12. Mathematical Induction

Exercise 12.1

1. Question

If P(n) is the statement "n(n + 1) is even", then what is P(3)?

Given. P(n) = n(n + 1) is even.

Find. P(3) ?

Answer

We have P(n) = n(n + 1).

= P(3) = 3(3 + 1)

= P(3) = 3(4)

Hence, P(3) = 12, So P(3) is also Even.

2. Question

If P(n) is the statement "n^3 + n is divisible by 3", prove that P(3) is true but P(4) is not true.

Answer

Given. P(n) = n^3 + n is divisible by 3

Find P(3) is true but P(4) is not true

We have P(n) = n^3 + n is divisible by 3

Let's check with P(3)

= P(3) = 3^3 + 3

= P(3) = 27 + 3

Therefore P(3) = 30, So it is divisible by 3

Now check with P(4)

= P(4) = 4^3 + 4

= P(4) = 64 + 4

Therefore P(4) = 68, So it is not divisible by 3

Hence, P(3) is true and P(4) is not true.

3. Question

If P(n) is the statement "$2^n \geq 3n$", and if P(r) is true, prove that P(r + 1) is true.

Answer

Given. P(n) = "$2^n \geq 3n$" and p(r) is true.

Prove. P(r + 1) is true

we have P(n) = $2^n \geq 3n$

Since, P(r) is true So,

= $2^r \geq 3r$

Now, Multiply both side by 2

= $2.2^r \geq 3r.2$

$= 2^{r+1} \geq 6r$

$= 2^{r+1} \geq 3r + 3r$ [since $3r>3 = 3r + 3r \geq 3 + 3r$]

Therefore $2^{r+1} \geq 3(r + 1)$

Hence, P(r + 1) is true.

4. Question

If P(n) is the statement "$n^2 + n$" is even", and if P(r) is true, then P(r + 1) is true

Given. P(n) = $n^2 + n$ is even and P(r) is true.

Prove. P(r + 1) is true

Answer

Given P(r) is true that means,

$= r^2 + r$ is even

Let Assume $r^2 + r = 2k$ - - - - - - (i)

Now, $(r + 1)^2 + (r + 1)$

$r^2 + 1 + 2r + r + 1$

$= (r^2 + r) + 2r + 2$

$= 2k + 2r + 2$

$= 2(k + r + 1)$

$= 2\mu$

Therefore, $(r + 1)^2 + (r + 1)$ is Even.

Hence, P(r + 1) is true

5. Question

Given an example of a statement P(n) such that it is true for all n ϵ N.

Answer

$P(n) = 1 + 2 + 3 + - - - - - + n = \frac{n(n+1)}{2}$

P(n) is true for all natural numbers.

Hence, P(n) is true for all n∈N

6. Question

If P(n) is the statement "$n^2 - n + 41$ is prime", prove that P(1), P(2) and P(3) are true. Prove also that P(41) is not true.

Given. P(n) = $n^2 - n + 41$ is prime

Prove. P(1),P(2) and P(3) are true and P(41) is not true.

Answer

$P(n) = n^2 - n + 41$

$= P(1) = 1 - 1 + 41$

$= P(1) = 41$

Therefore, P(1) is Prime

$= P(2) = 2^2 – 2 + 41$

$= P(2) = 4 - 2 + 41$

$= P(2) = 43$

Therefore, P(2) is prime

$= P(3) = 3^2 – 3 + 41$

$= P(3) = 9 – 3 + 41$

$= P(3) = 47$

Therefore P(3) is prime

Now, $P(41) = (41)^2 - 41 + 41$

$= P(41) = 1681$

Therefore, P(41) is not prime

Hence, P(1),P(2),P(3) are true but P(41) is not true.

Exercise 12.2

1. Question

Prove the following by the principle of mathematical induction:

$1+2+3+...+n=\frac{n(n+1)}{2}$ i.e., the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Answer

Let us Assume $P(n) = 1 + 2 + 3 + ------ + n = \frac{n(n+1)}{2}$

For n = 1

L.H.S of P(n) = 1

R.H.S of $P(n) = \frac{1(1+1)}{2} = \frac{2}{2} = 1$

Therefore, L.H.S =R.H.S

Since, P(n) is true for n = 1

Let assume P(n) be the true for n = k, so

$1 + 2 + 3 + ----- + k = \frac{k(k+1)}{2}$ --- (1)

Now

$(1 + 2 + 3 + -- + k) + (k + 1)$

$= \frac{k(k+1)}{2} + (k + 1)$

$= (k + 1)\left(\frac{k}{2} + 1\right)$

$= \frac{(k+1)(k+2)}{2}$

$= \frac{(k+1)[(k+1)+1]}{2}$

P(n) is true for n = k + 1

P(n) is true for all n∈N

So , by the principle of Mathematical Induction

Hence, P(n) = 1 + 2 + 3 + - - - + n = $\frac{n(n+1)}{2}$ is true for all n∈N

2. Question

Prove the following by the principle of mathematical induction:

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

To prove: Prove that by the Mathematical Induction.

Answer

Let Assume P(n):$1^2 + 2^2 + 3^2 + - - - + n^2 = \frac{n(n+1)(2n+1)}{6}$

For n = 1

P(1): 1 = $\frac{1(1+1)(2+1)}{6}$

1=1

= P(n) is true for n = 1

Let P(n) is true for n = k, so

P(k): $1^2 + 2^2 + 3^2 + - - - - - + k^2 = \frac{k(k+1)(2k+1)}{6}$

Let's check for P(n) = k + 1, So

P(k): $1^2 + 2^2 + 3^2 + - - - - - + k^2 + (k + 1)^2 = \frac{k+1(k+2)(2k+3)}{6}$

= $1^2 + 2^2 + 3^2 + - - - - - + k^2 + (k + 1)^2$

= $\frac{k+1(k+2)(2k+3)}{6} + (k+1)^2$

= $(k+1)\left[\frac{2k^2+k}{6} + \frac{(k+1)}{1}\right]$

= $(k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$

= $(k+1)\left[\frac{2k^2+7k+6}{6}\right]$

= $(k+1)\left[\frac{2k^2+4k+3k+6}{6}\right]$

= $(k+1)\left[\frac{2k(k+2)+3(k+2)}{6}\right]$

= $\frac{(k+1)(2k+3)(k+2)}{6}$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N by PMI

3. Question

Prove the following by the principle of mathematical induction:

$$1+3+3^2+\ldots+3^{n-1}=\frac{3^n-1}{2}$$

Answer

Let P(n) : $1 + 3 + 3^2 + - - - - + 3^{n-1} = \frac{3^n - 1}{2}$

Now, For n =1

P(1): $1 = \frac{3^1 - 1}{2} = \frac{2}{2} = 1$

Therefore, P(n) is true for n =1

Now , P(n) is true for n = k

P(k) : $1 + 3 + 3^2 + - - - - + 3^{k-1} = \frac{3^k - 1}{2}$ - - - - - (1)

Now, We have to show P(n) is true for n = k + 1

i.e P(k + 1): $1 + 3 + 3^2 + - - - - + 3^k = \frac{3^{k+1} - 1}{2}$

then, $\{1 + 3 + 3^2 + - - - - + 3^{k-1}\} + 3^{k+1-1}$

$= \frac{3k-1}{2} + 3^k$ using equation (1)

$= \frac{3k - 1 + 2.3^k}{2}$

$= \frac{3.3k - 1}{2}$

$= \frac{3^{k+1} - 1}{2}$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N

4. Question

Prove the following by the principle of mathematical induction:

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Answer

Let P(n): $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

For n = 1

P(1): $\frac{1}{1.2} = \frac{1}{1+1}$

$\frac{1}{2} = \frac{1}{2}$

= P(n) is true for n = 1

Let P(n) is true for n = k, So

$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$ - - - - - (1)

Now, Let P(n) is true for n = k + 1, So

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{k(k+1)} + \frac{k}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Then,

$$\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}$$

$$=\frac{1}{(k+1)(k+2)}+\frac{k}{k+1}$$

$$=\frac{1}{k+1}\left[\frac{k(k+2)+1}{k+2}\right]$$

$$=\frac{1}{k+1}\left[\frac{k^2+2k+1}{k+2}\right]$$

$$=\frac{1}{k+1}\left[\frac{(k+1)(k+1)}{k+2}\right]$$

$$=\frac{k+1}{k+2}$$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N

5. Question

Prove the following by the principle of mathematical induction:

$1+3+5+\ldots+(2n-1)=n^2$ i.e., the sum of first n odd natural numbers is n^2.

Answer

Let P(n): 1 + 3 + 5 + … + (2n - 1) = n^2

Let check P(n) is true for n = 1

P(1) = 1 = 1^2

1 = 1

P(n) is true for n = 1

Now, Let’s check P(n) is true for n = k

P(k) = 1 + 3 + 5 + … + (2k - 1) = k^2 - - - (1)

We have to show that

1 + 3 + 5 + … + (2k - 1) + 2(k + 1) - 1 = $(k + 1)^2$

Now,

= 1 + 3 + 5 + … + (2k - 1) + 2(k + 1) - 1

= k^2 + (2k + 1)

= k^2 + 2k + 1

= $(k + 1)^2$

Therefore, P(n) is true for n =k + 1

Hence, P(n) is true for all n∈N.

6. Question

Prove the following by the principle of mathematical induction:

$$\frac{1}{25}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$$

Answer

Let P(n): $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots\ldots\ldots.+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$

Step 1: Let us check if P(1) is true or not,

P(1): $\frac{1}{2.5}=\frac{1}{6.1+4}\Rightarrow\frac{1}{10}=\frac{1}{10}$

Therefore, P(1) is true.

Step 2: Let us assume that P(k) is true, now we have to prove that P(k + 1) is true.

P(k): $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots\ldots\ldots.+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4}$

P(k+1): $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots\ldots\ldots.+\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3k+3-1)(3k+3+2)}$

From P(k) we can see that,

P(k + 1): $\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$

P(k + 1): $\frac{k(3k+5)+2}{2(3k+2)(3k+5)}$

P(k + 1): $\frac{k+1}{6(k+1)+4}$

Therefore, P(k + 1) is true.

