CHAPTER 26

DEFINITE INTEGRATION AND AREA UNDER THE CURVE

26.1 AREA FUNCTION

If f(x) is continuous on [a, b], then the function, $\int_a^x f(x)dx = A(x); x \in [a,b]$, is called area function, and it represents the algebraic sum of areas bounded by function f(x); ordinates x = a and x = x, such that the area bounded by function above the x-axis is positive and that is bounded by the function below the x-axis is negative.

26.2 FIRST FUNDAMENTAL THEOREM

If f(x) is continuous function on [a, b] and $A(x) = \int_a^x f(x)dx; x \geq a$ is the area function, then $A'(x) = f(x)\ \forall\ x \in [a, b]$.

26.3 SECOND FUNDAMENTAL THEOREM

If f(x) is continuous function on [a, b], then $\int_a^b f(x)dx = F(b) - F(a)$; where $\int f(x)dx = F(x) + C$.

Definite Integral as limit of sum (Integrating by first principle or ab-initio):

(a) By using subinterval of equal length:

$$\int_a^b f(x)dx = \lim_{\substack{h\to 0\\ n\to\infty}} h\left[f(a)+f(a+h)+f(a+2h)+\ldots+f(a+\overline{n-1}h)\right];\text{ where } h = \frac{b-a}{n}; h\to 0 \text{ as } n\to\infty \text{ or}$$

$$\int_a^b f(x)dx = \lim_{\substack{h\to 0\\ n\to\infty}} h\left[f(a+h)+f(a+2h)+\ldots+f(a+nh)\right];\text{ where or}$$

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty\\ h\to 0}} h\sum_{r=1}^{n} f(a+(r-1)h = \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+(\ell-1)\frac{b-a}{n}\right) = \text{left and estimation of } \int_a^b f(x)dx$$

$$\text{and } \int_a^b f(x)dx = \lim_{\substack{n\to\infty\\ h\to 0}} h\sum_{r=1}^{n} f(a+rh) = \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+rh\right)$$

(b) By using subintervals of unequal length, such that their end point are forming a G.P.:
Let [a, b] be divided into n-subintervals with partition $\{a_0, a_1, a_2, a_3, \ldots, a_n\}$, such that $a_0 = a$ and $a_i = aR^i$ and $a_n = b$.

$\Rightarrow aR^n = b \Rightarrow R = \left(\frac{b}{a}\right)^{1/n}$ = common ratio, then

Length of rth subintervals = $\Delta r = a_r - a_{r-1} = aR^r - aR^{r-1} = aR^{r-1}(R-1)$

$= a = \left(\frac{b}{a}\right)^{\frac{r-1}{n}}(R-1) \to 0 \text{ as } n \to \infty$; then $\int_a^b f(x)dx = \lim_{n\to\infty}\left[f(a_1)\Delta_1 + f(a_2)\Delta_2 + \ldots + f(a_n)\Delta_n\right]$

$= \lim_{n\to\infty}\sum_{\ell=1}^{n} f(a_r)\Delta_r; \text{where } \Delta_r = a\left(\frac{b}{a}\right)^{\frac{\ell-1}{n}}(R-\ell).$

For Example: if $f(x) = \frac{1}{x}$; then $\int_2^3 \frac{1}{x}dx$ can be evaluated by above G.P. method.

Remark:

If $f(x) = \frac{1}{x^2}$*; then* $\int_a^b \frac{dx}{x^2}; (a < b)$ *can be evaluated by using the inequality.*

$$\frac{h}{[a+(r-1)h][a+rh]} < \frac{h}{[a+(r-1)h]^2} < \frac{h}{[a+(r-2)h][a+(r-1)h]};$$

Substituting $\ell = 1, 2, 3, \ldots, n$ *and adding we get,* $\frac{1}{a} - \frac{1}{b} < \sum_{\ell=1}^{n}\frac{h}{[a+(\ell-1)h]^2} < \frac{1}{a-h} - \frac{1}{b-h}$

$$\therefore \int_a^b \frac{1}{x^2}dx = \lim_{\substack{h\to 0\\ n\to\infty}}\sum_{\ell=1}^{n}\frac{h}{[a+(\ell-1)h]^2} = \frac{1}{a} - \frac{1}{b}$$

26.4 LINEARITY OF DEFINITE INTEGRAL

Suppose, f and g are integrable on [a, b] and that k is a constant, then kf and f + g are integrable and

(i) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$; (ii) $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx$; and consequently

(iii) $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

26.5 PROPERTIES OF DEFINITE INTEGRAL

Property 1: Mere change of variable does not change the value of integral, i.e., $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Property 2: By interchanging the limits of integration, the value of integral becomes negative, i.e., $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

Property 3: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided that 'c' lies in the domain of continuity of f(x).

26.5.1 Generalization

The property can be generalized into the following form:

$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots + \int_{c_n}^{b} f(x)dx$; where $c_1, c_2, c_3, .., c_n$ lies in the domain of continuity of f(x).

