

Chapter - 8

BINOMIAL THEOREM

Binomial theorem for any positive integer n

$$(a+b)^n = {}^n C_0 a^n + {}^n C_1 a^{n-1} b + {}^n C_2 a^{n-2} b^2 + {}^n C_3 a^{n-3} b^3 + \dots + {}^n C_n b^n$$

Recall

$$1) {}^n C_r = \frac{n!}{(n-r)! r!}$$

$$2) {}^n C_r = {}^n C_{n-r}$$

$${}^7 C_4 = {}^7 C_3 = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$$

$${}^8 C_6 = {}^8 C_2 = \frac{8 \times 7}{1 \times 2} = 28$$

$$3) {}^n C_n = {}^n C_0 = 1$$

$$4) {}^n C_1 = n$$

OBSERVATIONS/ FORMULAS

- 1) The coefficients ${}^n C_r$ occurring in the binomial theorem are known as binomial coefficients.
- 2) There are $(n+1)$ terms in the expansion of $(a+b)^n$, ie one more than the index.
- 3) The coefficient of the terms equidistant from the beginning and end are equal.
- 4) $(1+x)^n = {}^n C_0 + {}^n C_1 x + {}^n C_2 x^2 + {}^n C_3 x^3 + \dots + {}^n C_n x^n$. (By putting $a = 1$ and $b = x$ in the expansion of $(a + b)^n$).
- 5) $(1-x)^n = {}^n C_0 - {}^n C_1 x + {}^n C_2 x^2 - {}^n C_3 x^3 + \dots + (-1)^n {}^n C_n x^n$ (By putting $a = 1$ and $b = -x$ in the expansion of $(a + b)^n$).
- 6) $2^n = {}^n C_0 + {}^n C_1 + {}^n C_2 + {}^n C_3 + \dots + {}^n C_n$ (By putting $x = 1$ in (4))
- 7) $0 = {}^n C_0 - {}^n C_1 + {}^n C_2 - {}^n C_3 + \dots + (-1)^n {}^n C_n$. (By putting $x = 1$ in (5))

8**) $(r + 1)^{\text{th}}$ term in the binomial expansion for $(a+b)^n$ is called the general term which is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r.$$

i.e to find 4th term = T_4 , substitute $r = 3$.

9*) **Middle term** in the expansion of $(a+b)^n$

i) If **n is even**, middle term = $\left[\frac{n}{2} + 1\right]^{th}$ term.

ii) If **n is odd**, then 2 middle terms are, $\left[\frac{n+1}{2}\right]^{th}$ term and $\left[\frac{n+1}{2} + 1\right]^{th}$ term.

10*) To find the **term independent of x or the constant term**, find the coefficient of x^0 . (ie put power of $x = 0$ and find r)

Problems

eg 4** (4 marks)

Ex 8.1

Q 2,4,7,9 (1 mark)

10*, 11*,12* (4 marks)

13**,14** (4 marks)

13**) Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer

Or

$3^{2n+2} - 8n - 9$ is divisible by 64

Solution: $9^{n+1} - 8n - 9 = (1+8)^{n+1} - 8n - 9$

$$= {}^{n+1}C_0 + {}^{n+1}C_1 8 + {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + \dots + {}^{n+1}C_{n+1} 8^{n+1} - 8n - 9$$

$$= 1 + 8n + 8 + 8^2 [{}^{n+1}C_2 + {}^{n+1}C_3 \cdot 8 + \dots + 8^{n-1}] - 8n - 9$$

$$\text{(since } {}^{n+1}C_0 = {}^{n+1}C_{n+1} = 1, {}^{n+1}C_1 = {}^{n+1},$$

$$8^{n+1}/8^2 = 8^{n+1-2} = 8^{n-1})$$

$$= 8^2 [{}^{n+1}C_2 + {}^{n+1}C_3 \cdot 8 + \dots + 8^{n-1}] \text{ which is divisible by 64}$$

Problems

eg 5*,6**,7* (4 marks)

eg 8**,9** (6 marks)

Ex 8.2

Q 2,3* (1 mark)

Q 7**,8**,9**,11**,12** (4 marks), 10** (6 marks)

eg 10**,11 (HOT),12 (HOT), 13(HOT), eg 15*,17** (4 marks)

Misc ex

Q 1** (6 mark),2,3(HOT), 8* (4 marks)

Ex 8.2

Q 10**(6 marks)

The coefficients of the $(r-1)^{\text{th}}$, r^{th} and $(r+1)^{\text{th}}$ terms in the expansion of $(x+1)^n$ are in the ratio 1 : 3 : 5. Find n and r.