Hence, Proved by mathematical induction.

7. Question

Prove the following by the principle of mathematical induction:

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$$

Answer

Let P(n): $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

For n= 1 is true,

P(1): $\frac{1}{1.4}=\frac{1}{4}$

$\frac{1}{4}=\frac{1}{4}$

Since, P(n) is true for n = 1

Let P(n) is true for n = k, so

$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$ - - - - (1)

We have to show that,

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}=\frac{k+1}{3k+4}$$

Now,

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{1}{3k+1}\left[\frac{k}{1}+\frac{1}{3k+4}\right]$$

$$=\frac{1}{3k+1}\left[\frac{k(3k+4)+1}{3k+4}\right]$$

$$=\frac{1}{3k+1}\left[\frac{3k^2+4k+1}{3k+4}\right]$$

$$=\frac{1}{3k+1}\left[\frac{3k^2+3k+k+1}{3k+4}\right]$$

$$=\frac{3k(k+1)+(k+1)}{(3k+4)(3k+1)}$$

$$=\frac{(3k+1)(k+1)}{(3k+4)(3k+1)}$$

$$=\frac{(k+1)}{(3k+4)}$$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N

8. Question

Prove the following by the principle of mathematical induction:

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(1n-1)(2n+3)}=\frac{n}{3(2n+3)}$$

Answer

Let P(n): $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

Step1: Let us verify P(1).

P(1): $\frac{1}{3.5}=\frac{1}{3.(2.1+3)}$

P(1): $\frac{1}{15}=\frac{1}{15}$

Therefore, P(1) is true.

Step 2:

Let P(k) is true.

Therefore, P(k): $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)}$

Now we have to prove that P(k + 1) is also true.

So,

L.H.S $=\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2(k+1)+1)(2(k+1)+3)}$

L.H.S $=\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}$

Now from P(k) we can say that,

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)}$$

Putting this value, we get,

L.H.S = $\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)}$

L.H.S = $\frac{k(2k+5)+3}{3(2k+3)(2k+5)}$

L.H.S = $\frac{k+1}{3(2(k+1)+3)}$

L.H.S = R.H.S

Hence, Proved.

9. Question

Prove the following by the principle of mathematical induction:

$$\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4n-1)(4n+3)}=\frac{n}{3(4n+3)}$$

Answer

Let P(n): $\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4n-1)(4n+3)}=\frac{n}{3(4n+3)}$

For n= 1is true

P(1): $\frac{1}{3.7}=\frac{1}{(4.1-1)(4+3)}=\frac{1}{21}=\frac{1}{21}$

Since, P(n) is true for n =1

Let P(n) is true for n= k

P(n): $\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4k-1)(4k+3)}=\frac{k}{3(4k+3)}$ - - - - - - - (1)

We have to show that,

$$\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4k-1)(4k+3)}+\frac{1}{(4k+3)(4k+7)}=\frac{k+1}{3(4k+7)}$$

Now,

$$\left\{\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4k-1)(4k+3)}\right\}+\frac{1}{(4k+3)(4k+7)}$$

$=\frac{k}{(4k+3)}+\frac{1}{(4k+3)(4k+7)}$

$=\frac{1}{(4k+3)}\left[\frac{k(4k+7)+3}{3(4k+7)}\right]$

$=\frac{1}{(4k+3)}\left[\frac{4k^2+7k+3}{3(4k+7)}\right]$

$=\frac{1}{(4k+3)}\left[\frac{4k^2+3k+4k+3}{3(4k+7)}\right]$

$= \frac{1}{(4k+3)}\left[\frac{4k(k+1)+3(k+1)}{3(4k+7)}\right]$

$= \frac{1}{(4k+3)}\left[\frac{(4k+3)(k+1)}{3(4k+7)}\right]$

$= \frac{k+1}{3(4k+7)}$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈ N

10. Question

Prove the following by the principle of mathematical induction:

$1.2 + 2.2^2 + 3.2^3 + \ldots + n.2^n = (n-1)\,2^{n+1} + 2$

Answer

Let P(n): $1.2 + 2.2^2 + 3.2^3 + \ldots + n.2^n = (n-1)\,2^{n+1} + 2$

For n = 1

$= 1.2 = 0.2^0 + 2$

= 2 = 2

Since, P(n) is true for n = 1

Let P(n) is true for n = k, so

P(k): $1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k = (k-1)\,2^{k+1} + 2$ - - - - - - (1)

We have to show that,

$\{1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k + (k+1)\,2^{k+1} = k.2^{k+2} + 2$

Now,

$\{1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k\} + (k+1)2^{k+1}$

$= [(k-1)2^{k+1} + 2] + (k+1)2^{k+1}$ using equation (1)

$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$

$= 2^{k+1}(k - 1 + k + 1) + 2$

$= 2^{k+1}.2k + 2$

$= k.2^{k+2} + 2$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N by PMI

11. Question

Prove the following by the principle of mathematical induction:

2 + 5 + 8 + 11 + … + (3n – 1) = 1/2 n(3n + 1)

Answer

Let P(n): $2 + 5 + 8 + 11 + \ldots + (3n - 1) = \frac{1}{2}\,n(3n + 1)$

For n=1

P(1): $2 = \frac{1}{2}.1.(4)$

2 = 2

Since, P(n) is true for n = 1

Let P(n) is true for n = k, so

P(k): $2 + 5 + 8 + 11 + \ldots + (3k - 1) = \frac{1}{2} k(3k + 1)$ - - - - - - - (1)

We have to show that,

$2 + 5 + 8 + 11 + \ldots + (3k - 1) + (3k + 2) = \frac{1}{2}(k + 1)(3k + 4)$

Now,

$\{2 + 5 + 8 + 11 + \ldots + (3k - 1)\} + (3k + 2)$

$= \frac{1}{2}.k(3k + 1) + (3k + 2)$

$= \frac{3k^2 + k + 2(3k + 2)}{2}$

$= \frac{3k^2 + k + 6k + 2}{2}$

$= \frac{3k^2 + 7k + 2}{2}$

$= \frac{3k^2 + 4k + 3k + 2}{2}$

$= \frac{3k(k + 1) + 4(k + 1)}{2}$

$= \frac{(k + 1)(3k + 4)}{2}$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n ∈ N by PMI

12. Question

Prove the following by the principle of mathematical induction:

$1.3 + 2.4 + 3.5 + \ldots + n.(n + 2) = \frac{1}{6}n(n + 1)(2n + 7)$

Answer

Let P(n): $1.3 + 2.4 + 3.5 + \ldots + n.(n + 2) = \frac{1}{6}n(n + 1)(2n + 7)$

For n = 1

P(1): $1.3 = \frac{1}{6}.1.(2)(9)$

= 3 = 3

Since, P(n) is true for n = 1

Now,

For n = k

$= P(n): 1.3 + 2.4 + 3.5 + \ldots + k\,.\,(k + 2) = \frac{1}{6}k(k+1)(2k+7)$ - - - - - (1)

We have to show that

$= 1.3 + 2.4 + 3.5 + \ldots + k\,.\,(k + 2) + (k + 3) = \frac{k+1}{6}(k+2)(2k+9)$

Now,

$= \{1.3 + 2.4 + 3.5 + \ldots + k\,(k + 2)\} + (k + 1)(k + 3)$

$= \frac{1}{6}k(k + 1)(2k + 7) + (k + 1)(k + 3)$ using equation (1)

$= (k + 1)\left[\frac{k(2k+7)}{6} + \frac{k+3}{1}\right]$

$= (k + 1)\left[\frac{2k^2 + 7k + 6k + 18}{6}\right]$

$= (k + 1)\left[\frac{2k^2 + 13k + 18}{6}\right]$

$= (k + 1)\left[\frac{2k^2 + 9k + 4k + 18}{6}\right]$

$= (k + 1)\left[\frac{2k(k+2) + 9(k+2)}{6}\right]$

$= (k + 1)\left[\frac{(2k+9)(k+2)}{6}\right]$

$= \frac{1}{6}(k + 1)(k + 2)(2k + 9)$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all $n \in N$

13. Question

Prove the following by the principle of mathematical induction:

$1.3 + 3.5 + 5.7 + \ldots + (2n - 1)(2n + 1) = \frac{n\left(4n^2 + 6n - 1\right)}{3}$

Answer

Let P(n): $1.3 + 3.5 + 5.7 + \ldots + (2n - 1)(2n + 1) = \frac{n(4n^2 + 6n - 1)}{3}$

For n = 1

P(1): $(2.1 - 1)(2.1 + 1) = \frac{1(4.1^2 + 6.1 - 1)}{3}$

$= 1 \times 3 = \frac{1(4 + 6 - 1)}{3}$

= 3 = 3

Since, P(n) is true for n =1

Now, For n = k, So

$1.3 + 3.5 + 5.7 + \ldots + (2k - 1)(2k + 1) = \frac{k(4k^2 + 6k - 1)}{3}$ - - - - - - - (1)

We have to show that,

$1.3 + 3.5 + 5.7 + \ldots + (2k - 1)(2k + 1) + (2k + 1)(2k + 3) = \frac{(k+1)\left[\left(4(k+1)^2 + 6(k+1) - 1\right)\right]}{3}$

Now,

1.3 + 3.5 + 5.7 + … + (2k – 1) (2k + 1) + (2k + 1)(2k + 3)

$= \frac{k(4k^2 + 6k - 1)}{3}$ + (2k + 1)(2k + 3) using equation (1)

$= \frac{k(4k^2 + 6k - 1) + 3(4k^2 + 6k + 2k + 3)}{3}$

$= \frac{4k^3 + 6k^2 - k + 12k^2 + 18k + 6k + 9}{3}$

$= \frac{4k^3 + 18k^2 + 23k + 9}{3}$

$= \frac{4k^3 + 4k^2 + 14k^2 + 14k + 9k + 9}{3}$

$= \frac{(k+1)(4k^2 + 8k + 4 + 6k + 6 - 1)}{3}$

$= \frac{(k+1)[4(k+1)^2 + 6(k+1) - 1]}{3}$

Therefore, P(n) is true for n=k + 1

Hence, P(n) is true for all n∈ N by PMI

14. Question

Prove the following by the principle of mathematical induction:

