CHAPTER TWENTY SIX

Inverse Trigonometric Functions

INVERSE TRIGONOMETRIC FUNCTIONS AND FORMULAE

Function	Domain	Range
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\pi/2 < y < \pi/2$
$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \sec^{-1} x$	$-\infty < x \le -1$	$0 \le y \le \pi$, $y \ne \pi/2$
	or $1 \le x < \infty$	
$y = \text{cosec}^{-1} x$	$-\infty < x \le -1$	$-\pi/2 \le y \le \pi/2$, $y \ne 0$
	or $1 \le x < \infty$	

(a) (i) $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin\theta) = \theta$ if $-\pi/2 \le \theta \le \pi/2$

(ii) $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$ and $\cos^{-1}(\cos\theta) = \theta$ if $0 \le \theta \le \pi$

Illustration 1

Find the value of $\sin^{-1}(\sin 12)$ and $\cos^{-1}(\cos 12)$

Solution: $4\pi - \dfrac{\pi}{2} < 12 < 4\pi$

$$\Rightarrow \quad -\frac{\pi}{2} < 12 - 4\pi < 0 < \frac{\pi}{2}$$

So $\sin^{-1}(\sin 12) = \sin^{-1}(\sin(12 - 4\pi))$
$= 12 - 4\pi$

Also $0 < 4\pi - 12 < \pi$

So $\cos^{-1}(\cos 12) = \cos^{-1}(\cos(4\pi - 12)) = 4\pi - 12$

(iii) $\tan(\tan^{-1} x) = x$ if $-\infty < x < \infty$ and $\tan^{-1}(\tan\theta) = \theta$ if $-\pi/2 < \theta < \pi/2$

(iv) $\cot(\cot^{-1} x) = x$ if $-\infty < x < \infty$ and $\cot^{-1}(\cot\theta) = \theta$ if $0 < \theta < \pi$

(v) $\sec(\sec^{-1} x) = x$ if $|x| \ge 1$ and $\sec^{-1}(\sec\theta) = \theta$ if $0 \le \theta \le \pi$, $\theta \ne \pi/2$

(vi) $\text{cosec}(\text{cosec}^{-1} x) = x$ if $|x| \ge 1$ and $\text{cosec}^{1}(\text{cosec}\,\theta) = \theta$ if $-\pi/2 \le \theta \le \pi/2$, $\theta \ne 0$.

(b) (i) $\sin^{-1} x + \cos^{-1} x = \pi/2, \quad -1 \le x \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \pi/2, \quad -\infty < x < \infty$

(iii) $\sec^{-1} x + \text{cosec}^{-1} x = \pi/2, -\infty < x \le -1, \; 1 \le x < \infty.$

(c) (i) $\sin^{-1} x = \text{cosec}^{-1}(1/x), \quad -1 \le x \le 1, x \ne 0$

(ii) $\cos^{-1} x = \sec^{-1}(1/x), \quad -1 \le x \le 1, x \ne 0$

(iii) $\tan^{-1} x = \cot^{-1}(1/x)$, if $x > 0$ and $\tan^{-1} x = \cot^{-1}(1/x) - \pi$ if $x < 0$.

(d) (i) $\sin^{-1}(-x) = -\sin^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1} x \; (-1 \le x \le 1)$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x$, $\cot^{-1}(-x) = \pi - \cot^{-1} x \; (-\infty < x < \infty)$

(iii) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x \; (|x| \ge 1)$

(e) (i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ if $-1 \le x, y \le 1$, $x^2 + y^2 \le 1$ or if $xy < 0$, $x^2 + y^2 > 1$

$= \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ if $0 < x, y \le 1$, $x^2 + y^2 > 1$

$= -\sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] - \pi$ if $-1 \le x, y < 0$, $x^2 + y^2 > 1$.

Illustration 2

Find the value of

$$\sin^{-1}\frac{2}{3} + \sin^{-1}\frac{3}{4} + \sin^{-1}\frac{2\sqrt{7} + 3\sqrt{5}}{12}$$

Solution: $0 < \frac{2}{3}, \frac{3}{4} < 1$ and $\left(\frac{2}{3}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{145}{144} > 1$

So, $\sin^{-1}\frac{2}{3} + \sin^{-1}\frac{3}{4} + \sin^{-1}\frac{2\sqrt{7}+3\sqrt{5}}{12}$

$= \pi - \sin^{-1}\left[\frac{2}{3}\sqrt{1-\frac{9}{16}} + \frac{3}{4}\sqrt{1-\frac{4}{9}}\right] + \sin^{-1}\frac{2\sqrt{7}+3\sqrt{5}}{12}$

$= \pi - \sin^{-1}\frac{2\sqrt{7}+3\sqrt{5}}{12} + \sin^{-1}\frac{2\sqrt{7}+3\sqrt{5}}{12} = \pi$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$ if $-1 \le x, y \le 1, x^2 + y^2 \le 1$ or if $xy > 0$, and $x^2 + y^2 > 1$

$= \pi - \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$ if $0 < x \le 1, -1 \le y < 0, x^2 + y^2 > 1$

$= -\sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right] - \pi$ if $-1 \le x < 0, 0 < y \le 1, x^2 + y^2 > 1$

(iii) $\cos^{-1} x + \cos^{-1} y$

$= \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$ if $-1 \le x, y \le 1, x + y \ge 0$

$= 2\pi - \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$ if $-1 \le x, y < 0, x + y \le 0$

(iv) $\cos^{-1}x - \cos^{-1}y$

$= \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$ if $-1 \le x, y \le 1, x \le y$

$= -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$ if $-1 \le y < 0, 0 < x \le 1, x \ge y$

(v) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$

$= \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy > 1, x > 0, y > 0$

$= \tan^{-1}\left(\frac{x+y}{1-xy}\right) - \pi$ if $xy > 1, x < 0, y < 0$

Illustration 3

Find the value of

$\tan^{-1}(1/2) - \tan^{-1}(-3) - \tan^{-1}(-7)$

Solution: $-3 < 0, -7 < 0, (-3)(-7) > 1$

So $\tan^{-1}(1/2) - \tan^{-1}(-3) - \tan^{-1}(-7)$

$= \tan^{-1}(1/2) - \left[\tan^{-1}\frac{-3-7}{1-(-3)(-7)}\right] + \pi = \pi$

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ if $xy > -1$

$= \tan^{-1}\left(\frac{x-y}{1+xy}\right) - \pi$ if $x < 0, y > 0, xy < -1$

$= \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ if $x > 0, y < 0, xy < -1$

(vii) $\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x$ if $x \le 1$

$= \tan^{-1} x - \frac{3\pi}{4}$ if $x > 1$

(viii) $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{\pi}{4} - \tan^{-1} x$ if $x \ge -1$

$= \tan^{-1} x - \frac{\pi}{4}$ if $x < -1$

(f) (i) $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$ if $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

$= \pi - \sin^{-1}(2x\sqrt{1-x^2})$ if $\frac{1}{\sqrt{2}} < x \le 1$

$= \sin^{-1}(2x\sqrt{1-x^2}) - \pi$ if $-1 \le x < -\frac{1}{\sqrt{2}}$

(ii) $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$ if $0 \le x \le 1$

$= \pi - \cos^{-1}(2x^2 - 1)$ if $-1 \le x \le 0$

Illustration 4

Find the value of

$\cos\left(2\cos^{-1}\left(-\frac{3}{4}\right)\right)$

Solution: $\cos\left(2\cos^{-1}\left(-\frac{3}{4}\right)\right)$

$= \cos\left(\pi - \cos^{-1}\left(2 \times \frac{9}{16} - 1\right)\right)$

$= \cos\left(\pi - \cos^{-1}\left(\frac{1}{8}\right)\right)$

$= -\cos\left(\cos^{-1}\left(\frac{1}{8}\right)\right) = -\frac{1}{8}.$

(iii) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ if $-1 < x < 1$

$= \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ if $x > 1$

$= \tan^{-1}\left(\frac{2x}{1-x^2}\right) - \pi$ if $x < -1$

(iv) $2\tan^{-1} x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ if $-1 \le x \le 1$

$= \pi + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ if $x > 1$

$= \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) - \pi$ if $x < -1$

(v) $2\tan^{-1} x = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ if $0 \le x < \infty$

$= -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ if $-\infty < x \le 0$

(g) (i) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$ if $-\dfrac{1}{2} \le x \le \dfrac{1}{2}$

(ii) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$ if $\dfrac{1}{2} \le x \le 1$

(iii) $3\tan^{-1} x = \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right)$ if $-\dfrac{1}{\sqrt{3}} \le x \le \dfrac{1}{\sqrt{3}}$

Illustration 5

Find the values of

$\sin\left(3\sin^{-1}\left(\dfrac{1}{3}\right)\right), \cos\left(3\cos^{-1}\left(\dfrac{2}{3}\right)\right), \tan\left(3\tan^{-1}\left(\dfrac{1}{2}\right)\right)$

Solution:

$$\sin\left(3\sin^{-1}\left(\frac{1}{3}\right)\right) = \sin\left(\sin^{-1}\left(3\times\frac{1}{3} - 4\times\frac{1}{27}\right)\right) = \frac{23}{27}$$

$$\cos\left(3\cos^{-1}\left(\frac{2}{3}\right)\right) = \cos\left(\cos^{-1}\left(4\times\frac{8}{27} - 3\times\frac{2}{3}\right)\right)$$

$$= \cos\ \cos^{-1}\left(-\frac{22}{27}\right)$$

$$= \cos\left(\pi - \cos^{-1}\left(\frac{22}{27}\right)\right)$$

$$= -\frac{22}{27}.$$

$$\tan\left(3\tan^{-1}\left(\frac{1}{2}\right)\right) = \tan\left(\tan^{-1}\left(\frac{3\times\frac{1}{2} - \frac{1}{8}}{1 - 3\times\frac{1}{4}}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{11}{2}\right)\right) = \frac{11}{2}$$

If an expression contains

$$\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$$

simplify them by applying the following substitution.

Expression	Substitution	
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$(0 < \theta < \pi)$
$\sqrt{a^2 - x^2}$ and $\sqrt{a^2 + x^2}$ (Both)	$x^2 = a^2 \cos \theta$	$\left(-\frac{\pi}{4} < \theta < \frac{\pi}{4}\right)$

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: If $\cos^{-1}x + \cos^{-1}(2/3) = (\pi/2)$, the value of x is

(a) $1/3$ (b) $\sqrt{5}/3$

(c) $\sqrt{5}/2$ (d) $1/2$

Ans. (b)

Solution: $\cos^{-1} x = \pi/2 - \cos^{-1} 2/3$

$$\Rightarrow \quad x = \cos\left(\frac{\pi}{2} - \cos^{-1}\frac{2}{3}\right) = \sin\cos^{-1}\left(\frac{2}{3}\right)$$

$$= \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$$

Example 2: The value of

$\cos\left(3\cos^{-1}\frac{4}{5}\right) + \sin\left(3\sin^{-1}\frac{4}{5}\right)$ is equal to

(a) -1 (b) 0

(c) $4/5$ (d) 1

Ans. (b)

Solution: $\cos\left[\cos^{-1}\left(4 \times (4/5)^3 - 3 \times 4/5\right)\right] +$

$\sin\left[\sin^{-1}\left(3 \times 4/5 - 4 \times (4/5)^3\right)\right]$

$$= 4 \times (4/5)^3 - 3 \times (4/5) + 3 \times (4/5) - 4 \times (4/5)^3 = 0$$

Example 3: A value of x for which $\sin(\cot^{-1}(1 + x)) = \cos(\tan^{-1} x)$ is

(a) $-(1/2)$ (b) 1

(c) 0 (d) $1/2$

Ans. (a)

Solution: $\sin(\cot^{-1}(1 + x)) = \sin(\pi/2 + \tan^{-1} x)$

$$\Rightarrow \quad \cot^{-1}(1 + x) = \pi/2 + \tan^{-1} x$$

$$\Rightarrow \quad 1 + x = \cot(\pi/2 + \tan^{-1} x) = -\tan(\tan^{-1} x) = -x$$

$$\Rightarrow \quad 2x = -1 \Rightarrow x = -(1/2)$$

Example 4: The number of solutions of the equation $\sin^{-1} x = 2\tan^{-1} x$ (in principal values) is

(a) 1 (b) 4

(c) 2 (d) 3

Ans. (d)

Solution: Let $x = \tan\theta$, then $\sin^{-1} x = 2\tan^{-1} x = 2\theta$

$$\Rightarrow \quad x = \sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{2x}{1 + x^2}.$$

$$\Rightarrow \quad x(x^2 - 1) = 0 \Rightarrow x = 0, \pm 1$$

Example 5: The principal value of $\tan^{-1}(\cot 43\pi/4)$ is

(a) $-3\pi/4$ (b) $3\pi/4$
(c) $\pi/4$ (d) $-\pi/4$

Ans. (d)

Solution: $\tan^{-1}\cot(11\pi - \pi/4)$
$= \tan^{-1}\cot(-\pi/4) = \tan^{-1}(-1) = -(\pi/4)$.

Example 6: $2\cot^{-1}(7) + \cos^{-1}(3/5)$, in principal value is equal to

(a) $\cos^{-1}\left(\frac{44}{125}\right)$ (b) $\cot^{-1}\left(\frac{44}{117}\right)$
(c) $\tan^{-1}\left(\frac{41}{117}\right)$ (d) $\text{cosec}^{-1}\left(\frac{117}{125}\right)$

Ans. (b)

Solution: $2\cot^{-1}(7) + \cos^{-1}(3/5)$

$$= 2\tan^{-1}(1/7) + \tan^{-1}(4/3)$$

$$= \tan^{-1}\frac{2(1/7)}{1-(1/7)^2} + \tan^{-1}(4/3)$$

$$= \tan^{-1}(7/24) + \tan^{-1}(4/3).$$

$$= \tan^{-1}\frac{(7/24)+(4/3)}{1-(7/24)\times(4/3)} = \tan^{-1}\frac{117}{44} = \cot^{-1}\frac{44}{117}$$

Example 7: Number of solutions of the equation $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = 2\pi^3$ is

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (a)

Solution: $\sin^{-1}x \le \pi/2$, $\cos^{-1}x \le \pi$

$$\Rightarrow \quad (\sin^{-1}x)^3 + (\cos^{-1}x)^3 \le \frac{9\pi^3}{8}$$

So the given equation has no solution.

