5. PERMUTATIONS AND COMBINATIONS

1. INTRODUCTION

The main subject of this chapter is counting. Given a set of objects the problem is to arrange some or all of them according to some order or to select some or all of them according to some specification.

2. FUNDAMENTAL PRINCIPLE OF COUNTING

The rule of sum: If a first task can be performed in m ways, while a second task can be performed in n ways, and the tasks cannot be performed simultaneously, then performing either one of these tasks can be accomplished in any one of total m+n ways.

The rule of product: If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in a total of mn ways.

Illustration 1: There are three stations, A. B and C. Five routes for going from station A to station B and four routes for going from station B to station C. Find the number of different ways through which a person can go from A to C via B. **(JEE MAIN)**

Sol: This problem is an application of the Fundamental Principle of Counting. The rule of product can be used to solve this question easily.

Given there are five routes for going from A to B and four routes for going from B to C.

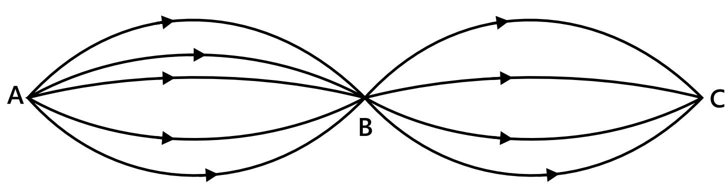


Figure 5.1

Hence, by the fundamental principle of multiplication the total number of different ways

= 5 x 4 (i.e., A to B and then B to C) = 20 ways

Illustration 2: A hall has 12 gates. In how many ways, can a man enter the hall through one gate and come out through a different gate. **(JEE MAIN)**

Sol: The rule of product can be used to solve this problem.

There are 12 ways of entering the hall. After entering the hall the man can come out through any of 11 different gates.

Hence, by the fundamental principle of multiplication, the total number of ways are 12 x 11 = 132 ways.

Illustration 3: How many numbers between 10 and 10,000 can be formed by using the digits 1, 2, 3, 4, 5 if

(i) no digit is repeated in any number. **(ii)** digits can be repeated. **(JEE MAIN)**

Sol: The numbers between 10 and 10,000 can be either two digit, three digit or four digit numbers. We consider each of these cases and try to find the number of possibilities using 1, 2, 3, 4 and 5. Finally, we add them up to get the desired result.

(i) Number of two digit numbers = 5 x 4 = 20

Number of three digit numbers = 5 x 4 x 3 = 60

Number of four digit numbers = 5 x 4 x 3 x 2 = 120

Total number of numbers = 20 + 60 +120 = 200

(ii) Number of two digit numbers = 5 x 5 = 25

Number of three digit numbers = 5 x 5 x 5 = 125

Number of four digit numbers = $5 \times 5 \times 5 \times 5 = 625$

Total number of numbers = 25 +125 +625 = 775

3. FACTORIAL NOTATION

An efficient way of writing a product of several consecutive integers is the factorial notation. The number n! (read as "n-factorial") is defined as follows :

For any positive integer n; $n! = n(n-1)(n-2) \ldots\ldots (3)(2)(1)$; For instance, $4! = 4.3.2.1 = 24$

Note: (i) $n! = n(n-1)(n-2) \ldots\ldots 3.2.1$; $n! = n.(n-1)!$; $0! = 1! = 1$; $(2n)! = 2^n.n![1.3.5.7 \ldots\ldots (2n-1)]$

(ii) $n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)!$

(iii) $n(n-1) \ldots\ldots (n-r+1) = \dfrac{n!}{(n-r)!}$

Illustration 4: Find the sum of n terms of the series whose n^{th} term is $n! \times n$. **(JEE ADVANCED)**

Sol: Represent the general term in this question as a difference of two terms and then add them up to find the answer.

The required sum $= (1)! + 2(2)! + 3(3)! + \ldots\ldots + n(n!) = (2-1)! + (3-1)2! + (4-1)3! + \ldots\ldots + [(n+1)-1]n!$

$= (2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots\ldots + [(n+1)! - n!] = (n+1)! - 1$

4. PERMUTATION

Each of the different arrangements, which can be made by taking some or all of a number of objects is called permutation. The number of permutations of n different objects taken r at a time is represented as

$${}^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2) \ldots\ldots (n-r+1) \text{ (where, } 0 \le r \le n)$$

Note: (i) In permutation, the order of the items plays an important role.

(ii) The number of all permutations of n distinct objects taken all at a time is n!

Illustration 5: If $^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, find $^{r}P_2$ **(JEE MAIN)**

Sol: Use the formula for ^nP_r.

We have , $\frac{^{56}P_{r+6}}{^{54}P_{r+3}} = \frac{30800}{1} = \frac{56!}{(50-r)!} \times \frac{(51-r)!}{54!} = \frac{30800}{1} \Rightarrow 56 \text{ x } 55\,(51-r) = 30800 \Rightarrow r = 41$

$\therefore \quad {}^{41}P_2 = 41 \text{ x } 40 = 1640$

Illustration 6: Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them? **(JEE ADVANCED)**

Sol: Use the concept and understanding of Permutation, i.e. arrangement to find the answer.

The total number of ways in which three men can wear 4 coats is the number of arrangements of 4 different coats taken 3 at a time. So, three men can wear 4 coats in $^4P_3$ ways. Similarly, 5 waist coats and 6 caps can be worn by three men in $^5P_3$ and $^6P_3$ ways respectively. Hence, the required no. of ways $= {}^4P_3 \times {}^5P_3 \times {}^6P_3 = (4!) \times (5 \times 4 \times 3) \times (6 \times 5 \times 4) = 172800$.

Illustration 7: Suppose 8 people enter an event in a swim meet. In how many ways could the gold, silver, and bronze prizes be awarded? **(JEE ADVANCED)**

Sol: Use the formula for ^nP_r. The required number of ways is an arrangement of 3 people out of 8 i.e.

$$^8P_3 = \frac{8!}{5!} = \frac{8.7.6.5.4.3.2.1}{5.4.3.2.1} = 8\,.\,7\,.\,6 = 336.$$

CONCEPTS

The following two steps are involved in the solution of a permutation problem:

Step 1: Recognizing the objects and the places involved in the problem.

Step 2: Checking whether the repetition of the objects is allowed or not.

Uday Kiran G (JEE 2012, AIR 102)

4.1 Permutation with Repetition

These kinds of problems occur with permutations of different objects in which some of the objects can be repeated. The no. of permutations of n different objects taken r at a time when each object may be repeated any number of times is n^r.

Illustration 8: A student appears in an objective test which contains 10 multiple choice questions. Each question has four choices in which one is the correct option. What maximum number of different answers can the student give? How will the answer change if each question may have more than one correct answers? **(JEE ADVANCED)**

Sol: Use the concept of Permutation with Repetition.

For the first part each question has four possible answers. So, the total possible answers $= 4 \times 4 \times \ldots$ 10 times $= 4^{10}$.

For the second part. Suppose the choices for each question are denoted by A, B, C and D. Now the choice A is either correct or incorrect (two ways) similarly the other choices are either correct or incorrect. Thus, this particular question can have $2 \times 2 \times 2 \times 2 = 16$ possible answers. But this includes the case when all the four choices are incorrect. Thus the total number of answers = 15. Now, as there are 10 questions and each question has 15 possible answers. Therefore the total number of answers $= 15^{10}$.

CONCEPTS

The m^n or n^m dilemma

Let us start with an example.

Q. There are 7 letters and 5 letter-boxes. In how many ways can you put the letters in the boxes?

Sol: This is the typically confusing question asked frequently from the P & C area. Let's see how you can solve this type of question.

The Exhaustive Approach: One way to solve this question is through (what we will call) The Exhaustive Approach. While solving such problems, first decide which of the items (letters and letter-boxes here) is exhaustive. Exhaustive here means the entity which is sure to be used up completely.

In this example, all the "letters" are sure to be placed in the boxes, whereas there is no such constraint as regards the "letter-boxes". Some boxes could go empty. Having decided this, just go by the options we have for all the instances of the exhaustive item and you have your answer.

As you can see here, every letter has 5 boxes to choose from. Thus the total would

be (5 x 5 x........7 times) = (5^7)

A similar question could be: In how many ways can 10 rings be worn on 5 fingers? Try it yourself.

Chinmay S Purandare (JEE 2012, AIR 698)

4.2 Permutation of Alike Objects

This kind of problems involve permutations of different objects in which some of them are similar.

The number of permutations of n objects taken all at a time in which, p are alike objects of one kind, q are alike objects of second kind & r are alike objects of a third kind and the rest (n – (p + q + r)) are all different is

$$\frac{n!}{p!q!r!}$$

Illustration 9: Determine the number of permutations of the letters of the word 'SIMPLETION' taken all at a time. **(JEE MAIN)**

Sol: In the given word the letter I occurs twice and the remaining letters occur only once. So, the concept of Permutation of Alike Objects is used to find out the answer.

There are 10 letters in the word 'SIMPLETION' and out of these 10 letters two are identical. So, just selecting all 10 objects at a time will give twice the actual result. Hence, the number of permutations of taking all the letters at a time = ${}^{10}P_{10}$ / 2! = 10! / 2! = 181440.

4.3 Permutation under Restriction

(a) The number of permutations of n different objects, taken r at a time, when a particular object is to be always included in each arrangement, is r. ${}^{n-1}P_{r-1}$

The number of permutations of n different objects, taken r at a time, when a particular object is never taken in each arrangement is ${}^{n-1}P_r$.

(b) **String method:** The number of permutations of n different objects, taken all at a time, when m specified objects always come together is m! × (n – m + 1)!.

(c) **Gap Method:** The number of permutations when no two given objects occur together.

In order to find the number of permutations when no two given objects occur together.

(a) First of all, put the m objects for which there is no restriction, in a line. These m objects can be arranged in m! ways.

(b) Then count the number of gaps between every two m objects for which there is no restriction, including the end positions. Number of such gaps will be (m + 1).

(c) If m is the number of objects for which there is no restriction and n is the number of objects, two of which are not allowed to occur together, then the required number of ways = $m! \times {}^{m+1}C_n \times n!$

The number of permutations when two types of objects are to be arranged alternately

(a) If their numbers differ by 1 put the object whose number is greater in the first, third, fifth.... places, etc. and the other object in the second, fourth, sixth.... places.

(b) If the number of two types of objects is same, consider two cases separately keeping the first type of object in the first, third, fifth places, etc. and the second type of object in the first, third, fifth places.... and then add.

4.4 Non-Consecutive Selection

The number of selections of r consecutive objects out of n objects in a row = n - r + 1

The number of selections of r consecutive objects out of n objects along a circle = $\begin{cases} n, & \text{when } r < n \\ 1, & \text{when } r = n \end{cases}$

Illustration 10: Find the numbers between 300 and 3000 that can be formed with the digits 0, 1, 2, 3, 4 and 5, where no digit is repeated in any number. **(JEE MAIN)**

Sol: The numbers between 300 and 3000 can either be a three digit number or a four digit number. The solution is divided into these two different cases and their sum will give us the desired result.

Any number between 300 and 3000 must be of three or four digits.

Case I: When number is of three digits: The hundreds place can be filled by any one of the three digits 3, 4 and 5 in 3 ways. The remaining two places can be filled by the remaining five digits in ${}^5P_2$, ways.

$\therefore$ The number of numbers formed in this case = $3 \times {}^5P_2 = 3 \times \frac{5!}{3!} = 60$

Case II: When number is of four digits: The thousands place can be filled by any one of the two digits 1 and 2 in 2 ways and the remaining three places can be filled by the remaining five digits in ${}^5P_3$ ways.

$\therefore$ The number of numbers formed in this case = $2 \times {}^5P_3 = 2 \times \frac{5!}{2!} = 120$

$\therefore$ Total numbers = 60 + 120 = 180

Illustration 11: How many words can be formed from the letters of the word ARTICLE, so that vowels occupy the even places? **(JEE MAIN)**

Sol: Clearly, this is an example of Permutation under Restriction. We identify the even places and the odd places and try to find the number of ways in which the vowels and consonants can fill the spaces.

There are seven places: 3 even and 4 odd in which we have to fill 3 vowels and 4 consonants.

$\therefore$ The number of words = ${}^3P_3 . {}^4P_4 = 3! \times 4! = 6 \times 24 = 144$.

Illustration 12: How many different words can be formed with the letters of the word ORDINATE so that

(a) Four vowels occupy the odd places (b) Beginning with O (c) Beginning with O and ending with E. **(JEE MAIN)**

Sol: The concept of Permutation under Restriction can be used to solve this problem.

There are 4 vowels and 4 consonants. Total 8 letters.

(a) No. of words = 4! × 4! = 24 × 24 = 576. Because 4 vowels are to be adjusted in 4 odd place and the 4 consonants in the remaining 4 even places.

(b) 7! ways, O being fixed.

(c) 6! ways, O fixed in first and E fixed in last.

Illustration 13: Find the number of ways in which 5 boys and 5 girls can be seated in a row so that

(a) No two girls may sit together.

(b) All the girls sit together and all the boys sit together

(c) All the girls are never together. **(JEE ADVANCED)**

Sol: Since the number of girls and the number of boys are equal they have to sit alternately. This can be used to solve (a). For (b), we keep the girls together and arrange the boys in five places. Also, the girls can be arranged amongst themselves in 5! ways. This gives us the number of arrangements. Use the answer of the second part to find (c).

(a) 5 boys can be seated in a row in ${}^5P_5$ = 5! ways. Now, in the 6 gaps between 5 boys, the 5 girls can be arranged in ${}^6P_5$ ways. Hence, the number of ways in which no two girls sit together = 5! × ${}^5P_5$ = 5! × 6!

(b) The two groups of girls and boys can be arranged in 2! ways. 5 girls can be arranged among themselves in 5! ways. Similarly, 5 boys can be arranged among themselves in 5! ways. Hence, by the fundamental principle of counting, the total number of requisite seating arrangements = 2!(5! × 5!) = $2(5!)^2$.

(c) The total number of ways in which all the girls are never together = Total number of arrangements – Total number of arrangements in which all the girls are always together = 10! – 5! × 6!

Illustration 14: The letters of the word OUGHT are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word TOUGH in this dictionary. **(JEE MAIN)**

Sol: The word TOUGH will appear after all the words that start with G, H and O. Then we look at the second letter of the words starting with T and then third. Hence, the rank of the word TOUGH will be one more than the sum of all the possibilities just mentioned.

Total number of letters in the word OUGHT is 5 and all the five letters are different, the alphabetical order of these letters is G, H, O, T, U.

Number of words beginning with G = 4! = 24

Number of words beginning with H = 4! = 24

Number of words beginning with O = 4! = 24

Number of words beginning with TG = 3! = 6

Number of words beginning with TH = 3! = 6

Number of words beginning with TOG = 2! = 2

Number of words beginning with TOH = 2! = 2

Next come the words beginning with TOU and TOUH is the first word beginning with TOU.

$\therefore$ Rank of 'TOUGH' in the dictionary = 24 + 24 + 24 + 6 + 6 + 2 + 2 + 1 = 89

Illustration 15: There are 21 balls which are either white or black and the balls of same color are alike. Find the number of white balls so that, the number of arrangements of these balls in a row is maximum. **(JEE ADVANCED)**

Sol: The property of a binomial coefficient can be used to solve this question.

Let there be r white balls so that the number of arrangements of these balls in a row be maximum. Number of arrangements of these balls is

$A = \dfrac{21!}{r!(21-r)!}$ A will be maximum when $r = \dfrac{21+1}{2}$ or $\dfrac{21-1}{2}$ i.e. 10 or 11

Illustration 16: The number plates of cars must contain 3 letters of the alphabet denoting the place and area to which its owner belongs. This is to be followed by a three-digit number. How many different number plates can be formed if:

(i) Repetition of letters and digits is not allowed. (ii) Repetition of letters and digits is allowed. **(JEE ADVANCED)**

Sol: This is a simple application of Permutation with and without repetition.

There are 26 letters of alphabet and 10 digits from 0 to 9.