1.2 + 2.3 + 3.4 + … + n(n + 1)$= \frac{n(n+1)(n+2)}{3}$

Answer

Let P(n): 1.2 + 2.3 + 3.4 + … + n(n + 1)$= \frac{n(n+1)(n+2)}{3}$

For n = 1

P(1): 1(1 + 1)$= \frac{1(1+1)(1+2)}{3}$

$= 1x2 = \frac{6}{3}$

= 2 = 2

Since, P(n) is true for n = 1

Let P(n) is true for n = k

= P(k): 1.2 + 2.3 + 3.4 + … + k(k + 1)$= \frac{k(k+1)(k+2)}{3}$ - - - - - (1)

We have to show that,

= 1.2 + 2.3 + 3.4 + … + k(k + 1) + (k + 1)(k + 2)$= \frac{(k+1)(k+2)(k+3)}{3}$

Now,

{1.2 + 2.3 + 3.4 + … + k(k + 1)} + (k + 1)(k + 2)

$= \frac{(k+1)(k+2)}{3} + \frac{(k+1)(k+2)}{1}$

$= (k + 2)(k + 1)\left[\frac{k}{2} + 1\right]$

$= \frac{(k+1)(k+2)(k+3)}{3}$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n ∈ N

15. Question

Prove the following by the principle of mathematical induction:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

Answer

Let P(n): $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^2}$

For n =1 is true,

P(1): $\frac{1}{2^1}=1-\frac{1}{2^1}$

$=\frac{1}{2}=\frac{1}{2}$

Since, P(n) is true for n =1

Now, For n = k

P(k): $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k}$ - - - - (1)

We have to show that,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}$$

Now,

$$\left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}\right\}+\frac{1}{2^{k+1}}$$

$= 1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$ using equation (1)

$= 1-\left(\frac{2-1}{2^{k+1}}\right)$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n ϵ N by PMI

16. Question

Prove the following by the principle of mathematical induction:

$$1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{1}{3}n\left(4n^2-1\right)$$

Answer

Let P(n): $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{1}{3}n(4n^2-1)$

For n = 1

$= (2.1 - 1)^2 = \frac{1}{3}1(4-1)$

= 1 = 1

Since, P(n) is true for n = 1

Let P(n) is true for n = k ,

P(k)): $1^2 + 3^2 + 5^2 + \ldots + (2k - 1)^2 = \frac{1}{3}k(4k^2 - 1)$ - - (1)

We have to show that,

$1^2 + 3^2 + 5^2 + \ldots + (2k - 1)^2 + (2k + 1)^2 = \frac{1}{3}(k + 1)[(4(k + 1)^2 - 1)]$

Now,

$\{1^2 + 3^2 + 5^2 + \ldots + (2k - 1)^2\} + (2k + 1)^2$

$= \frac{1}{3}k(4k^2 - 1) + (2k + 1)^2$ using equation (1)

$= \frac{1}{3}k(2k + 1)(2k - 1) + (2k + 1)^2$

$= (2k + 1)\left[\frac{k(2k-1)}{3} + (2k + 1)\right]$

$= (2k + 1)\left[\frac{2k^2 - k + 3(2k + 1)}{3}\right]$

$= (2k + 1)\left[\frac{2k^2 - k + 6k + 3}{3}\right]$

$= \left[\frac{(2k + 1)2k^2 + 5k + 3}{3}\right]$

$= \left[\frac{(2k + 1)(2k(k + 1) + 3(k + 1))}{3}\right]$

$= \left[\frac{(2k + 1)(2k + 3)(k + 1)}{3}\right]$

$= \frac{k + 2}{2}[4k^2 + 6k + 2k + 3]$

$= \frac{k + 2}{2}[4k^2 + 8k - 1]$

$= \frac{k + 2}{2}[4(k + 1)^2 - 1]$

Therefore, P(n)is true for n = k + 1

Hence, P(n) is true for all n∈ N

17. Question

Prove the following by the principle of mathematical induction:

$a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r - 1}\right), r \neq 1$

Answer

Let P(n): $a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r - 1}\right)$

For n =1

$a = a\left(\frac{r^1 - 1}{r - 1}\right)$

a = a

Since, P(n) is true for n = 1

Let P(n) is true for n = k , so

P(k): $a + ar + ar^2 + \ldots + ar^{k-1} = a\left(\frac{r^k-1}{r-1}\right)$ - - - - - - - (1)

We have to show that,

$a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = a\left(\frac{r^{k+1}-1}{r-1}\right)$

Now,

$\{a + ar + ar^2 + \ldots + ar^{k-1}\} + ar^k$

$= a\left(\frac{r^k-1}{r-1}\right) + ar^k$ using equation (1)

$= \frac{a[r^k-1+r^k(r-1)]}{r-1}$

$= \frac{a[r^k-1+r^{k+1}-r^k]}{r-1}$

$= \frac{a[r^{k+1}-1]}{r-1}$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n ∈ N

18. Question

Prove the following by the principle of mathematical induction:

$a + (a + d) + (a + 2d) + \ldots + (a + (n-1)d) = \frac{n}{2}\left[2a + (n-1)d\right]$

Answer

P(n): $a + (a + d) + (a + 2d) + \ldots + (a + (n-1)d) = \frac{n}{2}[2a + (n-1)d]$

For n = 1

$a = \frac{1}{2}[2a + (1 - 1)d]$

a = a

Since, P(n) is true for n =1,

Let P(n) is true for n = k, so

$a + (a + d) + (a + 2d) + \ldots + (a + (k-1)d) = \frac{k}{2}[2a + (k-1)d]$ - - - - - (1)

We have to show that,

$a + (a + d) + (a + 2d) + \ldots + (a + (k-1)d) + (a + (k)d) = \frac{(k+1)}{2}[2a + kd]$

Now,

$\{a + (a + d) + (a + 2d) + \ldots + (a + (k-1)d)\} + (a + kd)$

$= \frac{k}{2}[2a + (k - 1)d] + (a + kd)$ using equation

$= \frac{2ka + k(k-1)d + 2(a+kd)}{2}$

$= \frac{2ka + k^2d - kd + 2a + 2kd}{2}$

$= \frac{2ka + 2a + k^2d + kd}{2}$

$= \frac{2a(k+1) + d(K^2 + k)}{2}$

$= \frac{(k+1)}{2}[2a + kd]$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true all n∈ N by PMI

19. Question

Prove the following by the principle of mathematical induction:

$5^{2n} – 1$ is divisible by 24 for all n ϵ N

Answer

Let P(n): 5^{2n} - 1 is divisible by 24

Let’s check For n =1

P(1): 5^2 - 1 = 25 - 1

= 24

Since, it is divisible by 24

So, P(n) is true for n=1

Now, for n=k

5^{2k} - 1 is divisible by 24

P(k): 5^{2k} - 1 = 24λ - - - - - - - (1)

We have to show that,

$5^{2k + 1}$ - 1 is divisible by 24

$5^{2(k + 1)}$ - 1 = 24μ

Now,

$5^{2(k + 1)}$ - 1

= $5^{2k}.5^{2\,-}$ 1

= 25.5^{2k} - 1

= 25.(24λ + 1) - 1 using equation (1)

= 25.24λ + 24

= 24λ

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N by PMI

20. Question

Prove the following by the principle of mathematical induction:

3^{2n} + 7 is divisible by 8 for all n ϵ N

Answer

Let P(n): 3^{2n} + 7 is divisible by 8

Let's check For n =1

P(1): $3^2 + 7 = 9 + 7$

$= 16$

Since, it is divisible by 8

So, P(n) is true for n=1

Now, for n=k

P(k): $3^{2k} + 7 = 8\lambda$ - - - - - - - (1)

We have to show that,

$3^{2(k + 1)} + 7$ is divisible by 8

$3^{2k + 2} + 7 = 8\mu$

Now,

$3^{2(k + 1)} + 7$

$= 3^{2k}.3^2 + 7$

$= 9.3^{2k} + 7$

$= 9.(8\lambda - 7) + 7$

$= 72\lambda - 63 + 7$

$= 72\lambda - 56$

$= 8(9\lambda - 7)$

$= 8\mu$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N by PMI

21. Question

Prove the following by the principle of mathematical induction:

$5^{2n + 2} – 24n – 25$ is divisible by 576 for all n ϵ N.

Answer

Let P(n): $5^{2n + 2} – 24n – 25$

For n = 1

$= 5^{2.1 + 2} - 24.1 - 25$

$= 625 – 49$

$= 576$

Since, it is divisible by 576

Let P(n) is true for n=k, so

$= 5^{2k + 2} – 24k – 25$ is divisible by 576

$= 5^{2k + 2} – 24k – 25 = 576\lambda$ - - - - - (1)

We have to show that,

$= 5^{2k + 4} – 24(k + 1) – 25$ is divisible by 576

$= 5^{(2k + 2) + 2} – 24(k + 1) – 25 = 576\mu$

Now,

$= 5^{(2k + 2) + 2} – 24(k + 1) – 25$

$= 5^{(2k + 2)}.5^2 – 24k – 24– 25$

$= (576\lambda + 24k + 25)25 – 24k– 49$ using equation (1)

$= 25. 576\lambda + 576k + 576$

$= 576(25\lambda + k + 1)$

$= 576\mu$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N by PMI

22. Question

Prove the following by the principle of mathematical induction:

$3^{2n + 2} – 8n – 9$ is divisible by 8 for all n ϵ N.

