponding adjucent sides are equal that is AB = PQ, AD = PS. Like wise in triangles, can we finds some conditions for the congruency? In this chapter we will investigate about this in detail.

Congruency of triangle

In geometry, triangle is a closed figure which is made by the least number of line segments.

All the polygons are made up of triangles, that is why triangle congruency helps in checking the congruency of polygons.

We know that the triangles are congruent if their corresponding sides and angles are equal .

Corresponding Part of a Triangle

Look the triangles ABC and PQR, if we consider sides AB and PQ to be corresponding sides then the remaining corresponding parts of the triangles are as given in the table below.

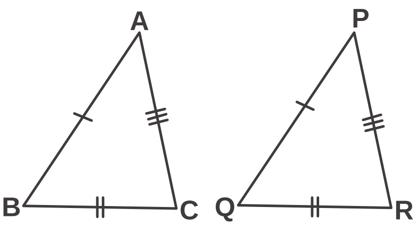


Fig. 6

Corresponding sides	**Corresponding Angle**	**Vertex**
$AB \leftrightarrow PQ$	$\angle A \leftrightarrow \angle P$	$A \leftrightarrow P$
$BC \leftrightarrow QR$	$\angle B \leftrightarrow \angle Q$	$B \leftrightarrow Q$
$AC \leftrightarrow PR$	$\angle C \leftrightarrow \angle R$	$C \leftrightarrow R$

$\leftrightarrow$ sign indicates correspondancy and $\cong$ sign relates to congruency.

If two triangles ΔABC and ΔPQR are congruent. That is $\Delta ABC \cong \Delta PQR$, than $AB = PQ$, $BC = QR$, $AC = PR$, $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$

We can also write $\Delta BCA \cong \Delta QRP$ or $\Delta CAB \cong \Delta RPQ$.

We know that ΔABC and ΔBCA or ΔCAB all are same.

We can write ΔBCA congruent to ΔQRP. But we shouldn't write-

$$\Delta ABC \cong \Delta RQP \quad \text{or} \quad \Delta BCA \cong \Delta RPQ \text{ (why?)}$$

In triangles congruency it is necessary to write angle and vertices in the correct sequence. An abbreviation for corresponding parts of congruent triangles is CPCT.

How to check triangle congruency

Is it necessary to show the equality of all parts of a triangle when looking for congruency in triangles? Let us try to find conditions for triangles congruency mathematically.

(i) **Side-angle-side congruency (SAS congruency)-** "Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle".

(ii) **Angle-side-angle congruency (ASA congruency)-** "Two triangles are congruent if two angles and the included side of the one are equal to the two angles and the included side of the other.

(iii) **Side-side-side congruency (SSS congruency)-** Two triangles are said to be congruent if three sides of one triangles are equal to the three sides of the other triangles.

These three self evident postulate , can be used to identify the congruence of triangles, and on the basis of these tests we can find new tests to check for congruency of triangles. But identification is only possible if we can prove these like the theorems.

AXIOM, POSTULATE AND THEOREM

While learning mathematics we come across words like Axioms, postulates, theorem, corollary. Let us understand these words in brief-

Axiom and Postulate : Both are self evident, based on this we make new statements and prove them. In general, logical self evident statements are used in all subjects. We call them Axioms. Like in euclidian geometry some axioms are (i) if a is equal to b, and a also equals c, then b is equal to c. (2) Whole is bigger than its part. Like wise we can take some others axioms.

Postulate : Those self evident truths which are related to some specific subjects are know as postulates. Often axioms and postulates are treated as synonyms. Some geometrical postulates are- A straight line can be drawn using two given points and the points will be on this line. Or any line segment can be extended on either side to form a line.

Theorem and corollary : All those mathematical statements which are logically proved by using axioms, postulate and definitions are called theorems. Like the sum of interior angles of a triangle is 180°

On the basis of theorems, axioms and postulate some more theorems can be proved, these are called corollaries. In geometry there is no clear distinction between corollary and theorems. They often used one instead of the other.

(iv) Angle-angle-side congruence (AAS congruence theorem):

Theorem-10.1 : Two triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal.

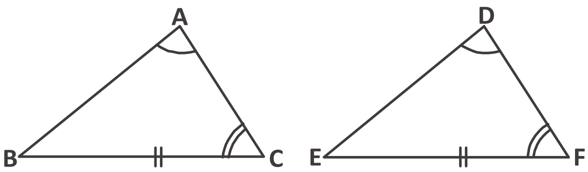


Fig. 7

Given $\angle A = \angle D$, $\angle C = \angle F$

and $\overline{BC} = \overline{EF}$

Two angles is in triangle ΔABC and ΔDEF are equal so the third angle is also equal.

$\because \angle A = \angle D$ and $\angle C = \angle F$ (given)

$\therefore \angle B = \angle E$ (sum of interior angles of a triangle is 180°)

Because side $\overline{BC}$ lies in between $\angle B$ and $\angle C$,

We can use ASA congruence rule to prove that ΔABC and ΔDEF are congruent.

Therefore $\Delta ABC \cong \Delta DEF$ (ASA congruency)

(v) Right angle hypotenuse side theorem:

Theorem-10.2 : Two triangles are congruent if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and corresponding side of the other.

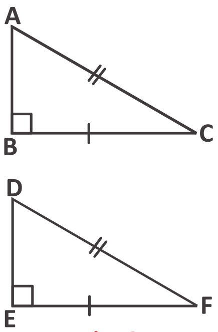


Fig. 8

Given in triangle ΔABC and ΔDEF

$\angle B = \angle E = 90°$, $AC = DF$ and $BC = EF$

We have to prove that $\Delta ABC \cong \Delta DEF$,

Produce DE to P so that EP = AB, join PF.

$\therefore \Delta ABC \cong \Delta PEF$ (by S.A.S.)

$\therefore \angle A = \angle P$ ... (1) C.P.C.T.

$AC = PF$ ...(2) C.P.C.T.

But AC = DF given.

$\therefore DF = PF$ and $\angle D = \angle P$ ... (3) (Angles opposite to equal sides of ΔDPF are equal)

From (1) and (3)

$\angle A = \angle D$

Now again in ΔABC and ΔDEF

BC = EF, AC = DF (known)

$\angle ACB = \angle DFE$

$\therefore \Delta ABC \cong \Delta DEF$ (SAS Congruency)

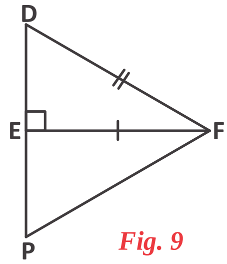


Fig. 9

THEOREMS AND SELF PROOFS

There are few statements in geometry, where it is not sure, whether they be considered as theorem or postulates. To prove theorem in a simple way and understand logical relationships in Geometry, we can select postulates in different ways. In some books AAS and ASA are taken as postulate to prove the theorem SSS. While in some others, AAS is taken as a postulate to prove the theorem ASA .

In this textbook we will consider ASA, SAS and SSS as postulates (self proved) and AAS, RHS as theorems.

Solution of the problems can be demonstrated in different ways. Some are shown below:-

EXAMPLE-1. In this example we use SAS condition to know more about this given figure.

In figure OA = OD and OB = OC.