(i) When repetition is not allowed

3 letters selected in $26 \times 25 \times 24$ ways

3 digit numbers are $= 9 \times 9 \times 8$ (as zero can't be in the hundreds place)

$\therefore$ The Number of plates $= 26 \times 25 \times 24 \times 9 \times 9 \times 8 = 10108800$.

(ii) When repetition is allowed

3 letters are selected $26 \times 26 \times 26$ ways

3 digit numbers are $= 9 \times 10 \times 10 = 900$

$\therefore$ The number of plates $= 26 \times 26 \times 26 \times 900 = 15818400$.

CONCEPTS

Constraint based arrangement

Let us start with an example first:

Q. In how many ways can the word VARIETY be arranged so that exactly 2 vowels are together?

The problem could be easier if none or all vowels were to be kept together. Isn't it? Well, we will do exactly that!

Whenever question involving "constraints that has choices (2 vowels could be IE or AI or AE)" are asked, "go for the backward approach." Rather than finding the favorable cases, subtract the unfavorable ones from the total possible cases. This method is more reliable.

So, the solution to the above question would be:

(Total arrangements of VARIETY) – (Arrangements with no vowels together + Arrangements with all the vowels together)

Vaibhav Krishnan (JEE 2009, AIR 22)

5. COMBINATION

Each of the different groups or selection which can be made by some or all of a number of given objects without reference to the order of the objects in each group is called a combination.

The number of all combinations of n objects, taken r at a time is generally denoted by C(n, r) or ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ $(0 \le r \le n) = \dfrac{{}^np_r}{r!}$

Note:

(a) The number of ways of selecting r objects out of n objects, is the same as the number of ways in which the remaining (n - r) can be selected and rejected.

(b) The combination notation also represents the binomial coefficient. That is, the binomial coefficient ${}^nC_r$ is the combination of n elements chosen r at a time.

(c) (a) ${}^{n}C_{r} = {}^{n}C_{n-r}$

(b) ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

(c) ${}^{n}C_{x} = {}^{n}C_{y} \Rightarrow x = y$ or $x + y = n$

(d) If n is even, then the greatest value of ${}^{n}C_{r}$ is ${}^{n}C_{n/2}$

(e) If n is odd, then the greatest value of ${}^{n}C_{r}$ is ${}^{n}C_{\frac{n+1}{2}}$

(f) ${}^{n}C_{0} + {}^{n}C_{r} + \ldots\ldots\ldots\ldots + {}^{n}C_{n} = 2^{n}$

(g) ${}^{n}C_{n} + {}^{n+1}C_{n} + {}^{n+2}C_{n} + \ldots\ldots\ldots\ldots + {}^{2n-1}C_{n} = {}^{2n}C_{n+1}$

(d) Comparison of permutation and combination

Permutations	**Combinations**
Different orderings or arrangement of the r objects are different permutations ${}^{n}P_{r} = \dfrac{n!}{(n-r)!}$ **Clue words:** arrangement, schedule, order	Each choice or subset of r object give one combination. Order within the group of r objects does not matter. ${}^{n}C_{r} = \dfrac{n!}{(n-r)!r!}$ **Clue words:** group, committee, sample, selection, subset.

Illustration 17: A basketball coach must select two attackers and two defenders from among three attackers and five defenders. How many different combinations of attackers and defenders can he select? **(JEE MAIN)**

Sol: The number of ways two attackers and two defenders can be selected is ${}^{3}C_{2}$ and ${}^{5}C_{2}$ respectively.

$\therefore$ He can select in ${}^{3}C_{2} \times {}^{5}C_{2} = \dfrac{5.4.3.2.1}{2.2} = 30$ different combinations.

Illustration 18: A soccer team of 11 players is to be chosen from 30 boys, of whom 4 can play only in goal, 12 can play only as forwards and the remaining 14 in any of the other positions. If the team is to include five forwards and of course, one goalkeeper, in how many ways can it be made up? **(JEE MAIN)**

Sol: Proceed according to the previous question.

There are: ${}^{4}C_{1}$ ways of choosing the goalkeeper. ${}^{12}C_{5}$ ways of choosing the forwards and ${}^{14}C_{5}$ ways of choosing the other 5 players. That is, ${}^{4}C_{1} \times {}^{12}C_{5} \times {}^{14}C_{5}$ combinations altogether = 4 x 792 x 2002 = 6342336.

5.1 Combinations under Restrictions

(a) Number of ways of choosing r objects out of n different objects if p particular objects must be excluded.

Consider the objects $A_1, A_2, A_3, \ldots, A_p, A_{p+1}, \ldots\ldots\ldots, A_n$. If the p objects $A_1, A_2 \ldots\ldots A_p$ are to be excluded then we will have to select r objects from the remaining n – p objects ($A_{p+1}, A_{p+2} \ldots\ldots\ldots A_n$).

Hence the required number of ways = ${}^{(n-p)}C_{r}$

(b) Number of ways of choosing r objects out of n different objects if p particular objects must be included ($p \le r$).

Consider the objects $A_1, A_2, A_3 \ldots\ldots A_p, A_{p+1}, \ldots\ldots A_n$. If the p particular objects $A_1, A_2, \ldots\ldots A_p$ (say) must be included in the selection then to complete the selection, we must select (r – p) more objects to complete the selection. These objects are to be selected from the remaining n – p objects.

Hence, the required number of ways = ${}^{n-p}C_{r-p}$

(c) The total number of combinations of n different objects taken one or more at a time = 2^n - 1.

5.2 Combinations of Alike Objects

(a) The number of combinations of n identical objects taking ($r \le n$) at a time is 1.

(b) The number of ways of selecting r objects out of n identical objects is n + 1.

(c) If out of (p + q + r + s) objects, p are alike of one kind, q are alike of a second kind, r are alike of the third kind and s are different, then the total number of combinations is $(p + 1)(q + 1)(r + 1)2^s - 1$

Note: The list of alike objects can be extended further

4. The number of ways in which r objects can be selected from a group of n objects of which p are identical, is $\sum_{0}^{t} {}^{n-p}C_r$, if $r \le p$ and $\sum_{r=p}^{t} {}^{n-p}C_r$ if $r > p$

Illustration 19: There are 4 oranges, 5 apples and 6 mangoes in a fruit basket. In how many ways can a person select fruits from among the fruits in the basket? **(JEE MAIN)**

Sol: Use the concept of Combination of Alike objects described above.

Here, we consider all fruits of the same type as identical.

Zero or more oranges can be selected out of 4 identical oranges in 4 + 1 = 5 ways.

Zero or more apples can be selected out of 5 identical apples in 5 + 1 = 6 ways.

Zero or more mangoes can be selected out of 6 identical mangoes in 6 + 1 = 7 ways

$\therefore$ The total number of selections when all the three types of fruits are selected (the number of any type of fruit may be zero) = $5 \times 6 \times 7 = 210$.

But in one of these selections number of each type of fruit is zero and hence there is no selection, this must be excluded.

$\therefore$ The required number = 210 – 1 = 209.

Caution: When all fruits of same type are different, the number of selections

$= ({}^4C_0 + {}^4C_1 + \dots + {}^4C_4)({}^5C_0 + {}^5C_1 + \dots + {}^5C_5)({}^6C_0 + \dots + {}^6C_6) - 1 = 2^4 \times 2^5 \times 2^6 - 1 = 2^{15} - 1$

Illustration 20: How many four digit numbers are there whose decimal notation contains not more than two distinct digits? **(JEE MAIN)**

Sol: A four digit number can consist of either only one digit or two digits as per the question. Clearly, there are nine four digit numbers with the same digit. Similarly, calculate the number of four digit numbers with two distinct digits and hence the sum gives us the desired result.

Evidently any number so formed of four digits contains

(i) Only one digit (like 1111, 2222,...) and there are 9 numbers. **(ii)** Two digits

(a) if zero is one of the two, then the one more can be anyone of the nine, and these two digits can be arranged in ${}^9C_1\,[{}^3C_1 + {}^3C_2 + {}^3C_2 + {}^3C_3] = 63$.

(b) if zero is not one of them, then two of the digits have to be selected from 9, and these two can be arranged in ${}^9C_2[{}^4C_1 + {}^4C_2 + {}^4C_3] = 504$

Hence, the total number of required numbers = 567.

Illustration 21: In how many ways can a cricket team of eleven players be chosen out of a batch of 15 players, if

(a) there is no restriction on the selection

(b) a particular player is always chosen

(c) a particular player is never chosen **(JEE MAIN)**

Sol: Using the concept of combination of alike objects we can get the answer.

(a) The total number of ways of selecting 11 players out of 15 is = ${}^{15}C_{11} = {}^{15}C_{15-11} = {}^{15}C_4 = \dfrac{15\times14\times13\times12}{4\times3\times2\times1} = 1365$

(b) A particular player is always chosen. This means that 10 players are selected out of the remaining 14 players.

$\therefore$ The required number of ways = ${}^{14}C_{10} = {}^{14}C_{14-10} = {}^{14}C_4 = 1001$

(c) The number of ways = ${}^{14}C_{11} = {}^{14}C_3 = \dfrac{14\times13\times12}{3\times2} = 364$

Illustration 22: In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Moreover, no three lines pass through one point, no line passes both points A and B. and no two are parallel. Find the number of points of intersection of the straight lines. **(JEE MAIN)**

Sol: Two non parallel straight lines give us a point of intersection. Using this idea we find the total number of points of intersection. Care should be taken that a point is not counted more than once.

The number of points of intersection of 37 straight lines is ${}^{37}C_2$. But 13 straight lines out of the given 37 straight lines pass through the same point A. Therefore, instead of getting ${}^{13}C_2$ points, we get merely one point A. Similarly, 11 straight lines out of the given 37 straight lines intersect at point B. Therefore, instead of getting ${}^{11}C_2$ points, we get only point B. Hence, the number of intersection points of the lines in ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535$.

Illustration 23: How many five-digit numbers can be made having exactly two identical digits? **(JEE MAIN)**

Sol: Note that zero cannot occupy the first place. So we divide the solution into two cases when the common digit is 0 and otherwise. Calculate the number of possibilities in these two cases and their sum gives us the desired result.

Case I: Two identical digits are 0, 0.

The number of ways to select three more digits is ${}^{9}C_3$. The number of arrangements of these five digits is $\left(5!/2!\right)-4!=60-24=36$.

Hence, the number of such numbers is ${}^{9}C_3 \times 36 = 3024$ ... (i)

Case II: Two identical digits are (1, 1) or (2, 2) or.... or (9, 9).

If 0 is included, then number of ways of selection of two more digits is ${}^{8}C_2$.

The number of ways of arrangements of these five digits is 5! / 2! -4! / 2! =48.

Therefore, the number of such numbers is ${}^{8}C_2 \times 48$.

If 0 is not included, then selection of three more digits is ${}^{8}C_3$.

Therefore, the number of such numbers is ${}^{8}C_3 \times 5!/2! = {}^{8}C_3 \times 60$.

Hence, the total number of five-digit numbers with identical digits (1.1)......(9.9) is

$9 \times ({}^{8}C_2 \times 48 + {}^{8}C_3 \times 60) = 42336$ ... (ii)

From Eqs. (i) and (ii), the required number of numbers is 3024 + 42336 = 45360.

Illustration 24: How many words can be made with letters of the word "INTERMEDIATE" if

(i) The words neither begin with I nor end with E,

(ii) The vowels and consonants alternate in the words,

(iii) The vowels are always consecutive,

(iv) The relative order of vowels and consonants does not change,

(v) No vowel is in between two consonants,

(vi) The order of vowels does not change? **(JEE MAIN)**

Sol: This is an application of Permutation under restriction and Permutation of Alike objects. Proceed according to the given conditions.

(i) The required number of words = (the number of words without restriction) – (the number of words beginning with I) – (the number of word ending with E) + (the number of words beginning with I and ending with E) (because words beginning with I as well as words ending with E contain some words beginning with I and ending with E).

The number of words without restriction = $\frac{12!}{2!\,3!\,2!}$

($\because$ There are 12 letters in which there are two I's, three E's and two T's).

The number of words beginning with I = $\frac{11!}{2!3!}$

($\because$ With E in the extreme left place we are left to arrange 11 letters INTERMEDIATE in which there are two T's and three E's).

The number of words ending with E = $\frac{11!}{2!2!2!}$

($\because$ With E in the extreme right place we are left to arrange 11 letters INTERMEDIATE in which there are two I's. two E's and two T's.)

The number of words beginning with I and ending with E = $\frac{10!}{2!2!}$

($\because$ With I in the extreme left and E in the extreme right places we are left to arrange 10 letters INTERMEDIATE in the which there are two T's and Two E's).

$\therefore$ the required number or words = $\frac{12!}{2!3!2!}-\frac{11!}{2!3!}-\frac{11!}{2!2!2!}+\frac{10!}{2!2!}=\frac{10!}{2!3!2!}(12\times 11-11\times 2-11\times 3+6) = \frac{83\times 10!}{24}$

(ii) There are 6 vowels and 6 consonants. So, the number of words in which vowels and consonants alternate = (the number of words in which vowels occupy odd places and consonants occupy even places) + (the number of words in which consonants occupy odd places and vowels occupy even places)

$= \frac{6!}{2!3!}\times\frac{6!}{2!}+\frac{6!}{2!}\times\frac{6!}{2!3!} = 2.\frac{6!}{2!3!}.\frac{6!}{2!} = 43200$

(iii) Considering the 6 vowels IEEIAE as one object, the number of arrangements of this with 6 consonants = $\frac{7!}{2!}$ ($\because$ there are two T's in the consonants).

For each of these arrangements, the 6 consecutive vowels can be arranged among themselves in $\frac{6!}{2!3!}$.

$\therefore$ The required number of words = $\frac{7!}{2!}\times\frac{6!}{2!3!}$ (as above) = 151200

(iv) The relative order of vowels and consonants will not change if in the arrangements of letters, the vowels occupy places of vowels, i.e.. 1^{st}, 4^{th}, 7^{th}, 9^{th}, 10^{th}, 12^{th} places and consonants occupy their places, i.e., 2nd. 3rd. 5th. 6th. 8th. 11th places, the required number of words

$\frac{6!}{2!3!}\times\frac{6!}{2!} = 21600$

(v) No vowel will be between two consonants if all the consonants become consecutive

$\therefore$ the required number of words = the number of arrangements when all the consonants are consecutive

$= \frac{7!}{2!3!}\times\frac{6!}{2!}$ (as above) = 151200

(vi) The order of vowels will not change if no two vowels interchange places, i.e., in the arrangement all the vowels are treated as identical.

(For example LATE. ATLE. TLAE. etc.. have the same order of vowels A. E. But LETA, ETLA.

TLEA. etc.. have changed order of vowels A. E. So. LATE is counted but LETA is not.

If A, E, are taken as identical say V then LVTV does not give a new arrangement by interchanging V, V.

The required number of words,

= The number of arrangements of 12 letters in which 6 vowels are treated as identical

= $\frac{12!}{6!2!}$ ($\because$ there are two T's also).

Illustration 25: India and South Africa play a one day international series until one team wins 4 matches. No match ends in a draw. Find, in how many ways can the series can be won. **(JEE ADVANCED)**

Sol: The team who wins the series is the team with more number of wins. The losing team wins either 0 or 1 or 2 or 3 matches. Using this we find the number of ways in which a team can win.

Let I for India and S for South Africa. We can arrange I and S to show the wins for India and South Africa respectively

Suppose India wins the series, then the last match is always won by India.

	Wins of S	Wins of I	No. of ways
(i)	0	4	1
(ii)	1	4	4! / 3! = 4
(iii)	2	4	$\frac{5!}{2!3!}=10$
(iv)	3	4	$\frac{6!}{3!3!} = 20$

$\therefore$ Total no. of ways = 35

In the same number of ways South Africa can win the series

$\therefore$ Total no. of ways in which the series can be won = 35 × 2 = 70

Illustration 26: There are p intermediate railway stations on a railway line from one terminal to another. In how many ways can a train stop at three of these intermediate stations, if no two of these stations (where it stops) are to be consecutive? **(JEE ADVANCED)**

Sol: The train stops only at three intermediate stations implies that the train does not stop at (p – 3) stations. Using this idea we proceed further to get the answer.