Answer

Let P(n): $3^{2n + 2} – 8n – 9$

For n = 1

$= 3^{2.1 + 2} - 8.1 - 9$

= 81 – 17

= 64

Since, it is divisible by 8

Let P(n) is true for n=k, so

$= 3^{2k + 2} – 8k – 9$ is divisible by 8

$= 3^{2k + 2} – 8k – 9 = 8\lambda$ - - - - - (1)

We have to show that,

$= 3^{2k + 4} – 8(k + 1) – 9$ is divisible by 8

$= 3^{(2k + 2) + 2} – 8(k + 1) – 9 = 8\mu$

Now,

$= 3^{2(k + 1)}.3^2 – 8(k + 1) – 9$

$= (8\lambda + 8k + 9)9 – 8k – 8 – 9$

$= 72\lambda + 72k + 81 - 8k - 17$ using equation (1)

$= 72\lambda + 64k + 64$

$= 8(9\lambda + 8k + 8)$

$= 8\mu$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N by PMI

23. Question

Prove the following by the principle of mathematical induction:

$(ab)^n = a^n b^n$ for all n ϵ N

Show that: $(ab)^n = a^n b^n$ for all n ϵ N by Mathematical Induction

Answer

Let P(n) : $(ab)^n = a^n b^n$

Let check for n = 1 is true

$= (ab)^1 = a^1b^1$

= ab = ab

Therefore, P(n) is true for n =1

Let P(n) is true for n=k,

$= (ab)^k=a^k.b^k$ - - - - - - (1)

We have to show that,

$= (ab)^{k+1}=a^{k+1}.b^{k+1}$

Now,

$= (ab)^{k+1}$

$=(ab)^k (ab)$

$= (a^kb^k)(ab)$ using equation (1)

$= (a^{k+1})(b^{K+1})$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N by PMI

24. Question

Prove the following by the principle of mathematical induction:

n(n + 1) (n + 5) is a multiple of 3 for alln ϵ N.

Show that: P(n): n(n + 1) (n + 5) is multiple by 3 for all n∈N

Answer

Let P(n): n(n + 1) (n + 5) is multiple by 3 for all n∈N

Let P(n) is true for n=1

P(1): 1(1 + 1) (1 + 5)

= 2 × 6

= 12

Since, it is multiple of 3

So, P(n) is true for n = 1

Now, Let P(n) is true for n = k

P(k): k(k + 1) (k + 5)

= k(k + 1) (k + 5) is a multiple of 3

Then, k(k + 1) (k + 5) = 3λ - - - - - (1)

We have to show,

= (k + 1)[(k + 1) + 1][(k + 1) + 5] is a multiple of 3

= (k + 1)[(k + 1) + 1][(k + 1) + 5] = 3μ

Now,

= (k + 1)[(k + 1) + 1][(k + 1) + 5]

= (k + 1)(k + 2)[(k + 1) + 5]

= [k(k + 1) + 2(k + 1)][(k + 5) + 1]

= k(k + 1)(k + 5) + k(k + 1) + 2(k + 1)(k + 5) + 2(k + 1)

= $3\lambda + k^2 + k + 2(k^2 + 6k + 5) + 2k + 2$

= $3\lambda + k^2 + k + 2k^2 + 12k + 10 + 2k + 2$

= $3\lambda + 3k^2 + 15k + 12$

= $3(\lambda + k^2 + 5k + 4)$

= 3μ

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n∈N

25. Question

Prove the following by the principle of mathematical induction:

$7^{2n} + 2^{3n-3}$. 3n – 1 is divisible by 25 for all n ϵ N

Answer

Let P(n): $7^{2n} + 2^{3n-3}.3^{n-1}$ is divisible by 25

For n=1

= $7^2 + 2^0.3^0$

= 49 + 1

= 50

Therefor it is divisible by 25

So, P(n) is true for n = 1

Now, P(n) is true For n = k,

So, we have to show that $7^{2n} + 2^{3n-3}.3^{n-1}$ is divisible by 25

= $7^{2k} + 2^{3k-3}.3^{k-1} = 25\lambda$ - - - - - - - (1)

Now, P(n) is true For n = k + 1,

So, we have to show that $7^{2k+1} + 2^{3k}.3^k$ is divisible by 25

= $7^{2k+2} + 2^{3k}.3^k = 25\mu$

Now,

= $7^{2(k+1)} + 2^{3k}.3^k$

= $7^{2k}.7^1 + 2^{3k}.3^k$

= $(25\lambda - 2^{3k-3}.3^{k-1})49 + 2^{3k}.3k$ from eq 1

$= 25\lambda.49 - \frac{2^{3k}}{8}.\frac{3^k}{3}.49 + 2^{3k}.3^k$

$= 24\times25\times49\lambda - 2^{3k}.3^{k}.49 + 24.2^{3k}.3^k$

$= 24\times25\times49\lambda - 25.2^{3k}.3^k$

$= 25(24.49\lambda - 2^{3k}.3^k)$

$= 25\mu$

Therefore, P(n) is true for n = k + 1

Hence, P(n) is true for all n ∈ N

26. Question

If P(n) is the statement "n(n + 1) is even", then what is P(3)?

$2.7^n + 3.5^n – 5$ is divisible by 24 for all n ϵ N

Answer

Let $P(n) = 2.7^n + 3.5^n – 5$

Now, P(n): $2.7^n + 3.5^n – 5$ is divisible by 24 for all n ϵ N

Step1:

P(1) = 2.7 + 3.5 – 5 = 1.2

Thus, P(1) is divisible by 24

Step2:

Let, P(m) be divisible by 24

Then, $2.7^m + 3.5^m – 5 = 24\lambda$, where λ ϵ N.

Now, we need to show that P(m+1) is true whenever P(m) is true.

So, $P(m+1) = 2.7^{m+1} + 3.5^{m+1} – 5$

$= 2.7^{m+1} + 5.(2.7^m + 3.5^m – 5) – 5$

$= 2.7^{m+1} + 5.(24\lambda + 5 - 2.7^m) – 5$

$= 2.7^{m+1} + 120\lambda + 25 - 10.7^m – 5$

$= 2.7^m.7 - 10.7^m + 120 \lambda +24 – 4$

$= 7^m(14 – 10) + 120 \lambda +24 – 4$

$= 7^m(4) + 120 \lambda +24 – 4$

$= 7^m(4) + 120 \lambda +24 – 4$

$= 4(7^m - 1) + 24(5\lambda +1)$

As, $7^m – 1$ is a multiple of 6 for all m ϵ N.

So, $P(m+1) = 4.6\mu +24(5\lambda +1)$

$= 24(\mu +5\lambda +1)$

Thus, P(m+1) is true.

So, by the principle of mathematical induction, P(n) is true for all n ϵN.

27. Question

If P(n) is the statement "n(n + 1) is even", then what is P(3)?

$11^{n+2} + 12^{2n+1}$ is divisible by 133 for all n ϵ N

Answer

Let P(n) = $11^{n+2} + 12^{2n+1}$

Now, P(n): $11^{n+2} + 12^{2n+1}$ is divisible by 133 for all n ϵ N

Step1:

P(1) = 1331 + 1728 = 3059

Thus, P(1) is divisible by 133

Step2:

Let, P(m) be divisible by 24

Then, $11^{m+2} + 12^{2m+1} = 133\lambda$, where λ ϵ N.

Now, we need to show that P(m+1) is true whenever P(m) is true.

So, P(m+1) = $11^{m+3} + 12^{2m+3}$

= $11^{m+2}.11 + 12^{2m+1}.12^2 + 11.12^{2m+1} – 11.12^{2m+1}$

= $11.(11^{m+2} + 12^{2m+1}) + 12^{2m+1}(144-11)$

= $11.133\lambda + 12^{2m+1}.133$

= $133.(11\lambda + 12^{2m+1})$

Thus, P(m+1) is true.

So, by the principle of mathematical induction, P(n) is true for all n ϵN.

28. Question

If P(n) is the statement "n(n + 1) is even", then what is P(3)?

1×1! + 2×2! + 3×3! +...+ n×n! = (n + 1)! – 1 for all nϵ N.

Answer

Let P(n) = 1×1! + 2×2! + 3×3! +...+ n×n

P(n): 1×1! + 2×2! + 3×3! +...+ n×n! = (n + 1)! – 1 for all nϵ N

Step1:

P(1) = 1×1! = (2)! – 1 = 1

Thus, P(n) is equal to (n + 1)! – 1 for n = 1

Step2:

Let, P(m) be equal to (m + 1)! – 1

Then, 1×1! + 2×2! + 3×3! +...+ m×m! = (m + 1)! – 1

Now, we need to show that P(m+1) is true whenever P(m) is true.

P(m+1) = 1×1! + 2×2! + 3×3! +...+ m×m! + (m+1)×(m+1)!

= (m+1)! – 1 + (m+1)×(m+1)!

= (m+1)!(m+1+1) – 1

= (m+1)!(m+2) – 1

= (m+2)! – 1

Thus, P(m+1) is true.

So, by the principle of mathematical induction, P(n) is true for all n∈N.

29. Question

If P(n) is the statement "n(n + 1) is even", then what is P(3)?

$n^3 – 7n + 3$ is divisible by 3 for all n ϵ N.

Answer

Let P(n) = $n^3 – 7n + 3$

Now, P(n): $n^3 – 7n + 3$ is divisible by 3 for all n ϵ N

Step1:

P(1) = 1 – 7 + 3 = -3

Thus, P(1) is divisible by 3

Step2:

Let, P(m) be divisible by 24

Then, $n^3 – 7n + 3 = 3\lambda$, where λ ϵ N.

Now, we need to show that P(m+1) is true whenever P(m) is true.

So, P(m+1) = $(n+1)^3 – 7(n+1) + 3$

= $n^3+3n^2+3n+1-7n-7+3$

= $n^3– 7n + 3 +3n^2+3n+1-7$

= $3\lambda+3(n^2+n-2)$

= $3(\lambda+n^2+n-2)$

Thus, P(m+1) is true.

So, by the principle of mathematical induction, P(n) is true for all n ϵN.

30. Question

If P(n) is the statement "n(n + 1) is even", then what is P(3)?

$1 + 2 + 2^2 + … + 2^n = 2^{n + 1} – 1$ for all n ϵ N

Answer

Let P(n) = $1 + 2 + 2^2 + … +$

P(n): $1 + 2 + 2^2 + … + 2^n = 2^{n + 1} – 1$ for all n ϵ N

Step1:

P(1) = 1 = (2) – 1 = 1

Thus, P(n) is equal to $2^{n + 1} – 1$ for n = 1

Step2:

Let, P(m) be equal to $2^{m + 1} – 1$

Then, $1 + 2 + 2^2 + … + 2^m = 2^{m + 1} – 1$

Now, we need to show that P(m+1) is true whenever P(m) is true.