Conclusion: Although we can break limit of integration at any point, but it is necessary to break limit at following points:

1. where f(x) is discontinuous.
2. where f(x) is not defined.
3. where f(x) changes its definition.

Property 4: $\int_a^b f(x)\,dx = 0$ and f(x) is continuous, then f(x) has at least one root $\in$ (a, b).

Remarks:

Converse of above property is not true, i.e., if f(x) has a root in (a, b), then $\int_a^b f(x)\,dx$ *need not be zero.*

Example, if f(x) = $x^2 - 2x$ *has a root x = 2* $\in$ *(1, 3), but* $\int_1^3 (x^2 - 2x)\,dx = \frac{1}{3}(27-1) - (9-1) = \frac{26}{3} - 8 = \frac{2}{3} \neq 0.$

Property 5: Substitution Property: To evaluate $\int_a^b f(x)\,dx$, if we decide to substitute g(x) = t, then $x = g^{-1}(t)$, then the following conditions must be kept in mind:

26.5.2 Condition of Substitution

- g(x) must be continuous and defined $\forall\, x \in [a, b]$.
- g(x) must be monotonic $\forall\, x \in [a, b]$ (to ensure invertibility).

  If the above two conditions are fulfilled, then we may take the following steps:

  Step 1: Change integrand g(x) = t, g'(x) dx = dt.

  Step 2: Change the limits of integration $\int_{g(a)}^{g(b)} f(g^{-1}(t)).\frac{dt}{g'(g^{-1}(t))}$

Property 6: $\int_a^b f(x)dx$ is called improper integral, if

- f(x) is discontinuous at at least one point c $\in$ (a, b), whether the discontinuity is of first kind or infinite discontinuity.
- If $\int_a^b f(x)dx$ is such that f(x) is unbounded as $x \to a^+$, then we take $\int_a^b f(x)dx = \lim_{t\to 0^+} \int_{a+t}^{b} f(x)dx$
- If $\int_a^b f(x)\,dx$ is such that f(x) is unbounded ∞ as $x \to b^-$, then we take $\int_a^b f(x)\,dx = \lim_{t\to 0^+} \int_{a}^{b-t} f(x)\,dx$

26.6 CONVERGENT AND DIVERGENT IMPROPER INTEGRALS

A definite integral having either or both limits infinite (improper integral) is said to be convergent if its value is finite, i.e., if the area bounded by the continuous function f(x), x-axis and between its limits is finite, otherwise it is said to be divergent. Thus,

(i) $\int_a^{\infty} f(x)\,dx$ is said to be divergent if $\lim_{b\to\infty}\int_a^b f(x)\,dx = L$ (finite)

(ii) $\int_{-\infty}^{b} f(x)\,dx$ is said to be divergent if $\lim_{a\to-\infty}\int_a^b f(x)\,dx = L$ (finite)

(iii) $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent if $\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx$, and each of the two integrals on right hand side is convergent. Note that if at least one of the two improper integrals on right side is divergent, then $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be divergent.

Property 7: Reflection Property: $\int_a^b f(x)dx = \int_{-b}^{-a} f(-x)dx$

Property 8: Shifting Property: $\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx$, i.e., area under a part of function and above x-axis remains same when graph of function is shifted horizontally without having any change in the shape of curve.

Property 9: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, i.e., area under a part of function above x-axis and that under its reversed part above x-axis are same.

Property 10: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Remark:

If a = 0 and we take b = a, then $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \Rightarrow \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$*, i.e., property 9.*

26.7 APPLICATIONS

Application I: If f(a + b – x) = f(x). Then to evaluate $I = \int_a^b x f(x)\,dx$ ....(i)

By above property

$\Rightarrow \quad I = \int_a^b (a+b-x) f(a+b-x)\,dx = \int_a^b (a+b-x) f(x)\,dx$ ......(ii)

as f(a + b –x) = f(x) $\Rightarrow I = \frac{(a+b)}{2}\int_a^b f(x)\,dx$

Application II: If f(x) + f(a + b – x) = λ, then evaluate $I=\int_a^b f(x)\,dx$ ……….(i)

By above property $I=\int_a^b f(a+b-x)\,dx$ …..(ii)

Adding (i) and (ii), we have $I=\frac{\lambda(b-a)}{2}$

Property 11: $\int_a^b f(x)dx=\underbrace{\frac{1}{k}\int_{ak}^{bk} f\left(\frac{x}{k}\right)dx}_{\text{(Streching)}}=\underbrace{k\int_{a/k}^{b/k} f(kx)dx}_{\text{(contraction)}},\ k>1$, i.e., when we stretch graph k times, area increases 'k' times. Therefore, we divide by 'k' to keep the value of integral unchanged.

Property 12: Transformation of a definite integral into other with new limits 0 to 1

Let $I=\int_a^b f(x)dx$ be the given definite integral.