Prove that

1. $\Delta AOB \cong \Delta DOC$
2. AB || CD

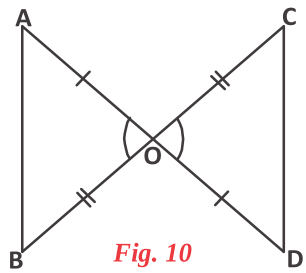


Fig. 10

SOLUTION : 1. In triangle ΔAOB and ΔDOC-

OA = OD given .....(1)

$\angle AOB = \angle DOC$ (vertically opposite angles are equal) ....(1)

OB = OC given .....(3)

Equation (1), (2) and (3) fulfill all the three conditions for congruence.

Therefore by SAS rule of congruency we have proved

$\Delta AOB \cong \Delta DOC$

2. The corresponding parts of the congruent triangles ΔAOB and ΔDOC are also equal.

Therefore $\angle OBA = \angle OCD$. Because they are the alternate angles between AB and CD line segment. Hence in this example AB || CD.

EXAMPLE-2. If in triangle ΔABC, $AP = PB$ and $CP \perp AB$. Then prove that-

1. $\Delta CPA \cong \Delta CPB$
2. $AC = BC$

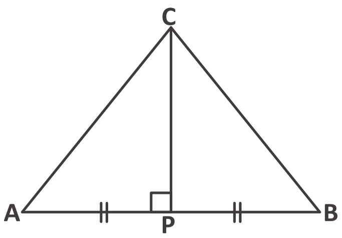


Fig. 11

SOLUTION : 1. In ΔCPA and ΔCPB

$AP = PB$ (given) .....(1)

$\angle APC = \angle BPC = 90°$ (given) .....(2)

$CP = CP$ (common side) ...(3)

Therefore by SAS congruence $\Delta CPA \cong \Delta CPB$

2. $\Delta CPA \cong \Delta CPB$ so $AC = BC$ (Corresponding parts of Congruent Triangle)

EXAMPLE-3. Angles opposite to equal sides of a triangle are equal.

SOLUTION : If we have a triangle ΔABC in which $AB = AC$

Construct a bisector of $\angle A$, which meets BC in point D.

In ΔABD and ΔACD

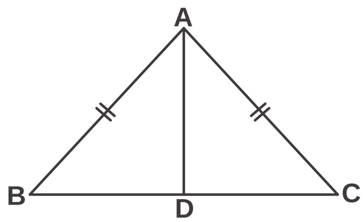


Fig. 12

$AB = AC$ (given)

$\angle BAD = \angle CAD$ (by construction)

$AD = AD$ (common)

$\therefore \Delta ABD \cong \Delta ACD$ (SAS congruence)

$\angle B = \angle C$ (C.P.C.T.)

EXAMPLE-4. If $\overrightarrow{BD}$, is a bisector of $\angle ABC$, and $\overline{AB} = \overline{BC}$ then with the help of SAS congruency prove that $\Delta ABD \cong \Delta CBD$

SOLUTION :

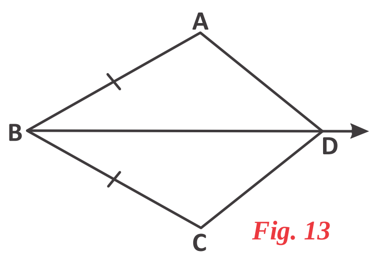


Fig. 13

Statement	Reason
$AB = BC$	given
$\overrightarrow{BD}$ is bisector of $\angle ABC$	given
$\angle ABD = \angle CBD$	By definition of angle bisector
$BD = BD$	common side
$\Delta ABD \cong \Delta CBD$	by SAS congruence

EXAMPLE-5. In figure AC = BC, and ∠DCA = ∠ECB and ∠DBC = ∠EAC

Prove that- ΔDBC ≅ ΔEAC in which DC = EC

SOLUTION : ∵ AC = BC (given)

∴ C is the mid point of AB

∠DCA = ∠ECB (given)

add ∠DCE on both side

∠DCA + ∠DCE = ∠ECB + ∠DCE

⇒ ∠ACE = ∠BCD

∠DBC = ∠EAC (given)

∴ ΔDBC ≅ ΔEAC (by ASA congruence)

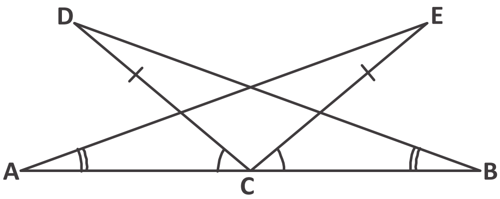


Fig. 14

EXAMPLE-6. Ray $\overrightarrow{AZ}$ bisects angle A, and B is any point on ray $\overrightarrow{AZ}$. BP and BQ are perpendiculars from B to the arms of angle A. Show that-

1. ΔAPB ≅ ΔAQB

2. BP = BQ that means point B is equidistant from the sides forming angle A.

SOLUTION : Given $\overrightarrow{AZ}$ is bisector of ∠QAP

∴ ∠QAB = ∠PAB

∠Q = ∠P = 90°

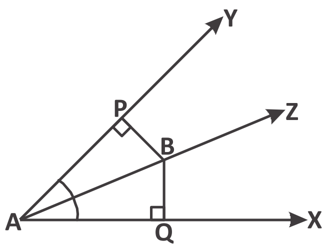


Fig. 15

1. Now in ΔAPB = ΔAQB

AB = AB (common)

∠APB = ∠AQB = 90° (given)

∠PAB = ∠QAB

∴ ΔAPB ≅ ΔAQB (by AAS congruence)

2. ∵ ΔAPB ≅ ΔAQB

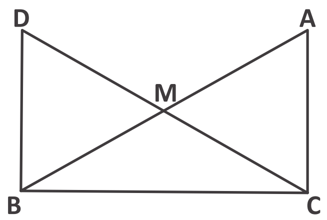
∴ BP = BQ (∵ corresponding parts are equal)

Hence, perpendicular distance of B from AP = perpendicular distance of B from AQ. Therefore, point B is equidistant from ∠A.

Try This

In a right angle triangle ΔABC. ∠C is right angle, M mid point of hypotenuse AB. Join C to M and extend it to D such that DM = CM. Jion Point D to B. Show that

1. ΔAMC ≅ ΔBMD 2. CM = ½ AB
3. ΔDBC ≅ ΔACB 4. ∠DBC = 90°



EXAMPLE-7. Given GE is a bisector of ∠DGF and ∠DEF.

Prove that- ΔGDE ≅ ΔGFE

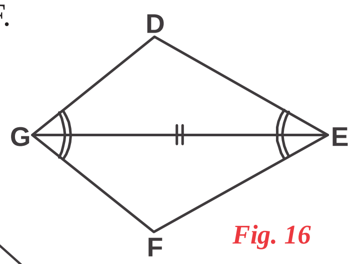


Fig. 16

SOLUTION :

GE is bisector of ∠DGF (Given) → ∠DGE = ∠FGE (By definition of Bisector)

GE bisector of ∠DEF (given) → ∠DEG = ∠FEG (By definition of Bisector)

GE = GE (Common side)

→ ΔGDE ≅ ΔGFE (ASA congruence)

EXAMPLE-8. In a triangle XYZ of ∠Y = ∠Z and XP is a bisector of ∠X, than prove that P is the midpoint of YZ and XP ⊥ YZ

SOLUTION : In ΔXYP and ΔXZP

∠Y = ∠Z (given)

∠YXP= ∠ZXP (XP is angle bisector)

XP = XP (Common side)

ΔXPY ≅ ΔXZP (AAS congruence)

∴ YP = PZ (by CPCT)

Hence P is the mid point of YZ

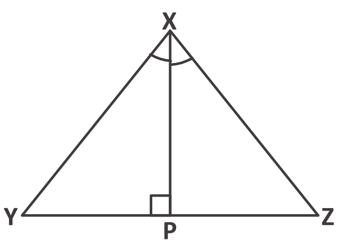


Fig. 17

$\angle YPX = \angle ZPX$ (by CPCT)

$\therefore$ $\angle YPX + \angle ZPX = 180°$ (linear pair)

$\angle YPX + \angle YPX = 180°$ ($\because \angle YPX = \angle ZPX$)

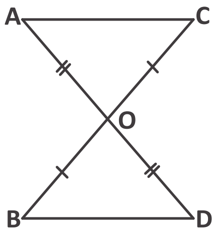
$\angle YPX = 90° = \angle ZPX$

$\therefore$ $XP \perp YZ$

Try This

Prove that the two triangles made by the diagonal of parallelogram are always congruent to each other.