P(m+1) = $1 + 2 + 2^2 + … + 2^m + 2^{m + 1}$

$= 2^{m+1} - 1 + 2^{m+1}$

$= 2.2^{m+1} - 1$

$= 2^{m+2} - 1$

Thus, P(m+1) is true.

So, by the principle of mathematical induction, P(n) is true for all n∈N.

31. Question

Prove that $7 + 77 + 777 + \ldots + \underbrace{777 \ldots\ldots 7}_{n-\text{digits}} = \frac{7}{81}\left(10^{n+1} - 9n - 10\right)$ for all n ϵ N

Answer

Let P(n) = 7 + 77 + 777 + ... + 777......n times......7

$$P(n): 7 + 77 + 777 + \ldots + 777 \ldots\ldots n \text{ times} \ldots\ldots 7 = \frac{7}{81}(10^{n+1} - 9n - 10) \text{ for all } n \in N$$

Step1:

$$P(1) = 7 = \frac{7}{81}(100 - 9 - 10) = 7$$

Thus, P(n) is equal to $\frac{7}{81}(10^{n+1} - 9n - 10)$ for $n = 1$

Step2:

Let, P(m) be equal to $\frac{7}{81}(10^{m+1} - 9m - 10)$

Then,

$$7 + 77 + 777 + \ldots + 777 \ldots\ldots m \text{ times} \ldots\ldots 7 = \frac{7}{81}(10^{m+1} - 9n - 10)$$

Now, we need to show that P(m+1) is true whenever P(m) is true.

This is a geometric progression with n = m+1

So, P(m+1) = 7 + 77 + 777 + ... + 777......(m+1) times......7

$$= \frac{7}{9}(9 + 99 + 999 \ldots\ldots + 999 \ldots\ldots (m+1) \text{ times} \ldots\ldots 9)$$

$$= \frac{7}{9}[(10 - 1) + (100 - 1) + (1000 - 1) \ldots\ldots + 111 \ldots\ldots (m+1) \text{ times} \ldots\ldots 1 - 1)]$$

$$= \frac{7}{9}(10 + 100 + 1000 \ldots\ldots + 100 \ldots\ldots (m+1) \text{ times} \ldots\ldots 0 - (1 + 1 + 1 \ldots m + 1 \text{ times})$$

$$= \frac{7}{9}\left[\frac{10(10^{m+1} - 1)}{9} - m + 1\right]$$

$$= \frac{7}{81}[10(10^{m+2} - 1) - 9m - 19]$$

Thus, P(m+1) is true.

So, by principle of mathematical induction, P(n) is true for all n∈N.

32. Question

Prove that $\frac{n^7}{7}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{n^2}{2}-\frac{37}{210}n$ is a positive integer for all n ϵ N

Answer

Let $P(n)=\frac{n^7}{7}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{n^2}{2}-\frac{37}{210}n$

$P(n):\frac{n^7}{7}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{n^2}{2}-\frac{37}{210}n$ is a positive integer for all n ϵ N

Step1:

$P(1)=\frac{1}{7}+\frac{1}{5}+\frac{1}{3}+\frac{1}{2}-\frac{37}{210}=1$

Thus, P(n) is a positive integer for $n = 1$

Step2:

Let, P(m) be equal to $\frac{m^7}{7}+\frac{m^5}{5}+\frac{m^3}{3}+\frac{m^2}{2}-\frac{37}{210}m$

Let $\frac{m^7}{7}+\frac{m^5}{5}+\frac{m^3}{3}+\frac{m^2}{2}-\frac{37}{210}m=\lambda$, where $\lambda \epsilon N$ is a positive integer

Now, we need to show that P(m+1) is true whenever P(m) is true.

$$P(m+1)=\frac{(m+1)^7}{7}+\frac{(m+1)^5}{5}+\frac{(m+1)^3}{3}+\frac{(m+1)^2}{2}-\frac{37}{210}(m+1)$$

$$=\frac{1}{7}(m^7+7m^6+21m^5+35m^4+35m^3+21m^2+7m+1)+\frac{1}{5}(m^5+5m^4+10m^3+10m^2+5m+1)+\frac{1}{3}(m^3+3m^2+3m+1)+\frac{1}{2}(m^2+2m+1)-\frac{37}{210}(m+1)$$

$$=\left[\frac{m^7}{7}+\frac{m^5}{5}+\frac{m^3}{3}+\frac{m^2}{2}-\frac{37}{210}m\right]+m^6+3m^5+5m^4+5m^3+3m^2+m+m^4+2m^3+2m^2+m+m^2+m+m+\frac{1}{7}+\frac{1}{5}+\frac{1}{3}+\frac{1}{2}-\frac{37}{210}$$

$$=\lambda+m^6+3m^5+5m^4+5m^3+3m^2+m+m^4+2m^3+2m^2+m+m^2+m+m+1$$

It is a positive integer.

Thus, P(m+1) is true.

So, by principle of mathematical induction, P(n) is true for all nϵN.

33. Question

Prove that $\frac{n^{11}}{11}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{62}{165}n$ is a positive integer for all n ϵ N

Answer

Let $P(n)=\frac{n^{11}}{11}+\frac{n^5}{5}+\frac{n^3}{3}-\frac{62}{165}n$

$P(n):\frac{n^{11}}{11}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{62}{165}n$ is a positive integer for all n ϵ N

Step1:

$$P(1) = \frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165} = 1$$

Thus, P(n) is a positive integer for $n = 1$

Step2:

Let, P(m) be equal to $\frac{m^{11}}{11} + \frac{m^5}{5} + \frac{m^3}{3} + \frac{62}{165}m$

Let $\frac{m^{11}}{11} + \frac{m^5}{5} + \frac{m^3}{3} + \frac{62}{165}m = \lambda$, where $\lambda \epsilon N$ is a positive integer

Now, we need to show that P(m+1) is true whenever P(m) is true.

$$P(m+1) = \frac{(m+1)^{11}}{11} + \frac{(m+1)^5}{5} + \frac{(m+1)^3}{3} + \frac{62}{165}(m+1)$$

$$= \frac{1}{11}(m^{11} + 11m^{10} + 55m^9 + 165m^8 + 330m^7 + 462m^6 + 462m^5 + 330m^4 + 165m^3 + 55m^2 + 11m + 1) + \frac{1}{5}(m^5 + 5m^4 + 10m^3 + 10m^2 + 5m + 1) + \frac{1}{3}(m^3 + 3m^2 + 3m + 1) + \frac{62}{165}(m+1)$$

$$= \left[\frac{m^{11}}{11} + \frac{m^5}{5} + \frac{m^3}{3} + \frac{62}{165}m\right] + (m^{10} + 5m^9 + 15m^8 + 30m^7 + 42m^6 + 42m^5 + 30m^4 + 15m^3 + 5m^2 + m) + (m^4 + 2m^3 + 2m^2 + m) + (m^2 + m) + \frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165}$$

$$= \lambda + m^6 + 3m^5 + 5m^4 + 5m^3 + 3m^2 + m + m^4 + 2m^3 + 2m^2 + m + m^2 + m + m + 1$$

It is a positive integer.

Thus, P(m+1) is true.

So, by principle of mathematical induction, P(n) is true for all nϵN.

34. Question

Prove that $\frac{1}{2}\tan\left(\frac{x}{2}\right) + \frac{1}{4}\tan\left(\frac{x}{4}\right) + \ldots + \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - \cot x$ for all n ϵ N and $0 < x < \frac{\pi}{2}$

Answer

$$\text{Let } P(n) = \frac{1}{2}\tan\left(\frac{x}{2}\right) + \frac{1}{4}\tan\left(\frac{x}{4}\right) + \ldots + \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - \cot x, \text{ for all } n\epsilon N \text{ and } 0 < x < \frac{\pi}{2}$$

Step1: For n=1

$$L.H.S = \frac{1}{2}\tan\left(\frac{x}{2}\right)$$

$$R.H.S = \frac{1}{2}\cot\left(\frac{x}{2}\right) - \cot x = \frac{1}{2\tan\left(\frac{x}{2}\right)} - \frac{1}{\tan x}$$

$$\Rightarrow \text{R.H.S} = \frac{1}{2\tan\left(\frac{x}{2}\right)} - \frac{1}{\frac{2\tan\frac{x}{2}}{1-\tan^2\left(\frac{x}{2}\right)}}$$

$$\Rightarrow \text{R.H.S} = \frac{1}{2\tan\left(\frac{x}{2}\right)} - \frac{1-\tan^2\left(\frac{x}{2}\right)}{2\tan\frac{x}{2}}$$

$$\Rightarrow \text{R.H.S} = \frac{1}{2}\tan\frac{x}{2}$$

So, it is true for n=1

Step2:

$$\text{Let, P(m) be equal to } \frac{1}{2}\tan\left(\frac{x}{2}\right) + \frac{1}{4}\tan\left(\frac{x}{4}\right) + \ldots + \frac{1}{2^m}\tan\left(\frac{x}{2^m}\right) = \frac{1}{2^m}\cot\left(\frac{x}{2^m}\right) - \cot x$$

Now, we need to show that P(m+1) is true whenever P(m) is true.

