DPP – 09

CLASS –12th

TOPIC – ADJOINT & INVERSE OF A MATRIX

Q.1 $Let\ A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\ and\ B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}.\ Find (AB)^{-1}$

Q.2 $Given\ A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix},\ compute\ A^{-1}\ and\ show\ that\ 2A^{-1} = 9I - A.$

Q.3 $If\ A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix},\ then\ show\ that\ A - 3I = 2(I + 3A^{-1}).$

Q.4 $Find\ the\ inverse\ of\ the\ matrix\ A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix},\ and\ show\ that\ aA^{-1} = (a^2 + bc + 1)I - aA.$

Q.5 $Given\ A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix},\ B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}.Compute (AB)^{-1}$

Q.6 $Let\ F(\alpha) = \begin{bmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\ and\ G(\beta) = \begin{bmatrix} cos\beta & 0 & sin\beta \\ 0 & 1 & 0 \\ -sin\beta & 0 & cos\beta \end{bmatrix}.Show\ that$

(i) $[F(\alpha)]^{-1} = F(-\alpha)$

(ii) $[G(\beta)]^{-1} = G(-\beta)$

(iii) $[F(\alpha)\,G(\beta)]^{-1} = G(-\beta)\,F(-\alpha)$

Q.7 $If\ A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\ verify\ that\ A^2 - 4A + I = 0,\ where\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ and\ O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.Hence\ find\ A^{-1}.$

Q.8 $Show\ that\ A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}\ satisfies\ the\ equation\ A^2 + 4A - 42I = 0.Hence\ find\ A^{-1}.$

Q.9 $If A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\ show\ that\ A^2 - 5A + 7I = 0.Hence\ find\ A^{-1}.$

SOLUTION

(MATHS) **DETERMINANTS**

DPP – 09

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TOPIC – ADJOINT & INVERSE OF A MATRIX

Sol.1 Given

$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

$|A| = 15 - 14 = 1 \neq 0$

Therefore adj A = $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$A^{-1} = \frac{\text{adj}}{|A|} = \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

$|B| = 54 - 56 = -2 \neq 0$

adj B = $\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$

$B^{-1} = \frac{\text{adj}B}{|B|} = \frac{1}{-2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$

Now, $(AB)^{-1} = B^{-1}A^{-1}$

$= \frac{1}{-2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$= \frac{1}{-2}\begin{bmatrix} 45 + 49 & -18 - 21 \\ -40 - 42 & 16 + 18 \end{bmatrix}$

$= \frac{1}{-2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

$= \frac{1}{-2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

$(AB)^{-1} = \begin{bmatrix} -47 & \frac{39}{2} \\ 41 & -17 \end{bmatrix}$

Sol.2 Given

$A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$|A| = 14 - 12 = 2 \neq 0$

adj A = $\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

To Show: $2A^{-1} = 9I - A$

We have

L.H.S = $2A^{-1} = 2.\frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

R.H.S = $9I - A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

Hence, $2A^{-1} = 9I - A$

Sol.3 Given

$A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$

$|A| = 4 - 10 = -6 \neq 0$

adj $A = \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{-6}\begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$

To Show: $A - 3I = 2(I + 3A^{-1})$

We have

LHS = $A - 3I$

$= \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix}$

R.H.S = $2(I + 3A^{-1}) = 2I + 6A^{-1}$

$= 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 6\frac{1}{6}\begin{bmatrix} -1 & 5 \\ 2 & -4 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ 2 & -4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix}$

Hence, $A - 3I = 2(I + 3A^{-1})$

Sol.4

$$A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$$

Now, $|A| = \frac{a + abc}{a} - bc = \frac{a + abc - abc}{a} = 1 \neq 0$

Hence, A^{-1} exists.

Cofactors of A are

$C_{11} = \frac{1+bc}{a}$

$C_{12} = -c$

$C_{21} = -b$

$C_{22} = a$

Since, adj A = $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

Adj A = $\begin{bmatrix} \frac{1+bc}{a} & -c \\ -b & a \end{bmatrix}^T$

$= \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}.\text{adj } A$

$A^{-1} = \frac{1}{1} \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

To show $a A^{-1} = (a^2 + bc + 1) I - aA$.

LHS = $a A^{-1}$

$= a\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

$= \begin{bmatrix} 1 + bc & -ab \\ -ac & a^2 \end{bmatrix}$

RHS = $(a^2 + bc + 1)\ I - a\ A$

$= \begin{bmatrix} a2 + bc + 1 & 0 \\ 0 & a2 + bc + 1 \end{bmatrix} - \begin{bmatrix} a^2 & ab \\ ac & 1 + bc \end{bmatrix} = \begin{bmatrix} 1 + bc & -ab \\ -ac & a^2 \end{bmatrix}$

Hence, LHS = RHS

Sol.5 Given

A =

$\begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ and B^{-1} =

$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Here, $(AB)^{-1} = B^{-1} A^{-1}$

$|A| = -5 + 4 = -1$

Cofactors of A are

$C_{11} = -1$

$C_{21} = 8$

$C_{31} = -12$

$C_{12} = 0$

$C_{22} = 1$

$C_{32} = -2$

$C_{13} = 1$

$C_{23} = -10$

$C_{33} = 15$

Adj A = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 8 & 1 & -10 \\ -12 & -2 & 15 \end{bmatrix}^T$$