The problem then reduces to the following:

In how many ways can three objects be placed among (p – 3) objects in a row such that no two of them are next to each other (at most 1 object is to be placed between any two of these (p – 3) objects). Since there are (p – 2) positions to place the three objects, the required number of ways = ${}^{p-2}C_3$.

Illustration 27: Five balls are to be placed in three boxes. Each can hold all the five balls. In how many different ways can we place the balls so that no box remains empty if

(i) balls and boxes are all different

(ii) balls are identical but boxes are different

(iii) balls are different but boxes are identical

(iv) balls as well as boxes are identical

(v) balls as well as boxes are identical but boxes are kept in a row? **(JEE MAIN)**

Sol: Use the different cases of combination to solve the question according to the given conditions

As no box is to remain empty, boxes can have balls in the following numbers:

Possibilities 1, 1, 3 or 1, 2, 2

(i) The number of ways to distribute the balls in groups of 1, 1, 3 = ${}^5C_1 \times {}^4C_1 \times {}^3C_3$.

But the boxes can interchange their content, no exchange gives a new way when boxes containing balls in equal number interchange.

$\therefore$ the total number of ways to distribute 1, 1, 3 balls to the boxes = ${}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!}$

Similarly, the total number of ways to distribute 1, 2, 2 balls to the boxes = ${}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!}$

$\therefore$ the required number of ways = ${}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!} + {}^5C_1 \times {}^4C_1 \times {}^2C_2 \times \frac{3!}{2!} = 5 \times 4 \times 3 + 5 \times 6 \times 3 = 60 + 90 = 150$

Note: Writing the whole answer in tabular form.

Possibilities	Combinations	Permutations
1, 1, 3	${}^5C_1 \times {}^4C_1 \times {}^3C_3$	${}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!} = 5 \times 4 \times 3 = 60$
1, 2, 2	${}^5C_1 \times {}^4C_2 \times {}^2C_2$	${}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!} = 5 \times 6 \times 3 = 90$

$\therefore$ the required number of ways = 60 + 90 = 150.

(ii) When balls are identical but boxes are different the number of combinations will be 1 in each case.

$\therefore$ the required number of ways = $1 \times \frac{3!}{2!} + 1 \times \frac{3!}{2!} = 3 + 3 = 6$

(iii) When the balls are different and boxes are identical, the number of arrangements will be 1 in each case.

$\therefore$ the required number of ways = ${}^5C_1 \times {}^4C_1 \times {}^3C_3 + {}^5C_1 \times {}^4C_2 \times {}^2C_2 = 5 \times 4 + 5 \times 6 = 20 + 30 = 50$

(iv) When balls as well as boxes are identical, the number of combinations and arrangements will be 1 each in both cases.

$\therefore$ the required number of ways = $1 \times 1 + 1 \times 1 = 2$

(v) When boxes are kept in a row, they will be treated as different. So, in this case the number of ways will be the same as in (ii).

Illustration 28: There are m points on one straight line AB and n points on another straight line AC, none of them being A. How many triangles can be formed with these points as vertices? How many can be formed if point A is also included? **(JEE MAIN)**

Sol: A triangle has three vertices, so we select two points on one line and one on the other and vice versa. Also, consider the case when one point of the triangle is the intersection of the two lines.

To get a triangle, we either take two points on AB and one point on AC. or one point on AB and two points on AC. Therefore, the number of triangles, we obtain

$$= ({}^mC_2)({}^nC_1) + ({}^mC_1)({}^nC_2) = \frac{m(m-1)}{2}n + m\frac{n(n-1)}{2} = \frac{1}{2}mn(m-1+n-1) = \frac{1}{2}mn(m+n-2)$$

If the point A is included, we get m n additional triangles. Thus, in this case we get

$$= \frac{mn}{2}(m+n-2) + mn = \frac{mn(m+n)}{2} \text{ triangles.}$$

5.3 Division into Groups

(a) The number of ways in which (m + n) different objects can be divided into two unequal groups containing m and n objects respectively is $\frac{(m+n)!}{m!n!}$.

If m = n, the groups are equal and in this case the number of division is $\frac{(2n)!}{n!n!2!}$; as it is possible to interchange the two groups without obtaining a new distribution.

(b) However, if 2n objects are to be divided equally between two persons then the number of ways

$$= \frac{(2n)!}{n!n!2!}2! = \frac{(2n)!}{n!n!}$$

(c) The number of ways in which (m + n + p) different objects can be divided into three unequal groups containing m , n and p objects respectively is $= \frac{(m+n+p)}{m!n!p!}$, $m \neq n \neq p$

If m = n = P then the number of groups = $\frac{(3n)!}{n!n!n!3!}$. However, if 3n objects are to be divided equally among three persons then the number of ways = $\frac{(3n)!}{n!n!n!3!}3! = \frac{(3n)!}{(n!)^3}$

For example, the number of ways in which 15 recruits can be divided into three equal groups is $\frac{15!}{5!5!5!3!}$ and the number of ways in which they can be drafted into three different regiments, five in each, is $\frac{15!}{5!5!5!}$

(d) The number of ways in which mn different objects can be divided equally into m groups if order of groups is not important is $\frac{mn!}{(n!)^m m!}$

(e) The number of ways in which mn different objects can be divided equally into m groups if the order of groups is important is $\frac{mn!}{(n!)^m m!} \times m! = \frac{(mn)!}{(n!)^m}$

Illustration 29: In how many ways can 12 balls be divided between 2 boys, one receiving 5 and the other 7 balls? **(JEE MAIN)**

Sol: Simple application of division of objects into groups. Since the order is important, the number of ways in which 12 different balls can be divided between two boys who each get 5 and 7 balls respectively, is

$$\frac{12!}{5!7!} \times 2! = \frac{12.11.10.9.8.7!}{(5.4.3.2.1)7!} \times 2 = 1584$$

Alternative:

The first boy can be given 5 balls out of 12 balls in ${}^{12}C_5$ ways. The second boy can be given the remaining 7 balls in one way. But the order is important (the boys can interchange 2 ways).

Thus, the required number of ways = ${}^{12}C_5 \times 1 \times 2! = \frac{12!}{5!7!} \times 2 = \frac{12.11.10.9.8.7!.2}{5.4.3.2.1.7!} = 1584$

Illustration 30: Find the number of ways in which 9 different toys can be distributed among 4 children belonging to different age groups in such a way that the distribution among the 3 elder children is even and the youngest one is to receive one toy more. **(JEE ADVANCED)**

Sol: Using the concept of division of objects into groups we can solve this problem very easily.

The distribution should be 2, 2, 2 and 3 to the youngest. Now, 3 toys for the youngest can be selected in ${}^9C_3$ ways, the remaining 6 toys can be divided into three equal groups in

$\frac{6!}{(2!)^3.3!}$ ways and can be distributed in 3 ! ways.

Thus, the required number of ways = ${}^9C_3 . \frac{6!}{(2!)^3 3!} 3! = \frac{9!}{3!(2!)^3}$

Illustration 31: Divide 50 objects in 5 groups of size 10, 10, 10, 15 and 5 objects. Also find the number of distributions? **(JEE MAIN)**

Sol: Same as the above question. Number of ways of dividing 50 objects into 5 groups as given = $\dfrac{50!}{(10!)^3(15)!(5)!(3)!}$

Number of ways of distributing 50 objects into above formed groups = $\dfrac{50!}{(10)!^3.(15)!(5)!3!}\times 5!$

CONCEPTS

Identical Objects and Distinct Choices

Questions involving identical objects tend to be tricky, especially when the choices that they have are distinct.

Many choices image 1

An example would be :

Q. In how many ways can we place 10 identical oranges in 3 distinct baskets, such that every basket has at least 2 oranges each?

One method is to place 2 oranges in every basket and make cases for the rest of them. However, in such questions, the other approach results in fewer cases, and hence, simpler calculations and a more efficient solution

For this question: Divide 10 into groups of 3 rather than placing 2 in each and dividing the remaining four.

Vaibhav Gupta (JEE 2009, AIR 54)

6. CIRCULAR PERMUTATION

Let us consider that persons A, B, C and D are sitting around a round table. If all of them (A, B, C, D) are shifted one place in an anticlockwise order, then we will get fig.(b) from fig.(a). Now, if we shift A, B, C, D in anticlockwise order again, we will get fig. (c). We shift them once more and we will get fig.(d); and in the next time fig(a).

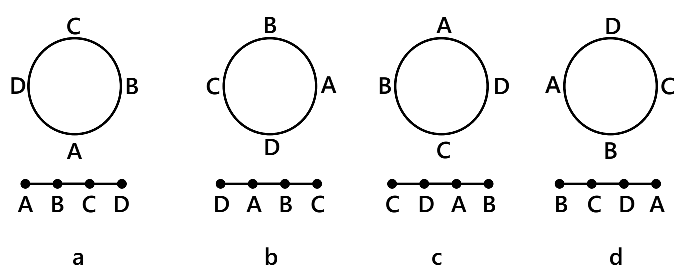


Figure 5.2

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements thus obtained will be same, because the anticlockwise order of A, B, C, D does not change. But if A, B, C and D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 objects, then for each circular arrangement number of linear arrangements is 4.

Similarly, if n different objects are arranged along a circle, for each circular arrangement the number of linear arrangements is n.

Therefore, the number of circular arrangements of n different objects = the number of linear arrangements of n different objects / n = n!/(n) = (n – 1)!

Clockwise and Anticlockwise Arrangements

Let the four persons A, B, C and D sit at a round table in anticlockwise as well as clockwise directions. These two arrangements are different. But if four flowers R (red), G (green), Y (yellow) and B (blue) are arranged to form a garland in anticlockwise and in clockwise order, then the two arrangements are same because if we view the garland from one side the four flowers R, G, Y, B will appear in anticlockwise direction and if seen from the other side the four flowers will appear in the clockwise direction. Here the two arrangements will be considered as one arrangement because the order of flowers does not change, rather only the side of observation changes. Here, two permutations will be counted as one.

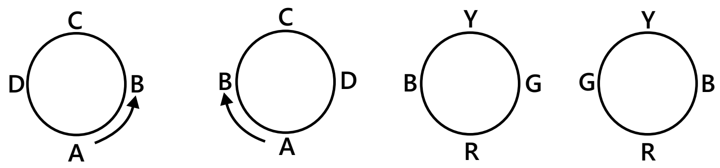


Figure 5.3

Therefore, when clockwise and anticlockwise arrangements are not different, i.e. when observations can be made from both sides, the number of circular arrangements of n different objects is $(n – 1)!/2$

Consider five persons A, B, C, D, E on the circumference of a circular table in an order which has no head. Now, shifting A, B, C, D, E one position in anticlockwise direction we will get arrangements as follows:

We see, that arrangements in all figures are different.

$\therefore$ The number of circular permutation of n different objects taken all at a time is $(n – 1)!$, if clockwise and anticlockwise orders are taken as different.

Note:

(a) The number of circular permutations of n different objects taken r at a time

${}^nP_r/r$, when clockwise and anticlockwise orders are treated as different.

${}^nP_r/2r$, when clockwise and anticlockwise orders are treated as same.

(b) The number of circular permutations of n different objects altogether

${}^nP_n/n = (n – 1)!$, when clockwise and anticlockwise order are treated as different,

${}^nP_n/2n = 1/2(n – 1)!$, when the above two orders are treated as same.

Illustration 32: In how many ways can 5 Indians and 4 Englishmen be seated at a round table if

(a) There is no restriction,
(b) All the four Englishmen sit together,
(c) All four Englishmen don't sit together,
(d) No two Englishmen sit together. **(JEE MAIN)**

Sol: Clearly, this is a case of Circular Permutation. Using the formula $(n – 1)!$, we can find the answer according to the given cases.

(a) Total number of persons = 5 + 4 = 9. These 9 persons can be seated at the round table in 8! Ways.

$\therefore$ Required number of ways = 8!

(b) Regarding 4 Englishmen as one person, we have only 5 + 1 i.e. 6 persons.

These 6 persons can be seated at the round table in 5! ways. Also, the 4 Englishmen can be arranged among themselves in 4! ways.

$\therefore$ the required number of ways = 5! 4!

(c) The total number of arrangements when there is no restriction = 8!; the number of arrangements when all the four English men sit together = 5! 4!

$\therefore$ The number of arrangements when all the four Englishmen don't sit together = 8! – 5! 4!

(d) As there is no restriction on Indians, we first arrange the 5 Indians.

Now, 5 Indians can be seated around a table in 4! ways. If an Englishman sits between two Indians, then no two Englishmen will sit together. Now, there are 5 places for 4 English men, therefore, 4 Englishmen can be seated in ${}^5P_4$ ways.

$\therefore$ The required number of ways = $4! \times {}^5P_4 = 4 \times 5!$

Illustration 33: Consider 21 different pearls on a necklace. How many ways can the pearls be placed in this necklace such that 3 specific pearls always remains together? **(JEE MAIN)**

Sol: This is the case of circular permutation when there is no distinction between clockwise and anticlockwise arrangements.

After fixing the places of three pearls. Treating 3 specific pearls = 1 unit. So we have now 18 pearls + 1 unit = 19 and the number of arrangements will be (19 – 1)! = 18!. Also the number of ways 3 pearls can be arranged between themselves is 3! = 6. As there is no distinction between the clockwise and anticlockwise arrangements, the required number of arrangements = $\frac{1}{2}18!.\ 6 = 3\ (18!)$.

Illustration 34: Six persons A, B, C, D, E and F are to be seated at a circular table. Find the number of ways this can be done if A must have either B or C on his right and B must have either C or D on his right. **(JEE MAIN)**

Sol: Fix the position of some of the persons relative to each other as per the question and arrange the remaining in the seats available.

When A has B or C to his right we have either AB or AC

When B has C or D to his right we have BC or BD.

Thus, we must have ABC or ABD or AC and BD.

For ABC, D, E, F in a circular number of ways = 3! = 6

For ABD, C, E, F in a circular number of ways = 3! = 6

For AC, BD E, F the number of ways = 3! = 6

Hence, the required number of ways = 18

7. LINEAR EQUATIONS WITH UNIT COEFFICIENTS

Consider the equation $x_1 + x_2 + x_3 + \dots + x_k = m$, in k variables whose sum must always be m,

The number of non-negative solutions to the above equation is given by the fictitious partition method which is stated as:

Method of fictitious partition: Number of ways in which n identical objects may be distributed among p persons if each person may receive none, one or more objects is = ${}^{n+p-1}C_n$.

Coefficient Method

(a) The number of non-negative integral solutions of equation $x_1 + x_2 + \dots + x_r = n$

= The number of ways of distributing n identical objects among r persons when each person can get zero, one or more objects = coeff. of x^n in $[(1 + x + x^2 + \dots + x^n)(1 + x + x^2 + \dots + x^n)(1 + x + x^2 + \dots + x^n)\dots$upto r factors]

= coeff. of x^n in $(1 + x + x^2 + \dots + x^n)^r$

= coeff. of x^n in $\left(\frac{1-x^{n+1}}{1-x}\right)^r$ = coeff. of x^n in $(1 - x^{n+1})^r (1 - x)^{-r}$ = coeff. of x^n in $(1 - x)^{-r}$

[leaving terms containing powers of x greater than n] = ${}^{n+r-1}C_{r-1}$

Note: If n is a positive integer, then

$$(1-x)^{-n} = 1 + \frac{(-n)}{1!}(x) + \frac{(-n)(-n-1)}{2!}(-x)^2 + \frac{(-n)(-n-1)(-n-2)}{3!}(-x)^3 + \ldots \text{ to } \infty$$

$$= 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots \text{. To } \infty = 1 + {}^{n}C_1x + {}^{n+1}C_2x^2 + {}^{2}C_3x^3 + \ldots \text{ to } \infty$$

Coeff. X^r in $(1-x)^{-n} = {}^{n+r-1}C \Rightarrow$ coeff. of x^n in $(1-x)^{-r} = {}^{n+r-1}C_n = {}^{n+r-1}C_{r-1}$

(b) The number of positive integral solutions for the equation $x_1 + x_2 + \ldots + x_r = n$

= The number of ways of distributing n identical objects among r persons when each person can get at least one object

= coeff. of x^n in $[x + x^2 + \ldots + x^n)(x + x^2 + \ldots + x^n)(x + x^2 + \ldots + x^n) \ldots$ Upto r factors]

= coeff. of x^n in $(x + x^2 + \ldots + x^n)^r$ = coeff. of x^n in $x^r\left(\frac{1-x^n}{1-x}\right)^r$

= coeff. Of x^{n-r} in $(1-x^n)^r(1-x)^{-r}$ = coeff. Of x^{n-r} in $(1-x)^{-r}$

[Leaving terms containing powers of x greater than n – r]

$= {}^{n-r+r-1}C_{r-1} = {}^{n-1}C_{r-1}$.