Solution

$$T_{r+1} = {}^n C_r x^{n-r}$$

$$T_r = T_{(r-1)+1} = {}^n C_{r-1} x^{n-r+1}$$

$$T_{r-1} = T_{(r-2)+1} = {}^n C_{r-2} x^{n-r+2}$$

$$\text{Given } {}^n C_{r-2} : {}^n C_{r-1} : {}^n C_r :: 1 : 3 : 5$$

$$\frac{{}^n C_{r-2}}{{}^n C_{r-1}} = \frac{1}{3}$$

$$\frac{\frac{n!}{(n-r+2)!(r-2)!}}{\frac{n!}{(n-r+1)!(r-1)!}} = \frac{1}{3}$$

$$\frac{(n-r+1)!}{(n-r+2)!} \times \frac{(r-1)!}{(r-2)!} = \frac{1}{3}$$

$$\frac{(n-r+1)!}{(n-r+1)!(n-r+2)} \times \frac{(r-2)!(r-1)}{(r-2)!} = \frac{1}{3}$$

$$\frac{r-1}{n-r+2} = \frac{1}{3}$$

$$3r-3 = n-r+2$$

$$n-4r = -5 \quad (1)$$

$$\frac{{}^n C_{r-1}}{{}^n C_r} = \frac{3}{5}$$

$$\frac{n!}{(n-r+1)!r} = \frac{3}{5}$$

simplify as above and get the equation $3n - 8r = -3$ _____(2)

solving (1) and (2) we get

$n = 7$ and $r = 3$.

EXTRA/HOT QUESTIONS

- 1) Using Binomial theorem show that $2^{3n} - 7n - 1$ or $8^n - 7n - 1$ is divisible by 49 where n is a natural number. (4 marks**)
- 2) Find the coefficient of x^3 in the equation of $(1+2x)^6 (1-x)^7$ (HOT)
- 3) Find n if the coefficient of 5^{th} , 6^{th} & 7^{th} terms in the expansion of $(1+x)^n$ are in A.P.
- 4) If the coefficient of x^{r-1} , x^r , x^{r+1} in the expansion of $(1+x)^n$ are in A.P. prove that $n^2 - (4r+1)n + 4r^2 - 2 = 0$. (HOT)
- 5) If 6^{th} , 7^{th} , 8^{th} & 9^{th} terms in the expansion of $(x+y)^n$ are respectively a, b, c & d then show that $\frac{b^2 - ac}{c^2 - bd} = \frac{4a}{3c}$ (HOT)
- 6) Find the term independent of x in the expansion of $\left[3x^2 - \frac{1}{2x^3}\right]^{10}$ (4 marks*)
- 7) Using Binomial theorem show that $3^{3n} - 26n - 1$ is divisible by 676. (4 marks**)
- 8) The 3^{rd} , 4^{th} & 5^{th} terms in the expansion of $(x+a)^n$ are 84, 280 & 560 respectively. Find the values of x , a and n . (6 marks**)
- 9) The coefficient of 3 consecutive terms in the expansion of $(1+x)^n$ are in the ratio 3 : 8 : 14. Find n . (6 mark**)
- 10) Find the constant term in the expansion of $(x-1/x)^{14}$
- 11) Find the middle term(s) in the expansion of
 - i) $\left[\frac{x}{a} - \frac{a}{x}\right]^{10}$
 - ii) $\left[2x - \frac{x^2}{4}\right]^9$
- 12) If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$
Prove that $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n \cdot 2^{n-1}$

Answers

2) -43

3) $n = 7$ or 14

6) $\frac{76545}{8}$

8) $x = 1$, $a = 2$, $n = 7$

9) 10

10) -3432

11) i) -252

ii) $\frac{-63}{32} x^{14}$

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