Exercise - 10.1



1. In a figure if OA = OD and OB = OC then which of the given statements is true-

   a) $\Delta AOC \cong \Delta BDO$

   b) $\Delta AOC \cong \Delta DOB$

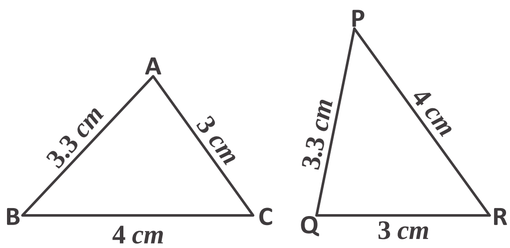
   c) $\Delta CAO \cong \Delta BOD$

2. See the given figure of ΔABC and ΔPQR and tell which statement is true-

   a) $\Delta ABC \cong \Delta PQR$

   b) $\Delta ABC \cong \Delta QPR$

   c) $\Delta ABC \cong \Delta PRQ$



3. From the following which is not the condition for congruence.

   a) SSS b) SAS c) AAA

4. Besides equivalance of two corresponding angle, which least element is necessary to state that the given two triangles are congruent.

   (a) No corresponding side is equal

   (b) At least one corresponding side is equal.

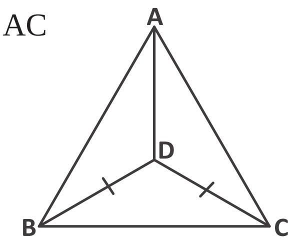
   (c) Third corresponding angle is equal

5. In the figure $\angle B = \angle C$, BD and CD are the bisector of them. Then AB : AC will be-

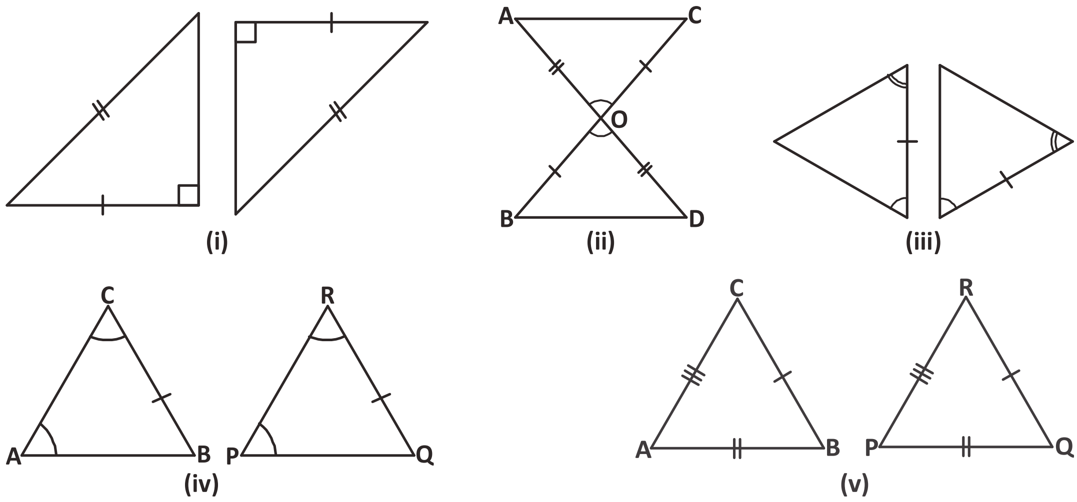
   a) 2 : 1

   b) 3 : 2

   c) 1 : 1

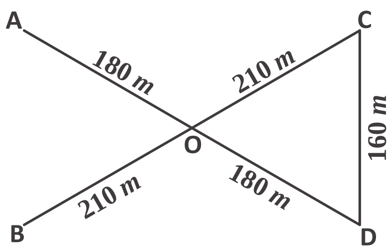


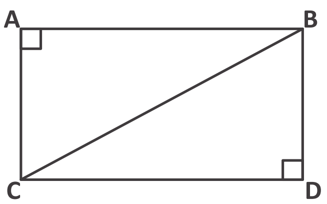
6. See the triangle pairs and state which condition of congruence applies to which pair.



7. If $\Delta ABC \cong \Delta PQR$, $AC = 3x + 2$, $PR = 6x - 13$ and $BC = 5x$ then find the value of QR.

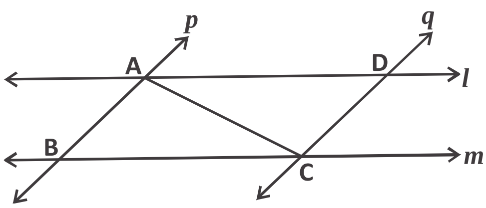
8. With the help of the given points, find the distance between A and B, also give reason to your statement.

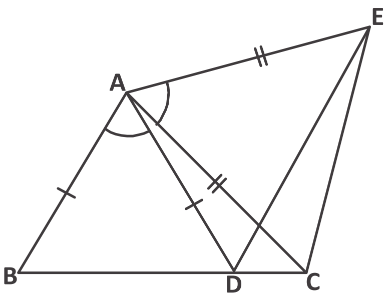




9. For a running competition there is a special arrangment made for two teams. One team runs from A to B and then from B to C and returns to the starting point A. While the other team starts from C and via D to B and then B to the point C again. If $\overline{AB} \parallel \overline{CD}$ and $\angle A = \angle D = 90°$, then the length of the travel done by the teams are equal. Also explain your answer.

10. l and m are parallel lines, which have been intersected by the parallel lines p and q. Show $\Delta ABC \cong \Delta CDA$ (write answer in flow-chart way).





11. In a figure $AC = AE$, $AB = AD$ and $\angle BAD = \angle EAC$. Then show that $BC = DE$.

Property of Isosceles Triangle

So far we have studied about the rules of congruency in triangles. Let us apply these rules to know some more characteristics of triangles.

A triangles which has two equal sides is called an isosceles triangle. Let us understand some characteristics of isosceles triangles.

Try This

Construct a triangle which has two sides of measure 4.5 cm and another side is of 6 cm.

Measure the angles opposite each side, are they equal. You will find, angle opposite to equal sides are equal.

Make some more isosceles triangles with different sides, which shows this important characteristic of isosceles triangle.

Theorem-10.3 : Any triangle which has equal sides will have equal angles opposite to these.

Let us prove this mathematical statement.

We have taken isosceles triangle ABC

Which has sides AB = AC

We have to prove that $\angle B = \angle C$

For this we draw angle bisector of $\angle A$ which meet to side BC at point D.