$$P(m+1) = \frac{1}{2}\tan\left(\frac{x}{2}\right) + \frac{1}{4}\tan\left(\frac{x}{4}\right) + \ldots + \frac{1}{2^m}\tan\left(\frac{x}{2^m}\right) + \frac{1}{2^{m+1}}\tan\left(\frac{x}{2^{m+1}}\right) = \frac{1}{2^{m+1}}\cot\left(\frac{x}{2^{m+1}}\right) - \cot x$$

$$\text{Let, L} = \frac{1}{2^m}\cot\frac{x}{2^m} - \cot x + \frac{1}{2^{m+1}}\tan\left(\frac{x}{2^{m+1}}\right)$$

$$\Rightarrow L = \frac{1}{2^m}\cot\frac{x}{2^m} + \frac{1}{2^{m+1}}\tan\left(\frac{x}{2^{m+1}}\right) - \cot x$$

$$\Rightarrow L = \frac{1}{2^m\tan\frac{2x}{2^{m+1}}} + \frac{1}{2^{m+1}}\tan\left(\frac{x}{2^{m+1}}\right) - \cot x$$

$$\Rightarrow L = \frac{1}{2^m \times \frac{2\tan\left(\frac{x}{2^{m+1}}\right)}{1-\tan^2\left(\frac{x}{2^{m+1}}\right)}} + \frac{1}{2^{m+1}}\tan\left(\frac{x}{2^{m+1}}\right) - \cot x$$

$$\Rightarrow L = \frac{1-\tan^2\left(\frac{x}{2^{m+1}}\right)}{2^{m+1}\times\tan\left(\frac{x}{2^{m+1}}\right)} + \frac{1}{2^{m+1}}\tan\left(\frac{x}{2^{m+1}}\right) - \cot x$$

$$\Rightarrow L = \frac{1-\tan^2\left(\frac{x}{2^{m+1}}\right) + \tan^2\left(\frac{x}{2^{m+1}}\right)}{2^{m+1}\times\tan\left(\frac{x}{2^{m+1}}\right)} - \cot x$$

$$\Rightarrow L = \frac{1}{2^{m+1}}\cot\left(\frac{x}{2^{m+1}}\right) - \cot x$$

Now,

$$\frac{1}{2}\tan\left(\frac{x}{2}\right) + \frac{1}{4}\tan\left(\frac{x}{4}\right) + \ldots + \frac{1}{2^m}\tan\left(\frac{x}{2^m}\right) + \frac{1}{2^{m+1}}\tan\left(\frac{x}{2^{m+1}}\right) = \frac{1}{2^{m+1}}\cot\left(\frac{x}{2^{m+1}}\right) - \cot x$$

Thus, P(m+1) is true.

$$\text{Thus, } \frac{1}{2}\tan\left(\frac{x}{2}\right) + \frac{1}{4}\tan\left(\frac{x}{4}\right) + \ldots + \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - \cot x\text{, for all } n \in N \text{ and } 0 < x < \frac{\pi}{2}$$

35. Question

Prove that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for all natural numbers, n ≥ 2.

Answer

Let P(n) = $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots.\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$

Let us find if it is true at n = 2,

P(2): $1-\frac{1}{2^2}=\frac{2+1}{2.2}$

P(2): $\frac{3}{4}=\frac{3}{4}$

Hence, P(2) holds.

Now let P(k) is true, and we have to prove that P(k + 1) is true.

Therefore, we need to prove that,

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots.\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2(k+1)}$$

P(k) = $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots.\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$ ....(1)

Taking L.H.S of P(k) we get,

P(k) = $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots.\left(1-\frac{1}{k^2}\right)$

P(k + 1) = $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots.\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$

From equation (1),

P(k + 1) = $\left(1-\frac{1}{(k+1)^2}\right)\frac{k+1}{2k}$

P(k + 1) = $\frac{k+1}{2k}.\frac{k^2+1+2k-1}{(k+1)^2}$

P(k + 1) = $\frac{k(k+2)}{2k(k+1)}$

P(k + 1) = $\frac{(k+2)}{2(k+1)}$

Therefore, P(k + 1) holds.

Hence, P(n) is true for all n ≥ 2.

36. Question

Prove that $\frac{(2n)!}{2^{2n}(n!)^2}\leq\frac{1}{\sqrt{3n+1}}$ for all n ϵ N

Answer

$$\text{Let } P(n)=\frac{(2n)!}{2^{2n}(n!)^2}\leq\frac{1}{\sqrt{3n+1}}$$

Step1:

$$P(1)=\frac{(2)!}{2^2(1!)^2}=\frac{1}{2}\leq\frac{1}{\sqrt{3+1}}$$

Thus, P(1)is true.

Step2:

Let, P(m) be equal to $\frac{(2m)!}{2^{2m}(m!)^2} \leq \frac{1}{\sqrt{3m+1}}$

Now, we need to show that P(m+1) is true whenever P(m) is true.

$$P(m+1) = \frac{(2m+2)!}{2^{2m+2}\left((m+1)!\right)^2}$$

$$\Rightarrow P(m+1) = \frac{(2m+1)(2m+1)(2m)!}{2^{2m}.2^2(m+1)^2(m!)^2}$$

$$\Rightarrow \frac{(2m+2)!}{2^{2m+2}\left((m+1)!\right)^2} = \frac{(2m)!}{2^{2m}(m!)^2} \times \frac{(2m+2)(2m+1)}{2^2(m+1)^2}$$

$$\Rightarrow \frac{(2m+2)!}{2^{2m+2}\left((m+1)!\right)^2} \leq \frac{(2m+1)}{2(m+1)\sqrt{3m+1}}$$

$$\Rightarrow \frac{(2m+2)!}{2^{2m+2}\left((m+1)!\right)^2} \leq \sqrt{\frac{(2m+1)^2}{4(m+1)^2(3m+1)}}$$

$$\Rightarrow \frac{(2m+2)!}{2^{2m+2}\left((m+1)!\right)^2} \leq \sqrt{\frac{(4m^2+4m+1)\times(3m+4)}{4(3m^3+7m^2+5m+1)(3m+4)}}$$

$$\Rightarrow \frac{(2m+2)!}{2^{2m+2}\left((m+1)!\right)^2} \leq \sqrt{\frac{(12m^3+28m^2+19m+4)}{(12m^3+28m^2+20m+4)(3m+4)}}$$

As $\frac{12m^3+28m^2+19m+4}{12m^3+28m^2+20m+4} < 1$

$$\therefore \frac{(2m+2)!}{2^{2m+2}\left((m+1)!\right)^2} \leq \sqrt{\frac{1}{(3m+4)}}$$

Thus, P(m+1) is true.

So, by the principle of mathematical induction, P(n) is true for all n∈N.

37. Question

Prove that $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{n^2}<2-\frac{1}{n}$ for all n > 2, n ∈ N.

Answer

Let the given statement be P(n)

Thus, $P(n) = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$, for all n > 2, n ∈ N

Step1:

$P(2): \frac{1}{2^2} = \frac{1}{4} < 2-\frac{1}{2}$

Thus, P(2) is true.

Let, P(m) be true,

Now,

Step2: $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{m^2}<2-\frac{1}{m}$

Now, we need to prove that P(m+1) is true whenever P(m) is true.

We have $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{m^2}<2-\frac{1}{m}$

Adding, $\frac{1}{(m+1)^2}$ on both sides

We have $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{m^2}+\frac{1}{(m+1)^2}<2-\frac{1}{m}+\frac{1}{(1+m)^2}$

$(m+1)^2>m+1 \Rightarrow \frac{1}{(m+1)^2}<\frac{1}{m+1} \Rightarrow \frac{1}{m}-\frac{1}{(1+m)^2}<\frac{1}{m+1}$

$\therefore P(m+1)<2-\frac{1}{m+1}$

Thus, Pm+1 is true. By the principle of mathematical induction, P(n) is true for all n∈N, n≥2.

38. Question

Prove that $x^{2n-1}+y^{2n-1}$ is divisible by x + y for all n ϵ N.

Answer

Let, P(n) be the given statement,

Now, P(n):$x^{2n-1}+y^{2n-1}$

Step1: P(1):x+y which is divisible by x+y

Thus, P(1) is true.

Step2: Let, P(m) be true.

Then, $x^{2m-1}+y^{2m-1}=\lambda(x+y)$

Now, P(m+1) = $x^{2m+1}+y^{2m+1}$

$= x^{2m+1}+y^{2m+1}-x^{2m-1}.y^2+x^{2m-1}.y^2$

$= x^{2m-1}(x^2-y^2)+y^2(x^{2m-1}+y^{2m-1})$

$= (x+y)(x^{2m-1}(x-y)+\lambda y^2)$

Thus, P(m+1) is divisible by x+y. So, by the principle of mathematical

induction P(n) is true for all n.

39. Question

Prove that sin x + sin 3x + … + sin (2n – 1) x $=\frac{\sin^2 nx}{\sin x}$ for all

nϵN.

Answer

Let, P(n) be the given statement,

Now, $P(n):\sin x+\sin 3x+\ldots+\sin(2n-1)x=\frac{\sin^2 nx}{\sin x}$

Step1: $P(1):\sin x=\frac{\sin^2 x}{\sin x}$

Thus, P(1) is true.

Step2: Let, P(m) be true.

$$\text{Then}, \sin x + \sin 3x + \dots + \sin(2m-1)x = \frac{\sin^2 mx}{\sin x}$$

Now, we need to show that P(m+1) is true when P(m) is true.

As P(m) is true

$$\therefore \sin x + \sin 3x + \dots + \sin(2m-1)x = \frac{\sin^2 mx}{\sin x}$$

$$\Rightarrow \sin x + \sin 3x + \dots + \sin(2m-1)x + \sin(2m+1)x$$
$$= \frac{\sin^2 mx}{\sin x} + \sin(2m+1)x$$

$$\Rightarrow P(m+1) = \frac{\sin^2 mx + \sin x\,[\sin mx \cos(m+1)x + \sin(m+1)x \cos mx]}{\sin x}$$

$$= \frac{\sin^2 mx + \sin x \begin{bmatrix} \sin mx \cos mx \cos x - \sin^2 mx \sin x + \\ \sin mx \cos x \cos mx + \cos^2 mx \sin x \end{bmatrix}}{\sin x}$$

$$= \frac{\sin^2 mx + 2\sin x \cos x \cos mx - \sin^2 x \sin^2 mx + \cos^2 mx \sin^2 x}{\sin x}$$

$$= \frac{\sin^2 mx\,(1 - \sin^2 x) + 2\sin x \cos x \cos mx + \cos^2 mx \sin^2 x}{\sin x}$$

$$= \frac{\sin^2 mx \cos^2 x + 2\sin x \cos x \cos mx + \cos^2 mx \sin^2 x}{\sin x}$$

$$= \frac{(\sin mx \cos x + \cos mx \sin x)^2}{\sin x}$$

$$= \frac{(\sin(m+1)x)^2}{\sin x}$$

Thus, P(m+1) is divisible by x+y. So, by the principle of mathematical induction P(n) is true for all n.