Example 8: If $\cot(\cot^{-1}x - \tan^{-1}x) = 24/7$, then the values of x are

(a) $-(3/4), (4/3)$ (b) $(3/4), (4/3)$
(c) $-(3/4), (4/3)$ (d) $(3/4), (4/3)$

Ans. (b)

Solution: $\cot(\cot^{-1}x - \tan^{-1}x) = 24/7$

$$\Rightarrow \quad \cot(\pi/2 - 2\tan^{-1}x) = 24/7$$

$$\Rightarrow \quad \tan(2\tan^{-1}x) = 24/7$$

$$\Rightarrow \quad \frac{2x}{1-x^2} = \frac{24}{7}$$

$$\Rightarrow \quad 12x^2 + 7x - 12 = 0$$

$$\Rightarrow \quad (3x+4)(4x-3) = 0$$

$$\Rightarrow \quad x = -(4/3), (3/4).$$

Example 9: $\cos(2\cos^{-1}x) + \cos(2\sin^{-1}x) = 0$ for

(a) all real values of x (b) two values of x
(c) no value of x (d) $-1 \le x \le 1$

Ans. (d)

Solution: $\cos(2\cos^{-1}x) + \cos(2\sin^{-1}x)$
$= 2x^2 - 1 + 1 - 2x^2 = 0$

for all x for which $\cos^{-1}x$ and $\sin^{-1}x$ are defined ie $-1 \le x \le 1$.

Example 10: The value of $\tan\left[\sin^{-1}(7/\sqrt{50}) + \cos^{-1}(1/\sqrt{50}) + \cot^{-1}(1/7)\right]$ is equal to

(a) 7 (b) $\sqrt{50}$
(c) 161/73 (d) 73/161

Ans. (c)

Solution: Let $\sin^{-1}(7/\sqrt{50}) = \theta \Rightarrow (7/\sqrt{50}) = \sin\theta$

$\Rightarrow \cos\theta = (1/\sqrt{50})$

$\Rightarrow \quad \tan\theta = 7 \Rightarrow \cot\theta = 1/7.$

So $\tan\left[\sin^{-1}(7/\sqrt{50}) + \cos^{-1}(1/\sqrt{50}) + \cot^{-1}(1/7)\right]$

$$= \tan(3\theta)$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$= \frac{3\times 7 - 7^3}{1 - 7^2} = \frac{7\times 46}{146} = \frac{161}{73}.$$

LEVEL 1

Straight Objective Type Questions

Example 11: If $x = 1/5$, the value of $\cos(\cos^{-1}x + 2\sin^{-1}x)$ is

(a) $-\sqrt{24/25}$ (b) $\sqrt{24/25}$
(c) $-1/5$ (d) $1/5$

Ans. (c)

Solution: The given expression is equal to

$$\cos(\cos^{-1}x + \sin^{-1}x + \sin^{-1}x) = \cos(\pi/2 + \sin^{-1}x)$$

$$= -\sin(\sin^{-1}x) = -x = -1/5.$$

Example 12: $\tan^{-1}(1/11) + \tan^{-1}(2/12)$ is equal to

(a) $\tan^{-1}(33/132)$ (b) $\tan^{-1}(1/2)$
(c) $\tan^{-1}(132/33)$ (d) none of these

Ans. (d)

Solution: $\tan^{-1}(1/11) + \tan^{-1}(1/6)$

$$= \tan^{-1}\frac{\frac{1}{11}+\frac{1}{6}}{1-\frac{1}{11}\times\frac{1}{6}} = \tan^{-1}\left(\frac{17}{65}\right)$$

Example 13: The value of $\sin^{-1}(\sin 10)$ is

(a) 10 (b) $3\pi - 10$
(c) $10 - 3\pi$ (d) none of these

Ans. (b)

Solution: $y = \sin^{-1}(\sin 10)$

$\Rightarrow \quad \sin y = \sin 10 = \sin(3\pi + (10 - 3\pi))$

$(\because 3\pi < 10 < 3\pi + \pi/2)$

$= -\sin(10 - 3\pi)$

$= \sin(3\pi - 10)$

$\Rightarrow \quad y = 3\pi - 10$

Example 14: If $\sin^{-1}x + \sin^{-1}y = 2\pi/3$, then $\cos^{-1}x + \cos^{-1}y$ is equal to

(a) $2\pi/3$ (b) $\pi/3$
(c) $\pi/6$ (d) π

Ans. (b)

Solution: Let $\cos^{-1}x + \cos^{-1}y = \theta$

then $\quad \sin^{-1}x + \cos^{-1}x + \sin^{-1}y + \cos^{-1}y = 2\pi/3 + \theta$

$$\Rightarrow \quad \frac{\pi}{2} + \frac{\pi}{2} = \frac{2\pi}{3} + \theta \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

Example 15: $\sin^{-1}(1 - x) - 2\sin^{-1}x = \pi/2$, then x is equal to

(a) 0, 1/2 (b) 1, 1/2
(c) 0 (d) 1/2

Ans. (c)

Solution: $\sin^{-1}(1 - x) = \pi/2 + 2\sin^{-1}x$

$\Rightarrow \quad 1 - x = \sin(\pi/2 + 2\sin^{-1}x) = \cos(2\sin^{-1}x)$

$= 2(1 - x^2) - 1$

$\Rightarrow \quad 2x^2 - x = 0 \quad \Rightarrow \quad x = 0$ or $1/2$

But $x = 1/2$ does not satisfy the given equation.
Hence $x = 0$.

Example 16: If x, y, z are in G.P., $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are in A.P., then

(a) $x = y = z$ or $y \neq 1$
(b) $x = 1/z$
(c) $x = y = z$, but their common value is not necessary 0
(d) $x + z = y$

Ans. (c)

Solution: x, y, z are in G.P $\Rightarrow \quad y^2 = xz$

$\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are in A.P.

$\Rightarrow \quad 2\tan^{-1}y = \tan^{-1}x + \tan^{-1}z$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$\Rightarrow \quad 2y = x + z$ as $y^2 = xz$.

$\Rightarrow \quad x, y, z$ are in A.P.

So x, y, z are in A.P. and are in G.P.

$\Rightarrow \quad x = y = z$ for any real value of x.

Example 17: If $\cos^{-1}x + \cos^{-1}(y/2) = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to

(a) $4\sin^2\alpha$ (b) $-4\sin^2\alpha$
(c) $2\sin 2\alpha$ (d) 4

Ans. (a)

Solution: $\cos\alpha = x(y/2) + \sqrt{(1-x^2)}\sqrt{1-(y^2/4)}$

$\Rightarrow \quad (2\cos\alpha - xy)^2 = (1 - x^2)(4 - y^2)$

$\Rightarrow \quad 4\cos^2\alpha - 4xy\cos\alpha = 4 - 4x^2 - y^2$

$\Rightarrow \quad 4x^2 - 4xy\cos\alpha + y^2 = 4(1 - \cos^2\alpha) = 4\sin^2\alpha.$

Example 18: If $\sin^{-1}(x/5) + \text{cosec}^{-1}(5/4) = \pi/2$ then a values of x is

(a) 1 (b) 3
(c) 4 (d) 5

Ans. (b)

Solution: $\sin^{-1}(x/5) + \text{cosec}^{-1}(5/4) = \pi/2$

$\Rightarrow \quad \sin^{-1}(x/5) + \sin^{-1}(4/5) = \pi/2$

$\Rightarrow \quad \sin^{-1}(x/5) + \cos^{-1}\sqrt{1 - 16/25} = \pi/2$

$\Rightarrow \quad \sin^{-1}(x/5) + \cos^{-1}(3/5) = \pi/2$

$\Rightarrow \quad x = 3$ as $\sin^{-1}x + \cos^{-1}x = \pi/2 \ \forall\ x$.

Example 19: $\tan^{-1}(1/4) + \tan^{-1}(2/9)$ is equal to

(a) $(1/2)\cos^{-1}(3/5)$ (b) $\sin^{-1}(4/5)$
(c) $(1/2)\tan^{-1}(3/5)$ (d) $\tan^{-1}(8/9)$

Ans. (a)

Solution: $\tan^{-1}(1/4) + \tan^{-1}(2/9)$

$$= \tan^{-1}\frac{1/4 + 2/9}{1 - (1/4)(2/9)}$$

$$= \tan^{-1}(17/34) = \tan^{-1}(1/2)$$

If $\theta = \tan^{-1}(1/2) \quad \Rightarrow \quad \tan\theta = (1/2)$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{4}{5},\ \cos 2\theta = \frac{3}{5}$$

$$\Rightarrow \quad \theta = \frac{1}{2}\cos^{-1}\frac{3}{5}$$

Example 20: $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$ for

(a) $|x| \geq 1$ (b) $x \geq 0$
(c) $|x| \leq 1$ (d) all $x \in \mathbb{R}$.

Ans. (c)

Solution: $-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{2x}{1+x^2}\right) \leq \frac{\pi}{2}$

$\Rightarrow \quad -(\pi/2) \leq 2\tan^{-1}x \leq (\pi/2)$

$\Rightarrow \quad -(\pi/4) \leq \tan^{-1}x \leq (\pi/4)$

$\Rightarrow \quad \tan(-\pi/4) \leq x \leq \tan(\pi/4)$

$\Rightarrow \quad -1 \leq x \leq 1 \quad \Rightarrow \quad |x| \leq 1$

Example 21: $\cot^{-1}[(\cos\alpha)^{1/2}] - \tan^{-1}[(\cos\alpha)^{1/2}] = x$ then $\sin x =$

(a) $\tan^2(\alpha/2)$ (b) $\cot^2(\alpha/2)$
(c) $\tan\alpha$ (d) $\cot\alpha$

Ans. (a)

Solution: $x = (\pi/2) - 2\tan^{-1}[(\cos\alpha)^{1/2}]$

$\Rightarrow \quad (\pi/2) - x = 2\tan^{-1}[(\cos\alpha)^{1/2}]$

$$\Rightarrow \quad \tan(\pi/2 - x) = \frac{2(\cos\alpha)^{1/2}}{1-\cos\alpha}$$

$$\Rightarrow \quad \cot x = \frac{2(\cos\alpha)^{1/2}}{1-\cos\alpha}$$

$$\Rightarrow \quad \operatorname{cosec}^2 x = 1 + \frac{4\cos\alpha}{(1-\cos\alpha)^2} = \left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)^2$$

$$\Rightarrow \quad \sin x = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{2\sin^2(\alpha/2)}{2\cos^2(\alpha/2)}$$

$$= \tan^2(\alpha/2)$$

Example 22: The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for

(a) all real values of a (b) $|a| \leq \frac{1}{\sqrt{2}}$
(c) $|a| \geq \frac{1}{\sqrt{2}}$ (d) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$

Ans. (b)

Solution: $\sin^{-1}x = \sin^{-1}2a\sqrt{1-a^2}$ if $|a| \leq 1/\sqrt{2}$

$\Rightarrow \quad x = 2a\sqrt{1-a^2}$ which is possible

if $\quad x^2 = 4a^2(1-a^2) \leq 1 \qquad [\because\ -1 \leq \sin^{-1}x \leq 1]$

or if $4a^4 - 4a^2 + 1 \geq 0$ if $(2a^2-1)^2 \geq 0$

which is true, so $|a| \leq 1/\sqrt{2}$

Example 23: If $0 \leq x \leq 1$ and $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x$, then

(a) $\theta \leq \pi/2$ (b) $\theta \geq \pi/4$
(c) $\theta = \pi/4$ (d) $\pi/4 \leq \theta \leq \pi/2$

Ans. (d)

Solution: $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x = \pi/2 - \tan^{-1}x$

Since $0 \leq x \leq 1 \quad \Rightarrow \quad 0 \leq \tan^{-1}x \leq \pi/4 \quad \Rightarrow \quad \pi/4 \leq \theta \leq \pi/2$.

Example 24: If $x > 0$, $y > 0$ and $x > y$, then $\tan^{-1}(x/y) + \tan^{-1}[(x+y)/(x-y)]$ is equal to

(a) $-\pi/4$ (b) $\pi/4$
(c) $3\pi/4$ (d) none of these

Ans. (c)

Solution: Since $\frac{x}{y}.\frac{x+y}{x-y} > 1$. The given expression is equal to

$$\pi + \tan^{-1}\left[\frac{\frac{x}{y}+\frac{x+y}{x-y}}{1-\frac{x}{y}\times\frac{x+y}{x-y}}\right]$$

$$= \pi + \tan^{-1}\frac{x^2+y^2}{-(x^2+y^2)} = \pi + \tan^{-1}(-1) = 3\pi/4.$$

Example 25: The principal value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}\cos(7\pi/6)$ is

(a) $5\pi/6$ (b) $\pi/2$
(c) $3\pi/2$ (d) none of these

Ans. (b)

Solution: $\sin^{-1}(-\sqrt{3}/2) = -\sin^{-1}(\sqrt{3}/2) = -\pi/3$

and $\quad \cos^{-1}(\cos(7\pi/6) = \cos^{-1}\cos(2\pi - 5\pi/6)$
$\qquad = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

Hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos 7\pi/6)$

$= -(\pi/3) + (5\pi/6) = (\pi/2)$.