After angle bisector we can see two triangles

In- ΔBAD and ΔCAD

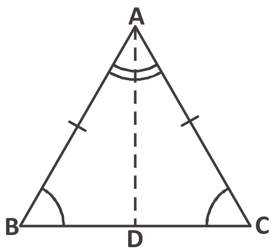


Fig. 18

$AB = AC$	(given)
$\angle BAD = \angle CAD$	(by construction)
$AD = AD$	(common side)
Therefore, $\Delta BAD \cong \Delta CAD$	(side-angle-side congruency law)

$\therefore\ \angle B = \angle C$ (by CPCT)

Therefore this statement is true for every isosceles triangle. Now let us consider its converse and think about it.

Theorem-10.4 : (Converse of theorem-10.3) If two angles of a triange are equal, then the sides opposite to these must be equal.

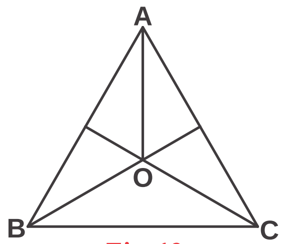


Fig. 19

EXAMPLE-9. In a isosceles triangle ABC, AB = AC. The angle bisectors of ∠B and ∠C and intersect at point O. Show that-

1. OB = OC   2. AO bisects angle A

SOLUTION : ΔABC

Statement	Reason
AB = AC	given
∴ ∠C = ∠B	angles opposite to equal sides are equal
∴ ½ ∠B = ½ ∠C	
∠OCB= ∠OBC	bisector
OB = OC	sides opposite to equal angles are equal

2. In ΔABO and ΔACO

Statement	Reason
AB = AC	given
OB = OC	already proved
∠OBA = ∠OCA	∵ ½∠C = ½∠B
∴ ΔABO ≅ ΔACO	by SAS congruency
⇒ ∠OAB = ∠OAC	C.P.C.T

Therefore AO bisect angle A.

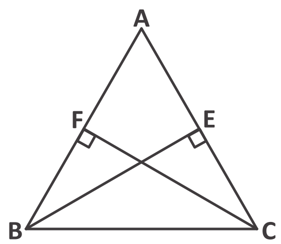


Fig. 20

EXAMPLE-10. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively. Show that these attitudes are equal.

SOLUTION : Given, In Δ ABC, AB and AC are equal.

We have to prove that altitude BE = CF

1. From the given reasons select the appropriate reason for each statement.

Angle opposite to equal sides are equal	**Each angle is of 90°**
Common side	**Given**
By ASA congruence	**Corresponding part of congruent triangles are equal**

Statement	Reason
AB = AC	
∠ACB = ∠ABC	
∠BFC = ∠BEC	
BC = BC	
∠BEC = ∠CFB	
∴ ΔBEC ≅ ΔCFB	
BF = CF	

EXAMPLE-11. In a given figure M is a mid point of AB. $\overline{CA} = \overline{CB}$ then prove that ΔACM ≅ ΔBCM.

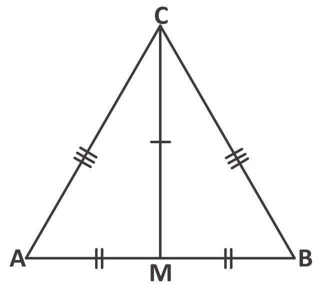
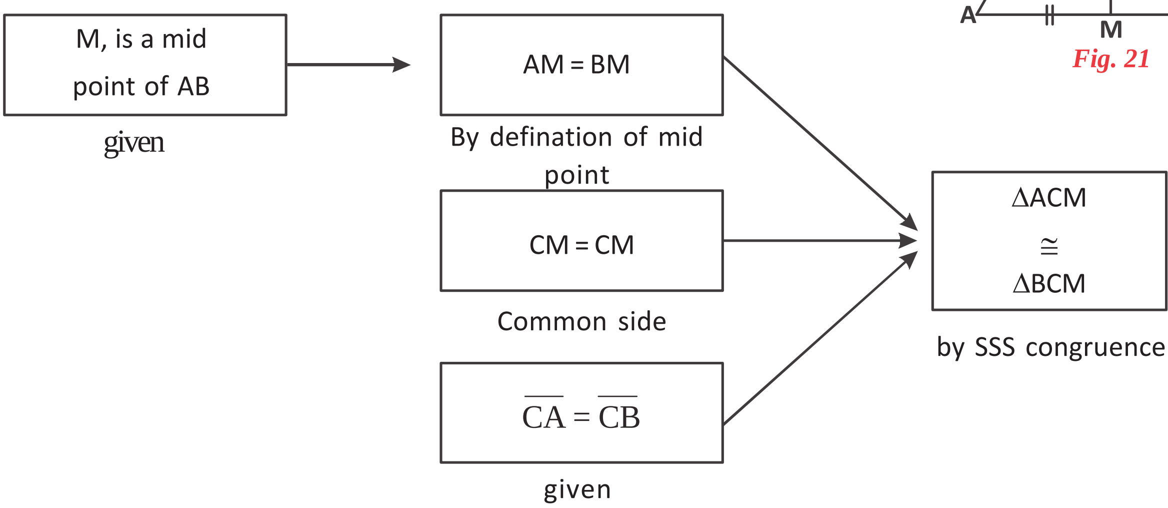


Fig. 21

SOLUTION : Given- M, is a mid point of AB and $\overline{CA} = \overline{CB}$

We have to prove that ΔACM ≅ ΔBCM.



EXAMPLE-12. Given ∠O = ∠P = 90°, $\overline{MN} = \overline{QR}$, $\overline{OM} = \overline{PQ}$

Prove that ΔMOR ≅ ΔQPN

SOLUTION :

Statement	Reason
∠O = ∠P = 90°	given
OM = PQ	$\overline{OM} = \overline{PQ}$ (given)
MN = QR	$\overline{MN} = \overline{QR}$ (given)
MN+NR = QR+NR	adding NR in both the sides
MR = NQ	by figure
ΔMOR ≅ ΔQPN	by RHS congruence theorem

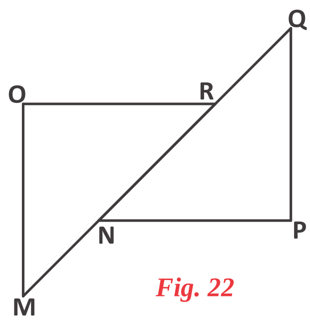
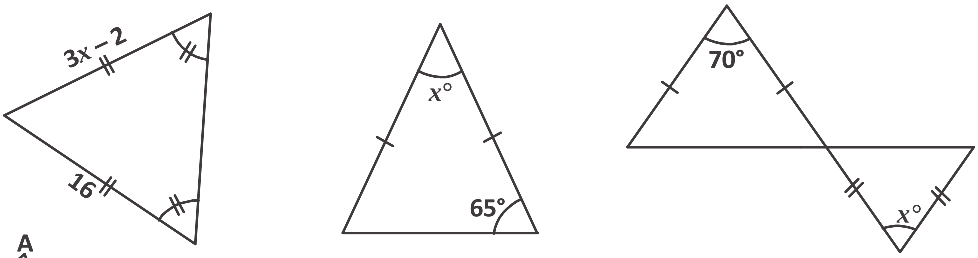
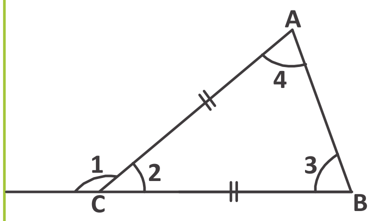


Fig. 22

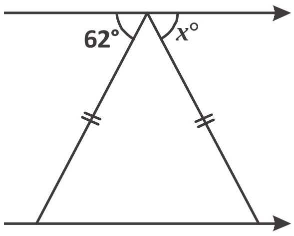
Try This

1. In a given isosceles triangle find the value of x.



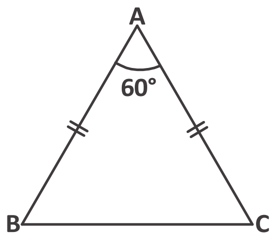


2. $BC \cong AC$ (given) and $\angle 1 = 140°$ then find the measurement of angle $\angle 2$, $\angle 3$ and $\angle 4$.