40. Question

Prove that cos α + cos (α + β) + cos (α + 2β) + … + cos (α + (n – 1)β) $= \dfrac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$ for all n ϵ N

Answer

$$\text{Let}, P(n) = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta)$$
$$= \frac{\cos\left\{\alpha + \frac{n-1}{2}\beta\right\} \sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \forall\, n \epsilon\, N.$$

Step1: For n=1

L.H.S = cos [α+(1-1)β] = cos α

$$\text{R.H.S} = \frac{\cos\left\{\alpha + \frac{1-1}{2}\beta\right\}\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \cos\alpha$$

As, L.H.S = R.H.S

So, it is true for n=1

Step2: For n=k

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(k-1)\beta) = \frac{\cos\left\{\alpha + \frac{k-1}{2}\beta\right\}\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}} \text{ be true.}$$

Now, we need to show that P(k+1) is true when P(k) is true.

Adding cos(α+kβ) both sides of P(k)

$$\text{L.H.S} = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(k-1)\beta) + \cos(\alpha+k\beta) = \frac{\cos\left\{\alpha + \frac{k-1}{2}\beta\right\}\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}} + \cos(\alpha+k\beta)$$

$$= \frac{\cos\left\{\alpha + \frac{k-1}{2}\beta\right\}\sin\frac{k\beta}{2} + + \cos(\alpha+k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{-\sin\left(\alpha - \frac{\beta}{2}\right) + \sin\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{2\cos\left(\frac{2\alpha + k\beta}{2}\right)\sin\left(\frac{k\beta+\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{(k+1)\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$\text{R.H.S} = \frac{\cos\left\{\alpha + \frac{k}{2}\beta\right\}\sin\frac{(k+1)\beta}{2}}{\sin\frac{\beta}{2}}$$

As, LHS = RHS

Thus, P(k+1) is true. So, by the principle of mathematical induction

P(n) is true for all n.

41. Question

Prove that $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$, for all natural numbers n > 1.

Answer

$$\text{Let, } P(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{13}{24}\ \forall \text{ natural numbers, } n > 1$$

Step1: For n=2

$$\frac{1}{2+1}+\frac{1}{2+2}=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}>\frac{13}{24}$$

So, it is true for n=2

Step2: For n=k

$$P(k)=\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}>\frac{13}{24}$$

Now, we need to show that P(k+1) is true when P(k) is true.

$$P(k)=\frac{1}{k+2}+\frac{1}{k+3}+\cdots\frac{1}{2k}+\frac{1}{2(k+1)}$$

As, LHS = RHS

Thus, P(k+1) is true. So, by the principle of mathematical induction

P(n) is true for all n.

42. Question

Given $a_1=\frac{1}{2}\left(a_0+\frac{A}{a_0}\right), a_2=\frac{1}{2}\left(a_1+\frac{A}{a_1}\right)$ and $a_{n+1}=\frac{1}{2}\left(a_n+\frac{A}{a_n}\right)$ for n ≥ 2, where a > 0, A > 0.

Prove that $\frac{a_n-\sqrt{A}}{a_n+\sqrt{A}}=\left(\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}\right)^{2^{n-1}}$.

Answer

Given, $a_1=\frac{1}{2}\left(a_0+\frac{A}{a_0}\right), a_2=\frac{1}{2}\left(a_1+\frac{A}{a_1}\right)$ and $a_{n+1}=\frac{1}{2}\left(a_n+\frac{A}{a_n}\right), a, A>0$

To prove: $\frac{a_n-\sqrt{A}}{a_n+\sqrt{A}}=\left(\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}\right)^{2^{n-1}}$

Let $P(n)=\frac{a_n-\sqrt{A}}{a_n+\sqrt{A}}=\left(\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}\right)^{2^{n-1}}$

Step1: For n=1

$$L.H.S=\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}$$

$$R.H.S=\left(\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}\right)^{2^{1-1}}=\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}$$

As LHS=RHS.

So, it is true for P(1)

For n=k, let P(k) be true.

$$\therefore \frac{a_k-\sqrt{A}}{a_k+\sqrt{A}}=\left(\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}\right)^{2^{k-1}}$$

Now, we need to show P(k+1) is true whenever P(k) is true.

P(k+1):

$$L.H.S = \frac{a_{k+1} - \sqrt{A}}{a_{k+1} + \sqrt{A}}$$

$$= \frac{\frac{1}{2}\left(a_k + \frac{A}{a_k}\right) - \sqrt{A}}{\frac{1}{2}\left(a_k + \frac{A}{a_k}\right) + \sqrt{A}}$$

$$= \frac{\frac{\frac{1}{2}(a_k^2 + A - 2a_k\sqrt{A})}{a_k}}{\frac{\frac{1}{2}(a_k^2 + A + 2a_k\sqrt{A})}{a_k}}$$

$$= \frac{(a_k - \sqrt{A})^2}{(a_k + \sqrt{A})^2}$$

$$= \left(\frac{a_k - \sqrt{A}}{a_k + \sqrt{A}}\right)^2$$

$$= \left[\left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{k-1}}\right]^2$$

$$= \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^k}$$

As L.H.S=R.H.S

Thus, P(k+1) is true. So, by the principle of mathematical induction

P(n) is true for all n.

43. Question

Let P(n) be the statement: $2^n \geq 3n$. If P(r) is true, show that P(r + 1) is true. Do you conclude that P(n) is true for all n∈N?

Answer

If P(r) is true then $2^r \geq 3r$

For, P(r+1)

$2^{r+1}=2.2^r$

For, x>3, 2x>x+3

So, $2.2^r>2^r+3$ for r>1

⇒ $2^{r+1}>2^r+3$ for r>1

⇒ $2^{r+1}>3r+3$ for r>1

⇒ $2^{r+1}>3(r+1)$ for r>1

So, if P(r) is true, then P(r+1) is also true.

For, n=1, P(1):

L.H.S=2

R.H.S=3

As L.H.S<R.H.S

So, it is not true for n=1

Hence, P(n) is not true for all natural numbers.

44. Question

Show by the Principle of Mathematical induction that the sum S_n of the n terms of the series $1^2+2\times 2^2+3^2+2\times 4^2+5^2+2\times 6^2+7^2+\ldots$ is given by

$$S_n=\begin{cases}\frac{n(n+1)^2}{2} & \text{,if n is even}\\ \frac{n^2(n+1)^2}{2} & \text{,if n is odd}\end{cases}$$

Answer

$$\text{Let, } P(n): S_k=1^2+2\times 2^2+3^2+2\times 4^2+5^2=\begin{cases}\frac{n(n+1)^2}{2}, \text{when n is even}\\ \frac{n^2(n+1)}{2}, \text{when n is odd}\end{cases}$$

Step1: For n=1, P(1):

LHS=S_1=1

RHS=S_1=1

So, P(1) is true.

Step2: Let P(n) be true for n=k

$$P(k): S_k=1^2+2\times 2^2+3^2+2\times 4^2+5^2=\begin{cases}\frac{k(k+1)^2}{2}, \text{when n is even}\\ \frac{k^2(k+1)}{2}, \text{when n is odd}\end{cases}$$

Now, we need to show P(k+1) is true whenever P(k) is true.

P(k+1):

Case1: When k is odd, then (k+1) is even

$P(k+1)$:

$$\text{LHS}=1^2+2\times 2^2+3^2+2\times 4^2+5^2+\cdots+k^2+2\times(k+1)^2$$

$$=\frac{k^2(k+1)}{2}+2\times(k+1)^2$$

$$=\frac{k^2(k+1)+4(k+1)^2}{2}$$

$$=\frac{(k+1)(k^2+4k+4)}{2}$$

$$=\frac{(k+1)(k+2)^2}{2}$$

$$\text{RHS}=\frac{(k+1)(k+1+1)^2}{2}$$

$$=\frac{(k+1)(k+2)^2}{2}$$

As LHS=RHS

So, it is true for n=k+1 when k is odd.

Case2: When k is even, then (k+1) is odd

$P(k+1)$:

$LHS = 1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + \cdots + 2 \times k^2 + (k+1)^2$

$= \frac{k(k+1)^2}{2} + (k+1)^2$

$= \frac{k(k+1)^2 + 2(k+1)^2}{2}$

$= \frac{(k+1)^2(k+2)}{2}$

$RHS = \frac{(k+1)^2(k+1+1)}{2}$

$= \frac{(k+1)^2(k+2)}{2}$

As LHS=RHS

So, it is true for n=k+1 when k is even.

Hence, by the principle of mathematical induction P(n) is true ∀ n∈N.

45. Question

Prove that the number of subsets of a set containing n distinct elements is 2^n for all n ϵ N.

Answer

Let the given statement be defined as

P(n): The number of subsets of a set containing n distinct

elements=2^n, for all n ϵ N.

Step1: For n=1,

L.H.S=As, the subsets of the set containing only 1 element are:

Φ and the set itself

i.e. the number of subsets of a set containing only element=2

R.H.S=2^1=2

As, LHS=RHS, so, it is true for n=1.

Step2: Let the given statement be true for n=k.

P(k): The number of subsets of a set containing k distinct

elements=2^k

Now, we need to show P(k+1) is true whenever P(k) is true.

P(k+1):

Let A={a_1, a_2, a_3, a_4,..., a_k, b} so that A has (k+1) elements.

So the subset t of A can be divided into two collections:

first contains subsets of A which don t have b in them and

the second contains subsets of A which do have b in them.

First collection: { }, $\{a_1\}$,$\{a_1, a_2\}$,$\{a_1, a_2, a_3\}$,...,$\{a_1, a_2, a_3, a_4,..., a_k\}$ and

Second collection: $\{b\}$, $\{a_1,b\}$,$\{a_1,a_2,b\}$,$\{a_1,a_2,a_3,b\}$,...,$\{a_1,a_2,a_3,a_4,...,a_k, b\}$

It can be clearly seen that:

The number of subsets of A in first collection

= The number of subsets of set with k elements i.e. $\{a_1, a_2, a_3, a_4,..., a_k\}=2^k$

Also it follows that the second collection must have

the same number of the subsets as that of the first = 2^k

So the total number of subsets of $A=2^k+2^k=2^{k+1}$

Thus, by the principle of mathematical induction P(n) is true.