Example 26: The value of $\cos^{-1}(-1/2) - 2\sin^{-1}(1/2) + 3\cos^{-1}(-1/\sqrt{2}) - 4\tan^{-1}(-1)$ is equal to

(a) $7\pi/4$ (b) $11\pi/4$
(c) $\pi/12$ (d) $25\pi/12$

Ans. (d)

◎ **Solution:** The given expression is equal to

$(2\pi/3) - 2\times(\pi/6) + 3\times(3\pi/4) - 4(-\pi/4) = 25\pi/12.$

◉ **Example 27:** If $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x} = 4$ then $x =$

(a) tan 2 (b) tan 4
(c) tan (1/4) (d) tan 8

Ans. (d)

◎ **Solution:** Taking $x = \tan\theta$, $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$

$$= \tan^{-1}\frac{\sec\theta - 1}{\tan\theta}$$

$$= \tan^{-1}\frac{1-\cos\theta}{\sin\theta} = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= (1/2)\theta = (1/2)\tan^{-1}(x)$$

so that according to the given condition,

$$(1/2)\tan^{-1}x = 4 \quad\Rightarrow\quad \tan^{-1}x = 8 \text{ or } x = \tan 8.$$

◉ **Example 28:** $\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3)$ is equal to

(a) 1 (b) 5
(c) 10 (d) 15

Ans. (d)

◎ **Solution:** The given expression is equal to

$$1 + \tan^2(\tan^{-1}2) + 1 + \cot^2(\cot^{-1}3)$$
$$= 1 + 4 + 1 + 9 = 15.$$

◉ **Example 29:** The equation $2\cos^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$ is valid for all values of x satisfying

(a) $-1 \le x \le 1$ (b) $0 \le x \le 1$
(c) $0 \le x \le 1/\sqrt{2}$ (d) $1/\sqrt{2} \le x \le 1$

Ans. (d)

◎ **Solution:** If we denote $\cos^{-1}x$ by y, then

$$\text{since } 0 \le \cos^{-1}x \le \pi \quad\Rightarrow\quad 0 \le 2y \le 2\pi \qquad (1)$$

Also since $-\pi/2 \le \sin^{-1}(2x\sqrt{1-x^2}) \le \pi/2$

$\Rightarrow -\pi/2 \le \sin^{-1}\sin(2y) \le \pi/2$

$\Rightarrow \quad -\pi/2 \le 2y \le \pi/2 \qquad (2)$

From (1) and (2) we find $0 \le 2y \le \pi/2$

$\Rightarrow \quad 0 \le y \le \pi/4 \quad\Rightarrow\quad 0 \le \cos^{-1}x \le \pi/4$

which holds if $1/\sqrt{2} \le x \le 1$.

◉ **Example 30:** If $\tan^{-1}\dfrac{1}{1+2} + \tan^{-1}\dfrac{1}{1+(2)(3)} + \tan^{-1}\dfrac{1}{1+(3)(4)} + \dots \tan^{-1}\dfrac{1}{1+n(n+1)} = \tan^{-1}\theta$, then $\theta =$

(a) $\dfrac{n}{n+1}$ (b) $\dfrac{n+1}{n+2}$
(c) $\dfrac{n}{n+2}$ (d) $\dfrac{n-1}{n+2}$

Ans. (c)

◎ **Solution:** $\tan^{-1}\dfrac{1}{1+n(n+1)} = \tan^{-1}\dfrac{n+1-n}{1+n(n+1)}$

$$= \tan^{-1}(n+1) - \tan^{-1}(n)$$

so that L.H.S. of the given equation is

$\tan^{-1}2 - \tan^{-1}1 + \tan^{-1}3 - \tan^{-1}2 + \dots + \tan^{-1}(n+1) - \tan^{-1}n.$

$$= \tan^{-1}(n+1) - \tan^{-1}1$$

$$= \tan^{-1}\frac{n+1-1}{1+(n+1)} = \tan^{-1}\frac{n}{n+2}$$

so that $\quad \tan^{-1}\dfrac{n}{n+2} = \tan^{-1}\theta \quad\Rightarrow\quad \theta = \dfrac{n}{n+2}.$

◉ **Example 31:** A value of x satisfying $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}(3/5)$ is

(a) 0 (b) 2
(c) 4 (d) 8

Ans. (c)

◎ **Solution:** $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}(3/5)$

$\Rightarrow \quad \tan^{-1}\dfrac{(x+3)-(x-3)}{1+(x+3)(x-3)} = \tan^{-1}(3/4)$

$\Rightarrow \quad \dfrac{6}{1+x^2-9} = \dfrac{3}{4} \quad\Rightarrow x^2 = 16$

$\Rightarrow \quad x = \pm 4.$

◉ **Example 32:** If $x = \text{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ $a \in [0, 1]$, then

(a) $x^2 - a^2 = 3$ (b) $x^2 + a^2 = 3$
(c) $x^2 - a^2 = 2$ (d) $x^2 + a^2 = 2$

Ans. (b)

◎ **Solution** $\quad x = \text{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\dfrac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$

$$= \text{cosec}(\tan^{-1}(\cos(\sec^{-1}(\sqrt{2-a^2}))))$$

$$= \text{cosec}\left(\tan^{-1}\frac{1}{\sqrt{2-a^2}}\right)$$

$$= \text{cosec}(\cot^{-1}\sqrt{2-a^2}) = \sqrt{3-a^2}$$

$\Rightarrow \quad x^2 + a^2 = 3$

Example 33: If $\sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\frac{-19\pi}{8}\right) = \frac{a\pi}{b}$

where a and b are in their lowest form, then $(a + b)$ is equal to

(a) 17 (b) 20
(c) 23 (d) none of these

Ans. (b)

Solution: We have L.H.S $= \sin^{-1}(\sin(5\pi - 2\pi/7)) + \cos^{-1}(\cos(7\pi - 3\pi/7)) + \tan^{-1}(-\tan(2\pi - 3\pi/8)) + \cot^{-1}(-\cot(3\pi - 5\pi/8))$

$$= \sin^{-1}(\sin(2\pi/7)) + \cos^{-1}(-\cos(3\pi/7)) + \tan^{-1}(\tan(3\pi/8)) + \cot^{-1}(\cot(5\pi/8))$$

$$= \frac{2\pi}{7} + \pi - \frac{3\pi}{7} + \frac{3\pi}{8} + \frac{5\pi}{8}$$

$$= \frac{13\pi}{7} = \frac{a\pi}{b} \quad \Rightarrow a + b = 20$$

Example 34: If $0 < x < 1$, the number of solutions of the equation $\tan^{-1}(x - 1) + \tan^{-1}(x) + \tan^{-1}(x + 1) = \tan^{-1}(3x)$ is

(a) 0 (b) 1
(c) 2 (d) 3

Ans. (b)

Solution: $\tan^{-1}(x - 1) + \tan^{-1}(x + 1) = \tan^{-1}3x - \tan^{-1}x$

$$\Rightarrow \quad \frac{(x-1)+(x+1)}{1-(x+1)(x-1)} = \frac{3x-x}{1+3x^2}$$

$\Rightarrow \quad 1 + 3x^2 = 2 - x^2, \quad x \neq 0$

$\Rightarrow \quad x = \pm 1/2$, so $x = 1/2$ as $0 < x < 1$.

Example 35: $\tan^{-1}(\tan 4) - \tan^{-1}(\tan(-6)) + \cos^{-1}(\cos 10)$ is equal to

(a) 0 (b) π
(c) $-\pi$ (d) 5π

Ans. (b)

Solution: $\pi < 4 < 3\pi/2 \quad \Rightarrow 0 < 4 - \pi < \pi/2$

$\Rightarrow \tan^{-1}(\tan 4) = 4 - \pi \quad \because \tan(4 - \pi) = \tan 4$

$2\pi - 6 \in (-\pi/2, \pi/2)$, $\tan(2\pi - 6) = -\tan 6$

$\Rightarrow \tan^{-1}(\tan(-6)) = \tan^{-1}(-\tan 6) = 2\pi - 6$ and

$4\pi - 10 \in (0, \pi)$, $\cos(4\pi - 10) = \cos 10$

so the given expression is equal to

$4 - \pi - (2\pi - 6) + 4\pi - 10 = \pi$

Example 36: If $x < -1/\sqrt{3}$, then the value of

$3\tan^{-1}x - \tan^{-1}\frac{3x - x^3}{1 - 3x^2}$ equals

(a) $-\pi$ (b) π
(c) $-\pi/2$ (d) 0

Ans. (a)

Solution: Let $x = \tan\theta$, then $x < -1/\sqrt{3}$

$\Rightarrow \quad \tan\theta < -1/\sqrt{3} \quad \Rightarrow -\pi/2 < \theta < -\pi/6$

$\Rightarrow \quad -3\pi/2 < 3\theta < -\pi/2$

$\Rightarrow \quad -\pi/2 < \pi + 3\theta < \pi/2$

so $\quad 3\tan^{-1}x - \tan^{-1}\frac{3x - x^3}{1 - 3x^2}$

$$= 3\tan^{-1}(\tan\theta) - \tan^{-1}(\tan 3\theta)$$
$$= 3\tan^{-1}(\tan\theta) - \tan^{-1}(\tan(\pi + 3\theta))$$
$$= 3\theta - (\pi + 3\theta) = -\pi$$

Example 37: $\sin^{-1}(4/5) + \sin^{-1}(5/13) + \sin^{-1}(16/65)$ is equal to

(a) 0 (b) $\pi/2$
(c) π (d) $\sin^{-1}(63/65)$

Ans. (b)

Solution

$$\sin^{-1}\left[(4/5)\sqrt{1-(5/13)^2} + \frac{5}{13}\sqrt{1-(4/5)^2}\right] + \sin^{-1}(16/65)$$

$$= \sin^{-1}[(4/5)\times(12/13) + (5/13)\times(3/5)] + \sin^{-1}(16/65)$$

$$= \sin^{-1}(63/65) + \sin^{-1}(16/65)$$

$$= \cos^{-1}\sqrt{1-(63/65)^2} + \sin^{-1}(16/65)$$

$$= \cos^{-1}(16/65) + \sin^{-1}(16/65) = \pi/2$$

Example 38: If $\tan^{-1}(1/4) + \tan^{-1}(2/9) = (1/2)\cos^{-1}x$

then x is equal to

(a) 1/2 (b) 2/5
(c) 3/5 (d) none of these

Ans. (c)

Solution: $(1/2)\cos^{-1}x$

$$= \tan^{-1}\frac{(1/4)+(2/9)}{1-(1/4)(2/9)} = \tan^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\times 2\tan^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\cos^{-1}\left[\frac{1-(1/4)}{1+(1/4)}\right]$$

$$\left(\text{using } 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \text{ for } x \geq 0\right)$$

$$= \frac{1}{2}\cos^{-1}\frac{3}{5}.$$

Assertion-Reason Type Questions

Example 39: Statement-1: The sum of

$\cot^{-1}\left(\frac{7}{4}\right)+\cot^{-1}\left(\frac{19}{4}\right)+\ldots.+\cot^{-1}\frac{4r^2+3}{4}+\ldots.$ upto infinity is $\tan^{-1}2$.

Statement-2: $\sum_{r=1}^{n}\tan^{-1}\frac{x_r-x_{r-1}}{1+x_{r-1}x_r}=\tan^{-1}x_n-\tan^{-1}x_0$

Ans. (a)

Solution: L.H.S in Statement-2 is

$$\sum_{r=1}^{n}\left(\tan^{-1}x_r-\tan^{-1}x_{r-1}\right)=\tan^{-1}x_n-\tan^{-1}x_0$$

$\Rightarrow$ Statement-2 is true.

In Statement-1, the given expression is equal to

$$\sum_{r=1}^{\infty}\tan^{-1}\frac{4}{4r^2+3}=\sum_{r=1}^{\infty}\tan^{-1}\frac{1}{1+(r^2-1/4)}$$

$$=\sum_{r=1}^{\infty}\tan^{-1}\frac{(r+1/2)-(r-1/2)}{1+(r+1/2)(r-1/2)}$$

$$=\lim_{n\to\infty}\left[\tan^{-1}(n+1/2)-\tan^{-1}(1/2)\right]$$

(using statement–2)

$$=\pi/2-\tan^{-1}(1/2)=\cot^{-1}(1/2)=\tan^{-1}2.$$

So Statement-1 is also true.

Example 40: Statement-1: The equation $(\sin^{-1}x)^3+(\cos^{-1}x)^3=a\pi^3$ has a solution if $\frac{1}{32}\le a\le\frac{7}{8}$.

Statement-2: $0\le(\sin^{-1}x-\pi/4)^2\le 9\pi^2/16$.

Ans. (a)

Solution: $-\pi/2\le\sin^{-1}x\le\pi/2$

$\Rightarrow\quad -3\pi/4\le\sin^{-1}x-\pi/4\le\pi/4$

$\Rightarrow\quad (\sin^{-1}x-\pi/4)^2\le 9\pi^2/16$

$\Rightarrow\quad$ Statement-2 is true.

$(\sin^{-1}x)^3+(\cos^{-1}x)^3=a\pi^3$

$\Rightarrow (\sin^{-1}x+\cos^{-1}x)\,[(\sin^{-1}x+\cos^{-1}x)^2-3\sin^{-1}x\cos^{-1}x]=a\pi^3$

$\Rightarrow\quad \pi^2/4-3\sin^{-1}x\cos^{-1}x=2a\pi^2$

$\Rightarrow\quad \sin^{-1}x(\pi/2-\sin^{-1}x)=\pi^2(1-8a)/12$

$\Rightarrow\quad (\sin^{-1}x)^2-(\pi/2)\sin^{-1}x+(\pi^2/12)(1-8a)=0$

$\Rightarrow\quad (\sin^{-1}x-\pi/4)^2=(\pi^2/48)(32a-1)$

using Statement-2,

$$0\le(\pi^2/48)(32a-1)\le(9\pi^2/16)$$

$\Rightarrow\quad 0\le 32a-1\le 27\quad\Rightarrow 1/32\le a\le 7/8.$

$\Rightarrow\quad$ Statement-1 is also true.