So, adj A = $\begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}.\text{adj } A = \frac{1}{-1}\begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$

$(AB)^{-1} = B^{-1} A^{-1}$

$$= \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-3 & -8-3+30 & 12+6-45 \\ 1+0-3 & -8-4+30 & 12+8-45 \\ 1+0-4 & -8-3+40 & 12+6-60 \end{bmatrix}$$

Hence, $(AB)^{-1} = \begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$

Sol.6 (i) Given

$F(\alpha) =$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|F(\alpha)| = \cos^2\alpha + \sin^2\alpha = 1 \neq 0$

Cofactors of A are

$C_{11} = \cos\alpha$

$C_{21} = \sin\alpha$

$C_{31} = 0$

$C_{12} = -\sin\alpha$

$C_{22} = \cos\alpha$

$C_{32} = 0$

$C_{13} = 0$

$C_{23} = 0$

$C_{33} = 1$

Adj F (α) = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$

So, adj F (α) = $\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|}$ adj F (α) $= \frac{1}{1}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ...... (i)

And, F (– α) = $\begin{bmatrix} \cos(-\alpha) & \sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ...... (ii)

Hence, $[F(\alpha)]^{-1} = F(-\alpha)$

(ii) We have

$|G(\beta)| = \cos^2\beta + \sin^2\beta = 1$

Cofactors of A are

$C_{11} = \cos\beta$

$C_{21} = 0$

$C_{31} = -\sin\beta$

$C_{12} = 0$

$C_{22} = 1$

$C_{32} = 0$

$C_{33} = \cos\beta$

Adj G (β) = $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

= $\begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}^T$

So, adj G (β) = $\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$

Now, $[G(\beta)]^{-1} = \frac{1}{1}\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$ ...... (i)

And, G (– β) = $\begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ \sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix}$

= $\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$ ...... (ii)

Hence, $[G(\beta)]^{-1} = G - \beta)$

(iii) Now we have to show that

$[F(\alpha)\, G(\beta)]^{-1} = G(-\beta)\, F(-\alpha)$

We have already know that

$[G(\beta)]^{-1} = G(-\beta)\ [F(\alpha)]^{-1} = F(-\alpha)$

And LHS = $[F(\alpha)\, G(\beta)]^{-1}$

= $[G(\beta)]^{-1}\,[F(\alpha)]^{-1}$

= $G(-\beta)\, F(-\alpha)$

Hence = RHS

Sol.7 Consider,

$A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+3 & 6+6 \\ 2+2 & 3+4 \end{bmatrix}$

$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$4A = 4\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 7-8+1 & 12-2+0 \\ 4-4+0 & 7-8+1 \end{bmatrix}$

Hence, $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Now, $A^2 - 4A + I = 0$

$A.A - 4A = -I$

Multiply by A^{-1} both sides we get

$A.A(A^{-1}) - 4A A^{-1} = -IA^{-1}$

$AI - 4I = -A^{-1}$

$A^{-1} = 4I - AI = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Sol.8 Given

$A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$

$A^2 = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 64+10 & -40+20 \\ -16+8 & 10+16 \end{bmatrix}$

$= \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix}$

$4A = 4\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -32 & 20 \\ 8 & 16 \end{bmatrix}$

$42I = 42\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}42 & 0\\0 & 42\end{bmatrix}$

Now,

$A^2 + 4A - 42I = \begin{bmatrix}74 & -20\\-8 & 26\end{bmatrix} + \begin{bmatrix}-32 & 20\\8 & 16\end{bmatrix} - \begin{bmatrix}42 & 0\\0 & 42\end{bmatrix}$

$= \begin{bmatrix}74-74 & -20+20\\-8+8 & 42-42\end{bmatrix}$

Hence, $= \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

Now, $A^2 + 4A - 42I = 0$

$= A^{-1} A. A + 4 A^{-1}.A - 42 A^{-1}I = 0$

$= IA + 4I - 42A^{-1} = 0$

$= 42A^{-1} = A + 4I$

$= A^{-1} = \frac{1}{42}[A + 4I]$

$= \frac{1}{42}\left[\begin{bmatrix}-8 & 5\\2 & 4\end{bmatrix} + \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\right]$

$A^{-1} = \frac{1}{42}\left[\begin{bmatrix}-4 & 5\\2 & 8\end{bmatrix}\right]$

Sol.9 Given

$A = \begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$

$A^2 = \begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix} = \begin{bmatrix}9-1 & 3+2\\-3-2 & -1+4\end{bmatrix}$

$= \begin{bmatrix}8 & 5\\-5 & 3\end{bmatrix}$

Now, $A^2 - 5A + 7I = \begin{bmatrix}8 & 5\\-5 & 3\end{bmatrix} - 5\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix} + 7\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$= \begin{bmatrix}8-15+7 & 5-5+0\\-5+5+0 & 3-10+7\end{bmatrix}$

$= \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

So, $A^2 - 5A + 7I = 0$

Multiply by A^{-1} both sides

$A.A A^{-1} - 5A. A^{-1} + 7I. A^{-1} = 0$

$A - 5I + 7 A^{-1} = 0$

$A^{-1} = \frac{1}{7}[5I - A]$

$A^{-1} = \frac{1}{7}.\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{7}.\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$