Illustration 35: How many integral solutions are there to x + y + z + w = 29, when $x \ge 1$, $y \ge 2$, $z \ge 3$ and $w \ge 0$? **(JEE ADVANCED)**

Sol: Application of multinomial theorem.

$x + y + z + w = 29$ ....(i)

$x \ge 1, y \ge 2, z \ge 3, w \ge 0 \quad \Rightarrow \quad x-1 \ge 0, y-2 \ge 0, z-3 \ge 0, w \ge 0$

Let $x_1 = x-1$, $x_2 = y-2$, $x_3 = z-3$

$\Rightarrow \quad x = x_1 + 1, y = x_2 + 2, z = x_3 + 3$ and then $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, w \ge 0$

From (i), $x_1 + 1 + x_2 + 2 + x_3 + 3 + w = 29$

$\Rightarrow \quad x_1 + x_2 + x_3 + w = 23$

Hence, the total number of solutions $= {}^{23+4-1}C_{4-1} = {}^{26}C_3 = 2600$

Illustration 36: Find the number of non-negative integral solution 3x + y + z = 24. **(JEE MAIN)**

Sol: Application of multinomial theorem.

$3x + y + z = 24, x \ge 0, y \ge 0, z \ge 0$

Let $x = k \quad \therefore y + z = 24 - 3k$ ....(i)

Here, $0 \le 24 - 3k \le 24$. Hence, $0 \le k \le 8$

The total number of integral solutions of (1) is ${}^{24-3k+2-1}C_{2-1} = {}^{25-3k}C_1 = 25 - 3k$

Hence, the total number of solutions of the original equation

$$= \sum_{k=0}^{8}(25-3k) = 25\sum_{k=0}^{8}1 - 3\sum_{k=0}^{8}k \Rightarrow 25.9 - 3.\frac{8.9}{2} = 225 - 108 = 117.$$

Illustration 37: Find the number of solutions of the equation x + y + z = 6, where $x, y, z \in W$. **(JEE MAIN)**

Sol: The number of solutions $= {}^{6+3-1}C_{3-1} = {}^{8}C_2 = 280.$

Illustration 38: How many integers are there between 1 and 1000000 having the sum of the digits as 18?

(JEE ADVANCED)

Sol: Let the digits be a_1, ..., a_6 and use multinomial theorem we get the answer.

Any number between 1 and 1000000 must be of less than seven digits. Therefore, it must be of the form $a_1\ a_2\ a_3\ a_4\ a_5\ a_6$

Where $a_1.\ a_2.a_3.a_4.a_5.a_6 \in \{0, 1, 2, \ldots 9\}$

Thus $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 18$, where $0 \le a_i \le 9$, i = 1, 2, ... 9

The required number of ways = coeff. of x^{18} in $(1 + x + x^2 + \ldots x^9)^6$

$$= \text{coeff. of } x^{18} \text{ in } \left(\frac{1-x^{10}}{1-x}\right)^6$$

= coeff. of x^{18} in $[(1 - x^{10})^6 (1 - x)^{-6}]$; = coeff. of x^{18} in $[(1 - {}^6C_1\, x^{10} \ldots) (1 - x)^{-6}]$

[leaving terms containing powers of x greater than 18]

= coeff. of x^{18} in $(1 - x)^{-6} - {}^6C_1$. coeff. of x^8 in $(1 - x)^{-6} = {}^{6+18-1}C_5 - 6.\ {}^{6+8-1}C_5 = {}^{23}C_5 - 6.{}^{13}C_5$

$$= \frac{23.22.21.20.19}{120} - 6.\ \frac{13.12.11.10.9}{120} = 33649 - 7722 = 25927$$

CONCEPTS

- m different white balls and n different red balls are to be arranged in a line such that the balls of the same colour are always together = (m! n! 2!)
- m different white balls and n different red balls are to be arranged in a line such that all the red balls are together = ((m + 1)! n!)
- m different white balls and n different red balls are to be arranged in a line such that no two red balls are together $(m \ge n - 1) = ({}^{m+1}C_n\ m!\ n!)$
- m different white balls and m different red balls are to be arranged in a line such that colour of the balls is alternating = $(2 \times (m!)^2)$
- m identical white balls and n different red balls are to be arranged in a line such that no two red balls are together $(m \ge n - 1) = ({}^{m+1}C_n)$
- m identical white balls and n different red balls are to be arranged in a line such that no two red balls are together $(m \ge n - 1) = ({}^{m+1}C_n\ n!)$
- If n objects are arranged in a line the number of selections of r objects $(n \ge 2r - 1)$ such that no two objects are adjacent is same number of ways of arranging n – r identical white balls and r identical red ball in a line such that no two balls are together = $({}^{n-r+1}C_r)$. e.g. suppose there are n stations on trains's route and a train has to stop at r stations such that no two stations are adjacent. The number of ways must be ${}^{n-r+1}C_r$.
- suppose there are N seats in a particular row of a theatre. The number of ways of making n people sit $(N \ge 2n - 1)$ such that no two people sit side by side is same as number of ways of arranging N – n identical white balls (empty seats) and n different red balls (n people) such that no two red balls are together. The required number of ways are ${}^{N-n+1}C_n \times n!$.

Nitish Jhawar (JEE 2009, AIR 7)

8. DIVISIBILITY OF NUMBERS

The following table shows the conditions of divisibility of some numbers

Divisible by	Condition
2	Whose last digit is even
3	sum of whose digits is divisible by 3
4	whose last two digits number is divisible by 4
5	whose last digit is either 0 or 5
6	which is divisible by both 2 and 3
7	If you double the last digit and subtract it from the rest of the number, answer is a multiple of 7
8	whose last three digits number is divisible by 8
9	sum of whose digits is divisible by 9
10	Whose last digit is 0
11	If you sum every second digit and then subtract sum of all other digits, answer is a multiple of 11
25	whose last two digits are divisible by 25

Illustration 39: How many four digit numbers can be made with the digits 0, 1, 2, 3, 4, 5 which are divisible by 3 (digits being unrepeated in the same number)? How many of these will be divisible by 6? **(JEE ADVANCED)**

Sol: A number is divisible by 3 if the sum of the digits is divisible by 3. This reduces the problem to the number of non-negative integral solutions of equation $x_1 + x_2 + \ldots + x_r = n$.

Here, 0 + 1 + 2 + 3 + 4 + 5 = 15; so two digits are to be omitted whose sum is 3 or 6 or 9.

Hence, the number of four digits can be made by either

1, 2, 4, 5 or 0, 3, 4, 5 (omitting two digits whose sum is 3)

0, 1, 3, 5 or 0, 2, 3, 4 (omitting two digits whose sum 6)

0, 1, 2, 3 (omitting two digits whose sum is 9)

The number of 4-digit numbers that can be made with 1, 2, 4, 5 = ${}^4P_4 = 4!$

The number of 4-digit numbers that can be made by the digits in any one of remaining four groups (each containing 0) = $4! - 3!$

$\therefore$ The required number of 4-digit numbers divisible by 3 = $4! + 4(4! - 3!) = 24 + 4(24 - 6) = 96$

Now, a number is divisible by 6 if it is even as well as divisible by 3.

So, the number of 4-digit numbers divisible by 6 that can be made with 1, 2, 4, 5 = $2 \times 3!$ ($\because$ the number should have an even digit in the units places).

The number of numbers of 4 digits, divisible by 6, that can be made with 0, 3, 4, 5 = $(3! - 2!) + 3!$

($\because$ The number should have 4 or 0 in units place and 0 should not come in thousands place).

Similarly, the number of numbers of 4 digits, divisible by 6, that can be made with 0, 1, 2, 3 = $(3! - 2!) + 3!$

The number of 4-digit numbers divisible by 6 that can be made with the digits 0, 1, 3 = 3!

The number of numbers of 4 digits, divisible by 6, that can be made with 0, 2, 3, 4 = $(3! - 2!) + (3! - 2!) + 3!$

($\because$ The number should have 4 or 2 or 0 in units place and 0 should not come in thousands place)

$\therefore$ the required 4-digit numbers divisible by 6

$= 2 \times 3! + (3! - 2!) + (3! - 2!) + 3! + 3! + (3! - 2!) + (3! - 2!) + (3! - 2!) + 3! = 12 + 4 + 6 + 4 + 6 + 6 + 4 + 4 + 6 = 52.$

9. SUM OF NUMBERS

(a) For given n different digits $a_1, a_2, a_3 \ldots a_n$ the sum of the digits in the units place of all numbers formed (if numbers are not repeated) is

$(a_1 + a_2 + a_3 + \ldots + a_n)(n - 1)!$ i.e. (sum of the digits) $(n - 1)!$

(b) Sum of the total numbers which can be formed with given different digits $a_1, a_2, a_3 \ldots\ldots a_n$ is

$(a + a_2 + a_3 + \ldots + a_n)(n - 1)!$ (111. .... n times)

Illustration 40: Find the sum of all 4 digit numbers formed using the digits 1, 2, 4 and 6. **(JEE MAIN)**

Sol: Use formula, Sum $= (a_1 + a_2 + a_3 + \ldots + a_n)\ (n-1)!\ (111 \ldots . N$ times)

Using formula, Sum $= (1 + 2 + 4 + 6)\ 3!\ (1111) = 13 \times 6 \times 1111 = 86658$

Alternate:

Here, the total 4-digit numbers will be $4! = 24$. So, every digit will occur 6 times at every one of the four places. Since the sum of the given digits $= 1 + 2 + 4 + 6 = 13$. So, the sum of all the digits at every place of all the 24 numbers $= 13 \times 6 = 78$.

The sum of the values of all the digits

At first place = 78

At the tens place = 780

At the hundreds place = 7800

At the thousands place = 78000

$\therefore$ The required sum 78 + 780 + 7800 + 78000 = 86658

10. FACTORS OF NATURAL NUMBERS

Let $N = p^a \cdot q^b \cdot r^c \ldots$ where p, q, r ... are distinct primes & a, b, c ... are natural numbers, then:

(a) The total number of divisors of N including 1 and N are $= (a + 1)(b + 1)(c + 1) \ldots$

(b) The sum of these divisors is $= (p^0 + p^1 + p^2 + \ldots + p^a)(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)\ldots$

(c) The number of ways in which N can be resolved as a product of two factors is

$$= \begin{cases} \frac{1}{2}(a_1+1)(a_2+1)(a_3+1)\ldots & \text{if N is not a perfect square} \\ \frac{1}{2}\left[(a_1+1)(a_2+1)(a_3+1)\ldots+1\right] & \text{if N is a perfect square} \end{cases}$$

(d) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N

Illustration 41: Find the number of factors of the number 38808 (excluding 1 and the number itself). Find also the sum of these divisors. **(JEE MAIN)**

Sol: Factorise 38808 into its product of primes and then use the concept of combination to find the answer.

$38808 = 2^3 \cdot 3^2 \cdot 7^2 \cdot 11$

Hence, the total number of divisors (excluding 1 and itself) = (3 + 1) (2 + 1) (2 + 1) (1 + 1) – 2 = 70

The sum of these divisors = $(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)(7^0 + 7^1 + 7^2)(11^0 + 11^1)$

= (15) (13) (57) (12) – 1 – 38808 = 94571

Illustration 42: In how many ways can the number 10800 be resolved as a product of two factors? **(JEE MAIN)**

Sol: Check whether the number is a perfect square or not and accordingly use the formula to find the desired result.

$10800 = 2^4 \cdot 3^3 \cdot 5^2$

Here 10800 is not a perfect square ($\because$ power of 3 is odd).

Hence, the number of ways = $\frac{1}{2}(4+1)(3+1)(2+1) = 30$.

Illustration 43: Find the number of positive integral solutions of $x_1.x_2.x_3 = 30$. **(JEE ADVANCED)**

Sol: Factorise 30 into primes and then use combination to get the desired result.

$x_1x_2x_3 = 2 \times 3 \times 5$. If we treat 2, 3, 5 as objects and x_1, x_2, x_3 as distinct boxes then finding the number of positive integer solution is the same as finding the number of ways of distributing 3 distinct objects in 3 distinct boxes. Thus, the required number of solutions is $3^5 = 27$

(For example, if all the objects are held by x_1 the corresponding solution is $x_1 = 30$. $x_2 = 1$, $x_3 = 1$, if 2 and 3 are held by x_1 and 5 by x_3 then $x_1 = 6$, $x_2 = 1$, $x_3 = 5$ etc)

11. EXPONENT OF A PRIME P IN N!

Consider a prime p and we want to know its exponent in n!.

The number of multiples of p in n! is given by $\left[\frac{n}{p}\right]$. Even if a number k between 1 and n has two factors of p, this formula counts it as only one. Hence we need to evaluate $\left[\frac{n}{p^2}\right]$ also. Similarly, for three and four to infinity. Hence, the exponent of prime p in n! is given by $e_p(n) = \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$ e_p is called Legendre's function.

Even though this is an infinite sum result, it is finite since for all p^i greater than n, step function becomes zero.

Let's understand this method by an example of finding exponent of 2 in 100!

Sol: $\left[\frac{100}{2}\right] = 50; \left[\frac{100}{2^2}\right] = 25; \left[\frac{100}{2^3}\right] = 12; \left[\frac{100}{2^4}\right] = 6; \left[\frac{100}{2^5}\right] = 3; \left[\frac{100}{2^6}\right] = 1; \left[\frac{100}{2^7}\right] = 0$

From this result we can infer that there are 50 numbers between 1 and 100 which have a factor of 2.

Out of these 50, there are 25 numbers that have a factor of 2^2.

Out of these 25, there are 12 numbers that have a factor of 2^3.

Out of these 12, there are 6 numbers that have a factor of 2^4.

Out of these 6, there are 3 numbers that have a factor of 2^5.

Out of these 3, there is 1 number that has a factor of 2^6.

But there is no number that has a factor of 2^7 or higher.

$\Rightarrow$ 100! can be written as $r.2^n$ where $2^n = (2^6)^1 (2^5)^{(3-1)} (2^4)^{(6-3)} (2^3)^{(12-6)} (2^2)^{(25-12)} (2)^{(50-25)}$ and r being a natural number which doesn't have 2 as a factor.

$\Rightarrow 2^n = (2^5.2)^1\ (2^4.2)^{(3-1)}(2^3.2)^{(6-3)}\ (2^2.2)^{(12-6)}\ (2.2)^{(25-12)}\ (2)^{(50-25)}$

$= (2^5)^1(2^4)^{(3-1)}\ (2^3)^{(n-3)}(2^2)^{(12-6)}\ (2)^{(25-12)}\ (2)^{50}$

$= (2^4.2)^1\ (2^3.2)^{(3-1)}\ (2^2.2)^{(6-3)}\ (2.2)^{(12-6)}\ (2)^{(25-12)}\ (2)^{50}$

$= (2^3.2)^1\ (2^2.2)^{(3-1)}\ (2.2)^{(6-3)}\ (2)^{(12-6)}\ (2)^{25}\ (2)^{50}$

$= (2^2.2)^1\ (2.2)^{(3-1)}\ (2)^{(6-3)}\ (2)^{12}\ (2)^{25}\ (2)^{50}$

$= (2.2)^1\ (2)^{(3-1)}\ (2)^6\ (2)^{12}\ (2)^{25}\ (2)^{50}$

$= (2)^1\ (2)^3\ (2)^6\ (2)^{12}\ (2)^{25}\ (2)^{50}$

Hence, the exponent of 2 in 100! is = 50 + 25 + 12 + 6 + 3 + 1 = 97

CONCEPTS

Following method involves part of an advanced topic in mathematics called Modular Arithmetic. Legendre's function also has another result, which is

$$e_p(n) = \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right] = \frac{n - S_p(n)}{p-1}$$

Where $S_p(n)$ is the sum of digits of n when written in base p.