Example 41: Statement-1: If $1/2\le x\le 1$, then

$$\cos^{-1}x+\cos^{-1}\left[\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right]=\pi/3$$

Statement-2: $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x$ if $x\in\left[-1/\sqrt{2},1/\sqrt{2}\right]$

Ans. (b)

Solution: In Statement-1, put $x=\cos\theta$ then $0\le\theta\le\pi/3$

L.H.S $=\cos^{-1}(\cos\theta)+\cos^{-1}\left[\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right]$

$=\theta+\pi/3-\theta=\pi/3\quad\Rightarrow\quad$ Statement-1 is true

In Statement-2, put $x=\sin\theta$, then $-\pi/4\le\theta\le\pi/4$

L.H.S. $=\sin^{-1}(2\sin\theta\cos\theta)=2\theta=2\sin^{-1}x.$

$\Rightarrow$ Statement-2 is also true but does not lead to Statement-1

Example 42: Statement-1: $\text{cosec}^{-1}(3/2)+\cos^{-1}(2/3)-2\cot^{-1}(1/7)-\cot^{-1}7$ is equal to $\cot^{-1}7$.

Statement-2: $\sin^{-1}x+\cos^{-1}x=\pi/2$, $\tan^{-1}x+\cot^{-1}x=\pi/2$
$\text{cosec}^{-1}(x)=\sin^{-1}(1/x)$,
$\cot^{-1}x=\tan^{-1}(1/x)$, $-1\le x\le 1$, $x\ne 0$

Ans. (d)

Solution: Statement-2 is true, using in Statement-1

L.H.S $=\sin^{-1}(2/3)+\cos^{-1}(2/3)-(\tan^{-1}7+\cot^{-1}7)-\cot^{-1}(1/7)$

$=\pi/2-\pi/2-\tan^{-1}7=-\tan^{-1}7.$

$\Rightarrow$ Statement-1 is false.

Example 43: Statement-1: $\cos^{-1}x=2\sin^{-1}\sqrt{\frac{1-x}{2}}=2\cos^{-1}\sqrt{\frac{1+x}{2}}$

Statement-2: $1+\cos\theta=2\cos^2(\theta/2)$, $1-\cos\theta=2\sin^2(\theta/2)$

Ans. (a)

Solution: Statement-2 is true. In Statement-1, put $x=\cos\theta$

then $\cos^{-1}(\cos\theta)=2\sin^{-1}\sqrt{\frac{1-\cos\theta}{2}}=2\cos^{-1}\sqrt{\frac{1+\cos\theta}{2}}$.

Using Statement-2, $\theta=2\times\sin^{-1}(\sin(\theta/2))$
$=2\cos^{-1}(\cos(\theta/2))$

which is true $\Rightarrow$ Statement-1 is also true.

Example 44: Statement-1: If $x = 1/5\sqrt{2}$, then

$\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 = 51/50$.

Statement-2: $\tan\left[\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right] = \frac{29}{3}$.

Ans. (c)

Solution: Put $x = \cot y$ in Statement-1,

then L.H.S $= (\cot y\cos y + \sin y)^2 = \left[\frac{\cos^2 y + \sin^2 y}{\sin y}\right]^2$

$= 1 + \cot^2 y = 1 + x^2 = 1 + 1/50 = 51/50$

$\Rightarrow$ Statement-1 is true.

In Statement-2, L.H.S $= \tan[\tan^{-1}7 - \tan^{-1}4]$

$[\because \cos\theta = 1/5\sqrt{2} \Rightarrow \tan\theta = 7, \sin\theta = 4/\sqrt{17}$

$\Rightarrow \tan\theta = 4]$

$= \frac{7-4}{1+7\times 4} = \frac{3}{29}$ Statement-2 is false.

Example 45: Statement-1: If $\sin^{-1}x + \cos^{-1}(1-x) = \sin^{-1}(-x)$, then x satisfies the equation $2x^2 - 3x = 0$

Statement-2: $\sin^{-1}(x) + \sin^{-1}(-x) = 0$

Ans. (a)

Solution: Statement-2 is true as $\sin^{-1}(-x) = -\sin^{-1}x$

In Statement-1,

$\cos^{-1}(1-x) = \sin^{-1}(-x) - \sin^{-1}(x) = -2\sin^{-1}x$ (using Statement-2)

$\Rightarrow 1 - x = \cos(-2\sin^{-1}x) = \cos(2\sin^{-1}x)$

$= 1 - 2\sin^2(\sin^{-1}x)$

$\Rightarrow 1 - x = 1 - 2x^2 \Rightarrow x = 0$ or $x = 1/2$

But $x = 1/2$ does not satisfy the equtaion.

so $x = 0$, which satisfies $2x^2 - 3x = 0$

$\Rightarrow$ Statement-1 is true.

LEVEL 2

Straight Objective Type Questions

Example 46: If $\text{cosec}^{-1}x = 2\cot^{-1}7 + \cos^{-1}(3/5)$ then the value of x is

(a) 44/117 (b) 125/117
(c) 24/7 (d) 5/3

Ans. (b)

Solution: $2\cot^{-1}7 + \cos^{-1}(3/5) = \cot^{-1}\frac{7^2-1}{2\times 7} + \cot^{-1}\frac{3}{4}$

$[\because$ If $\theta = \cos^{-1}(3/5)$, $\cos\theta = 3/5$, $\cot\theta = 3/4]$

$= \cot^{-1}(24/7) + \cot^{-1}(3/4)$

$$= \cot^{-1}\left[\frac{\frac{24}{7}\times\frac{3}{4} - 1}{\frac{24}{7} + \frac{3}{4}}\right].$$

$$= \cot^{-1}\frac{44}{117} = \operatorname{cosec}^{-1}\frac{125}{117}$$

Example 47: $\theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}((1/3)\tan\theta)$ if $\tan\theta$ is equal to

(a) – 2 (b) – 1
(c) 2/3 (d) 2

Ans. (a)

Solution: $\theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}((1/3)\tan\theta)$

$$\Rightarrow \tan\theta = \frac{2\tan^2\theta - (1/3)\tan\theta}{1 + (2/3)\tan^3\theta}$$

$$\Rightarrow \tan\theta\left[\frac{2\tan\theta - (1/3)}{1 + (2/3)\tan^3\theta} - 1\right] = 0$$

which is true if $\tan\theta = 0$

or $\frac{2\tan\theta - (1/3)}{1 + (2/3)\tan^3\theta} = 1 \Rightarrow \tan^3\theta - 3\tan\theta + 2 = 0$

$\Rightarrow (\tan\theta - 1)^2(\tan\theta + 2) = 0$ which holds if $\tan\theta = 1$ or $\tan\theta = -2$.

Example 48: If $u = \cot^{-1}\sqrt{\tan\alpha} - \tan^{-1}\sqrt{\tan\alpha}$, then $\tan\left(\frac{\pi}{4} - \frac{u}{2}\right) =$

(a) $\sqrt{\tan\alpha}$ (b) $\sqrt{\cot\alpha}$
(c) $\tan\alpha$ (d) $\cot\alpha$

Ans. (a)

Solution: Let $\sqrt{\tan\alpha} = \tan x$, then $u = \cot^{-1}(\tan x) - \tan^{-1}(\tan x)$

$$= (\pi/2) - x - x = (\pi/2) - 2x$$

$$\Rightarrow \quad 2x = (\pi/2) - u \quad \Rightarrow \quad x = (\pi/4) - (u/2)$$

$$\Rightarrow \quad \tan x = \tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$$

$$\Rightarrow \quad \sqrt{\tan\alpha} = \tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$$

Example 49: If $\tan^{-1} y = 4\tan^{-1} x$, then $1/y$ is zero for

(a) $x = 1 \pm \sqrt{2}$ (b) $x = \sqrt{2} \pm \sqrt{3}$

(c) $3 \pm 2\sqrt{2}$ (d) all values of x

Ans. (a)

Solution: If we put $x = \tan\theta$, the given equality becomes $\tan^{-1} y = 4\theta$.

$$\Rightarrow \quad y = \tan 4\theta = \frac{2\tan 2\theta}{1 - \tan^2 2\theta} = \frac{2\left[\dfrac{2\tan\theta}{1 - \tan^2\theta}\right]}{1 - \left(\dfrac{2\tan\theta}{1 - \tan^2\theta}\right)^2}$$

$$= \frac{2 \times 2x(1 - x^2)}{(1 - x^2)^2 - 4x^2} = \frac{4x(1 - x^2)}{1 - 6x^2 + x^4}$$

so that $1/y$ is zero if $x^4 - 6x^2 + 1 = 0$

$$\Rightarrow \quad x^2 = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2} = (1 \pm \sqrt{2})^2$$

Example 50: $3\cos^{-1} x - \pi x - \pi/2 = 0$ has

(a) one solution
(b) one and only one solution
(c) no solution
(d) more than one solution

Ans. (b)

Solution: $x = 1/2$ is clearly a solution of the given equation which can be obtained by trial and error method. The given equation can be written as

$$3\cos^{-1} x = \pi x + \pi/2 \qquad (1)$$

since the L.H.S. of (1) is a decreasing function and R.H.S. of (1) is an increasing function of x. The equation (1) has either no solution or only one solution. So $x = 1/2$ is one and only one solution of the given equation.

Example 51: If $\cos^{-1} x = \tan^{-1} x$, then $\sin(\cos^{-1} x) =$

(a) x (b) x^2

(c) $1/x$ (d) $1/x^2$

Ans. (b)

Solution: $\cos^{-1} x = \tan^{-1} x = \theta$ (say)

$$\Rightarrow \quad x = \cos\theta = \tan\theta$$

$$\Rightarrow \quad \cos^2\theta = \sin\theta \Rightarrow \sin^2\theta + \sin\theta - 1 = 0$$

$$\Rightarrow \quad \sin\theta = \frac{-1 \pm \sqrt{1+4}}{2} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{5} - 1}{2}$$

So $\quad x^2 = \cos^2\theta = \dfrac{\sqrt{5} - 1}{2}$

and $\quad \sin(\cos^{-1} x) = \sin\theta = \dfrac{\sqrt{5} - 1}{2} = x^2.$

Example 52: $x = n\pi - \tan^{-1} 3$ is a solution of the equation $12\tan 2x + \dfrac{\sqrt{10}}{\cos x} + 1 = 0$ if

(a) n is any integer (b) n is an even integer
(c) n is a positive integer (d) n is an odd integer

Ans. (d)

Solution: $x = n\pi - \tan^{-1} 3 \quad \Rightarrow \quad \tan^{-1} 3 = n\pi - x$

$$\Rightarrow \quad \tan(n\pi - x) = 3 \quad \Rightarrow \quad -\tan x = 3$$

$$\Rightarrow \quad \tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{3}{4}$$

and $\quad \cos x = \pm\dfrac{1}{\sqrt{1 + \tan^2 x}} = \pm\dfrac{1}{\sqrt{10}}$

on substituting these value in the given equation we find only $\cos x = -1/\sqrt{10}$ satisfies the equation.

So that the given equation holds for values of x for which $\tan x = -3$ and $\cos x = -1/\sqrt{10}$. Which is possible if x lies in the second quadrant only and so n must be an odd integer.

Example 53: If $\dfrac{1}{2}\sin^{-1}\left[\dfrac{3\sin 2\theta}{5 + 4\cos 2\theta}\right] = \tan^{-1} x$, then $x =$

(a) $\tan 3\theta$ (b) $3\tan\theta$
(c) $(1/3)\tan\theta$ (d) $3\cot\theta$

Ans. (c)

Solution: $\dfrac{3\sin 2\theta}{5 + 4\cos 2\theta} = \dfrac{6\tan\theta}{9 + \tan^2\theta} = \dfrac{2\tan\phi}{1 + \tan^2\phi} = \sin 2\phi$

where $\tan\theta = 3\tan\phi$

$$\Rightarrow \quad \left(\frac{1}{2}\right)\sin^{-1}\left[\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right] = \phi = \tan^{-1}\left[\left(\frac{1}{3}\right)\tan\theta\right]$$

so that $x = (1/3)\tan\theta$.

Example 54: If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = 3\pi/2$, then the value of

$$x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}} \text{ is}$$

(a) -1 (b) 0
(c) 1 (d) 3

Ans. (b)

Solution: Since $|\sin^{-1} x| \le \pi/2$, from the given relation we have $\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \pi/2$

$\Rightarrow \quad x = y = z = \sin(\pi/2) = 1$

So that the required value is 0.