Let x = λt + μ; (λ, μ constants), i.e., we can always choose a linear substitution, such that t = 0 at x = a and t = 1 at x = b.

∴ a = λ(0) + μ and b = λ(1) + μ ⇒ μ = a and λ = b – μ = b – a

∴ x = (b – a)t + a ⇒ dx = (b – a)dt

Thus $\int_a^b f(x)dx=(b-a)\int_0^1 f\left[(b-a)x+a\right]dx$

Property 13: $\int_{-a}^{a} f(x)\,dx=\begin{cases}0;\ \text{if } f(-x)=-f(x);\ \text{i.e., f is odd function}\\ 2\int_0^a f(x)dx;\ \text{if } f(-x)=f(x);\ \text{i.e., f is even function}\end{cases}$

Property 14: (a) $\int_0^{2a} f(x)\,dx=\int_0^a f(x)\,dx+\int_0^a f(2a-x)\,dx$

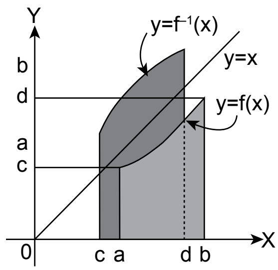
(b) $\int_0^{2a} f(x)\,dx=\int_0^a f(a-x)\,dx+\int_0^a f(a+x)\,dx$

Property 15: $\int_0^{2a} f(x)dx=\begin{cases}0; & \text{if } f(2a-x)=-f(x)\\ 2\int_0^a f(x)dx; & \text{if } f(2a-x)=f(x)\end{cases}$

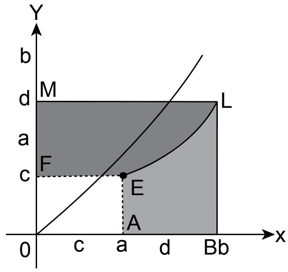
or $\int_0^{2a} f(x)dx=\begin{cases}0; & \text{if } f(a+x)=-f(a-x)\\ 2\int_0^a f(x)dx; & \text{if } f(a+x)=f(a-x)\end{cases}$

Equivalently $\int_0^{2a} f(x)dx=\begin{cases}0;\ \text{if graph of } f(x) \text{ is symmetric about point } (a,0).\\ 2\int_0^a f(x)dx;\ \text{if graph of } f(x) \text{ is symmetric about line } x=a.\end{cases}$

Property 16: Integral of an Inverse Function: If f is an invertible function and f' is continuous, then definite integral of f^{-1} can be expressed in terms of definite integral of function f(x), i.e., $\int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a) - \int_a^b f(x)\,dx.$



$= \int_c^d f^{-1}(x)dx;$



$= \int_a^b f(x)dx$

26.7.1 Evaluation of Limit Under Integral Sign

The limit of a function expressed in the form of definite integral can also be evaluated by first finding the limit of the integrand function w.r.t. a quantity, of which the limit of integration are independent and subsequently integrating the result thus obtained, e.g., $\lim_{x\to k}\int_{\alpha}^{\beta} f(x,t)dt = \int_{\alpha}^{\beta}\left(\lim_{x\to k} f(x,t)\right)dt$

26.7.2 Leibnitz's Rule for the Differentiation Under the Integral Sign

(a) If f is continuous on [a, b] and $\phi(x)$ and $\psi(x)$ are differentiable functions of x whose values lie in [a, b], then $\frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)dt = f\{\psi(x)\}.\frac{d\psi}{dx} - f\{\phi(x)\}.\frac{d\phi}{dx}$.

(b) If the function $\phi(x)$ and $\psi(x)$ are defined on [a, b] and differentiable at each point $x \in (a, b)$ and f(x, t) is continuous, then $\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(x,t)dt\right) = \int_{\phi(x)}^{\psi(x)} \frac{\partial}{\partial x} f(x, t)\,dt + \frac{d\psi(x)}{dx}.\ f(x, \psi(x)) - \frac{d\phi(x)}{dx}.\ f(x, \phi(x))$.

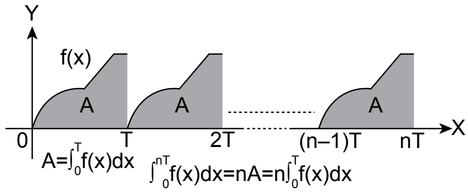
(c) If $f(x, \alpha)$ be a continuous function of x for $x \in [a, b]$ and $\alpha \in [c, d]$, let $I(\alpha) = \int_a^b f(x,\alpha)dx$ is a function of α, then $I'(\alpha) = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)dx$

Property 17: If f(x) is an odd function of x, then $\int_a^x f(t)\,dt$ is an even function of x.

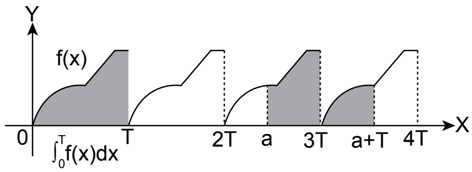
Property 18: If f (x) is an even function of x, then $\int_a^x f(t)\,dt$ is an odd function of x iff $\int_0^a f(t)dt = 0$.