46. Question

A sequence a_1, a_2, a_3, ... is defined by letting $a_1 = 3$ and $a_k = 7\ a_{k-1}$ for all natural numbers k ≥ 2. Show that $a_n = 3.7^{\ n-1}$ for all n ϵ N

Answer

Let P(n): $a_n=3.7^{n-1}$ for all n ϵ N

Step1: For n=1,

$a_1=3.7^{1-1}=3$

So, it is true for n=1

Step2: For n=k,

Let P(k) be true.

So, $a_k=3.7^{k-1}$

Now, we need to show P(k+1) is true whenever P(k) is true.

P(k+1):

$a_{k+1}=7.a_k$

$=7.3.7^{k-1}$

$=3.7^{k-1+1}$

$=3.7^{(k+1)-1}$

So, it is true for n=k+1

Hence, by the principle of mathematical induction P(n) is true.

47. Question

A sequence x_1, x_2, x_3, ... is defined by letting $x_1 = 2$ and $x_k = \frac{x_{k-1}}{n}$ for all natural numbers k, k ≥ 2. Show that $x_n = \frac{2}{n!}$ for all n∈N

Answer

Given: A sequence $x_1, x_2, x_3, \ldots$ is defined by letting $x_1 = 2$ and $x_k = \frac{x_{k-1}}{2}$

for all natural numbers k, k $\geq$ 2.

Let $P(n): x_n = \frac{2}{n!}$ For all $n \epsilon N$

Step1: For n=1

$P(1): x_1 = \frac{2}{1!} = 2$

So, it is true for n=1.

Step2: For n=k,

Let, $x_k = \frac{2}{k!}$ be true.

Now, we need to show P(k+1) is true whenever P(k) is true.

P(k+1):

$$x_{k+1} = \frac{x_k}{k+1}$$

$$= \frac{2}{(k+1) \times k!}$$

$$= \frac{2}{(k+1)!}$$

So, it is true for n=k+1.

Thus, by the principle of mathematical induction P(n) is true.

48. Question

A sequence x_0, x_1, x_2, x_3, ... is defined by letting $x_0 = 5$ and

$x_k = 4 + x_{k-1}$ for all natural numbers k. Show that $x_n = 5$ for all

$n \epsilon N$ using mathematical induction.

Answer

Let P(n): $x_n = 5+4n$ for all $n \epsilon N$

Step1: For n=0,

$P(0): x_0 = 5+4\times0 = 5$

So, it is true for n=0.

Step2: Let P(k) be true

Thus, $x_k = 5+4k$

Now, we need to show P(k+1) is true whenever P(k) is true.

P(k+1):

$x_{k+1} = 4 + x_{k+1-1}$

$= 4 + x_k$

$= 4+5+4k$

$= 5+4(k+1)$

=RHS

Thus, P(k+1) is true, so by mathematical induction P(n) is true.

49. Question

Using principle of mathematical induction prove that

$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}$ for all natural numbers n ≥ 2.

Answer

Let $P(n) = \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$ for all $n \geq 2$

Step1: For n=2, P(n):

$\text{LHS} = \sqrt{2} = 1.414$

$\text{RHS} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + 0.707 = 1.707$

Therefore, it is true for n=2.

Step2: Let P(n) be true for n=k.

$\text{Then}, \sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}}$

Now, we need to show P(k+1) is true whenever P(k) is true.

P(k+1):

$\text{LHS} = \sqrt{k+1}$

$\text{RHS} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$\Rightarrow \frac{k}{\sqrt{k+1}} < \sqrt{k}$

$\Rightarrow \frac{k+1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+1}} < \sqrt{k}$

$\Rightarrow \sqrt{k+1} - \frac{1}{\sqrt{k+1}} < \sqrt{k}$

$\Rightarrow \sqrt{k+1} < \sqrt{k} + \frac{1}{\sqrt{k+1}}$

$\text{So}, \text{LHS} < \text{RHS}$

So, it is true for n=k+1, thus by the principle of mathematical induction P(n) is true for all n ≥ 2

50. Question

The distributive law from algebra states that for real numbers

c, a_1 and a_2, we have $c(a_1 + a_2) = c\,a_1 + ca_2$

Use this law and mathematical induction to prove that, for all

natural numbers, n ≥ 2, if c, a_1, a_2, ... a_n are any real numbers,

then $c(a_1 + a_2 + \ldots + a_n) = c\,a_1 + c\,a_2 + \ldots + c\,a_n$.

Answer

Let $P(n): c(a_1+a_2+\ldots+a_n) = ca_1+ca_2+\ldots+ca_n$,for all natural

numbers, n ≥ 2.

Step1: For n=2,

P(2)

LHS= $c(a_1 + a_2)$

RHS= $c\ a_1 + ca_2$

As, it is given that $c(a_1 + a_2) = c\ a_1 + ca_2$

Thus, P(2) is true.

Step2: For n=k,

Let P(k) be true

So, $c(a_1+a_2+...+a_k) = ca_1+ca_2+...+ca_k$

Now, we need to show P(k+1) is true whenever P(k) is true.

P(k+1):

LHS= $c(a_1+a_2+...+a_K+ak_{+1})$

$=c[(a_1+a_2+...+a_K)+a_{k+1}]$

$=c(a_1+a_2+...+a_K)+ca_{k+1}$

$=ca_1+ca_2+...+ca_K+ca_{k+1}$

=RHS

Thus, P(k+1) is true, so by mathematical induction P(n) is true.

Very Short Answer

1. Question

State the first principle of mathematical induction.

Answer

The first principle of mathematical induction states that if the basis step and the inductive step are proven, then P(n) is true for all natural numbers.

2. Question

Write the set of value of n for which the statement P(n): 2n < n! is true.

Answer

The set of value of n for which the statement P(n): 2n < n! is true can be written as $\{n \in N : n \geq 4\}$.

3. Question

State the second principle of mathematical induction.

Answer

Let M be an integer. Suppose we want to prove that P(n) is true for all positive integers ≥M. Then if we show that:

Step 1: P(M) is true, and

Step 2: for an arbitrary positive integer k≥M, if P(M).P(M+1).P(M+2)……P(k) are true then P(k+1) is true,

Then P(n) is true for all positive integers greater than or equal to M.

4. Question

If P(n): $2 \times 4^{2n + 1} + 3^{3n + 1}$ is divisible by λ for all n ∈ N is true, then find the value of λ.

Answer

for n=1,

$2\times4^{2\times1+1} + 3^{3\times1+1}=2\times4^3+3^4$

= 2×64+81

= 128+81

= 209

For n=2,

$2\times4^{2\times2+1} + 3^{3\times2+1} = 2\times4^5+3^7$

= 2×1024+2187

= 2048+2187

= 4235

Now, the H.C.F of 209 and 4235 is 11.

Hence, λ=11.

MCQ

1. Question

Mark the Correct alternative in the following:

If $x^n - 1$ is divisible by x - λ, then the least positive integral value of λ is

A. 1

B. 2

C. 3

D. 4

Answer

Given x^n-1 is divisible by x-λ

⇒ x=λ is the root of the eqn x^n-1

$\Rightarrow\lambda^n-1=0$

$\Rightarrow\lambda^n=1$

Least value of λ=1

2. Question

Mark the Correct alternative in the following:

For all n ∈ N, $3 \times 5^{2n + 1} + 2^{3n + 1}$ is divisible by

A. 19

B. 17

C. 23

D. 25

Answer

Given for all n€ N $3 \times 5^{2n+1} + 2^{3n+1}$

For n=1,

$3 \times 5^3 + 2^4$

$3 \times 125 + 16$

$375 + 16 = 391$

For n=2,

$3 \times 5^5 + 2^7$

$3 \times 3125 + 128$

$9375 + 128 = 9503$

H.C.F of 391, 9503 = 17

3. Question

Mark the Correct alternative in the following:

If $10^n + 3 \times 4^{n+2} + \lambda$ is divisible by 9 for all $n \in N$, then the least positive integral value of λ is

A. 5

B. 3

C. 7

D. 1

Answer

Given $10^n + 3 \times 4^{n+2} + \lambda$ is divisible by 9

For n=1,

$10 + 3 \times 4^3 + \lambda$

$10 + 3 \times 64 + \lambda$

$= 202 + \lambda$

202 when divided by 9 gives remainder 4

For n=2,

$10^2 + 3 \times 4^4 + \lambda$

$=100 + 3 \times 256 + \lambda$

$=868 + \lambda$

868 when divided by 9 gives remainder 4

∴ $\lambda = 4 + 1 = 5$

4. Question

Mark the Correct alternative in the following:

Let P (n): $2n < (1 \times 2 \times 3 \times \ldots \times n)$. Then the smallest positive integer for which P(n) is true is

A. 1

B. 2

C. 3

D. 4

Answer

Given P(n): $2n < (1 \times 2 \times \ldots \times n)$

For n=1, 2<2

For n=2, 4<4

For n=3, 6<6

For n=4, 8<24

∴ the smallest positive integer for which P(n) is true is 4.

5. Question

Mark the Correct alternative in the following:

A student was asked to prove a statement P(n) by induction. He proved P (k + 1) is true whenever P(k) is true for all k > 5 ∈ N and also P(5) is true. On the basis of this he could conclude that P(n) is true.

A. for all n ∈ N

B. for all n > 5

C. for all n ≥ 5

D. for all n < 5

Answer

Since given P(5) is true and P(k) is true for all k>5€N,

then we can conclude that P(n) is true for all n≥5

6. Question

Mark the Correct alternative in the following:

If P(n) : $49^n + 16^n + \lambda$ is divisible by 64 for n ∈ N is true, then the least negative integral value of λ is

A. -3

B. -2

C. -1

D. -4

Answer

For n=1,

49 16 + λ

⇒ 65 + λ

Now we can see that if λ = -1, then it is divisible by 64

 λ = -1