Example 55: The number of real solutions of

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \pi/2 \text{ is}$$

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (c)

Solution: $\cos^{-1}\frac{1}{\sqrt{1+(x^2+x)}} = \pi/2 - \sin^{-1}\sqrt{x^2+x+1}$

$= \cos^{-1}\sqrt{x^2+x+1}$

$\Rightarrow \frac{1}{\sqrt{1+(x^2+x)}} = \sqrt{x^2+x+1}$

$\Rightarrow (x^2+x+1) = 1$

$\Rightarrow x^2 + x = 0 \Rightarrow x = -1, 0.$

EXERCISES

Concept-based Straight Objective Type Questions

1. Value of $\sin(\sin^{-1}(3/5) + \cos^{-1}(4/5))$ is equal to
(a) 12/25 (b) 24/25
(c) 6/25 (d) 7/25

2. $2\tan^{-1} 7 - \sin^{-1}\left(14/\sqrt{50}\right)$ is equal to
(a) 0 (b) $\pi/2$
(c) π (d) 2π

3. $2\sin^{-1} x + 3\cos^{-1} y = \alpha$, $3\cos^{-1} x + 2\sin^{-1} y = \beta$. If $\alpha + \beta = \frac{7\pi}{3}$, then $xy - \sqrt{(1-x^2)(1-y^2)}$ is equal to
(a) 1 (b) 1/2
(c) $\sqrt{3}/2$ (d) $1/\sqrt{2}$

4. $2\sin^{-1}(-1/2) + 3\cos^{-1}\left(-\sqrt{3}/2\right) + 4\sin^{-1}\left(1/\sqrt{2}\right) + 5\tan^{-1}(-1)$ is equal to (considering the principal values)
(a) $15\pi/6$ (b) $27\pi/12$
(c) $23\pi/12$ (d) $13\pi/6$

5. If $x + y + z = xyz$ and $x, y, z, > 0$, then the value of $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$ is equal to
(a) $\pi/2$ (b) $\pi/4$
(c) π (d) $-\pi/4$

6. If $\tan(\cos^{-1} x) = \sin\cot^{-1}(1/2)$, then the value of x is
(a) $\sqrt{5}$ (b) $-\sqrt{5}$
(c) $\sqrt{5}/3$ (d) $-\sqrt{5}/3$

7. The value of $\sin^{-1}(\cos(\cos^{-1}(\cos x) + \sin^{-1}(\sin x)))$ where $x \in \left(\frac{\pi}{2}, \pi\right)$ is equal to
(a) $-\pi/2$ (b) $\pi/2$
(c) $-\pi$ (d) π

8. If $\tan(x + y) = 33$ and $x = \tan^{-1} 3$, the value of y is
(a) 0.3 (b) $\tan^{-1}(0.3)$
(c) $\tan^{-1}(1.3)$ (d) $\tan^{-1}(1/3)$

9. If $\sin^{-1}(x-1) + \cos^{-1}(x-3) + \tan^{-1}\frac{x}{2-x^2} = \cos^{-1} k + \pi$, then the value of k is
(a) 1/2 (b) $1/\sqrt{2}$
(c) $\sqrt{3}/2$ (d) 1

10. $\sin\left[2\sin^{-1}(3/5) + 2\sin^{-1}(4/5)\right]$ is equal to
(a) 24/125 (b) 336/625
(c) 7/125 (d) 168/625

LEVEL 1

Straight Objective Type Questions

11. If $x = 2/3$, then $\sin^2(\tan^{-1} x) + \cos^2(\sin^{-1} x)$ is equal to
(a) 99/101 (b) 107/117
(c) 101/117 (d) none of these

12. A root of the equation
$17x^2 + 17x\tan[2\tan^{-1}(1/5) - \pi/4] - 10 = 0$ is
(a) 10/17 (b) -1
(c) $-7/17$ (d) 1

13. $\cos^{-1}\sqrt{\dfrac{a-x}{a-b}} = \sin^{-1}\sqrt{\dfrac{x-b}{a-b}}$ is possible if
   (a) $a > x > b$
   (b) $a < b$ and x takes any value
   (c) $a > b$ and x takes any value
   (d) $a = x = b$

14. The value of $\cos^{-1}(\cos(-17\pi/5))$, is equal to
   (a) $-17\pi/5$ (b) $3\pi/5$
   (c) $2\pi/5$ (d) none of these

15. The value of $(1/2)\cos^{-1}(3/5)$ is
   (a) $\sin^{-1}(1/2)$ (b) $\cos^{-1}(1/2)$
   (c) $\cot^{-1}(1/2)$ (d) $\tan^{-1}(1/2)$

16. The value of $\sin(\cot^{-1}(\cos(\tan^{-1}x)))$ is
   (a) $\sqrt{\dfrac{x^2+2}{x^2+1}}$ (b) $\sqrt{\dfrac{x^2+1}{x^2+2}}$
   (c) $\dfrac{x}{\sqrt{x^2+2}}$ (d) $\dfrac{1}{\sqrt{x^2+2}}$

17. If $\tan^{-1}(a/x) + \tan^{-1}(b/x) = \pi/2$, then x is equal to
   (a) ab (b) a/b
   (c) b/a (d) $\sqrt{ab}$

18. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, where $-1 \le x, y \le 1$ and $x + y \ge 0$, then $x^2 + y^2 + z^2 + 2xyz$ is equal to
   (a) 0 (b) 1
   (c) $(x + y + z)^2$ (d) $xy + yz + zx$

19. If $\cos^{-1}(x/2) + \cos^{-1}(y/3) = \theta$, then $9x^2 - 12xy\cos\theta + 4y^2$ is equal to
   (a) 0 (b) 36
   (c) $36\sin^2\theta$ (d) $36\cos^2\theta$

20. $\tan^{-1}\sqrt{\dfrac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\dfrac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\dfrac{c(a+b+c)}{ab}}$ is equal to
   (a) $\pi/4$ (b) $\pi/2$
   (c) π (d) 0

21. If $a_1, a_2, a_3, \ldots, a_n$ is an A.P. with common ratio d, then

$$\tan\left[\tan^{-1}\frac{d}{1+a_1a_2} + \tan^{-1}\frac{d}{1+a_2a_3} + \cdots + \tan^{-1}\frac{d}{1+a_{n-1}a_n}\right] =$$

   (a) $\dfrac{(n-1)d}{a_1+a_n}$ (b) $\dfrac{(n-1)d}{1+a_1a_n}$
   (c) $\dfrac{nd}{1+a_1a_n}$ (d) $\dfrac{a_n-a_1}{a_n+a_1}$

22. Two angles of a triangle are $\sin^{-1}(1/\sqrt{5})$ and $\sin^{-1}(1/\sqrt{10})$ then the third angle is
   (a) $\pi/4$ (b) $3\pi/4$
   (c) $\pi/6$ (d) $\pi/3$

23. If $x + 1/x = 5/2$, then the principal value of $\sin^{-1}x$ is
   (a) $\pi/6$ (b) $\pi/4$
   (c) $\pi/3$ (d) $5\pi/6$

24. The number of positive integral pairs (a, b) satisfying the equation $\tan^{-1}a + \tan^{-1}b = \tan^{-1}7$ is
   (a) 0 (b) 2
   (c) 4 (d) infinite

25. The value of

$$\sin^{-1}\left\{\cot\left[\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{12}}{4} + \sec^{-1}\sqrt{2}\right]\right\}$$

is equal to
   (a) 0 (b) $\pi/12$
   (c) $\pi/6$ (d) $\pi/4$

26. The sum of the infinite series

$\cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + \ldots$ is equal to
   (a) $\pi/4$ (b) $\pi/2$
   (c) $3\pi/4$ (d) none of these

27. The inequality $\sin^{-1}(\sin 5) > x^2 - 4x$ holds if
   (a) $x = 2 - \sqrt{9-2\pi}$
   (b) $x = 2 + \sqrt{9-2\pi}$
   (c) $x \in \left(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi}\right)$
   (d) $x > 2 + \sqrt{9-2\pi}$

28. $\sin^{-1}x > \cos^{-1}x$ holds for
   (a) all values of x (b) $x \in (0, 1/\sqrt{2})$
   (c) $x \in (1/\sqrt{2}, 1)$ (d) $x = 1.75$

29. $2\sin^{-1}x = \cos^{-1}(1 - 2x^2)$ is true for
   (a) all values of x (b) $-1 \le x \le 0$
   (c) $0 \le x \le 1$ (d) no value of x

30. If $\cos^{-1}x + \cos^{-1}y = \pi/2$ and $\tan^{-1}x - \tan^{-1}y = 0$ then $x^2 + xy + y^2$ is equal to
   (a) 0 (b) $1/\sqrt{2}$
   (c) 3/2 (d) 1/8

31. $x = \tan^{-1}3 + \tan^{-1}2$
   $y = \tan^{-1}3 - \tan^{-1}2$

If $a \tan x + b \tan y = 0$, then $a + b$ is equal to ka where k is equal to

(a) –1 (b) 1/7
(c) 1 (d) 8

32. If $x > 1$ and $2 \tan^{-1}x - \sin^{-1}(7/25) = \pi$ then the value of x is
(a) 7 (b) 2
(c) 5 (d) 25

33. Number of values of x satisfying $\sin^{-1}(1 - x) - 2 \sin^{-1}x = \pi/2$ is
(a) 0 (b) 1
(c) 2 (d) infinite

34. $\sin(3 \sin^{-1}(0.4))$ is equal to
(a) 8.624 (b) 0.944
(c) 0.256 (d) 1.2

35. If $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\theta\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\theta\right)$ is equal to $\frac{14}{\pi}$, then θ is equal to
(a) $\pi/2$ (b) π
(c) $\pi/7$ (d) none of these

36. $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$, $0 < x < \pi/2$ is equal to
(a) $x/2$ (b) x
(c) $\pi/4 + x/2$ (d) $\pi/2$

37. If $x < 0$, then $\tan^{-1}x + \tan^{-1}(1/x)$ equals
(a) $\pi/4$ (b) $\pi/3$
(c) $-\pi/2$ (d) π

38. If $\sin^{-1}x + \sin^{-1}2x = \pi/3$, then $x =$
(a) $\sqrt{(3/7)}$ (b) $(1/2)\sqrt{(3/7)}$
(c) $-\sqrt{(3/7)}$ (d) $-(1/2)\sqrt{(3/7)}$

Assertion-Reason Type Questions

39. **Statement-1:** $\tan^{-1}\left\{x+\sqrt{1+x^2}\right\}=\frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$

Statement-2: $\sin^2\left[2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right]=1-x^2$

40. **Statement-1:** $\tan^{-1}\frac{2}{5}+\tan^{-1}\frac{3}{7}=\frac{\pi}{4}$

Statement-2: $\tan^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y-x}{y+x}\right)=\frac{\pi}{4}$

$(x, y > 0)$

41. **Statement-1:** If $x^2 - px + q = 0$, where p is twice the tangent of the arithmetic mean of $\sin^{-1}x$ and $\cos^{-1}x$, q is the geometric mean of $\tan^{-1}x$ and $\cot^{-1} x$, then $x = 1$

Statement-2: $\tan(\sin^{-1}x + \cos^{-1}x) = 1$

42. **Statement-1:** If α, β are the roots of the equation $18(\tan^{-1}x)^2 - 9\pi \tan^{-1}x + \pi^2 = 0$, then $\alpha + \beta = 4/\sqrt{3}$.

Statement-2: $\sec^2(\cos^{-1}(1/4)) + \operatorname{cosec}^2(\sin^{-1}(1/5)) = 41$.

43. **Statement-1:** $\sin^{-1}\tan(\tan^{-1}x + \tan^{-1}(1 - x)) = \pi/2$ has no non-zero integral solution.

Statement 2 : The number of positive integral solutions of the equation $\tan^{-1} x + \cos^{-1}\frac{y}{\sqrt{1-y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$ is 2.

44. **Statement-1:** $\cos\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)=\frac{3}{4}$.

Statement-2: If $\cos\theta = x$, then $\cos(\theta/2) = \sqrt{\frac{1+x}{2}}$

45. **Statement-1:** $\sin\cot^{-1}\tan\cos^{-1}\left(3/\sqrt{13}\right) = 3/\sqrt{13}$

Statement-2: If $\alpha = \cos^{-1}(4/5)$, then $\cos^{-1}(44/125) = \pi - 3\alpha$.

LEVEL 2

Straight Objective Type Questions

46. If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = 5\pi^2/8$, then x is equal to
(a) – 1 (b) 0
(c) 1 (d) none of these

47. $\sum_{m=1}^{n}\tan^{-1}\frac{2m}{m^4+m^2+2}=$

(a) $\tan^{-1}(n^2 + n + 1)$ (b) $\tan^{-1}(n^2 - n + 1)$

(c) $\tan^{-1}\dfrac{n^2+n}{n^2+n+2}$ (d) none of these

48. If $\cos^{-1}(x/a) + \cos^{-1}(y/b) = \alpha$. Then $x^2/a^2 + y^2/b^2$ is equal to
(a) $(2xy/ab)\cos\alpha + \sin^2\alpha$
(b) $(2xy/ab)\sin\alpha + \cos^2\alpha$
(c) $(2xy/ab)\cos^2\alpha + \sin\alpha$
(d) $(2xy/ab)\sin^2\alpha + \cos\alpha$

49. If $\alpha = 2\tan^{-1}\left(2\sqrt{2}-1\right)$ and $\beta = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$ then
(a) $\alpha < \beta$ (b) $\alpha = \beta$
(c) $\alpha > \beta$ (d) none of these

50. If $y = \tan^{-1}\dfrac{1-x}{1+x}$, $0 \le x \le 1$, then
(a) $0 \le y \le \pi$ (b) $0 \le y \le \pi/4$
(c) $-\pi/4 \le y \le \pi/4$ (d) $\pi/4 \le y \le \pi/2$

51. $\dfrac{\alpha^3}{2}\operatorname{cosec}^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\alpha}{\beta}\right) + \dfrac{\beta^3}{2}\sec^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\beta}{\alpha}\right)$ is equal to
(a) $(\alpha+\beta)(\alpha^2+\beta^2)$ (b) $(\alpha-\beta)(\alpha^2+\beta^2)$
(c) $(\alpha-\beta)(\alpha^2-\beta^2)$ (d) $(\alpha+\beta)(\alpha^2-\beta^2)$

52. The value of $\sin^{-1}x - \sin^{-1}\left(\dfrac{x}{2} - \dfrac{\sqrt{3-3x^2}}{2}\right)$; $0 \le x \le 1/2$ is equal to
(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) 0