Property 19: If f(x) is a periodic function with period T, i.e., f(x) = f(x + T), then the following properties hold good.

- $\int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx$; where n is a positive integer.



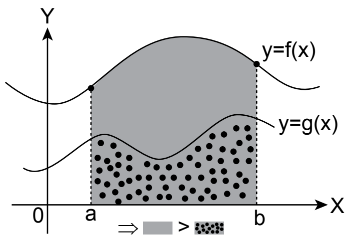
Property 20: If f (x) is a periodic function with period T, then $\int_a^{a+T} f(x)\,dx$ is independent of a. Hence, prove that $\int_a^{a+T} f(x)\,dx = \int_0^{T} f(x)\,dx$.



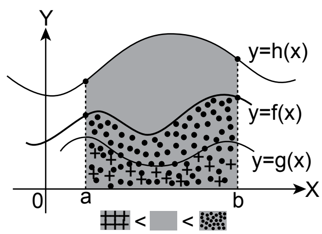
Corollary: $\int_a^{a+nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$; Where $n \in \mathbb{Z}^+$.

Property 21: If f(x) is a function such that $f(x) \geq 0$. $\forall\, x \in [a, b]$, then $\int_a^b f(x)dx \geq 0$.

Property 22: If $f(x) > g(x)\ \forall\, x \in [a, b]$, then $\int_a^b f(x)dx > \int_a^b g(x)dx$



Property 23: If f(x), g(x), h(x) are continuous functions such that $g(x) \leq f(x) \leq h(x)$ in [a, b], then $\int_a^b g(x)dx \leq \int_a^b f(x)dx \leq \int_a^b h(x)dx$



Application: To prove that $k_1 < \int_a^b f(x)\,dx < k_2$ where $k_1, k_2 \in \mathbb{R}$. It is suggested to find two functions g(x) and h(x). Such that $\int_a^b g(x)\,dx \geq k_1$ and $\int_a^b h(x)\,dx \leq k_2$, then prove that $g(x) \leq f(x) \leq h(x)$

$$\Rightarrow \int_a^b g(x)dx < \int_a^b f(x)dx < \int_a^b h(x)dx \Rightarrow k_1 \leq \int_a^b g(x)dx < \int_a^b f(x)dx < \int_a^b h(x)dx \leq k_2 \Rightarrow k_1 < \int_a^b f(x)dx < k_2$$

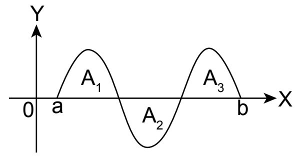
Property 24: $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$; where f(x) is continuous and bounded on (a, b).

Discussion: This is derived from generalized form of polygonal inequality and can be understood as below.

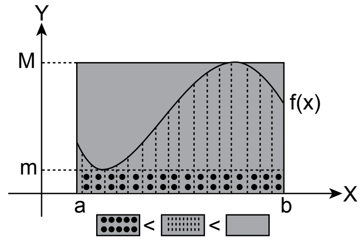
$$\left|\int_a^b f(x)dx\right| = |A_1 - A_2 + A_3| \le \int_a^b |f(x)|\,dx = A_1 + A_2 + A_3;$$

where A_1, A_2, A_3 are magnitudes of areas as shown above.

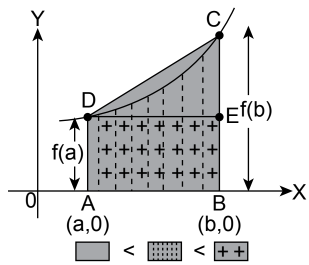
Here $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$ $\quad\therefore\quad$ $\left|\sum_{k=1}^{n} f(x_k^*)\Delta x\right| \le \sum_{k=1}^{n} |f(x_k^*)\Delta x|$



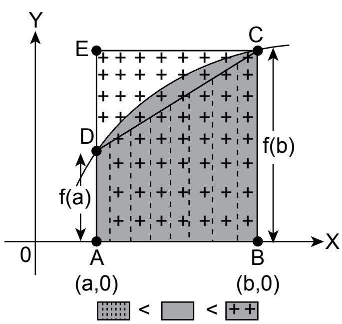
Property 25: (Max-Min inequality) If m and M are respectively the global min/max values of f(x) in [a, b], then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$



Property 26: If the function f(x) increases and has a concave graph in the interval [a, b], that is f'(x) and f''(x) both positive (+ve), then $(b-a)f(a) < \int_a^b f(x)\,dx < (b-a)\left(\frac{f(a)+f(b)}{2}\right)$.