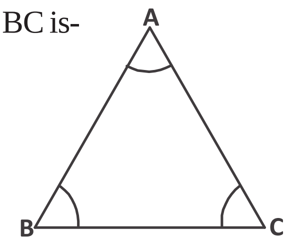


3. In the given figure find the value of x.

Exercise - 10.2



1. In the given figure AB = AC and $\angle A = 60°$ then the measurement of $\angle C$ is -
   (i) 35° (ii) 45°
   (iii) 60° (iv) 180°

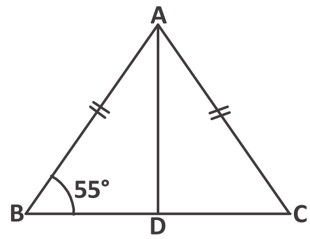


2. In the given figure if $\angle A = \angle B$ then AC : BC is-
   (i) 1:1 (ii) 1:2
   (iii) 2:1
   (iv) None of these

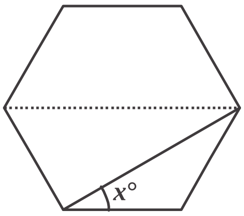
3. In ΔABC if AB = AC and $\angle B = 50°$ then find the measurement of $\angle A$.
   (i) 50° (ii) 180° (iii) 100° (iv) 80°

4. In triangle ΔABC if $\angle C = \angle A$ and AB= 4 cm, AC= 5 cm then BC would be-
   (i) 2 cm. (ii) 3 cm. (iii) 4 cm. (iv) 9 cm.

5. In the given figure $\angle B = 55°$. If D is a mid point of BC and AB = AC. Then the measurement of $\angle BAD$ is-

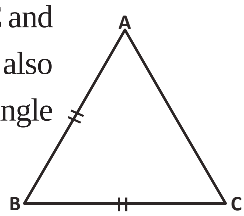


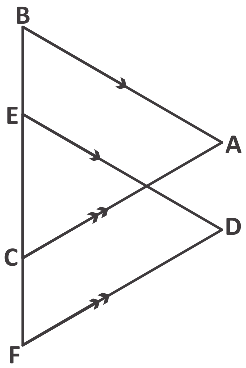
(i)	70°	(ii)	55°
(iii)	35°	(iv)	180°



6. For the given hexagon find the value of x.

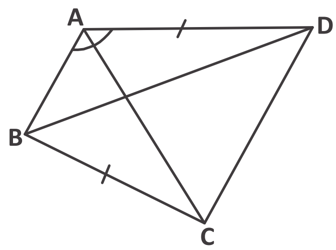
7. In an isosceles triangle ABC, AB = BC with base AC and $\angle A = 2x + 8$, $\angle B = 4x - 20$ then find the value of x and also verify whether this triangle is acute, obtuse or a right angle triangle.



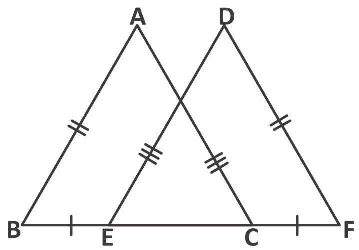


8. In the given figure AB || ED, CA || FD and BC = EF prove that $\Delta ABC \cong \Delta DEF$.

9. ABCD is a quadrilateral in which AD = BC and $\angle DAB = \angle CBA$. Prove that-



1. $\Delta ABD \cong \Delta BAC$
2. BD = AC
3. $\angle ABD = \angle BAC$



10. If AB = DF, AC = DE, BE = FC then prove that $\Delta ABC \cong \Delta DFE$

Application of Congruency

Two figures of same shape and size are congruent. There are some conditions under which the given two triangles can be called congruent. Like side-side-side equality, angle-side-angle equality etc. Here we shall see the relationship of congruency of a figure and its area.

Will the Area of the Congruent Figures be Equal?

Look at the triangles ABC and PQR drawn on the graph paper. Are they congruent? Which congruence conditions do they satisfy?

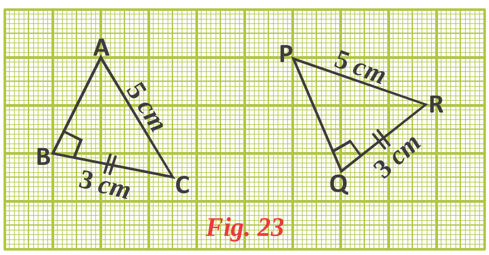


Fig. 23

In Δ ABC and ΔPQR-

$\angle B = \angle Q = 90°$, AC = PR and BC = QR

That means by RHS congruency theorem ΔABC and ΔPQR are congruent.

Now we will find the area of these triangles.

In ΔABC, BC = 3 *cm* and AC = 5 *cm*.

Then, $AB = \sqrt{(AC)^2 - (BC)^2} = \sqrt{(5)^2 - (3)^2} = \sqrt{16} = 4$ (why?)

$\therefore$ AB = 4 cm.

Therefore, the area of $\Delta ABC = \frac{1}{2} \times 3 \times 4 = 6\ cm^2$

Likewise in ΔPQR, PQ = 4 *cm*

Then, the area of ΔPQR is 6 cm^2

$\therefore$ Area of ΔABC = Area of ΔPQR

Make some more triangles congruent to ΔABC and ΔPQR. Are they all are equal in area?

You will find that all the congruent triangles have the same area.

Now, Look at *Fig.*23.

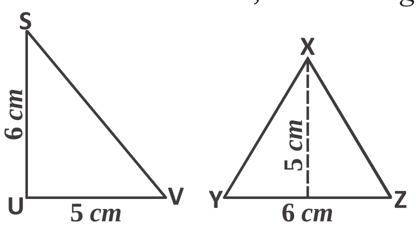


Fig. 24

Area of ΔSUV and ΔXYZ is 15 cm^2 (How?)

Are ΔSUV and ΔXYZ congruent? These two triangles are not congruent because they are not of same shape and size.

Make triangles with area of 15 cm^2 and test their congruency.

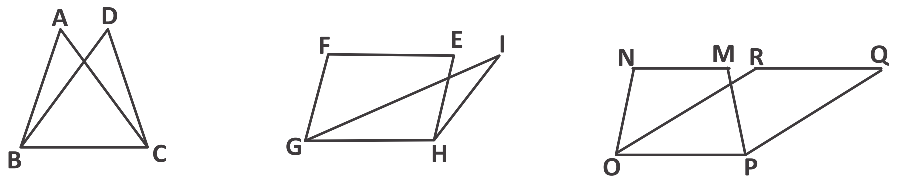
We can say that if two figures are congruent then their area must be equal. But if the area of two figures is the same they may and may not be congruent.