53. The equation $\sin^{-1}6x + \sin^{-1}6\sqrt{3}x = -\pi/2$ has
(a) only integral solutions
(b) two integral solutions
(c) no integral solution
(d) two real solutions

54. If $2\tan^{-1}\left[\sqrt{\dfrac{a-b}{a+b}}\cdot\tan\left(\dfrac{\theta}{2}\right)\right] = \cos^{-1}\left[\dfrac{2a+3b}{3a+2b}\right]$, then $\cos\theta$ is equal to
(a) 1/2 (b) 1/3
(c) 2/3 (d) $2a/3b$

55. $\tan^{-1}\left(\dfrac{x\cos\theta}{1-x\sin\theta}\right) - \cot^{-1}\left(\dfrac{\cos\theta}{x-\sin\theta}\right)$ is equal to
(a) θ (b) $\pi/2 - \theta$
(c) $\theta/2$ (d) $\pi/4 - \theta/2$

Previous Years' AIEEE/JEE Main Questions

1. $\cot^{-1}[(\cos\alpha)^{1/2}] - \tan^{-1}[(\cos\alpha)^{1/2}] = x$, then $\sin x =$
(a) $\tan^2(\alpha/2)$ (b) $\cot^2(\alpha/2)$
(c) $\tan\alpha$ (d) $\cot(\alpha/2)$ **[2002]**

2. The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for
(a) all real values of a
(b) $|a| \le 1/\sqrt{2}$
(c) $|a| \ge 1/\sqrt{2}$
(d) $1/2\,|a| < 1/\sqrt{2}$ **[2003]**

3. If $\cos^{-1}x - \cos^{-1}(y/2) = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to
(a) $4\sin^2\alpha$ (b) $-4\sin^2\alpha$
(c) $2\sin 2\alpha$ (d) 4 **[2005]**

4. If $\sin^{-1}(x/5) + \operatorname{cosec}^{-1}(5/4) = \pi/2$ then a value of x is
(a) 1 (b) 3
(c) 4 (d) 6 **[2007]**

5. The value of $\cot(\operatorname{cosec}^{-1}(5/3) + \tan^{-1}(2/3))$ is
(a) 5/17 (b) 6/17
(c) 3/17 (d) 4/17 **[2008]**

6. If x, y, z are in A.P and $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are also in A.P, then
(a) $2x = 3y = 6z$ (b) $6x = 3y = 2z$
(c) $6x = 4y = 3z$ (d) $x = y = z$ **[2013]**

7. A value of x for which $\sin(\cot^{-1}(1 + x)) = \cos(\tan^{-1}x)$, is:
(a) $-1/2$ (b) 1
(c) 0 (d) 1/2. **[2013, online]**

8. The number of solutions of the equation $\sin^{-1} x = 2\tan^{-1} x$ (in principal values) is:
   (a) 1 (b) 4
   (c) 2 (d) 3 **[2013, online]**

9. If $S = \tan^{-1}\dfrac{1}{x^2+x+1} + \tan^{-1}\dfrac{1}{x^2+3x+3} + \ldots + \tan^{-1}\dfrac{1}{1+(x+19)(x+20)}$, then tan S is equal to
   (a) $\dfrac{20}{401+20x}$ (b) $\dfrac{x}{x^2+20x+1}$
   (c) $\dfrac{20}{x^2+20x+1}$ (d) $\dfrac{x}{401+20x}$
   [2013, online]

10. **Statement I:** The equation $(\sin^{-1} x)^3 + (\cos^{-1} x)^3 - a\pi^3 = 0$ has a solution for all $a \geq \dfrac{1}{32}$. For any $x \in \mathbf{R}$.
    Statement II: $\sin^{-1}x + \cos^{-1}x = \pi/2$ and $0 \leq (\sin^{-1}x - \pi/4)^2 \leq 9\pi^2/16$
    (a) Both statements I and II are true
    (b) Both statements I and II are false
    (c) Statement I is true and statement II is false
    (d) Statement I is false and statement II is true
    [2014, online]

11. The principal value of $\tan^{-1}\left(\cot\dfrac{43\pi}{4}\right)$ is:
    (a) $-(3\pi/4)$ (b) $3\pi/4$
    (c) $\pi/4$ (d) $-(\pi/4)$ **[2014, online]**

12. Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ where $|x| < \dfrac{1}{\sqrt{3}}$, then a value of y is:
    (a) $\dfrac{3x-x^3}{1-3x^2}$ (b) $\dfrac{3x+x^3}{1-3x^2}$
    (c) $\dfrac{3x-x^3}{1+3x^2}$ (d) $\dfrac{3x+x^3}{1+3x^2}$ **[2015]**

13. If $f(x) = 2\tan^{-1}x + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, $x > 1$, then $f(5)$ is equal to:
    (a) $\dfrac{\pi}{2}$ (b) π
    (c) $4\tan^{-1}(5)$ (d) $\tan^{-1}\left(\dfrac{65}{156}\right)$
    [2015, online]

Previous Years' B-Architecture Entrance Examination Questions

1. $2\cot^{-1}(7) + \cos^{-1}(3/5)$, in principal value, is equal to
   (a) $\cos^{-1}\left(\dfrac{44}{125}\right)$ (b) $\cot^{-1}\left(\dfrac{44}{117}\right)$
   (c) $\tan^{-1}\left(\dfrac{41}{117}\right)$ (d) $\text{cosec}^{-1}\left(\dfrac{117}{125}\right)$ **[2013]**

2. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is:
   (a) $\dfrac{20}{19}$ (b) $\dfrac{19}{21}$
   (c) $\dfrac{21}{19}$ (d) $\dfrac{19}{20}$ **[2016]**

Answers

Concept-based

1. (b)	**2.** (c)	**3.** (b)	**4.** (c)
5. (c)	**6.** (c)	**7.** (a)	**8.** (a)
9. (b)	**10.** (b)		

Level 1

11. (c)	**12.** (d)	**13.** (a)	**14.** (b)
15. (d)	**16.** (b)	**17.** (d)	**18.** (b)
19. (c)	**20.** (d)	**21.** (b)	**22.** (b)
23. (a)	**24.** (b)	**25.** (a)	**26.** (a)
27. (c)	**28.** (c)	**29.** (c)	**30.** (c)
31. (d)	**32.** (a)	**33.** (b)	**34.** (b)
35. (c)	**36.** (a)	**37.** (c)	**38.** (b)
39. (b)	**40.** (a)	**41.** (c)	**42.** (b)
43. (d)	**44.** (a)	**45.** (b)	

Level 2

46. (a)	**47.** (c)	**48.** (a)	**49.** (c)
50. (b)	**51.** (a)	**52.** (c)	**53.** (c)
54. (c)	**55.** (a)		

Previous Years' AIEEE/JEE Main Questions

1. (a)	**2.** (b)	**3.** (a)	**4.** (b)
5. (b)	**6.** (d)	**7.** (a)	**8.** (d)

9. (c) **10.** (b) **11.** (d) **12.** (c)
13. (b)

Previous Years' B-Architecture Entrance Examination Questions

1. (b) **2.** (c)

Hints and Solutions

Concept-based

1. $\sin(\sin^{-1}(3/5) + \cos^{-1}(4/5)) = \sin(2\sin^{-1}(3/5)) = 2 \times (3/5) \times (4/5) = 24/25$

2. $2\tan^{-1} 7 = \pi + \sin^{-1} \dfrac{2 \times 7}{\sqrt{1+7^2}}$

   (use $2\tan^{-1} x = \pi + \sin^{-1} \dfrac{2x}{\sqrt{1+x^2}}$ if $x > 1$)

3. $\dfrac{7\pi}{3} = \alpha + \beta = 2 \times \dfrac{\pi}{2} + \cos^{-1} x + 2 \times \dfrac{\pi}{2} + \cos^{-1} y$

   $\Rightarrow \cos^{-1} x + \cos^{-1} y = \dfrac{7\pi}{3} - 2\pi = \dfrac{\pi}{3}$

   $\Rightarrow \cos(\cos^{-1} x + \cos^{-1} y) = \cos(\pi/3) = 1/2$

   $\Rightarrow xy - \sqrt{(1-x^2)(1-y^2)} = 1/2.$

4. $2\left(-\dfrac{\pi}{6}\right) + 3\left(\dfrac{5\pi}{6}\right) + 4\left(\dfrac{\pi}{4}\right) + 5\left(-\dfrac{\pi}{4}\right)$

   $= \dfrac{23\pi}{12}$

5. $xy = 1 + \dfrac{x}{z} + \dfrac{y}{z} > 1$

   So $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$

   $= \pi + \tan^{-1}\left[\dfrac{x+y}{1-xy}\right] + \tan^{-1} z$

   $= \pi + \tan^{-1}\left[\dfrac{xyz - z}{1-xy}\right] + \tan^{-1} z$

   $= \pi + \tan^{-1}(-z) + \tan^{-1} z = \pi$

6. $\tan(\cos^{-1} x) = \sin(\cot^{-1} \dfrac{1}{2})$

   $\Rightarrow \tan\left(\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)\right) = \sin\left(\sin^{-1}\left(\dfrac{2}{\sqrt{5}}\right)\right)$

   $\Rightarrow \dfrac{1-x^2}{x^2} = \dfrac{4}{5} \Rightarrow x = \dfrac{\sqrt{5}}{3}$

   [**Note:** $x = -\dfrac{\sqrt{5}}{3}$ makes L.H.S < 0 but R.H.S > 0]

7. $\sin^{-1}(\cos(x + \pi - x)) = \sin^{-1}(\cos \pi) = \sin^{-1}(-1) = -(\pi/2)$

8. $x = \tan^{-1}(3) \Rightarrow \tan x = 3$

   So $\tan(x + y) = 33$

   $\Rightarrow \dfrac{\tan x + \tan y}{1 - \tan x \tan y} = 33$

   $\Rightarrow \dfrac{3 + \tan y}{1 - 3\tan y} = 33 \Rightarrow \tan y = \dfrac{30}{100} = 0.3$

9. $\sin^{-1}(x-1)$ is defined for $-1 \le x - 1 \le 1 \Rightarrow 0 \le x \le 2$

   $\cos^{-1}(x-3)$ is defined for $-1 \le x - 3 \le 1 \Rightarrow 2 \le x \le 4$.

   so we get $x = 2$ and

   $\cos^{-1} k + \pi = \sin^{-1}(2-1) + \cos^{-1}(2-3) + \tan^{-1}(-1)$

   $= \dfrac{\pi}{2} + \pi - \dfrac{\pi}{4}$

   $\Rightarrow \cos^{-1} k = \pi/4 \Rightarrow k = 1/\sqrt{2}$

10. $\sin[2\sin^{-1}(3/5) + 2\sin^{-1}(4/5)]$

    $= \sin[\sin^{-1} 2 \times \dfrac{3}{5} \times \dfrac{4}{5} + \sin^{-1} 2 \times \dfrac{4}{5} \times \dfrac{3}{5}]$

    $= \sin[2\sin^{-1} \dfrac{24}{25}] = \sin[\sin^{-1} 2 \times \dfrac{24}{25} \times \dfrac{7}{25}]$

    $= \dfrac{336}{625}.$

Level 1

11. $\tan^{-1} x = \sin^{-1} \dfrac{x}{\sqrt{1+x^2}}$, $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$

    So the required value is

    $\dfrac{x^2}{1+x^2} + 1 - x^2 = \dfrac{4/9}{1+4/9} + 1 - \dfrac{4}{9} = \dfrac{101}{117}.$

12. $\tan[2(\tan^{-1}(1/5) - \pi/4]$

    $= \dfrac{\tan\left(2\tan^{-1}\left(\dfrac{1}{5}\right)\right) - 1}{1 + \tan\left(2\tan^{-1}\left(\dfrac{1}{5}\right)\right)} = \dfrac{\dfrac{2 \times (1/5)}{1 - (1/25)} - 1}{1 + \dfrac{10}{24}} = -\dfrac{7}{17}.$

13. $a \ne b$, $\dfrac{a-x}{a-b} \le 1$, $\dfrac{x-b}{a-b} \le 1$

    $\Rightarrow b \le x$, $x \le a \Rightarrow a > x > b$.