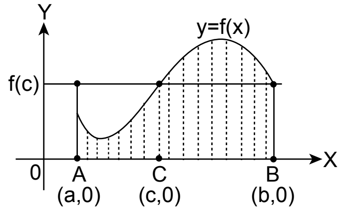


Property 27: If the function f(x) increases and has a convex upwards (or concave downwards) graph in the interval [a, b], that is f'(x) is positive (+ve) and f''(x) is negative (–ve), then $(b-a)\left(\frac{f(a)+f(b)}{2}\right) < \int_a^b f(x)\,dx < (b-a)f(b)$



Property 28: Schwarz–Bunyakovsky Inequality: If f(x) and g(x) are two functions, such that $f^2(x)$ and $g^2(x)$ are integrable, then $\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)}$.

Property 29: If f(x) is continuous in [a, b], then there exists a point $c \in (a, b)$ such that $\int_a^b f(x)dx = f(c)(b-a)$ and the number $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$ is called mean value of the function f(x) on the interval [a, b].



Evaluating Integrals Dependent on a Parameter:

Property 30: Suppose f (x, α) and f′(x, α) are continuous functions: when $c \le \alpha \le d$ and $a \le x \le b$, then $I'(\alpha) = \int_a^b f'(x,\alpha)dx$; (where $I'(\alpha)$) is the derivative of $I(\alpha)$ w.r.t., α and f′(x, α) is the derivative of f(x, α) w.r.t., α, keeping x constant; $I(\alpha) = \int_a^b f(x,\alpha)dx$, then $\frac{dI}{d\alpha} = I'(\alpha) = \int_a^b \frac{\partial}{\partial\alpha}\left(f(x,\alpha)\right)dx$.

26.7.3 Evaluate of Limit of Infinite Sum Using Integration

To evaluate $\lim_{n\to\infty} g(n)$ (when g(n) can be expressed as infinite sum) using definite integral, follow the steps given here:

Step I: Express the function g(n) in terms of infinite summation, using sigma notation.

i.e., $g(n) = \sum_{r=1}^{n} f\left(a + r\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right)$

Step II: Replace $a + r\left(\frac{b-a}{n}\right) \to x$ and $\left(\frac{b-a}{n}\right) \to dx$

Step III: $\lim_{n\to\infty}\sum_{r=1}^{n}$ converts to $\int_{\alpha}^{\beta}$; where $\beta = \lim_{n\to\infty}\left(a + r_{max}\frac{b-a}{n}\right) = b$; $\alpha = \lim_{n\to\infty}\left(a + r_{min}\frac{b-a}{n}\right) = a$

Step IV: $\lim_{n\to\infty}\sum_{r=1}^{n}\left(a + r\left(\frac{b-a}{n}\right)\right).\left(\frac{b-a}{n}\right) = \int_a^b f(x)dx$

- When domain of f(x) is divided into unit length sub-intervals each of which further divided into n subintervals Interval [α, β] contains $p(n)^{th}$ to $q(n)^{th}$ stripes. Then algorithm becomes:

Step I: Express the function g(n) in terms of infinite summation, using sigma notation.

Step II: Replace $\frac{r}{n} \to x$ and $\left(\frac{1}{n}\right) \to dx$

Step III: $\lim_{n\to\infty}\sum_{r=p(n)}^{r=q(n)}$ converts to $\int_{\alpha}^{\beta}$; where $\beta = \lim_{n\to\infty}\left(\frac{r_{max}}{n}\right) = \lim_{n\to\infty}\left(\frac{q(n)}{n}\right)$; $\alpha = \lim_{n\to\infty}\left(\frac{r_{min}}{n}\right) = \lim_{n\to\infty}\left(\frac{p(n)}{n}\right)$

Step IV: $\lim_{n\to\infty}\sum_{r=p(n)}^{q(n)} f\left(\frac{r}{n}\right).\left(\frac{1}{n}\right) = \int_{\alpha}^{\beta} f(x)dx$

26.8 WALLI'S FORMULAE

1. For $n \in \mathbb{N}, \int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \left\{\dfrac{(n-1)\times(n-3)\times(n-5)}{n\times(n-2)\times(n-4)}...\right\}\dfrac{\pi}{2}; & \text{If } n \text{ is even} \\ \left\{\dfrac{(n-1)\times(n-3)\times(n-5)}{n\times(n-2)\times(n-4)}...\right\}; & \text{If } n \text{ is odd} \end{cases}$

2. For $m, n \in \mathbb{N}, \int_0^{\pi/2} \sin^m x \cos^n x\, dx = \dfrac{\{(m-1)\times(m-3)...\}\{(n-1)\times(n-3)....\}}{(m+n)(m+n-2)........}.p;$

Where p = π/2 if both m and n are even otherwise p = 1.

26.8.1 Walli's Product

We can express π/2 in the form of infinite product given by $\dfrac{\pi}{2} = \lim_{n\to\infty}\left[\left(\dfrac{2.4.6....2n}{1.3.5.....(2n-1)}\right)^2 . \dfrac{1}{(2n+1)}\right]$.