Converting n to base p is done by repeated division of n by p and by noting the remainders to form a number starting with units place.

This procedure is similar to converting a decimal number to binary. In binary, the base is equal to 2. Let us solve an example using this method.

Example: Determine the exponent of 3 in ((3!)!)!

Sol: ((3!)!)! = (6!)! = 720!

Let us convert 720 to base 3

$$\begin{array}{l|ll} 3 & 720 & \\ 3 & 240 & R\ 0 \\ 3 & 80 & R\ 0 \\ 3 & 26 & R\ 2 \\ 3 & 8 & R\ 2 \\ 3 & 2 & R\ 2 \\ 3 & 0 & R\ 2 \end{array}$$

Hence 720 = $(222200)_3$

$S_3(720) = 2 + 2 + 2 + 0 + 0 = 8$

$$\Rightarrow e_3(720) = \frac{720-8}{3-1} = \frac{712}{2} = 356$$

You can verify this answer using previous method.

If you are confused by base conversion then do not use this method.

Akshat Kharaya (JEE 2009, AIR 235)

Illustration 44: Find the exponent of 7 in 400!. **(JEE MAIN)**

Sol: Apply Legendre's formula.

$$e_7(400) = \left[\frac{400}{7}\right]+\left[\frac{400}{7^2}\right]+\left[\frac{400}{7^3}\right] = 57 + 8 + 1 = 66.$$

Illustration 45: Find all positive integers of n such that n! ends in exactly 1000 zeros. **(JEE ADVANCED)**

Sol: 10 is a multiple of 2 and 5. In order to get 1000 zeroes we must have 1000 as the exponent of 5 in n!. Now use the definition of the GIF to find the range of numbers satisfying the given condition.

There are clearly more 2's than 5's in the prime factorization of n!, hence it suffices to solve the equation $\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]$ + ... = 1000.

But $\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right] + \ldots < \frac{n}{5} + \frac{n}{5^2} + \ldots = \frac{n}{5}\left(1+\frac{1}{5}+\ldots\right)$ (as [x] < x) $= \frac{n}{5}.\frac{1}{1-\frac{1}{5}} = \frac{n}{4}.$

Hence, n > 4000.

On the other hand, using the inequality [x] > x – 1, we have

$$1000 > \left(\frac{n}{5}-1\right)+\left(\frac{n}{5^2}-1\right)+\left(\frac{n}{5^3}-1\right)+\left(\frac{n}{5^4}-1\right)+\left(\frac{n}{5^5}-1\right) = \frac{n}{5}\left(1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}\right) - 5 = \frac{n}{5}.\frac{1-\left(\frac{1}{5}\right)^5}{1-\frac{1}{5}} - 5.$$

So, $N < \frac{1005\cdot 4\cdot 3125}{3124} < 4022.$

We narrowed n down to {4001, 4002, ... , 4021}. Using Legendre's formula we find that 4005 is the first positive integer with the desired property and that 4009 is the last. Hence, n = 4005, 4006, 4007, 4008, 4009.

Second solution: It suffices to solve the equation $e_5(n) = 1000$. Using the second form of Legendre's formula, this becomes $n - s_5(n) = 4000$. Hence n > 4000. We work our way upward from 4000 looking for a solution. Since $e_5(n)$ can change only at multiples of 5 (why?), we step up 5 each time:

$$e_5(4000) = \frac{4000-4}{5-1} = 999.$$

$$e_5(4005) = \frac{4005-5}{5-1} = 1000.$$

$$e_5(4010) = \frac{4010-6}{5-1} = 1001.$$

Any n > 4010 will clearly have $e_5(n) \ge e_5(4010) = 1001$. Hence the only solutions are n = 4005, 4006, 4007, 4008, 4009.

12. INCLUSION-EXCLUSION PRINCIPLE

In its general form, the principle of inclusion-exclusion states that for finite sets $A_1,\ldots A_n$. One has the identity

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}\left|A_i\right| - \sum_{1\le i<j\le n} |A_i \cap Aj| + \sum_{1\le i<j<k\le n} |A_i \cap Aj \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap \ldots \cap A_n| \cdot$$

This can be compactly written as $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\le i_1<\ldots\ldots<i_k\le n} |A_{i_1} \cap \ldots \cap A_{i_k}|\right)$

In words, to count the number of elements in a finite union of finite sets, first sum the cardinalities of the individual sets, then subtract the number of elements which appear in more than one set, then add back the number of elements which appear in more than two sets, then subtract the number of elements which appear in more than three sets, and so on. This process naturally ends since there can be no elements which appear in more than the number of sets in the union.

In applications it is common to see the principle expressed in its complementary form. That is, taking S to be a finite universal set containing all of the A_i and letting $\overline{A_i}$ denote the complement of A_i in S. By De Morgan's laws.

We have, $\left|\bigcap_{i=1}^{n}\overline{A_i}\right| = \left|S - \bigcup_{i=1}^{n} A_i\right| = |S| - \sum_{i=1}^{n}|A_i| + \sum_{1\le i<j\le n}|A_i \cap A_j| - ... + (-1)^n |A_1 \cap ... \cap A_n|$.

Illustration 46: 105 students take an examination of whom 80 students pass in English. 75 students pass in Mathematics and 60 students pass in both subjects. How many students fail in both subjects? **(JEE MAIN)**

Sol: A simple application of Inclusion-Exclusion Principle.

Let X = the set of students who take the examination.

A = the set of students who pass in English

B = the set of students who pass in Mathematics

We are given that n(X) = 105, n(A) = 80, n(B) = 75, $n(A \cap B) = 60$

Since, $n(A \cup B) = n(A) + n(B) - n(A \cup B)$.

Therefore, $n(A \cup B) = 80 + 75 - 60 = 95$.

The required number = $n(X) - n(A \cup B) = 105 - 95 = 10$.

Thus, 10 students fail in both subjects.

Illustration 47: Find the number of permutations of the 8 letters AABBCCDD, taken all at a time, such that no two adjacent letters are alike. **(JEE ADVANCED)**

Sol: Divide the question into cases when A's are adjacent, B's are adjacent and so on. Similarly proceed to find the number of ways in which two alike objects are adjacent and so on. Then use Inclusion-Exclusion Principle to find the result.

First disregard the restriction that no two adjacent letters be alike.

The total number of permutation is then N = $\frac{8!}{2!2!2!2!}$ = 2250

Now, apply the inclusion exclusion principle. Where a permutation has property α in case the A's are adjacent, property β in case the B's are adjacent, etc. It can be calculated that

$N(\alpha) = \frac{7!}{2!2!2!} = 630$. $N(\alpha, \beta) = \frac{6!}{2!2!} = 180$

$N(\alpha, \beta, \gamma) = 60$. $N(\alpha, \beta, \gamma, \delta) = 24$.

Hence, the answer is $N - 4N(\alpha) + 6N(\alpha, \beta) - 4N(\alpha, \beta, \gamma) + N(\alpha, \beta, \gamma, \delta) = 864$.

13. DERANGEMENTS THEOREM

Derangements theorem is an important application of inclusion exclusion principle.

Suppose, there are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is $n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + ... + \frac{(-1)^n}{n!}\right]$

Proof: n letters are denoted by 1, 2, 3, ... , n. Let, A_i denote the set of distribution of letters in envelopes (one letter in each envelope) so that the i^{th} letter is placed in the corresponding envelope. Then, $n(A_i) = (n-1)!$ [since the remaining n – 1 letters can be placed in n – 1 envelopes in (n – 1)! Ways] Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then $N(A_i \cap A_j) = (n-2)!$

Also, $n(A_i \cap A_j \cap A_k) = (n-3)!$

Hence, the required number is $n(A'_1 \cup A'_2 \cup \ldots \cup A'_n) = n! - n(A_i \cup A_2 \cup \ldots \cup A_n)$

$= n! - \left[\Sigma n(A_i) - \Sigma n(A_i \cap A_j) + \Sigma n(A_i \cap A_j \cap A_k) + \ldots + (-1)^n \Sigma n(A_1 \cap A_2 \ldots \cap A_n)\right]$

$= n! - [({}^nC_1)(n-1)! - {}^nC_2\,(n-2)! + {}^nC_3\,(n-3)! + \ldots + (-1)^{n-1} \times {}^nC_n 1]$

$= n! - \left[\frac{n!}{1!(n-1)}(n-1)! \frac{n!}{2!(n-2)!}(n-2)! + \ldots + (-1)^{n-1}\right] = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^n}{n!}\right]$

Remark: If r objects go to wrong place out of n object then (n – r) objects goes to original place.

$A'_r = r!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^r \frac{1}{r!})$

CONCEPTS

Number of derangements $D_n = n!\sum_{i=0}^{n} \frac{(-1)^i}{i!}$; Interestingly, as $n \to \infty$, $D_n = \frac{1}{e}$

This results in an interesting relating $D_n = \left[\frac{n!}{e}\right]$.Where [x] is the nearest integer function.

Use this formula only if the given options are wide apart from one another.

ExampleYou have 6 ball in 6 different colors, and for every ball you have a box of the same color. How many derangements do you have, if no ball is in a box of the same color?

Sol: We know that e = 2.71828. To make division simple let's round it to 2.7. You have to keep in mind that we have reduced the value of e, so the result which we get is greater than the actual result.

$\therefore\ D_n = \left[\frac{6!}{e}\right] = \left[\frac{720}{2.7}\right] = \left[\frac{800}{3}\right] = [266.66] = 267$

Hence, the result will be close to 266.

This is a pretty good approximation as the actual answer is 265.

But, if the given options are all close to 266, then it is advised to calculate using the original formula or by rounding the value of e to the number of significant digits equal to that of n! (numerator).

So if we use e = 2.72 we get 34

$D_n = \left[\frac{6!}{e}\right] = \left[\frac{720}{2.72}\right] = \left[\frac{4500}{17}\right] = [264.70] = 265$

Vaibhav Krishnan (JEE 2009, AIR 22)

Illustration 48: A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) atleast two of them are in the wrong envelopes. (ii) All the letters are in the wrong envelopes. **(JEE MAIN)**

Sol: Application of Derangement theorem.

(i) The number of ways in which at least two of them in the wrong envelopes = $\sum_{r=2}^{6} {}^6C_{6-r} D_r$

$= {}^6C_{6-2}\,D_2 + {}^6C_{6-3} D_3 + {}^6C_{6-4}\,D_4 + {}^6C_{6-5} D_5 + {}^6C_{6-6}\,D_6$

$= {}^6C_4.2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3.3!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\right) + {}^6C_2.4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$

$$+{}^6C_1.5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}\right)+{}^6C_0.6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

(ii) The number of ways in which all letters be placed in wrong envelopes

$$= 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right); = 720\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}\right); = 360 - 120 + 30 - 6 + 1 = 265$$

14. MULTINOMIAL THEOREM

(a) If there are l objects of one kind, m objects of a second kind, n objects of a third kind and so on; then the number of ways of choosing r objects out these (i.e., l + m + n ...) is the coefficient of x^r in the expansion of

$(1 + x + x^2 + x^3 + \ldots + xl)\ (1 + x + x^2 + x^3 + \ldots + x^m)\ (1 + x + x^2 + x^3 + \ldots + x^n)$

Further, if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., l + m + n + ...) is the coefficient of x^r in the expansion of

$(x + x^2 + x^3 + \ldots + xl)\ (x + x^2 + x^3 + \ldots + x^m)\ (x + x^2 + x^3 + \ldots + x^n)\ldots$

(b) If there are objects of one kind, m objects of a second kind, n objects of a third kind and so on; then the number of possible arrangements/permutations of r objects out of these object (i.e., l + m + m + ...) is the coefficient of x^r in the expansion of

$$r!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\ldots\frac{x^f}{l!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\frac{x^m}{m!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\frac{x^n}{n!}\right)$$

Illustration 49: In an examination, the maximum marks for each of three papers is n and that for the fourth paper is 2n. Prove that the number of ways in which candidate can get 3n marks is

$\frac{1}{6}$ (n + 1) ($5n^2$ + 10n + 6). **(JEE ADVANCED)**

Sol: The maximum marks in the four papers are n, n, n and 2n. Consider a polynomial $(1 + x + x^2 + \ldots + x^n)^3 (1 + x + \ldots + x^{2n})$. The number of ways of securing a total of 3n is equal to the co-efficient of the term containing x^{3n}.

The number of ways of getting 3n marks

= coefficient of x^{3n} in $(1 + x + x^2 + \ldots + x^n)^3 (1 + x + \ldots + x^{2n})$

= coefficient of x^{3n} in $(1 - x^{n+1})^3 (1 - x^{2n+1}) (1 - x)^{-4}$

= coefficient of x^{3n} in $(1 - 3x^{n+1} + 3x^{2n+2} - x^{3n+3}) (1 - x^{2n+1}) \times (1 + {}^4C_1 x + {}^5C_2x^2 + {}^6C_3 x^3 + \ldots.)$

= coefficient of x^{3n} in $(1 - 3x^{n+1} - x^{2n+1} + 3x^{2n+2}) (1 + {}^4C_1 x + {}^5C_2 x^2 \ldots)$

$= {}^{3n+3}C_{3n} - 3.{}^{2n+2}C_{2n-1} + 3 . {}^{n+1}C_{n-2} - {}^{n+2}C_{n-1}$

$$= \frac{(3n+3)!}{3!(3n)!} - 3 . \frac{(2n+2)!}{3!(2n-1)!} + 3\frac{(n-1)!}{3!(n-2)!} - \frac{(n+2)!}{3!(n-1)!}$$

$= 1/6\ (n + 1)\ (27n^2 + 27n + 6 - 24n^2 - 12n + 3n^2 - 3n - n^2 - 2n) = 1/6\ (n + 1)\ (5n^2 + 10n + 6)$

PROBLEM-SOLVING TACTICS

In any given problem, first try to understand whether it is a problem of permutations or combinations. Now, think if repetition is allowed and then try solving problem.

A simple method to solve these problems where repetition is not allowed is as follows -

First draw series of dashes representing the number of places you want to fill or number of items you want to select.

Now start filling dashes by the number of objects available to choose from and multiply the numbers. This is the final answer for a permutations problem.

If it is a combination problem then divide the answer with the factorial or number of items.

This calculation becomes complex if repetition is allowed.

FORMULAE SHEET

(a) Permutation (Arrangement of Objects): Each of the different arrangement, which can be made by taking some or all of a number of objects is called permutation.

(i) The number of permutations of n different objects taken r at a time is ${}^nP_r = \frac{n!}{(n-r)!}$.

(ii) The number of all permutations of n distinct objects taken all at a time is n!.

Permutation with Repetition: The number of permutations of n different objects taken r at a time when each object may be repeated any number of times is n^r.

Permutation of Alike Objects: The number of permutations of n objects taken all at a time in which, p are alike objects of one kind, q are alike objects of second kind & r are alike objects of a third kind and the rest $(n - (p + q + r))$ are all different, is $\frac{n!}{p!q!r!}$.

Permutation under Restriction: The number of permutations of n different objects, taken all at a time, when m specified objects always come together is $m! \times (n - m + 1)!$.

(b) Combination (Selection of Objects): Each of the different groups or selection which can be made by some or all of a number of given objects without reference to the order of the objects in each group is called a combination.

The number of all combinations of n objects, taken r at a time is generally denoted by C(n, r) or ${}^nC_r = \frac{n!}{r!(n-r)!}$ $(0 \le r \le n) = \frac{{}^nP_r}{r!}$

Note:

(a) The number of ways of selecting r objects out of n objects, is the same as the number of ways in which the remaining (n - r) can be selected and rejected.