14. $\cos^{-1}(\cos(-17\pi/5))$

    $= \cos^{-1}\left(\cos\left(\dfrac{17\pi}{5}\right)\right) = \cos^{-1}\left(\cos\left(4\pi - \dfrac{3\pi}{5}\right)\right)$

    $= \cos^{-1}\left(\cos\left(\dfrac{3\pi}{5}\right)\right) = \dfrac{3\pi}{5}$

15. $2\theta = \cos^{-1}(3/5) \Rightarrow \cos 2\theta = 3/5$

$\Rightarrow \tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta} = \dfrac{1}{4}.$

16. $x = \tan\theta \Rightarrow \sin(\cot^{-1}(\cos(\tan^{-1}x)))$

$= \sin(\cot^{-1}(\cos\theta)) = \sin\left(\cot^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)\right)$

$= \sin\left(\tan^{-1}\left(\sqrt{1+x^2}\right)\right)$

$= \sin\left(\sin^{-1}\dfrac{\sqrt{1+x^2}}{\sqrt{1+1+x^2}}\right) = \sqrt{\dfrac{x^2+1}{x^2+2}}.$

17. $\tan^{-1}(b/x) = \cot^{-1}(a/x) \Rightarrow x^2 = ab.$

18. $\cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right] = \pi - \cos^{-1}z = \cos^{-1}(-z)$

$\Rightarrow \sqrt{1-x^2}\sqrt{1-y^2} = xy + z$

$\Rightarrow (1-x^2)(1-y^2) = (xy+z)^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

19. $\cos\theta = \dfrac{x}{2}\times\dfrac{y}{3} - \sqrt{1-\dfrac{x^2}{2^2}}\sqrt{1-\dfrac{y^2}{3^2}}$

$\Rightarrow (4-x^2)(9-y^2) = (xy - 6\cos\theta)^2$

$\Rightarrow 36 - 9x^2 - 4y^2 = 36\cos^2\theta - 12xy\cos\theta$

$\Rightarrow 9x^2 - 12xy\cos\theta + 4y^2 = 36(1-\cos^2\theta)$
$= 36\sin^2\theta$

20. Use $\tan(A_1 + A_2 + A_3) = \dfrac{S_1 - S_3}{1 - S_2}.$

If the given expression is θ, then $\tan\theta = \dfrac{S_1 - S_3}{1 - S_2}$

where $S_1 = \dfrac{\sqrt{a+b+c}}{\sqrt{abc}}(a+b+c)$ and

$S_3 = \dfrac{(a+b+c)^{3/2}}{\sqrt{abc}}.$

$\Rightarrow S_1 - S_3 = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = \tan^{-1}0 = 0$

21. $\tan\left[\tan^{-1}\dfrac{a_2-a_1}{1+a_1a_2} + \tan^{-1}\dfrac{a_3-a_2}{1+a_2a_3} + \cdots + \tan^{-1}\dfrac{a_n-a_{n-1}}{1+a_na_{n-1}}\right]$

$= \tan[\tan^{-1}a_2 - \tan^{-1}a_1 + \tan^{-1}a_3 - \tan^{-1}a_2 + \ldots + \tan^{-1}a_n - \tan^{-1}a_{n-1}]$

$= \tan[\tan^{-1}a_n - \tan^{-1}a_1] = \dfrac{a_n - a_1}{1+a_1a_n} = \dfrac{(n-1)d}{1+a_1a_n}.$

22. Let $A = \sin^{-1}\left(1/\sqrt{5}\right)$, $B = \sin^{-1}\left(1/\sqrt{10}\right)$

then $\sin(A+B) = \dfrac{1}{\sqrt{5}}\sqrt{1-\dfrac{1}{10}} + \dfrac{1}{\sqrt{10}}\sqrt{1-\dfrac{1}{5}}$

$= \dfrac{3}{\sqrt{5}}\sqrt{1-\dfrac{1}{10}} + \dfrac{1}{\sqrt{10}}\sqrt{1-\dfrac{1}{5}}$

$\Rightarrow A + B = \pi/4 \Rightarrow C = 3\pi/4$

23. Let $y = \sin^{-1}x \Rightarrow \sin y + \dfrac{1}{\sin y} = \dfrac{5}{2}$

$\Rightarrow 2\sin^2 y - 5\sin y + 2 = 0$

$\Rightarrow \sin y = (1/2) \Rightarrow y = \sin^{-1}(1/2) = \pi/6$

24. $\pi + \tan^{-1}\dfrac{a+b}{1-ab} = \tan^{-1}7$ as $a > 0$, $b > 0$, $ab > 1$

$\Rightarrow \dfrac{a+b}{1-ab} = 7 \Rightarrow a = \dfrac{7-b}{1+7b}.$

which does not hold for any positive integral pairs (a, b).

25. $\alpha = \sin^{-1}\left[\cot\left(\theta + \dfrac{\pi}{6} + \dfrac{\pi}{4}\right)\right]$ where $\sin^2\theta = \dfrac{2-\sqrt{3}}{4}.$

$= \sin^{-1}\left[\dfrac{\cot\theta\cot(\pi/4+\pi/6)-1}{\cot\theta + \cot(\pi/4+\pi/6)}\right]$

Now,

$\cot\theta = \sqrt{\dfrac{4}{2-\sqrt{3}} - 1} = \sqrt{\dfrac{2+\sqrt{3}}{2-\sqrt{3}}} = 2+\sqrt{3}$ and

$\cot\left(\dfrac{\pi}{4}+\dfrac{\pi}{6}\right) = \dfrac{\cot\dfrac{\pi}{6}-1}{1+\cot\dfrac{\pi}{6}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} =$

$\dfrac{\left(\sqrt{3}-1\right)^2}{2} = 2-\sqrt{3}$

So $\alpha = \sin^{-1}0 = 0.$

26. $u_n = \cot^{-1}(2n^2) = \tan^{-1}\dfrac{1}{2n^2}$

$= \tan^{-1}\dfrac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}$

$= \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

$\Rightarrow S_n = \tan^{-1}(2n+1) - \tan^{-1}1$

$\lim\limits_{n\to\infty} S_n = \dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4}.$

27. $3\pi/2 < 5 < 2\pi$ so $\sin^{-1}(\sin 5) = \sin^{-1}\sin(5 - 2\pi)$ $= 5 - 2\pi$

$\Rightarrow -(2\pi - 5) > x^2 - 4x \Rightarrow x^2 - 4x + (2\pi - 5) < 0$

$$\Rightarrow \left[x - \frac{4-\sqrt{16-4(2\pi-5)}}{2}\right] \times \left[x - \frac{4+\sqrt{16-4(2\pi-5)}}{2}\right] < 0$$

$\Rightarrow x \in \left(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi}\right)$

28. $x \in \left(1/\sqrt{2}, 1\right) \Rightarrow \sin^{-1}(x) \in (\pi/4, \pi/2)$,

$\cos^{-1} x \in (0, \pi/4)$

so $\sin^{-1} x > \cos^{-1} x$ if $x \in \left(1/\sqrt{2}, 1\right)$

29. Let $x = \sin\theta$, then $2\sin^{-1} x = 2\theta$

$-\pi/2 \le \theta \le \pi/2$

Also $\cos^{-1}(1 - 2x^2) = \cos^{-1}(1 - 2\sin^2\theta) = 2\theta$.

$\Rightarrow 0 \le 2\theta \le \pi \Rightarrow 0 \le \theta \in \pi/2$

$\therefore\ 2\sin^{-1} x = \cos^{-1}(1 - 2x^2)$ holds for $0 \le \theta \le \pi/2$

$\Rightarrow 0 \le x \le 1$

30. $\tan^{-1} x - \tan^{-1} y = 0 \Rightarrow x = y$.

$\cos^{-1} x + \cos^{-1} y = \pi/2 \Rightarrow 2\cos^{-1} x = \pi/2 \Rightarrow \cos^{-1} x = \pi/4$

$\Rightarrow x = \cos\pi/4 = 1/\sqrt{2} \Rightarrow x^2 = 1/2$

and $x^2 + xy + y^2 = 3x^2 = 3/2$.

31. $\tan x = -1$, $\tan y = 1/7$

32. If $x > 1$, $2\tan^{-1}x = \pi + \sin^{-1}\dfrac{2x}{1+x^2}$

33. $1 - x = \sin(\pi/2 + 2\sin^{-1}x)$

$= \cos(2\sin^{-1}x) = \cos(\cos^{-1}(1 - 2x^2))$

$= 1 - 2x^2 \Rightarrow x = 0$ or $1/2$

But $x = 1/2$ does not satisfy the equation

34. Apply $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

35. $\tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)+\tan\left(\dfrac{\pi}{4}-\dfrac{\alpha}{2}\right)$ $[\theta = \cos\alpha]$

$$= \frac{(1+\tan(\alpha/2))^2+(1-\tan(\alpha/2))^2}{1-\tan^2(\alpha/2)}$$

$$= \frac{2}{\cos\alpha} = \frac{2}{\theta} = \frac{14}{\pi} \Rightarrow \theta = \pi/7.$$

36. $1 \pm \sin x = (\cos x/2 \pm \sin x/2)^2$

37. If $x < 0$, $\tan^{-1}(1/x) = \cot^{-1}x - \pi$ and $\tan^{-1}x + \cot^{-1}x = \pi/2$

38. $\sin^{-1}2x = \pi/3 - \sin^{-1}x$

$\Rightarrow 2x = \sin(\pi/3 - \sin^{-1}x) = \dfrac{\sqrt{3}}{2}\sqrt{1-x^2} - \dfrac{1}{2}x$

$\Rightarrow (5x)^2 = 3(1 - x^2) \Rightarrow 28x^2 = 3$

$\Rightarrow x = \pm\dfrac{1}{2}\cdot\sqrt{\dfrac{3}{7}}$.

$x \ne -\dfrac{1}{2}\cdot\sqrt{\dfrac{3}{7}}$ as the sum of two negative angles can not be positive.

39. Put $x = \cot\theta$, then L.H.S

$$= \tan^{-1}\left[\cot\theta + \sqrt{1+\cot^2\theta}\right] = \tan^{-1}\frac{1+\cos\theta}{\sin\theta}$$

$= \tan^{-1}(\cot(\theta/2)) = \tan^{-1}(\tan(\pi/2 - \theta/2))$

$= \pi/2 - \theta/2$.

$\Rightarrow$ Statement-1 is true,

for Statement-2 L.H.S $= \sin^2 2\theta$, $\theta = \tan^{-1}\sqrt{\dfrac{1+x}{1-x}}$

$$= \left(\frac{2\tan\theta}{1+\tan^2\theta}\right)^2 = \frac{4\tan^2\theta}{\left(1+\tan^2\theta\right)^2}$$

$$= \frac{4\left(\frac{1+x}{1-x}\right)}{\left[1+\left(\frac{1+x}{1-x}\right)\right]^2} = 1 - x^2$$

$\Rightarrow$ Statement-2 is also true.

40. In Statement-2, L.H.S $= \tan^{-1}\dfrac{\frac{x}{y}+\frac{y-x}{y+x}}{1-\frac{x(y-x)}{y(y+x)}}$

$$= \tan^{-1}\frac{x^2+y^2}{x^2+y^2} = \tan^{-1}1 = \pi/4$$

$\Rightarrow$ Statement-2 is true and which shows that Statement–1 is true by taking $x = 2$, $y = 5$.

41. $p = 2\tan\dfrac{\sin^{-1}x+\cos^{-1}x}{2} = 2\tan\dfrac{\pi}{4} = 2$,

$q = \sqrt{\tan^{-1}x\cot^{-1}x} = 1$

so $x^2 - 2x + 1 = 0 \Rightarrow x = 1$, the Statement-1 is true.

Statement-2 is false.

42. In Statement-1 $\tan^{-1}x = \pi/6$ or $\pi/3$

$\Rightarrow x = 1/\sqrt{3}$ or $\sqrt{3}$.

$\Rightarrow \alpha + \beta = 4/\sqrt{3} \Rightarrow$ Statement-1 is true.

In Statement-2, L.H.S = $\sec^2(\sec^{-1} 4) + \text{cosec}^2 (\text{cosec}^{-1} 5)$

= 16 + 25 = 41 and the Statement-2 is true but does not lead to Statement-1

43. In Statement-1 $\frac{x+1-x}{1-x(1-x)} = \sin(\pi/2) = 1$

$\Rightarrow$ $1 - x(1 - x) = 1$

$\Rightarrow$ $x = 1$ is a non-zero solution $\Rightarrow$ Statement-1 is false

In Statement-2:

$$\tan^{-1} x + \cos^{-1} \frac{y}{\sqrt{1-y^2}} = \sin^{-1} \frac{3}{\sqrt{10}}$$

$\Rightarrow$ $\tan^{-1} x + \tan^{-1}(1/y) = \tan^{-1}(3)$

$\Rightarrow$ $\tan^{-1}(1/y) = \tan^{-1} 3 - \tan^{-1} x$

$\Rightarrow$ $y = \frac{1+3x}{3-x}$.

As x, y are positive integers

$x = 1, 2 \Rightarrow y = 2, 7$ and the solutions are (1, 2), (2, 7)

$\Rightarrow$ Statement-2 is true.

44. Statement-2 is true, Taking $x = 1/8$, Statement-1 is true.

45. Put $\cos\theta = 3/\sqrt{3}$ in Statement-1,

L.H.S = $\sin(\cot^{-1}(\tan\theta)) = \sin\left(\frac{\pi}{2} - \theta\right)$

$= \cos\theta = 3/\sqrt{13}$

$\Rightarrow$ Statement-1 is true.

In Statement-2, $\cos\alpha = 4/5$

$\Rightarrow \cos(\pi - 3\alpha) = -\cos 3\alpha$

$= 3\cos\alpha - 4\cos^3\alpha = 44/125$

$\Rightarrow$ Statement-2 is also true but does not lead to Statement-1.