26.8.2 Some Important Expansion

1. $\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+.....\infty = \dfrac{\pi^2}{6}$

2. $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+.....\infty = \dfrac{\pi^2}{8}$

3. $\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+.....\infty = \dfrac{\pi^2}{12}$

4. $\dfrac{1}{2^2}+\dfrac{1}{4^2}+\dfrac{1}{6^2}+\dfrac{1}{8^2}+....\infty = \dfrac{\pi^2}{24}$

5. $\ln 2 = 1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+....\infty$

26.8.2.1 Root mean square value (R.M.S.V)

R.M.S.V. of a function y = f(x) in the range (a, b) is given by $\sqrt{\dfrac{\int_a^b [f(x)]^2 dx}{(b-a)}}$

26.9 BETA FUNCTION

It is denoted by B (m, n) and is given by $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$; where m, n > 0. It can be proved that for m, n ∈ (0, 1) the above improper integral is convergent, however the proof is beyond the scope of this book. Clearly, B (m, n) is proper for m, n ≥ 1.

If (2m–1) and (2n–1) are positive integers, then $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Let us substitute x = sin²θ

$\Rightarrow B(m, n) = \int_0^{\pi/2} \sin^{2m-2}\theta(\cos\theta)^{2n-2} . 2\sin\theta \cos\theta d\theta$

$\Rightarrow B(m, n) = \int_0^{\pi/2} \sin^{2m-1}\theta.\cos^{2n-1}\theta d\theta$

∴ By Walli's formula

$$B(m,n) = \frac{2\{(2m-2)(2m-4)...\}\{(2n-2)(2n-4...\}}{(2m+2n-2)(2m+2n-4)......}.p$$

Where p = π/2, if both (2m–1) and (2n–1) are even integers, otherwise p = 1.

26.10 GAMMA FUNCTION

The improper integral $\int_0^{\infty} e^{-x} x^{n-1} dx$; where n is a positive rational number is called gamma function and is denoted by $\overline{|n}$.

Thus, $\overline{|n} = (n-1)\overline{|(n-1)} = (n-1)(n-2)\overline{|(n-2)}$ and, so on (By previous illustration).

26.10.1 Properties of Gamma Function

(i) $\overline{|(n)} = (n-1)!$ if n is a positive integer

(ii) $\overline{|1} = (1-1)! = 0! = 1$

(iii) $\overline{|(n)} = (n-1)\overline{|n-1}$, e.g., $\overline{|5} = 4\overline{|4} = 4(3)\overline{|3} = 4(3)(2)\overline{|2}$

(iv) $\overline{|0} = \infty$

(v) $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$

26.10.2 Relation Between Beta and Gamma Functions

For $m, n > 0, B(m, n) = \dfrac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$

Remark

If m, n are positive integers, then $\beta(m, n) = \dfrac{\underline{|m-1}\,\underline{|n-1}}{\underline{|m+n-1}}$ *as* $\overline{|n} = \underline{|n-1}$ *for* $n \in \mathbb{N}$.

26.11 WEIGHTED MEAN VALUE THEOREM

If f(x) and g(x) are two continuous functions on [a, b], such that g(x) does not change its sign in [a, b], then there exists $c \in [a, b]$ such that $\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$.

26.11.1 Generalized Mean Value Theorem

If g(x) is continuous [a, b] and f(x) has derivative function which is continuous and never changes its sign in [a, b]. Then, there exists some $c \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)\,dx$.

26.12 DETERMINATION OF FUNCTION BY USING INTEGRATION

Let f(x) be a given continuous and differentiable function. Sometimes we are given a functional equation connecting the functional values at different points or function with some definite integral having integrand as f(x) or f'(x) or any other algebraic or trigonometric or exponential function. Then, by differentiating and integrating we can find the function f(x).

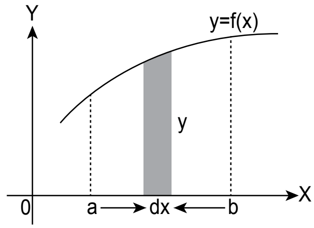
AREA UNDER THE CURVE

26.13 AREA BOUNDED BY SINGLE CURVE WITH X-AXIS

(a) If f(x) is a continuous function in [a, b] then area bounded by f(x) with x-axis in between the ordinates x = a and x = b is given by $A=\int_a^b |f(x)|\,dx$.

(b) If f(x) is discontinuous function in [a, b] say at x = c ∈ (a, b) then,

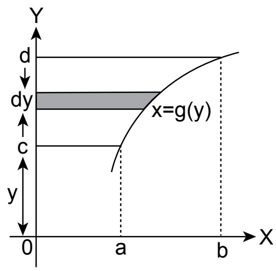
$$A=\int_a^c |f(x)|dx+\int_c^b |f(x)|dx$$



26.13.1 Area Bounded by Single Curve with y-axis

(a) If f(x) is a continuous function in [a, b]; such that f(a) = c and f(b) = d, then the area bounded by the function f(x), with y-axis and abscissa y = c and y = d is given by $A=\int_c^d |(x)|\,dy=\int_c^d f^{-1}|(y)|\,dy$.