(b) The combination notation also represents the binomial coefficient. That is, the binomial coefficient ${}^nC_r$ is the combination of n elements chosen r at a time.

(c) (i) ${}^nC_r = {}^nC_{n-r}$

(ii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iv) If n is even, then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$

(v) If n is odd, then the greatest value of ${}^nC_r$ is ${}^nC_{(n+1)/2}$

(vi) ${}^nC_0 + {}^nC_r + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n$

(vii) ${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots\ldots\ldots\ldots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

Combinations under Restrictions

(a) The number of ways of choosing r objects out of n different objects if p particular objects must be excluded $= {}^{(n-p)}C_r$

(b) The number of ways of choosing r objects out of n different objects if p particular objects must be included $(p \le r) = {}^{n-p}C_{r-p}$

(c) The total number of combinations of n different objects taken one or more at a time $= 2^n - 1$.

Combinations of Alike Objects

(a) The number of combinations of n identical objects taking $(r \le n)$ at a time is 1.

(b) The number of ways of selecting r objects out of n identical objects is n + 1.

(c) If out of (p + q + r + s) objects, p are alike of one kind, q are alike of a second kind, r are alike of the third kind and s are different, then total number of combinations is $(p + 1)(q + 1)(r + 1)2^s - 1$

(d) The number of ways in which r objects can be selected from a group of n objects of which p are identical, is $\sum_{0}^{t} {}^{n-p}C_r$, if $r \le p$ and $\sum_{r=p}^{t} {}^{n-p}C_r$ if $r > p$

Division into Groups

(a) The number of ways in which (m + n) different objects can be divided into two unequal groups containing m and n objects respectively is $\frac{(m+n)!}{m!n!}$.
If m = n, the groups are equal and in this case the number of divisions is $\frac{(2n)!}{n!n!2!}$; as it is possible to interchange the two groups without obtaining a new distribution.

(b) However, if 2n objects are to be divided equally between two persons then the number of ways $= \frac{(2n)!}{n!n!2!}2! = \frac{(2n)!}{n!n!}$

(c) The number of ways in which (m + n + p) different objects can be divided into three unequal groups containing m , n and p objects respectively is $= \frac{(m+n+p)}{m!n!p!}$, $m \ne n \ne p$
If m = n = P then the number of groups $= \frac{(3n)!}{n!n!n!3!}$. However, if 3n objects are to be divided equally among three persons then the number of ways $= \frac{(3n)!}{n!n!n!3!}3! = \frac{(3n)!}{(n!)^3}$

(d) The number of ways in which mn different objects can be divided equally into m groups if the order of groups is not important is $\frac{mn!}{(n!)^m m!}$

(e) The number of ways in which mn different objects can be divided equally into m groups if the order of groups is important is $\frac{mn!}{(n!)^m m!} \times m! = \frac{(mn)!}{(n!)^m}$

Circular Permutation

(a) The number of circular permutations of n different objects taken r at a time

${}^nP_r/r$, when clockwise and anticlockwise orders are treated as different.

${}^nP_r/2r$, when clockwise and anticlockwise orders are treated as same.

(b) The number of circular permutations of n different objects altogether

${}^nP_n/n = (n-1)!$, when clockwise and anticlockwise order are treated as different

${}^{n}P_{n}/2n = 1/2(n-1)!$, when above two orders are treated as same

The number of non-negative integral solutions of equation $x_1 + x_2 + \dots + x_r = n$

= The number of ways of distributing n identical objects among r persons when each person can get zero or one or more objects = ${}^{n+r-1}C_{r-1}$

The number of positive integral solutions for the equation $x_1 + x_2 + \dots + x_r = n$

= The number of ways of distributing n identical objects among r persons when each person can get at least one object = ${}^{n-r+r-1}C_{r-1} = {}^{n-1}C_{r-1}$.

(c) For given n different digits $a_1, a_2, a_3 \dots a_n$ the sum of the digits in the units place of all the numbers formed (if numbers are not repeated) is

$(a_1 + a_2 + a_3 + \dots + a_n)(n-1)!$ i.e. (sum of the digits) $(n-1)!$

(d) The sum of the total numbers which can be formed with given different digits $a_1, a_2, a_3 \dots\dots a_n$ is

$(a + a_2 + a_3 + \dots + a_n)(n-1)!$ (111. .... n times)

Factors of Natural Numbers

Let $N = p^a . q^b . r^c \dots$ where p, q, r ... are distinct primes & a, b, c ... are natural numbers, then:

(a) The total number of divisors of N including 1 and N are = $(a + 1)(b + 1)(c + 1) \dots$

(b) The sum of these divisors is = $(p^0 + p^1 + p^2 + \dots + p^a)(q^0 + q^1 + q^2 + \dots + q^b)(r^0 + r^1 + r^2 + \dots + r^c)\dots$

(c) The number of ways in which N can be resolved as a product of two factors is

$$= \begin{cases} \dfrac{1}{2}(a+1)(b+1)(c+1) & \text{if N is not a perfect square} \\ \dfrac{1}{2}\left[(a+1)(b+1)(c+1)\dots+1\right] & \text{if N is a perfect square} \end{cases}$$

(d) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N

Exponent of a Prime P in N! = $\displaystyle\sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$

Inclusion-Exclusion Principle: The principle of inclusion-exclusion states that for finite sets $A_1, \dots A_n$. One has the identity

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}|A_i| - \sum_{1 \le i < j \le n} |A_i \cap Aj| + \sum_{1 \le i < j < k \le n} |A_i \cap Aj \cap A_k| - \dots + (-1)^{n-1} |A_1 \cap \dots \cap A_n|.$$

This can be compactly written as $\displaystyle\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1 \le i_1 < \dots\dots < i_k \le n} |A_{i_1} \cap \dots \cap A_{i_k}|\right)$

Derangements Theorem: The number of ways in which letters n can be placed in n envelopes (one letter in each envelope) so that no letter is placed in the correct envelope is $n!\left[1 - \dfrac{1}{1!} + \dfrac{1}{2!} + \dots + \dfrac{(-1)^n}{n!}\right]$

If n objects are arranged at n places then the number of ways to rearrange exactly r objects at right places is =

$$\frac{n!}{r}\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^{n-r}\frac{1}{(n-r)!}\right]$$

Some Important results

(a) The number of totally different straight lines formed by joining n points on a plane of which m(<n) are collinear is ${}^nC_2 - {}^mC_2 + 1$.

(b) The number of total triangles formed by joining n points on a plane of which m(<n) are collinear is ${}^nC_3 - {}^mC_3$.

(c) The number of diagonals in a polygon of n sides is ${}^nC_2 - n$.

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed are ${}^mC_2 \times {}^nC_2$.

(e) Given n points on the circumference of a circle, then

the number of straight lines between these points are ${}^nC_2$

the number of triangles between these points are ${}^nC_3$

the number of quadrilaterals between these points are ${}^nC_4$

(f) If n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then, the number of parts into which these lines divide the plane is = 1 + Sn

Solved Examples

JEE Main/Boards

Example 1: Find the number of ways in which 5 identical balls can be distributed among 10 different boxes, if exactly one ball goes into a box.

Sol: It is same as selecting 5 boxes from 10 boxes and distributing the balls in those 5 boxes.

Number of boxes = 10 and Number of balls = 5.

$\therefore$ Possible number of ways = ${}^{10}C_5$

Example 2: There are n intermediate stations on a railway line from one terminal to another. In how many ways can the train stop at 3 of these intermediate stations if

(i) all the three stations are consecutive.

(ii) at least two of the stations are consecutive.

Sol: The first part is very trivial. For the second part consider a pair of consecutive stations and then select a station such that it is not consecutive. Check for multiple counting.

Let the intermediate stations be $S_1, S_2, \ldots\ldots, S_n$

(i) The number of triplets of consecutive stations, as $S_1S_2S_3, S_2S_3S_4, S_3S_4S_5, \ldots. S_{n-2}S_{n-1}S_n$, is (n – 2).

(ii) The total number of consecutive pairs of stations, as $S_1S_2, S_2S_3, \ldots\ldots\ldots\ldots, S_{n-1}S_n$ is (n – 1).

Each of the above pairs can be associated with a third station in (n – 2) ways. Thus, choosing a pair of stations and any third station can be done in (n – 1) (n – 2) ways.

The above count also includes the case of three consecutive stations. However, we can see that each such case has been counted twice. For example, the pair S_4S_5 combined with S_6 and the pair S_5S_6 combined with S_4 are identical.

Hence, subtracting the excess counting, the number of ways in which three stations can be chosen so that at least two of them are consecutive

$= (n-1)(n-2) - (n-2) = (n-2)^2$

Example 3: How many ways are there to invite 1 of 3 friends for dinner on 6 successive nights such that no friend is invited more than 3 times?

Sol: Divide the solution in different possible cases. 6 can be partitioned in the following ways

1 + 2 + 3

0 + 3 + 3

2 + 2 + 2

Using this we can form different possibilities and calculate the number of ways the friends can be invited.

Let x, y, z be the friends and let (a, b, c) denote the case where x is invited a times, y, b times and z, c times. For example, one possible arrangement corresponding to the triplet (3, 2, 1) is x, x, y, x, y, z

Then we have the following possibilities:

(i) (a, b, c) = (1, 2, 3); (1, 3, 2); (2, 3, 1); (2, 1, 3):

(ii) (a, b, c) = (3, 3,0); (3,0,3) ; (0, 3, 3).

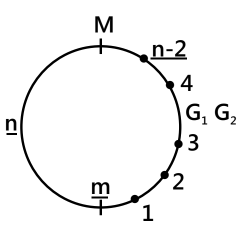
(iii) (a, b, c) = (2, 2, 2). So the total number of ways is 6 × 6!/1! 2! 3! + 3 × 6!/3!3! + 6!/2!2!2!

Note: We can also solve this problem using linear equations.

Example 4: There are 2n guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another. And that there are two specified guests who must not be placed next to one another. Show that the number of ways in which the company can be placed is $(2n-2)!\,(4n^2-6n+4)$

Sol: This is an application of division into groups. Find the total number of ways of arrangement of the guests and then subtract the number of ways in which the two mentioned guests are together.

Excluding the two specified guests, 2n – 2 persons can be divided into two groups, one containing n and the other (n – 2) in $\frac{(2n-2)!}{n!(n-2)!}$ and can sit on either side of mister and mistress in 2! ways and can arrange themselves in n!(n – 2)!



Now, the two specified guests where (n – 2) guests are seated will have (n – 1) gaps and can arrange themselves in 2! Ways. Number of ways when G_1 G_2 will always be together

$$= \frac{(2n-2)!}{n!(n-2)!}\, 2!\, n!\, (n-2)!\, (n-1)\, 2! = (2n-2)!\, 4(n-1)$$

Hence, the number of ways when G_1 G_2 are never together

$$= \frac{2!}{n!\, n!\, 2!}\, 2!\, n!\, n! - 4(n-1)\,(2n-2)!$$

$$= (2n-2)!\,[2n(2n-1) - 4\,(n-1)] = (2n-2)!\,[4n^2 - 6n + 4]$$

Example 5: Find the number of words of 5 letters that can be formed with the letters of the word Proposition.

Sol: Divide the cases into words having 5 distinct letters, 2 alike of one kind and 3 alike of different kind and so on. Count the number of words in these cases and their sum gives us the answer.

Proposition contains 11 letters PP, R, OOO, S, II, T, N.

Following table given the number of words.

Repeated letters: O(2), P(2), I(2)

Different letters: R, S, T, N

	Letters	No. of Words	Total
A	5 Distinct	${}^7C_5.5!$	2520
B	3 Alike 2 Alike	1.2C1.(51/3!2!)	20
C	3 Alike 2 Different	$1.{}^6C_2.(5!/3!)$	300
D	2 Alike 2Other Alike 1 Different	${}^3C_2.{}^5C_1.(5!/2!2!)$	450
E	2 Alike 3 Different	${}^3C_1.{}^6C_3.(5!/2!)$	3600

Total no. of words = 6890

Example 6: There are 10 points in a plane where no three points are collinear except for 4 points which are collinear. Find the number of triangles formed by the points as vertices.

Sol: A triangle is formed from three non-collinear points. Select 3 points from 10 points in ${}^{10}C_3$ ways and subtract the cases when the points are collinear, as they would not form a triangle.

Let us suppose that the 10 points are such that no three of them are collinear. Now, a triangle will be formed by any three of these ten points. Thus forming a triangle amount to selecting any three of the 10 points.

Now 3 points can be selected out of 10 point in ${}^{10}C_3$ ways.

∴ Number of triangles formed by 10 points when no three of them are collinear = ${}^{10}C_3$.

Similarly, the number of triangles formed by 4 points then no 3 of them are collinear = ${}^4C_3$

∴ Required number of triangle formed = ${}^{10}C_3 - {}^4C_3$ = 120 – 4 = 116.

Example 7: From 6 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done if the committee is to include at least one lady?

Sol: According to the question, the committee should include atleast one lady. Consider cases when the committee consists of 1, 2, 3 or 4 ladies and find the number of ways for all these cases.

Different combinations are listed below:

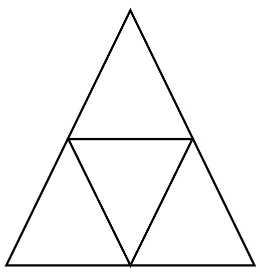
No. of Ladies	No. of Gentlemen	No. of Committees
1	4	${}^4C_1\,{}^6C_4$
2	3	${}^4C_2\,{}^6C_3$
3	2	${}^4C_3\,{}^6C_2$
4	1	${}^4C_4\,{}^6C_1$

Total number of committees

$= {}^4C_1\,{}^6C_4 + {}^4C_2\,{}^6C_3 + {}^4C_3\,{}^6C_2 + {}^4C_4\,{}^6C_1 = 246$

Example 8: (a) In how many ways can the following diagram be coloured, subject to two conditions: Each of the smaller triangle is to be painted with one of three colours: red, blue, green and no two adjacent regions should have the same color?

(b)How many numbers of four digits can be formed with the digits 1, 2, 3, 4 and 5?



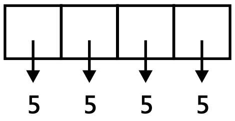
(c) A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them if he has 3 servants to carry the cards?

(d) Find the number of arrangements of the letters of the word 'BENEVOLENT'.

Sol: For each of the parts (a), (b), (c) and (d), identify the number of ways a particular cell can be coloured or filled in and then use permutation / combination to get the result.

(a) These conditions are satisfied if we proceed as follows: Just color the central triangle by one color, this can be done in three ways. Next paint other three triangles with remaining 2 colors. By the fundamental principle of counting. This can be done in $3 \times 2 \times 2 \times 2 = 24$ ways.

(b)



Each place can be filled by any of the 5 numbers. Therefore, the total number of arrangements is 5^4.

(c) Each card can be given to any of the 3 servants.

$\therefore$ No. of ways $= 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 729$.

(d) There are ten letters in the word BENEVOLENT of which three are E and two are N, and the rest five are different.

$\therefore$ Total number of arrangements $= \dfrac{10!}{3!\,2!}$

Example 9: n_1 and n_2 are five-digit numbers. Find the total number of ways of forming n_1 and n_2, so that n_2 can be subtracted from n_1 without borrowing at any stage.

Sol: Two numbers can be subtracted without borrowing if all the digits in n_1 is greater than all the corresponding digits in the number n_2. Using this information, find the number of ways for different possible cases and add them up to get the answer.

Let $n_1 = x_1\,x_2\,x_3\,x_4\,x_5$ and $n_2 = y_1\,y_2\,y_3\,y_4\,y_5$ be two numbers. n_1 and n_2 can be subtracted without borrowing at any stage if $x_i \geq y_i$.

Here, x_i and y_i denotes the digits at various places in the number n_1 and n_2 respectively.