Level 2

46. We can write $(\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1} x \cot^{-1} x = 5\pi^2/8$

$\Rightarrow \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1} x(\pi/2 - \tan^{-1}x) = 5\pi^2/8$

$\Rightarrow (\tan^{-1} x)^2 - (\pi/2)\tan^{-1} x - 3\pi^2/16 = 0$

$\Rightarrow \tan^{-1} x = 3\pi/4$ or $-\pi/4 \Rightarrow x = -1$

47. $\sum_{m=1}^{n} \tan^{-1} \frac{2m}{m^4 + m^2 + 2}$

$$= \sum_{m=1}^{n} \tan^{-1} \frac{1+m+m^2-(1-m+m^2)}{1+(1+m+m^2)(1-m+m^2)}$$

$$= \sum_{m=1}^{n}\left[\tan^{-1}(1+m+m^2) - \tan^{-1}(1-m+m^2)\right]$$

$$= \sum_{m=1}^{n}\left[\tan^{-1}(1+m+m^2) - \tan^{-1}(1+(m-1)+(m-1)^2)\right]$$

$= \tan^{-1}(1 + n + n^2) - \tan^{-1} 1$

$= \tan^{-1}\frac{n+n^2}{1+1+n+n^2} = \tan^{-1}\frac{n+n^2}{2+n+n^2}$

48. We have $\cos\alpha = \frac{xy}{ab} - \sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)}$

$\Rightarrow \left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right) = \left(\frac{xy}{ab} - \cos\alpha\right)^2$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \cos^2\alpha + \frac{2xy}{ab}\cos\alpha\cdot$

49. $\beta = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$

$= \sin^{-1}(3 \times (1/3) - 4(1/3)^3) + \sin^{-1}(3/5)$

$= \sin^{-1}(23/27) + \sin^{-1}(3/5)$

$< \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

[$\because$ $\frac{23}{27} = 0.85$, $\frac{3}{5} = 0.6$ and $\frac{\sqrt{3}}{2} = 1.7$

$\Rightarrow \beta < 2\pi/3$

$\alpha = 2\tan^{-1}\left(2\sqrt{2} - 1\right) > 2\tan^{-1}\sqrt{3} = 2\pi/3$

[$\because$ $2\sqrt{2} - 1 = 1.8$, $\sqrt{3} = 1.7$]

$\Rightarrow \alpha > 2\pi/3$

Hence $\alpha > \beta$

50. Let $x = \tan\theta$, then $0 \le x \le 1 \Rightarrow 0 \le \theta \le \pi/4$

$y = \tan^{-1}\frac{1-\tan\theta}{1+\tan\theta} = \pi/4 - \theta$

$\Rightarrow \theta = \pi/4 - y$ so $0 \le y \le \pi/4$

51. $\frac{\alpha^3}{2}\text{cosec}^2(\theta/2) + \frac{\beta^3}{2}\sec^2(\pi/4 - \theta/2)$

(Taking $\frac{\alpha}{\beta} = \tan\theta \Rightarrow \sin\theta = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$,

$\cos\theta = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$)

$= \frac{\alpha^3}{2}[1 + \cot^2(\theta/2)] + \frac{\beta^3}{2}[1 + \tan^2(\pi/4 - \theta/2)]$

$$= \frac{\alpha^3+\beta^3}{2} + \frac{\alpha^3}{2}\frac{1+\cos\theta}{1-\cos\theta} + \frac{\beta^3}{2}\left[\frac{1-\tan\theta/2}{1+\tan\theta/2}\right]^2$$

$$= \frac{\alpha^3+\beta^3}{2} + \frac{\alpha^3}{2}\left[\frac{1+\cos\theta}{1-\cos\theta}\right] + \frac{\beta^3}{2}\left[\frac{1-\sin\theta}{1+\sin\theta}\right]$$

$$= \frac{\alpha^3+\beta^3}{2} + \frac{\alpha^3}{2}\frac{\sqrt{\alpha^2+\beta^2}+\beta}{\sqrt{\alpha^2+\beta^2}-\beta} + \frac{\beta^3}{2}\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{\sqrt{\alpha^2+\beta^2}+\alpha}$$

$$= \frac{\alpha^3+\beta^3}{2} + \frac{\alpha^3}{2}\frac{\alpha^2+\beta^2+\beta^2+2\beta\sqrt{\alpha^2+\beta^2}}{\alpha^2+\beta^2-\beta^2}$$

$$+ \frac{\beta^3}{2}\frac{\alpha^2+\beta^2+\alpha^2-2\alpha\sqrt{\alpha^2+\beta^2}}{\alpha^2+\beta^2-\alpha^2}$$

$$= \frac{\alpha^3+\beta^3}{2} + \frac{\alpha}{2}\left[\alpha^2+2\beta^2+2\beta\sqrt{\alpha^2+\beta^2}\right] +$$

$$\frac{\beta}{2}\left(2\alpha^2+\beta^2-2\alpha\sqrt{\alpha^2+\beta^2}\right)$$

$= \alpha^3 + \beta^3 + \alpha\beta^2 + \alpha^2\beta$

$= (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2 + \alpha\beta)$

$= (\alpha+\beta)(\alpha^2+\beta^2)$

52. Put $x = \sin\theta$, then $0 \le x \le 1/2 \Rightarrow 0 \le \theta \le \pi/6$.

So, the given expression is equal to

$\sin^{-1}(\sin\theta) - \sin^{-1}\left[\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\right]$

$= \theta - \sin^{-1}(\sin(\theta - \pi/3))$

$= \theta - (\theta - \pi/3) = \pi/3$

53. $\sin^{-1} 6x = -\pi/2 - \sin^{-1} 6\sqrt{3}x$

$\Rightarrow 6x = -\sin(\pi/2 + \sin^{-1}6\sqrt{3}x)$

$= -\cos(\sin^{-1} 6\sqrt{3}x)$

$= \sqrt{1-108x^2}$

$\Rightarrow 36x^2 = 1 - 108x^2 \Rightarrow 144x^2 = 1 \Rightarrow x = \pm 1/12.$

But $x = 1/12$ does not satisfy the equation.

So the given equation has only one non-integral solution

54. Let $\frac{2a+3b}{3a+2b} = \cos\alpha$

$\Rightarrow \frac{a}{2\cos\alpha - 3} = \frac{b}{2-3\cos\alpha}$

$\Rightarrow \sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{5(1-\cos\alpha)}{1+\cos\alpha}} = \sqrt{5}\tan(\alpha/2)$

So $2\tan^{-1}[\sqrt{5}\tan(\alpha/2)\tan(\theta/2)] = \alpha$.

$\Rightarrow \sqrt{5}\tan(\alpha/2)\tan(\theta/2) = \tan(\alpha/2)$

$\Rightarrow \tan(\theta/2) = 1/\sqrt{5} \Rightarrow \cos\theta = \frac{1-(1/5)}{1+(1/5)} = \frac{2}{3}$

55. $\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x-\sin\theta}\right)$

$= \tan^{-1}\frac{x\cos\theta}{1-x\sin\theta} - \tan^{-1}\frac{x-\sin\theta}{\cos\theta}$

$$= \tan^{-1}\frac{\frac{x\cos\theta}{1-x\sin\theta} - \frac{x-\sin\theta}{\cos\theta}}{1+\frac{x\cos\theta}{1-x\sin\theta}\times\frac{x-\sin\theta}{\cos\theta}}$$

$= \tan^{-1}\frac{\sin\theta(x^2-2x\sin\theta+1)}{\cos\theta(x^2-2x\sin\theta+1)}$

$= \tan^{-1}(\tan\theta) = \theta$

Previous Years' AIEEE/JEE Main Questions

1. $x = (\pi/2) - 2\tan^{-1}[(\cos\alpha)^{1/2}]$

$\Rightarrow \quad (\pi/2) - x = 2\tan^{-1}[(\cos\alpha)^{1/2}]$

$\Rightarrow \quad \tan(\pi/2 - x) = \frac{2(\cos\alpha)^{1/2}}{1-\cos\alpha}$

$\Rightarrow \quad \cot x = \frac{2(\cos\alpha)^{1/2}}{1-\cos\alpha}$

$\Rightarrow \quad \operatorname{cosec}^2 x = 1 + \frac{4\cos\alpha}{(1-\cos\alpha)^2} = \left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)^2$

$\Rightarrow \quad \sin x = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{2\sin^2(\alpha/2)}{2\cos^2(\alpha/2)}$

$= \tan^2(\alpha/2)$

2. $\sin^{-1} x = \sin^{-1} 2a\sqrt{1-a^2}$ if $|a| \le 1/\sqrt{2}$

$\Rightarrow \quad x = 2a\sqrt{1-a^2}$ which is possible

if $x^2 = 4a^2(1-a^2) \le 1 \qquad [\because -1 \le \sin^{-1} x \le 1]]$

or if $4a^4 - 4a^2 + 1 \ge 0$ if $(2a^2-1)^2 \ge 0$

which is true, so $|a| \le 1/\sqrt{2}$

3. $\cos\alpha = x\left(\frac{y}{2}\right) + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}$

$\Rightarrow (2\cos\alpha - xy)^2 = (1-x^2)(4-y^2)$

$\Rightarrow 4\cos^2\alpha - 4xy\cos\alpha = 4 - 4x^2 - y^2$

$\Rightarrow 4x^2 - 4xy\cos\alpha + y^2 = 4(1-\cos^2\alpha) = 4\sin^2\alpha$

4. $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$\Rightarrow\ \sin^{-1}\left(\frac{x}{5}\right)+\sin^{-1}\left(\frac{4}{5}\right)=\frac{\pi}{2}$

$\Rightarrow\ \sin^{-1}\left(\frac{x}{5}\right)+\cos^{-1}\frac{3}{5}=\frac{\pi}{2}$

$\Rightarrow\ x = 3$ as $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$.

5. $\operatorname{cosec}^{-1}\left(\frac{5}{3}\right)=\cot^{-1}\left(\frac{4}{3}\right)$

so $\cot\left(\cot^{-1}\left(\frac{4}{3}\right)+\cot^{-1}\left(\frac{3}{2}\right)\right)=\dfrac{\frac{4}{3}\times\frac{3}{2}-1}{\frac{4}{3}+\frac{3}{2}}=\frac{6}{17}$

6. We have $2y = x + z$ and $2\tan^{-1}y=\tan^{-1}x+\tan^{-1}z$

$\Rightarrow\ \tan^{-1}\left(\frac{2y}{1-y^2}\right)=\tan^{-1}\left(\frac{x+z}{1-xz}\right)$

$\Rightarrow\ \frac{2y}{1-y^2}=\frac{x+z}{1+xz}\Rightarrow y^2=xz$

$\Rightarrow\ 4xz=(2y)^2=(x+z)^2$

$\Rightarrow\ (x-z)^2=0\quad\Rightarrow x=z.$

Thus, $z = y = x$

7. We can write $\cot^{-1}(1+x)=\sin^{-1}(\cos(\tan^{-1}x)$

$=\sin^{-1}\left(\sin\left(\frac{\pi}{2}+\tan^{-1}x\right)\right)$

$=\frac{\pi}{2}+\tan^{-1}x$

$\Rightarrow\ 1+x=\cot\left(\frac{\pi}{2}+\tan^{-1}x\right)=-x$

$\Rightarrow\ 2x=-1\quad\Rightarrow\quad x=\frac{-1}{2}$

8. Let $x=\tan\theta$.

Then we have $\tan\theta=\sin 2\theta=2\sin\theta\cos\theta$

$\Rightarrow\ \sin\theta(1-2\cos^2\theta)=0$

$\Rightarrow\ \sin\theta=0$ or $\cos\theta=\pm\frac{1}{\sqrt{2}}$ which gives 3 values of θ.

9. We can write

$S=\tan^{-1}(x+1)-\tan^{-1}x+\tan^{-1}(x+2)$

$-\tan^{-1}(x+1)+\cdots\tan^{-1}(x+20)-\tan^{-1}(x+19)$

$=\tan^{-1}(x+20)-\tan^{-1}x$

$\Rightarrow\ \tan S=\frac{x+20-x}{1+(x+20)x}=\frac{20}{x^2+20x+1}.$

10. We have $\sin^{-1}x\le\frac{\pi}{2}$, $\cos^{-1}x\le\pi$

$\Rightarrow(\sin^{-1}x)^3+(\cos^{-1}x)^3\le\frac{9}{8}\pi^3$

Thus, $(\sin^{-1}x)^3+(\cos^{-1}x)^3=2\pi^3$ has no solution.

Also, $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1\le x\le 1$, not for each real x.

$\therefore$ Both the Statements are false.

11. $\tan^{-1}\left(\cot\left(\frac{43}{4}\pi\right)\right)=\tan^{-1}(\cot(11\pi-\pi/4))$

$=\tan^{-1}(-\cot(\pi/4))=\tan^{-1}(-1)$

$=-\pi/4$

12. As $\tan^{-1}\left(\frac{2x}{1-x^2}\right)=2\tan^{-1}x$ for $|x|<1$ we get

$\tan^{-1}y=\tan^{-1}x+2\tan^{-1}x=3\tan^{-1}x$

$=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$\therefore\ y=\frac{3x-x^3}{1-3x^2}$

13. As $x>1$, $\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\pi-2\tan^{-1}(x)$

$\therefore\ f(x)=2\tan^{-1}(x)+\pi-2\tan^{-1}(x)$

$\Rightarrow\pi\ \forall\ x>1$

$\Rightarrow f(5)=\pi$

Previous Years' B-Architecture Entrance Examination Questions

1. $2\cot^{-1}(7)+\cos^{-1}\left(\frac{3}{5}\right)$

$=2\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{4}{3}$

$=\tan^{-1}\dfrac{\frac{2}{7}}{1-\frac{1}{49}}+\tan^{-1}\frac{4}{3}$

$=\tan^{-1}\frac{7}{24}+\tan^{-1}\frac{4}{3}$

$=\tan^{-1}\dfrac{\frac{7}{24}+\frac{4}{3}}{1-\frac{7\times4}{24\times3}}=\tan^{-1}\frac{117}{44}=\cot^{-1}\frac{44}{117}$

2. $1+\sum_{p=1}^{n}(2p)=1+\frac{2n(n+1)}{2}=1+n^2+n$

$$\therefore\ \cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)=\cot^{-1}(1+n(n+1))$$

$$=\ \tan^{-1}\left(\frac{n+1-n}{n+(n+1)n}\right)$$

$$=\ \tan^{-1}\ (n\ +\ 1)\ -\ \tan^{-1}(n)$$

$$\Rightarrow\ S=\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)=\tan^{-1}(20)-\frac{\pi}{4}$$

Thus, $\cot(S)=\cot\left(\tan^{-1}(20)-\frac{\pi}{4}\right)$

$$=\frac{\cot(\tan^{-1}(20))\cot\left(\frac{\pi}{4}\right)+1}{\cot\left(\frac{\pi}{4}\right)-\cot(\tan^{-1}(20))}$$

$$=\frac{\frac{1}{20}+1}{1-\frac{1}{20}}=\frac{21}{19}$$