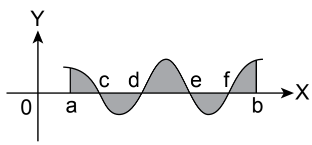
(b) If f(x) is discontinuous function in [a, b] at x = c, then $f^{-1}(y)$ is also discontinuous at y = f(c) = e(say), then $A=\int_c^e |f^{-1}(y)|dy+\int_e^d |f^{-1}(y)|dy$.



26.13.2 Sign Conversion for Finding the Area Using Integration

For the intervals where f(x) ≥ 0; take integrand f(x) and for the intervals where f(x) ≤ 0; take integrand –f(x), e.g., as given in the figure given below:

$$A=\int_a^b |f(x)|dx=\int_a^c f(x)dx+\int_c^d -f(x)dx+\int_d^e f(x)dx+\int_e^f -f(x)dx+\int_f^b f(x)dx$$

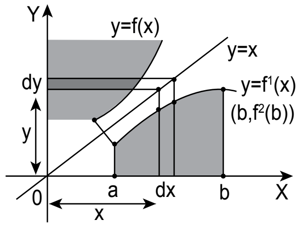


26.13.3 Area Bounded Between Two Curves

(a) If f(x) and g(x) are two continuous function functions on [a, b]; then the area bounded between two curves and the ordinates x= a and x = b is given by $A=\int_a^b |f(x)-g(x)|dx$.

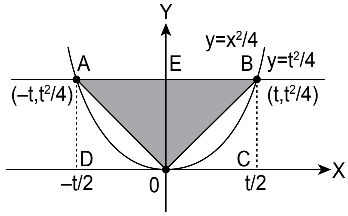
(b) Area bounded between the curves f(x), g(x) and the abscissa y = c and y = d are given by $A=\int_c^d \left||f^{-1}(y)-g^{-1}(y)|\,dy\right|$.

26.13.4 Area Enclosed by Inverse Function



Area enclosed by $y = f^{-1}(x)$ and x-axis between ordinate x = a and x = b is same as area enclosed y = f(x) and y-axis from y = a to y = b. Clearly, from above figure, the area bounded by y = f(x) with y-axis from y = a to y = b and $f^{-1}(x)$ with x-axis from x = a to x = b are same as y = f(x), and $f^{-1}(x)$ are reflection of each other on line y = x.

26.13.5 Variable Area its Optimization and Determination of Parameters

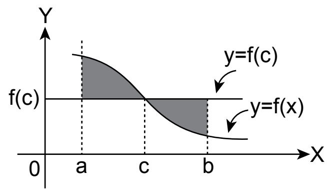


If the region bounded by curve, is continuously changing due to some variable ordinate or abscissae or any other parameter present in the boundary curve, then we obtain a variable area function that can be optimized with respect to involved parameters, e.g.,

Area $OAB = A_1 = \frac{t^3}{4}$, and Area of parabolic region $A_2 = \frac{t^3}{3}$.

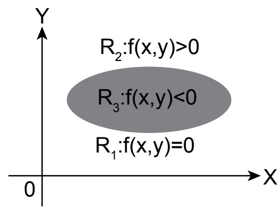
Thus, A_1 and A_2 can be optimized for parameter t.

26.13.5.1 Least value of variable area



Let f(x) be a monotonic function with $f'(x) \neq 0$ in (a, b), then the area bounded by function y = f(x), y = f(c); (a < c < b). And ordinates x = a, x = b is minimum for $c = \frac{a+b}{2}$.

26.13.5.2 Method of tracing the region represented by inequality



Each curve f(x, y) = 0 divides the entire x – y plane into three set of points as given in figure.

$\mathbf{R_1}$ **= {(x, y):** f(x, y) = 0, x, y $\in \mathbb{R}$}, i.e., the points lying on the curve f(x, y) = 0.

$\mathbf{R_2}$ = {(x, y): f(x, y) > 0, x, y $\in \mathbb{R}$}, i.e., the points lying on one side of the curve f(x, y) = 0 (outside the curve if closed).

$\mathbf{R_3}$ = {(x, y): f(x, y) < 0, x, y $\in \mathbb{R}$}, i.e., the points lying on other side of f(x, y) = 0 (inside the curve if closed).

Steps to Identify the Region Represented by a Given Inequality (say) f(x, y) > **0**.

Step I: Consider the equality and draw the curve using the symmetry and other concepts of curve sketching and transformation of graphs.

Step II: Consider any points (α, β) not lying on the curve preferably (0, 0) {or point on coordinate axis} and determine the sign of $f(\alpha, \beta)$.