This characteristic is not limited to triangles only but we can also see this in other geometrical shapes like circle, quadrilateral, pentagon etc.

This charateristic of congruent figures can be used in finding the area of different figures in different contexts. Now we will consider some situations where we use this characteristic to find some new information or some new relationships.

Figures on the Same Base & between Same Parallels

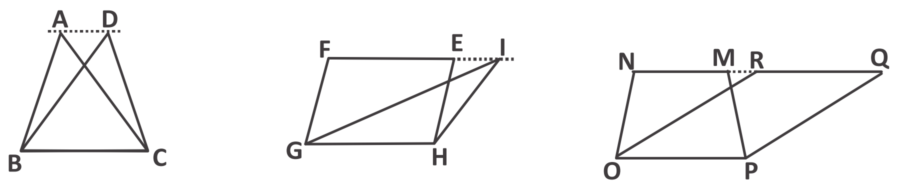
See the figures given below :



(i) (ii) (iii)

Fig. 25

You can see that in figure (i), ΔABC and ΔDBC have the same common base BC. In figure (ii) quadrilateral EFGH and ΔIGH have the common base GH. Likewise in figure (iii) trapezium MNOP and parallelogram QROP have a common base OP. Now if we construct in figure (i), (ii) and (iii) we find some new situations.



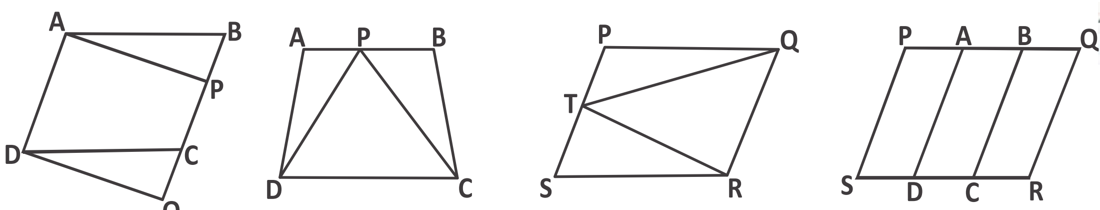
(i) (ii) (iii)

Fig. 26

After the construction we see that in figure (i) AD || BC ΔABC and ΔDBC are on the same base and in between same parallel lines AD and BC. Likewise EFGH and GHI are on the same base GH and in between same parallel lines EI and GH. In figure (iii) Trapezium MNOP and parallelogram OPQR are on the same base OP and in between same parallel lines NQ and OP.

Try This

Find the figures which are situated on the same base and in between the same parallel lines?



(i) (ii) (iii) (iv)

Area of the Figures which are on the Same Base and between Same Parallel Lines.

Now we will see the relationship between the area of those figures which are on the same base and between same parallel lines.

Suppose two parallelograms ABRS and PQRS are on the base SR and in between same parallel line AQ and SR.

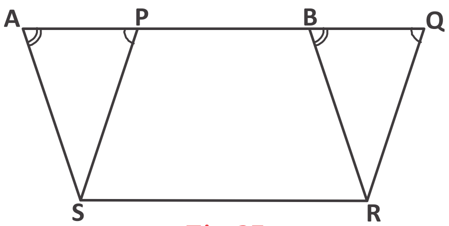


Fig. 27

In ΔAPS and ΔBQR, $AS \| BR$ and AQ is transversal.

$\angle SAP = \angle RBQ$ (corresponding angle)

and $PQ \| SR$ and AQ is transversal

$\angle SPA = \angle RQB$ (corresponding angle)

and $AS = BR$ ($\because$ ABRS is parallelogram)

$\therefore \Delta SPA \cong \Delta RQB$

therefore, area of ΔSPA = area of ΔRQB

So, area of ABRS = area of ΔAPS + area of trapezium PBRS (why?)

Area of parallelogram PQRS = area of ΔBQR + area of trapezium PBRS

Area of parallelogram ABRS = area of parallelogram PQRS

Clearly the area of parallelograms which are on the same base and in between the same parallel lines is equal.

So, parallelograms which are drawn on same base and lie between same parallel lines have an equal area. This is clearly a theorem which can be written as follows:

Theorem-10.5 : Prallelograms which are on the same base and between the same parallels have an equal area.

EXAMPLE-13. PQRS is a parallelogram and PQTV is a rectangle. SU is a perpendicular on PQ.

Prove that (i) area of PQRS = area of PQTV

(ii) area of PQRS = PQ × SU

SOLUTION : (i) Rectangle is also a parallelogram, and we have to prove that area of parallelogram PQRS = area of rectangle PQTV.

Can we prove this through the help of *Fig.*27?

Yes, we can see in the figure that parallelogram PQRS and rectangle PQTV are on the same base PQ and these two figures lie between parallel lines PQ and VR.

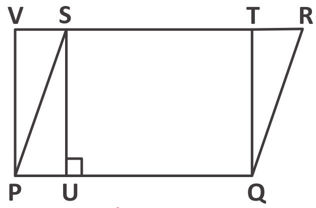


Fig. 28

We already know that the area of the parallelograms which are on the same base and between the same parallels are equal to one another, therefore,

area of parallelogram PQRS = area of rectangle PQTV

(ii) area of PQRS $= $ area of PQTV

$= PQ \times TQ$

$= PQ \times SU$ (SU is a perpendicular on PQ , so SU = TQ why?)

Therefore, area of PQRS $= PQ \times SU$

So, the area of a parallelogram is the product of any parallel side to the height with reference to these parallel lines.

EXAMPLE-14. If triangle ABC and parallelogram ABEF are on the same base AB and in between parallel lines AB and EF, then prove that-

area of $\Delta ABC = \frac{1}{2} \times$ area of parallelogram ABEF

SOLUTION : According to the question construct ΔABC and parallelogram ABEF on the same base AB and in between parallel lines AB and EF.

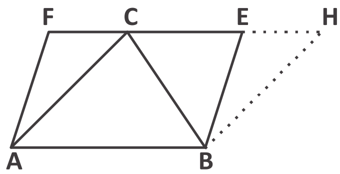


Fig. 29

Area of $\Delta ABC = \frac{1}{2} \times$ area of parallelogram ABEF to prove this

Construct BH parallel to AC which intersect FE extended in H.

Through construction we get parallelogram ABHC. BC is a diagonal which divides it into two triangles ΔABC and ΔBCH.

area of $\therefore$ ΔABC = area of ΔBCH (why?)

You know that diagonal of parallelogram divides it into two congruent triangles.

therefore, area of parallelogram ABHC = area of ΔABC + area of ΔBCH

area of parallelogram ABHC = area of ΔABC + area of ΔABC

area of parallelogram ABHC = 2 area of ΔABC

or $\frac{1}{2} \times$ area of parallelogram ABHC = area of ΔABC

Here area of ABHC = area of ABEF (why?)

(Because ABHC and ABEF are on the same base and lie between same parallel lines)

therefore, area of triangle $\Delta ABC = \frac{1}{2} \times$ area of ABEF

Triangles on the Same Base and lying between Same parallel Lines

Let there be two triangle ABC and DBC on the same base BC and between parallel lines AD and BC.