Value of x_5	Value of y_5
9	0,1,2,...9
8	0,1,2,...8
7	0,1,2,...7
6	0,1,2,3,4,5,6
5	0,1,2,3,4,5
4	0,1,2,3,4
3	0,1,2,3
2	0,1
1	0
0	

Thus, x_5 and y_5 can be selected collectively by $10 + 9 + 8 + \ldots 1 = 55$ ways. Similarly, each pair (x_4, y_4), (x_3, y_3), (x_2, y_2) can be selected in 55 ways. But, pair (x_1, y_1) can be selected in $1 + 2 + 3 + \ldots + 9 = 45$ ways as in this pair we cannot have 0.

Therefore total number of ways $= 45(55)^4$.

Example 10: Prove that the product of r consecutive positive integers is divisible by r!.

Sol: Simple application of the definition of ${}^nP_r$.

Let P be the product of r consecutive positive integers ending with n; then

$P = n(n-1) \ldots (n-r+1)$

$$\frac{P}{r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$$

$$\frac{[n(n-1)(n-2)\ldots(n-r+1)][(n-r)\ldots3.2.1]}{r!(n-r)\ldots3.2.1}$$

$$= \frac{n!}{r!\,n-r!} = {}^nC_r = \text{an integer}$$

$\therefore$ P is divisible by r!.

JEE Advanced/Boards

Example 1: How many numbers of n digits can be made with the non-zero digits in which no two consecutive digits are the same?

Sol: Using Permutation under Restriction we can easily find the answer.

There are nine non-zero digits, namely 1, 2, 3, … and 9.

9	8	8	8		8

In order the make an n-digit number we have to fill n places by using the nine digits. As no two consecutive digits are to be the same, a digit used in a place cannot be used in the next place but it can be used again in the place coming after the next place.

So, the first place can be filled in 9 ways;

the second place can be filled in 8 ways (rejecting the digit used in the first place)

the third place can be filled in 7 + 1, i.e., 8 ways (rejecting the digit used in the second place but including the digit used in the first place) and so on.

$\therefore$ The total number of desired numbers

$= 9 \times 8 \times 8 \times 8 \times \ldots$ to n factors $= 9 \times 8^{n-1}$.

Example 2: A dice is a six-faced cube, with the faces reading 1, 2, 3, 4, 5 and 6. When two dice are thrown, we add the digits they show on top and take that sum as the result of the throw. In how many different ways the first throw of the 2 dice shows a total of 5, and second throw of the 2 dice shows a total of 4?

Sol: List down different ways in which we get the sum of 5 and 4 and get the answer.

Event E (the first throw resulting in 5) can happen in one of four ways as:

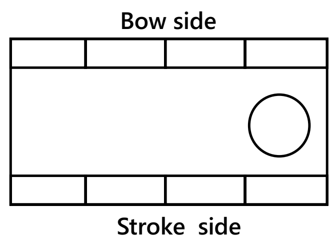
3 + 2; 4 + 1; 2 + 3; 1 + 4.

Event F (the second throw resulting in 4) can happen in one of three ways as:

2 + 2; 1 + 3; 3 + 1.

The two events can together happen in 4 × 3 = 12 ways.

Example 3: An eight-oared boat is to be manned by a crew chosen from 11 men of whom 3 can steer but cannot row and the rest cannot steer. In how many ways can the crew be arranged if two of the men can only row in bow side?



Sol: Find the number of ways we can select for steering, rowing and arranging the remaining men. Their product gives us the required result.

The total number of men = 11

The number of men who can only steer = 3

The number of other men = 8

The number of ways of selecting one man for steering out of 3 = ${}^3C_1$.

The number of ways in which the two particular men who only row on bow side

Can be arranged on bow side = ${}^4P_2$

The number of ways in which remaining 6 men can be arranged in remaining 6 places = 6!

$\therefore$ The required number = ${}^3C_1 . {}^4P_2 . 6!$

$= 3 \times 4 \times 3 \times 6! = 25920$

Example 4: The members of a chess club took part in a round robin competition in which each plays every one else once. All members scored the same number of points, except four juniors whose total score were 17.5. How many members were there in the club? (Assume that for each win a player scores 1 point, for draw $\frac{1}{2}$ point and zero for losing.)

Sol: Form an equation of the total number of points scored using the given information. Solve the equation to find the answer.

Let the number of members be n.

Total number of point = ${}^{n}C_2$.

$\therefore$ ${}^{n}C_2 - 17.5 = (n - 4)\,x$ (where x is the number of point scored by each player)

$n(n - 1) - 35 = 2(n - 4)x$

$2x = \dfrac{n(n-1)-35}{n-4}$ (where x takes the values 0.5, 1, 1.5 etc.)

$= \dfrac{n^2 - n - 35}{n-4}$ (must be an integer)

$= \dfrac{n(n-4)+3(n-4)-23}{n-4} = (n + 3) - \dfrac{23}{n-4}$

$\Rightarrow \dfrac{23}{n-4}$ must be an integer

$\Rightarrow$ n = 27 is the only possibility.

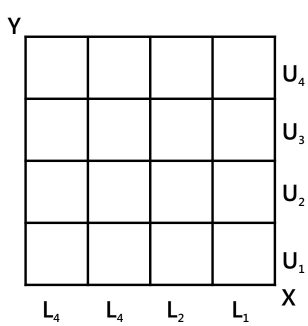
Example 5: If p, q, r, s, t are prime numbers. Find the number of ways in which the product, pq^2r^3st can be expressed as product of two factors, excluding 1 as a factor.

Sol: Use the standard result to find the answer.

Total factors = 2 × 3 × 4 × 2 × 2 = 96

Hence, the total ways = $\dfrac{96}{2}$ = 48. but this includes 1 and the number itself also. Hence, the required number of ways = 48 – 1 = 47

Example 6: In the given figure you have the road plan of a city. A man standing at X wants to reach the cinema hall at Y by the shortest path. What is the number of different paths that he can take?



Sol: If the man moves only in the upward and the leftward direction, then the path will be the shortest. Use this idea to calculate total number of shortest paths.

As the man wants to travel by one of the many possible shortest paths, he will never turn to the right or turn downward. So a travel by one of the shortest paths is to take 4 horizontal pieces and 4 vertical pieces of roads.

$\therefore$ A shortest path is an arrangement of eight objects $L_1, L_2, L_3, L_4, U_1, U_2, U_3, U_4$ so that the order of L's and U's do not change.

($\because$ Clearly L_2 cannot be taken without taking L_1, U_2 cannot be taken without taking U_1, etc.)

Hence, the number of shortest paths

= The number of arrangements of $L_1, L_2, L_3, L_4, U_1, U_2, U_3, U_4$ where the order of Ls as well as the order of Us do not change

= The number of arrangement treating Ls as identical and Us as identical

$= \dfrac{8!}{4!4!} = \dfrac{8.7.6.5}{24} = 2.7.5 = 70.$

Example 7: A condolence meeting being held in a hall which has 7 doors, by which mourners enter the hall. One can use any of the 7 doors to enter and can come at any time during the meeting. At each door, a register is kept in which mourner has to affix his signature while entering the hall. If 200 people attend the meeting, how many different sequences of 7 lists of signatures can arise?

Sol: Clearly, the total number of people is 200, hence the sum of the entries is 200. Apply Multinomial theorem to find the total number of ways list can be made and hence the answer.

There are 7 lists, say 1, 2,.... 7. Suppose, that list i has x_i names; then,

$x_1 + \ldots.. + x_7 = 200$ where $x_i \geq 0$ is an integer.

We need to first find the number of solutions of this equation.

(Note that this does not complete the solution to the questions as list 1 may contain 7 names which would remain the same in 7!, arrangements of the names)

The number of solutions are = ${}^{200+7-1}C_{7-1} = {}^{206}C_6$

But corresponding to any one solution $(x_1 \ldots x_7)$ (i.e. list f contains x_f names) we can have 200! arrangements consistent with distribution of x_j names to j^{th} list

$\therefore$ The number of different sequences of 7 lists

$= {}^{206}C_6 \times 200! = \dfrac{206}{6!}$.

JEE Main/Boards

Exercise 1

Q.1 How many odd numbers less than 1000 can be formed using the digits 0, 1, 4 and 7 if repetition of digits is allowed?

Q.2 In how many ways can five people be seated in a car with two people in the front seat and three in the rear, if two particular persons out of the five cannot drive?

Q.3 A team consisting of 7 boys and 3 girls play singles matches against another team consisting of 5 boys and 5 girls. How many matches can be scheduled between the two teams if a boy plays against a girl and a girl plays against a boy?

Q.4 Prove that $\frac{(2n+1)}{n!} = 2^n[1.3.5...(2n-1)(2n+1)]$

Q.5 If ${}^nP_4 = 360$, find n.

Q.6 Find the number of numbers between 300 and 3000 which can be formed with the digits 0, 1, 2, 3, 4 and 5, with no digit being repeated in any number.

Q.7 How many even numbers are there with three digits such that if 5 is one of the digits in a number then 7 is the next digit in that number?

Q.8 Find the sum of 3 digit numbers formed by digits 1, 2, 3 is

Q.9 A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that can be made?

Q.10 In telegraph communication, the Morse code is used in which all the letters of the English alphabet, digits 0 to 9 and even the punctuation marks, all usually referred as characters, are represented by 'dots' and 'dashes'

For example, E is represented by a dot (.), T by a dash (–), O by three dashes (- - -), S by three dots (. . .) and so on. Thus, SOS is represented by (. . . – – – . . .).

(i) How many characters can be transmitted using one symbol (dot or dash), two symbols, three symbols, four symbols? Also find the total number of characters which can be transmitted using at most four symbols.

(ii) How many characters can be transmitted by using (a) exactly five symbols? (b) at most five symbols?

Q.11 In how many of the distinct permutation of the letter in MISSISSIPPI do the four I's not come together?

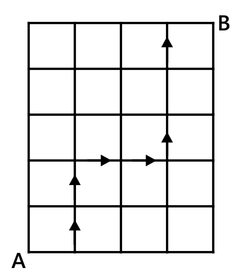
Q.12 In how many ways 4 boys and 3 girls can be seated in a row so that they are alternate?

Q.13 A biologist studying the genetic code is interested to know the number of possible arrangements of 12 molecules in a chain. The chain contains 4 different molecules represented by the initials A (for adenine), C (for Cytosine), G(for Guanine) and T (for Thymine) and 3 molecules of each kind. How many different such arrangements are possible in all?

Q.14 Find the number of rearrangement of the letters of the word 'BENEVOLENT'. How many of them end in L?

Q.15 How many words can be formed with the letters of the word PATALIPUTRA' without changing the relative order of the vowels and consonants?

Q.16 A person is to walk from A to B. However, he is restricted to walk only to the right of A or upwards of A, but not necessarily in this order. One such path is shown in the given figure Determine the total number of paths available to the person from A to B.



Q.17 In how many ways can three jobs I, II and III be assigned to three persons A, B and C, if one person is assigned only one job and all are capable of doing each job? Which assignment of jobs will take the least time to complete the jobs, if time taken (in hours) by an individual on each job as follows?

Job persons	I	II	III
A	5	4	4
B	$4\frac{1}{4}$	$3\frac{1}{2}$	4
C	5	3	5

Q.18 If ${}^{15}C_{3r} = {}^{15}C_{r+3'}$ find r.

Q.19 Prove that ${}^{n}C_{r} \times {}^{r}C_{s} = {}^{n}C_{s} \times {}^{n-s}C_{r-s}$.

Q.20 Find the value of the expression

$${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3 \ .$$

Q.21 Prove that the product of r consecutive integers is divisible by r!.

Q.22 From a class of 25 students, 10 are to be chosen for a field trip. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can the field trip members be chosen?

Q.23 There are ten points in a plane. Of these ten points, four points are in a straight line and with the exception of these four points, no three points are in the same straight line. Find-

(i) The number of triangles formed.

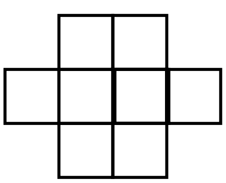
(ii) The number of straight lines formed

(iii) The number of quadrilaterals formed, by joining these ten points.

Q.24 In an examination a minimum of is to be secured in each of 5 subjects for a pass. In how many ways can a student fail?

Q.25 In how many ways 50 different objects can be divided in 5 sets three of them having 12 objects each and two of them having 7 objects each.

Q.26 Six "X"s (crosses) have to be placed in the squares of the figure given below, such that each row contains at least one X. In how many different ways can this be done?



Q.27 Five balls of different colors are to be placed in three boxes of different sizes. Each box can hold all five balls. In how many different ways can we place the balls so that no box remains empty?

Q.28 How many different words of 4 letters can be formed with the letters of the word "EXAMINATION"?

Exercise 2

Single Correct Choice Type

Q.1 If the letters of the word "VARUN" are written in all possible ways and then are arranged as in a dictionary, then rank of the word VARUN is:

(A) 98 (B) 99 (C) 100 (D) 101

Q.2 Number of natural numbers between 100 and 1000 such that at least one of their digits is 7, is

(A) 225 (B) 243 (C) 252 (D) none

Q.3 The 120 permutations of MAHES are arranged in dictionary order, as if each were an ordinary five-letter word. The last letter of the 86th word in the list is

(A) A (B) H (C) S (D) E

Q.4 A new flag is to be designed with six vertical strips using some or all of the colors yellow green, blue and red. Then the number of ways this can be done such that no two adjacent strips have the same color is

(A) 12 × 81 (B) 16 × 192 (C) 20 × 125 (D) 24 × 216

Q.5 The number of 10-digit numbers such that the product of any two consecutive digits in the number is a prime number, is

(A) 1024 (B) 2048 (C) 512 (D) 64

Q.6 Consider the five points comprising of the vertices of a square and the intersection point of its diagonals. How many triangles can be formed using these points?

(A) 4 (B) 6 (C) 8 (D) 10

Q.7 How many of the 900 three digit numbers have at least one even digit?

(A) 775 (B) 875 (C) 100 (D) 101

Q.8 A 5 digit number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 & 5 without repetition. The total number of ways in which this can be done is:

(A) 3125 (B) 600 (C) 240 (D) 216

Q.9 The number of different seven digit numbers that can be written using only three digits 1, 2 & 3 under the condition that the digit 2 occurs exactly twice in each number is

(A) 672 (B) 640 (C) 512 (D) none

Q.10 Out of seven consonants and four vowels, the number of words of six letters, formed by taking four consonants and two vowels is (Assume that each ordered group of letter is a word):

(A) 210 (B) 462 (C) 151200 (D) 332640

Q.11 All possible three digits even numbers which can be formed with the condition that if 5 is one of the digit, then 7 is the next digit is:

(A) 5 (B) 325 (C) 345 (D) 365

Q.12 Number of 5 digit numbers which are divisible by 5 and each number containing the digit 5, digits being all different is equal to k(4!), the value of k is

(A) 84 (B) 168 (C) 188 (D) 208

Q.13 The number of six digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6, & 7 so that digits do not repeat and the terminal digit are even is:

(A) 144 (B) 72 (C) 288 (D) 720

Q.14 The number of natural numbers from 1000 to 9999 (both inclusive) that do not have all 4 different digits is

(A) 4048 (B) 4464 (C) 4518 (D) 4536

Q.15 Number of positive integers which have no two digits having the same value with sum of their digits being 45, is

(A) 10! (B) 9! (C) 9.9! (D) 17.8!

Q.16 Number of 3 digit number in which the digit at hundredth's place is greater than the other two digit is

(A) 285 (B) 281 (C) 240 (D) 204

Q.17 Number of permutation of 1, 2, 3, 4, 5, 6, 7, 8 and 9 taken all at a time, such that the digit 1 appearing somewhere to the left of 2, 3 appearing to somewhere the left of 4 and 5 somewhere to the left of 6, is (e.g. 815723946 would be one such permutation)

(A) 9.7! (B) 8! (C) 5!.4! (D) 8!.4!