Step III: If $f(\alpha, \beta) > 0$, then f(x, y) > 0 represents the region containing (α, β). If $f(\alpha, \beta) < 0$, then the region which does not contains point (α, β) will be represented by inequality f(x, y) > 0.

Note that the region represented by inequality f(x, y) > 0 or f(x, y) < 0 does not contain the points on the curve whereas the region represented by inequality f(x, y) ≥ 0 and f(x, y) ≤ 0 contains the points on the curve.

26.13.6 Determination of Curve When Area Function is Given

If the area bounded by some function and x-axis between x = a and x = b is given g(a, b) ∀ a > b, where a is a given real number and b is a real parameter, then the function can be obtained as described below. Let the unknown function be y = f(x).

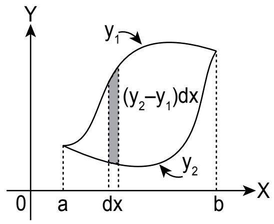
∵ $\int_a^b |f(x)|dx$ represents area enclosed between f(x) and x-axis between the semi-variable boundaries x = a and x = b as b is a real parameter, and it is given as g(a, b). Of course, area changes by variation in 'b', but always the value of area shall be represented by a function g(a, b).

Thus, $\int_a^b |f(x)|dx = g(a, b)$. Now differentiating both sides w.r.t. b, we get $|f(b)| = \frac{d}{db}(g(a,b)) = g'(a,b)$

⇒ f(b) = ± g'(a, b); consequently determining two curves f(x) = g'(a, x) or f(x) = −g'(a, x).

26.14 AREA ENCLOSED IN CURVED LOOP

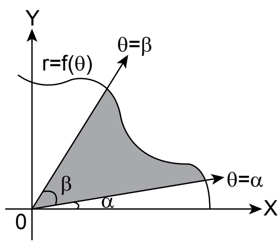
Any curve forming loop is multi-valued function, so first of all solve the equation of curve for y to find its functional branches and obtain the domain of function, say [α, β], e.g., $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ (say) solving for y, we get $y_1 = \frac{f(x)+\sqrt{g(x)}}{2}$; $y_2 = \frac{f(x)-\sqrt{g(x)}}{2}$.



Clearly, there two functions are forming the loop. Area of loop = $\int_\alpha^\beta |y_2 - y_1|dx = \int_\alpha^\beta \sqrt{g(x)}\,dx$.

Area enclosed by curve between two radius vectors when its equation is given in polar form:

If r = f(θ) is the equation of curve in polar form, where f(θ) is a continuous function of θ, then the area enclosed by curve r = f(θ), and the radius vectors r = f(α) and r = f(β); (α < β) is given by $A = \frac{1}{2}\int_\alpha^\alpha r^2\, d\theta = \frac{1}{2}\int_\alpha^\alpha [f(\theta)]^2\, d\theta$.



Note:

*In order to transform the Cartesian equation of a curve to polar form, we replace x by r cos*θ *and y by rsin*θ

$$\therefore \quad \underbrace{f(x,y)}_{\text{Catesian equation}} \xrightarrow[y=r\sin\theta]{x=r\cos\theta} \frac{f(r\cos\theta, r\sin\theta)=0}{\text{polar equation}}$$

26.14.1 Graphical Solution of the Intersection of Polar Curves

The following steps are taken to find the points of intersection of polar curves:

Step 1: Find all simultaneous solutions of the given system of equations.

Step 2: Determine whether the pole lies on the two graphs.

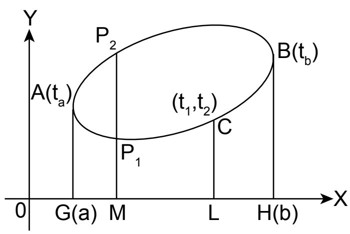
Step 3: Graph the curves to look for other points of intersection.

Area enclosed by curve having their equations in parametric form:
Let y = f(x) be a continuous function on closed interval [a, b] and let x = g(t) and y = h(t) be its parametric equations with domain $t \in [t_1, t_2]$ such that $g(t_1) = a$ and $g(t_2) = b$. Let the traced curve be simple. Its derivative function g'(t) is continuous on $[t_1, t_2]$, then the area under the curve is given by

$$A = \int_a^b y\,dx = \int_a^b h(t)d(g(t)) = \int_{t_1}^{t^2} h(t)g'(t)dt \ ; y \ge 0 \text{ for } t \in [t_1, t_2].$$

Area bounded by a closed curve defined in parametric form:

Consider a closed curve represented by the parametric equations x = f (t), y = ϕ (t). 't' being the parameter. We suppose that the curve does not intersect itself. Also, suppose that as the parameter't' increases from value t_1 to the value t_2, the point P(x, y) describes the curve completely in the counter clockwise sense. The curve being closed, the point on it corresponding to the value t_2 of the parameter is the same as the point corresponding to the value t_1 of the parameter. Let this point be C.



It will now be shown that the area of the region bounded by such a curve is $\frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt.$