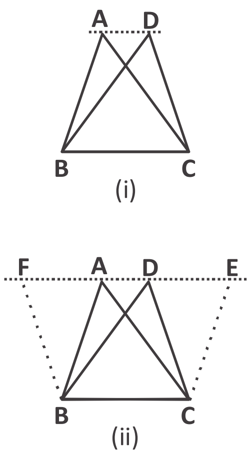


Fig. 30

Now we construct CE||BA and BF||CD so we get parallelograms AECB and FDCB on the same base BC and between parallel lines BC and EF.

Where the area of AECB = area of FDCB (why?)

area of $\Delta ABC = \frac{1}{2} \times$ area of AECB ....(1)

(diagonal of parallelogram divides it into two congruent triangle)

and area of $\Delta DBC = \frac{1}{2} \times$ area of FDCB

($\because$ area of AECB = area of FDCB)

area of $\Delta DBC = \frac{1}{2} \times$ area of AECB ....(2)

therefore, from eq. (1) and (2) we can say that-

area of ΔABC = area of ΔDBC

It is clear now that triangles which are on the same base and between the same parallel lines are equal in area. This is a theorem and can be written as-

Theorem-10.6 : Triangles which are on the same base and in between same parallel lines are equal in area.

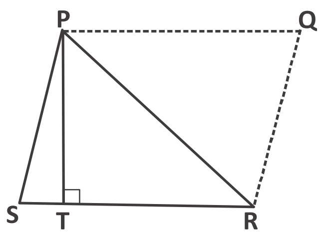


Fig. 31

Now we will find the relation of the area of a triangle to its base and the corresponding height (altitude)

Assume PSR is a triangle, where SR is base and PT its height.

PT $\perp$ SR now construct PQ and RQ such that PQ||RS and RQ||SP so that we get PQRS as parallelogram in which area of ΔPSR = area of ΔPQR (why?)

The diagonal divides a parallelogram into two congruent triangles.

Area of parallelogram PQRS = area of ΔPSR+ area of ΔPQR

Area of parallelogram PQRS = area of ΔPSR + area of ΔPSR

$$\text{Area of } \Delta PSR = \frac{1}{2} \times \text{areaof PQRS}$$

$$= \frac{1}{2} \times SR \times PT$$

Clearly the area of a triangle is half of the products of base and its corresponding height.

Theorem-10.7 : The area of a triangle is half of the product of its base and its corresponding height.

We know that triangles on the same base and between same parallels are equal in area. Now can we say that two triangles with same base and equal areas lie between the same parallel?

EXAMPLE-15. According to the figure XA || YB || ZC. Prove that the area of (ΔXBZ) = area of (ΔAYC).

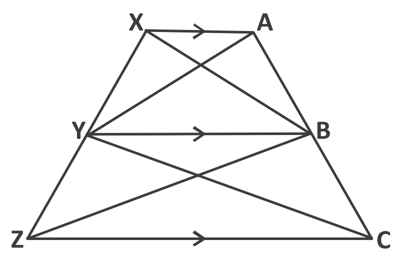


Fig. 32

SOLUTION : ΔXYB and ΔABY, on the same base YB and between same parallel lines XA and YB.

$\therefore$ area (ΔXYB) = area (ΔABY) .....(i)

Similarly,

ΔYBZ and ΔBYC on the same base YB and between same parallel lines YB and ZC.

$\therefore$ area of (ΔYBZ) = area of (ΔBYC) .....(ii)

Here area of (ΔXBZ) = area of (ΔXYB) + area of (ΔYBZ)

and area of (ΔAYC) = area of (ΔAYB) + area of (ΔBYC)

by adding equation (i) and (ii),

area of (ΔXYB) + area of (ΔYBZ) = area of (ΔABY) + area of (ΔBYC)

therefore, area of (ΔXBZ) = area of (ΔAYC)

Inequalities in Triangles

We have learnt the relation between triangles based on the equality of sides and angles, but there are a lot of geometrical shapes which are not same but we still compare them for example, the length of segment AB is more than segment CD or $\angle A$ is bigger than $\angle B$.

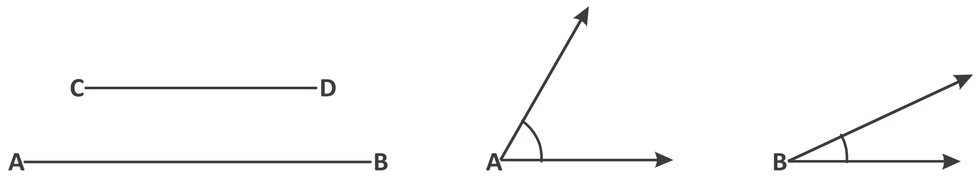


Fig. 33

We will learn about the relation between unequal sides and the angles of a triangle through this activity.

Activity- On a drawing board take two points B and C, put a thread using pin on B and C, now we have a side BC of a triangle.

Put one end of a different thread on C and fix its second end to a pencil. Draw ray $\overrightarrow{CQ}$.

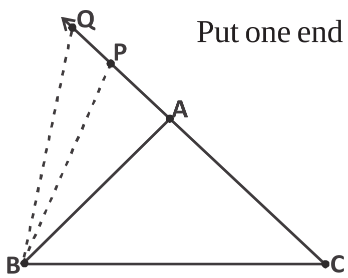


Fig. 34

Mark a point A with a pencil. Join A to B. Now mark another point P on the same ray. Join P to B, and also Q to B. Now compare the length of PC and AC.

Is PC > AC? Yes ...... (comparing the length)

Comparing triangle ΔABC and ΔPBC $\angle PBC > \angle ABC$

Likewise if we mark points on CA and keep joining them to B, we will see that as we increase the length of AC, the measurement of $\angle B$ also increases.

Do this with different triangles. We can see other important and interesting inequilities of triangles, some of them are given below in the form of theorems.

Theorem-10.8 : **If in a triangle two sides are unequal then the angle opposite to the longer side is greater.**

Theorem-10.9 : **Side opposite to greater angle is longer in a triangle.**

Theorem-10.10 : **Sum of lengths of any two sides of a triangle is always greater then the third side.**

We will understand theorem 10.10 through an activity.

Fix nails (A, B and C) on a drawing board such that they make a triangular shape.

Now join these three points by a thread and compare the length of the thread of any one side of a triangle to the other two side threads together, you will always find the length of two threads together is always greater then the third thread.

Measure the three sides AB, BC and CA and compare the sum of any two sides in different group to the third side. You will see that-

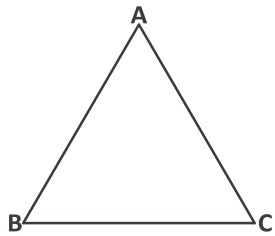


Fig. 35

(i) AB + BC > CA

(ii) BC + CA > AB

(iii) CA + AB > BC

Similarly we can find more results and prove them in the form of theorems.

Let us some examples based on these theorems.

EXAMPLE-16. Prove that hypotenuse is the longest side of any right angles triangle.

SOLUTION : Given in ΔABC

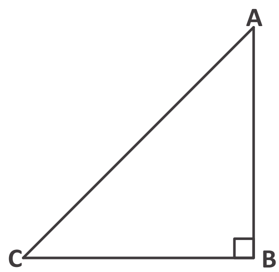


Fig. 36

$\angle B = 90°$

We have to prove that AC > AB

and AC > BC

In ΔABC

$\angle B = 90°$ (given)

So $\angle A + \angle C = 90°$ (sum of internal angles of triangle is 180°)

$\therefore \angle A + \angle C = \angle B$

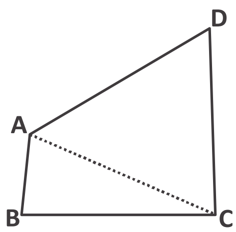
that means $\angle A < \angle B$ and $\angle C < \angle B$

So we can say that $\angle B$ is the biggest angle of ΔABC

We know that the side opposite to the biggest angle is the largest side.

Therefore AC > AB and AC > BC

Think and discuss



AB and CD are respectively the smallest and largest side of a quadrilateral ABCD. Show that $\angle A > \angle C$ and $\angle B > \angle D$.

EXAMPLE-17. In triangle ABC prove that- $\angle ABC > \angle ACB$

Construction- Take point D on AC. such that AB = AD, join B to D.

Corollary : In ΔABD

$AB = AD$ (by construction)

$\angle ABD = \angle ADB$ ....(i) (angle opposite to equal sides)

But $\angle ADB$, is a exterior angle of $\angle BCD$

$\angle ADB > \angle BCD$ ....(ii) (by exterior angle theorem)

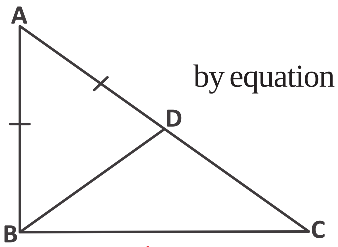


Fig. 37

by equation (i) and (ii)

$\angle ABD > \angle BCD$

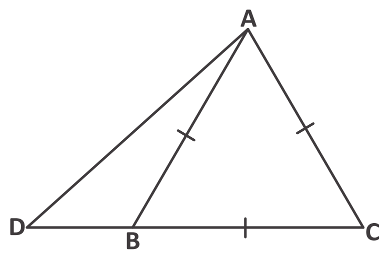
$\angle ABC > \angle ABD$ (by construction)

$\angle ABC > \angle ACB$

Try This

Choose right option.

1. From the following given measurement, which make a triangle-
   (i) $10cm, 5cm, 4cm$ (ii) $8cm, 6cm, 3cm$
   (iii) $5cm, 8cm, 3cm$ (iv) $14cm, 6cm, 7cm$
2. In triangle ABC if $\angle C > \angle B$ then which of the following is right-
   (i) $EF > DF$ (ii) $AB > AC$
   (iii) $AB < AC$ (iv) $BC > CA$
3. From the following given measurement, by which can you make a triangle-
   (i) 35°, 45°, 95° (ii) 40°, 50°, 100°
   (iii) 21°, 39°, 120° (iv) 110°, 80°, 20°
4. If in a triangle ABC, AD is a median, then which of the following statement is false-
   (i) $AB + BC > AD$ (ii) $AC + BC > AD$
   (iii) $AB + BC < AD$ (iv) $AB + BD > DC$
5. In a given figure if $AB = AC = BC$ then which of the following statement is true.
   (i) $AD = AC$
   (ii) $AD < AB$
   (iii) $BC = BD$
   (iv) $AD > AB$



EXAMPLE-18. In a given figure PR > PQ and PS is a angle bisector of ∠QPR, then prove that ∠PSR > ∠PSQ.

SOLUTION : Because PR > PQ

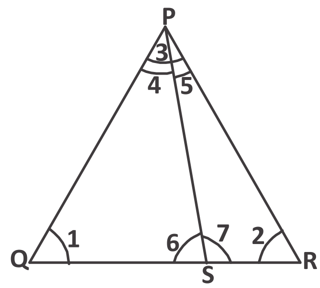


Fig. 38

∴ ∠1 > ∠2

in ΔPQS ∠1 + ∠4 + ∠6 = 180° .....(i)

in ΔPRS ∠2 + ∠5 + ∠7 = 180° .....(ii)

therefore in these two triangles

∠4 = ∠5 .....(iii) (angle bisector of ∠3)

∠1 > ∠2 .....(iv)

by adding (iii) and (iv)

⇒ ∠4 + ∠1 > ∠5 + ∠2 .....(v)

by equation (i) ∠1 + ∠4 = 180° – ∠6

by equation (ii) ∠5 + ∠2 = 180° – ∠7

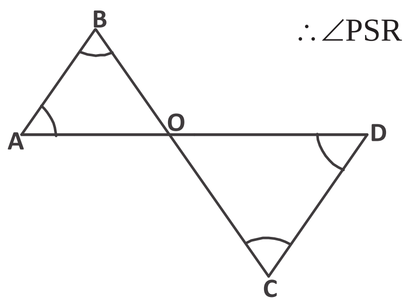
by putting value in equation (v)

180° – ∠6 > 180° – ∠7

– ∠6 > – ∠7 (by changing side)

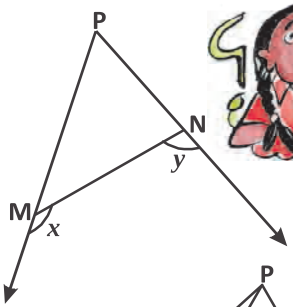
or ∠7 > ∠6

∴ ∠PSR > ∠PSQ

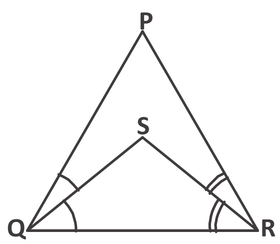


Exercise-10.3

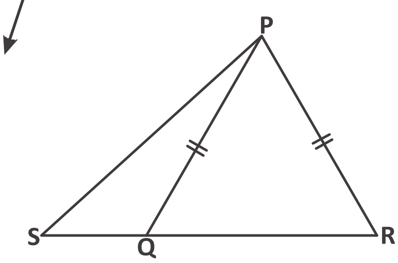
1. In figure ∠B < ∠A and ∠C < ∠D. Show that AD < BC.



2. In a given figure if $x > y$ then prove that MP > NP.



3. In a given figure PQ > PR and QS and RS are angle bisectors of ∠Q and ∠R respectively. Prove that SQ>SR



4. In a given figure PQ = PR then prove that PS > PQ.

Uses of Congruency

Congruency and congruent figures are not only useful in our daily life but this is also seen in the field of engineering when constructing bridges, buildings and towers.

What Have We Learnt

1. Geometrical shapes are congruent if they have the same shape and size.
2. Circle with the same length of radii are congruent.
3. Two squares are congruent if they have sides of equal length.
4. Two triangles are congruent when their corresponding sides and corresponding angles are equal.
5. If two sides and the included angle of one triangle are equal to the corresponding two sides and included angle of the triangle, then the triangles are congruent. (SAS)
6. If any two angles and their included side of triangle are equal to the corresponding two angles and included side of the other triangle, then the triangles are congruent. (ASA)
7. If two angles and a non-included side of one triangle are equal to the corresponding parts of another triangle, then the triangles are congruent.
8. Angles opposite to equal sides are equal.
9. Sides opposite to equal angles are equal.
10. If in two triangles, all three sides in one triangle are equal to the corresponding sides in the other, then the triangles are congruent.
11. If in two right triangles the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and corresponding side of the other triangle, the triangles are congruent.
12. Angle opposite to greater side of a triangle is greater.
13. Side opposite to greater angle of a triangle is greater.
14. In a triangle, the sum of any two sides of the triangle is greater than the third side.
15. Each angle of an equilateral triangle is 60°.
16. If ΔABC and ΔPQR are congruent then we write it like $\Delta ABC \cong \Delta PQR$.
17. We represent the corresponding parts of congruent triangles as CPCT.