Q.18 Number of odd integers between 1000 and 8000 which have none of their digit repeated, is

(A) 1014 (B) 810 (C) 690 (D) 1736

Q.19 The number of ways in which 5 different books can be distributed among 10 people if each person can get at most one book is:

(A) 252 (B) 10^5 (C) 5^{10} (D) ${}^{10}C_5.5!$

Q.20 A students have to answer 10 out of 13 questions in an examination. The number of ways in which he can answer if he must answer at least 3 of the first five questions is

(A) 276 (B) 267 (C) 80 (D) 1200

Q.21 The number of three digit numbers having only two consecutive digits identical is:

(A) 153 (B) 162 (C) 180 (D) 161

Q.22 The interior angles of a regular polygon measure 150° each. The number of diagonals of the polygon is

(A) 35 (B) 44 (C) 54 (D) 78

Q.23 The number of n digit numbers which consists of the digits 1 & 2 only if each digit is to be used at least once, is equal to 510 then n is equal to:

(A) 7 (B) 8 (C) 9 (D) 10

Q.24 Number of four digit numbers with all digits different and containing the digit 7 is

(A) 2016 (B) 1828 (C) 1848 (D) 1884

Q.25 An English school and a Vernacular school are both under one superintendent. Suppose that the superintendentship, the four teachership of English and Vernacular school each, are vacant, if there be altogether 11 candidates for the appointments, 3 of whom apply exclusively for the superintendentship and 2 exclusively for the appointment in the English school, the number of ways in which the different appointment can be disposed of is :

(A) 4320 (B) 268 (C) 1080 (D) 25920

Q.26 A committee of 5 is to be chosen from a group of 9 people. Number of ways in which it can be formed if two particular persons either serve together or not at all and two other particular persons refuse to serve with each other, is

(A) 41 (B) 36 (C) 47 (D) 76

Q.27 A question paper on mathematics consists of twelve questions divided into three parts A, B and C, each containing four questions, in how many ways can an examinee answer five questions, selecting at least one from each part?

(A) 624 (B) 208 (C) 1248 (D) 2304

Q.28 Number of ways in which 7 green bottles and 8 blue bottles can be arranged in a row if exactly 1 pair of green bottles is side by side, is (Assume all bottles to be alike except for the color).

(A) 84 (B) 360 (C) 504 (D) None

Q.29 The kindergarden teacher has 25 kids in her class. She takes 5 of them at a time, to zoological garden as often as she can, without taking the same 5 kids more than once. Then the number of visits, the teacher makes to the garden exceeds that of a kid by:

(A) ${}^{25}C_5 - {}^{24}C_5$ (B) ${}^{24}C_5$ (C) ${}^{24}C_4$ (D) None

Q.30 A rack has 5 different pairs of shoes. The number of ways in which 4 shoes can be chosen from it so that there will be no complete pair is:

(A) 1920 (B) 200 (C) 110 (D) 80

Q.31 Number of ways in which 9 different toys can be distributed among 4 children belonging to different age groups in such a way that distribution among the 3 elder children is even and the youngest one is to receive one toy more, is:

(A) $\frac{(5!)^2}{8}$ (B) $\frac{9!}{2}$ (C) $\frac{9!}{3!(2!)^3}$ (D) None

Q.32 There are 10 red balls of different shades & 9 green balls of identical shades. Then the number of such arrangements such that no two green balls are together in the row is:

(A) $(10!).{}^{11}P_9$ (B) $(10!).{}^{11}C_9$ (C) 10! (D) 10! 9!

Q.33 A shelf contains 20 different books of which 4 are in single volume and the others form sets of 8, 5 and 3 volumes respectively. Number of ways in which the books may be arranged on the shelf, if the volumes of each set are together and in their due order is

(A) $\frac{20!}{8!5!3!}$ (B) 7! (C) 8! (D) 7.8!

Q.34 Number of ways in which 3 men and their wives can be arranged in a line such that none of the 3 men stand in a position that is ahead of his wife, is

(A) 3!.3! (B) 2.3!.3! (C) 3! (D) $\frac{6!}{2!2!2!}$

Q.35 The number of different ways in which five 'dashes' and eight 'dots' can be arranged, using only seven of these 13 'dashes' & 'dots' is

(A) 1287 (B) 119 (C) 120 (D) 1235520

Q.36 Number of different words that can be formed using all the letters of the word "DEEPMALA" if two vowels are together and the other two are also together but separated from the first two is

(A) 960 (B) 1200 (C) 2160 (D) 1440

Q.37 In a unique hockey series between India & Pakistan, they decide to play on till a team wins 5 matches. The number of ways in which the series can be won by India, if no match ends in a draw is:

(A) 126 (B) 252 (C) 225 (D) None

Q.38 Sameer has to make a telephone call to his friend Harish, Unfortunately he does not remember the 7 digit phone number. But he remembers that the first three digits are 635 or 674, the number is odd and there is exactly one 9 in the number. The maximum number of trials that Sameer has to make to be successful is

(A) 10,000 (B) 3402 (C) 3200 (D) 5000

Q.39 There are 12 guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another, and that there are two specified guests who must always, be placed next to one another; the number of ways in which the company can be placed is :

(A) 20 . 10 ! (B) 22 . 10 ! (C) 44 . 10 ! (D) None

Q.40 In a conference 10 speakers are present. If S_1 wants to speak before S_2 and S_2 wants to speak after S_3, then the number of ways all the 10 speakers can give their speeches with the above restriction if the remaining seven speakers have no objection to speak at any number is

(A) ${}^{10}C_3$ (B) ${}^{10}P_8$ (C) ${}^{10}P_3$ (D) $\frac{10!}{3}$

Q.41 The number of all possible selection of one or more questions from 10 given questions, each question having an alternative is:

(A) 3^{10} (B) $2^{10}-1$ (C) $3^{10}-1$ (D) 2^{10}

Q.42 Number of 7 digit numbers the sum of whose digits is 61 is:

(A) 12 (B) 24 (C) 28 (D) None

Q.43 There are 2 identical white balls, 3 identical red balls and 4 green balls of different shades. The number of ways in which they can be arranged in a row so that at least one ball is separated from the balls of the same color, is:

(A) 6 (7! – 4!) (B) 7 (6! – 4!) (C) 8 ! – 5! (D) None

Q.44 Product of all the even divisors of N = 1000, is

(A) $32 . 10^2$ (B) $64 . 2^{14}$ (C) $64 . 10^{18}$ (D) $128 . 10^6$

Q.45 A lift with 7 people stops at 10 floors. People varying from zero to seven go out at each floor. The number of ways in which the lift can get emptied, assuming each way only differs by the number of people leaving at each floor, is

(A) ${}^{16}C_6$ (B) ${}^{17}C_7$ (C) ${}^{16}C_7$ (D) None

Q.46 You are given an unlimited supply of each of the digits 1, 2, 3 or 4. Using only these four digits, you construct n digit numbers. Such n digit numbers will be called LEGITIMATE if it contains the digit 1 either an even number times or not at all. Number of n digit legitimate numbers are

(A) $2^n + 1$ (B) $2^{n+1} + 2$ (C) $2^{n+2} + 4$ (D) $2^{n-1}(2^n + 1)$

Q.47 Distinct 3 digit numbers are formed using only the digits 1, 2, 3 and 4 with each digit used at most once in each number thus formed. The sum of all possible numbers so formed is

(A) 6660 (B) 3330 (C) 2220 (D) None

Q.48 An ice cream parlor has ice creams in eight different varieties. Number of ways of choosing 3 ice creams taking at least two ice creams of the same variety, is (Assume that ice creams of the same variety to be identical & available in unlimited supply)

(A) 56 (B) 64 (C) 100 (D) None

Q.49 There are 12 books on Algebra and Calculus in our library, the books of the same subject being different. If the number of selection each of which consists of 3 books on each topic is greatest then the number of books of Algebra and Calculus in the library are respectively:

(A) 3 and 9 (B) 4 and 8 (C) 5 and 7 (D) 6 and 6

Q.50 A person writes letters to his 5 friends and addresses the corresponding envelopes. Number of ways in which the letters can be placed in the envelope, so that at least two of them are in the wrong envelopes, is,

(A) 1 (B) 2 (C) 118 (D) 119

Q.51 For a game in which two partners oppose two other partners, 8 men are available. If every possible pair must play with every other pair, the number of games played is

(A) ${}^8C_2 . {}^6C_2$ (B) $8C2 . {}^6C_2 . 2$ (C) ${}^8C_4 . 3$ (D) None

Q.52 The number 916238457 is an example of nine digit number which contains each of the digit 1 to 9 exactly once. It also has the property that the digits 1 to 5 occur in their natural order, while the digits 1 to 6 do not. Number of such numbers are

(A) 2268 (B) 2520 (C) 2975 (D) 1560

Q.53 Number of functions defined from f : {1, 2, 3, 4, 5, 6} $\to$ {7, 8, 9, 10} such that the sum f(1) + f(2) + f(3) + f(4) +f(5) + f(6) is odd, is

(A) 2^{10} (B) 2^{11} (C) 2^{12} (D) $2^{12} - 1$

Multiple Correct Choice Type

Q.54 The continued product, 2.6.10.14... to n factors is equal to:

(A) ${}^{2n}C_n$ (B) ${}^{2n}P_n$

(C) ${}^{2n+1}C_n$ (D) None

Q.55 The maximum number of permutations of 2n letters in which there are only a's & b's, taken all at a time is given by :

(A) ${}^{2n}C_n$

(B) $\frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdots \frac{4n-6}{n-1} \cdot \frac{4n-2}{n}$

(C) $\frac{n+1}{1} \cdot \frac{n+2}{2} \cdot \frac{n+3}{3} \cdot \frac{n+4}{4} \cdots \frac{2n-1}{n-1} \cdot \frac{2n}{n}$

(D) $\frac{2^n\left[1.3.5...(2n-3)(2n-1)\right]}{n!}$

(E) All of the above

Q.56 Number of ways in which 3 numbers in A.P. can be selected from 1, 2, 3,... n is :

(A) $\left(\frac{n-1}{2}\right)^2$ if n is even (B) $\frac{n(n-2)}{4}$ if n is odd

(C) $\frac{(n-1)}{4}$ if n is odd (D) $\frac{n(n-2)}{4}$ if n is even

Previous Years' Questions

Q.1 The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to ***(1982)***

(A) ${}^{47}C_5$ (B) ${}^{52}C_5$

(C) ${}^{52}C_4$ (D) None of these

Q.2 Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4, and then the men select the chairs from amongst the remaining. The number of possible arrangements is ***(1982)***

(A) ${}^6C_3 \times {}^4C_2$ (B) ${}^4P_2 \times {}^4P_3$

(C) ${}^4C_2 + {}^4P_3$ (D) None

Q.3 A five digits number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 and 5, without repetition. The total number of ways this can be done, is ***(1989)***

(A) 216 (B) 240 (C) 600 (D) 3125

Q.4 Number of divisors of the form $(4n + 2)$, $n \geq 0$ of integer 240 is ***(1998)***

(A) 4 (B) 8 (C) 10 (D) 3

Q.5 If r, s, t are prime numbers and p, q are the positive integers such that LCM of p, q is $r^2s^4t^2$, then the number of ordered pairs (p, q) is ***(2006)***

(A) 252 (B) 254 (C) 225 (D) 224

Q.6 The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before the word COCHIN is ***(2007)***

(A) 360 (B) 192 (C) 96 (D) 48

Q.7 The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is ***(2009)***

(A) 55 (B) 66 (C) 77 (D) 88

JEE Advanced/Boards

Exercise 1

Q.1 Consider all the six digit numbers that can be formed using the digits 1, 2, 3, 4, 5 and 6, each digit being used exactly once. Each of such six digit numbers have the property that for each digit, not more than two digits, smaller than that digit, appear to the right of that digit. Find the number of such six digit numbers having the desired property

Q.2 Find the number of five digit number that can be formed using the digits 1, 2, 3, 4, 5, 5, 7, 9 in which one digit appears once and two digits appear twice (e.g 41174 is one such number but 75355 is not.)

Q.3 Find the number of ways in which 3 distinct numbers can be selected from the set $\{3^1, 3^2, 3^3, ... 3^{100}, 3^{101}\}$ so that they form a G.P.

Q.4 Find the number of odd numbers between 3000 to 6300 that have all different digits.

Q.5 A man has 3 friend. In how many ways he can invite one friend every day for dinner on 6 successive nights so that no friend is invited more than 3 times.

Q.6 In an election for the managing committee of a reputed club, the number of candidates contesting elections exceeds the number of members to be elected by r(r > 0). If a voter can vote in 967 different ways to elect managing committee by voting at least 1 of them & can vote in 55 different ways to elect (r – 1) candidates by voting in the same manner. Find the number of candidates contesting the election & the number of candidates losing the elections.

Paragraph for question nos. 7 to 9:

2 American men; 2 British men; 2 Chinese men and one each of Dutch, Egyptian, French and German persons are to be seated for a round table conference.

Q.7 If the number of ways in which they can be seated if exactly to pairs of persons of same nationality are together is p(6!), then find p.

Q.8 If the number of ways in which only American pair is adjacent is equal to q(6!), then find q.

Q.9 If the number of ways in which no two people of the same nationality are together given by r (6!), find r.

Q.10 For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence 1, 4, 7, 10 ..... Find the number of values of k for which S_k contain the term 36!

Q.11 A shop sells 6 different flavors of ice-cream. In how many ways can a customer choose 4 ice-cream cones if

(i) They are all of different flavors

(ii) They are not necessarily of different flavors

(iii) They contain only 3 different flavors

(iv) They contain only 2 or 3 different flavors?

Q.12 (a) How many divisors are there of the number 21600. Find also the sum of these divisors.

(b) In how many ways the number 7056 can be resolved as a product of 2 factors.

(c) Find the number of ways in which the number 300300 can be split into 2 factors which are relatively prime.

(d) Find the number of positive integers that are divisors of at least one of the number 10^{10}; 15^7; 18^{11}.

Q.13 How many 15 letter arrangement of 5A's, 5 B's and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters, and to C's in the last 5 letters.

Q.14 Determine the number of paths from the origin to the point (9, 0) in the Cartesian plane which never pass through (5, 5) in paths consisting only of steps going 1 unit North and 1 unit East.

Q.15 There are n triangles of positive area that have one vertex A(0, 0) and the other two vertices whose coordinates are drawn independently with replacement from the set {0, 1, 2, 3, 4} e.g. (1. 2), (0, 1) (2, 2) etc. Find the value of n.

Q.16 How many different ways can 15 Candy bars be distributed between Ram, Shyam, Ghanshyam and Balram, if Ram cannot have more than 5 candy bars and Shyam must have at least two. Assume all Candy bars to be alike

Q.17 Find the number of three digits number from 100 to 999 inclusive which have any one digit that is the average of the other two.

Q.18 (a) Find the number of non-empty subsets S of {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} such that if, S contains k elements, then S contains no number less than k.

(b) If the number of ordered pairs (S, T) of subsets of {1, 2, 3, 4, 5, 6} are such that $S \cup T$ contains exactly three elements 10λ, then find the value of λ.

Q.19 Find the number of permutation of the digits 1, 2, 3, 4 and 5 taken all at a time so that the sum of the digits at the first two places is smaller than the sum of the digit at the last two places.

Q.20 In a league of 8 teams, each team played every other team 10 times. The number of wins of the 8 teams formed an arithmetic sequence. Find the least possible number of games won by the champion.

Q.21 Find the sum of all numbers greater than 10000 formed by using the digits 0, 1, 2, 4, 5 no. digit being repeated in any number.

Q.22 There are 3 cars of different make available to transport 3 girls and 5 boys on a field trip. Each car can hold up to 3 children. Find

(a) the number of ways in which they can be accommodated.

(b) the numbers of ways in which they can be accommodated if 2 or 3 girls are assigned to one of the cars.

In both the cases internal arrangement of children inside the car is considered to be immaterial.

Q.23 Find the number of three elements sets of positive integers {a, b, c} such that $a \times b \times c = 2310$.

Q.24 Find the number of integer between 1 and 10000 with a least one 8 nd at least one 9 as digits

Q.25 Let N be the number of ordered pairs of non-empty sets A and B that have the following properties:

(a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(b) $A \cap B = \phi$

(c) The number of elements of A is not the element of B.

(d) The number of elements of B is not an element of A.

Find N.

Q.26 In how many other ways can be letters of the word MULTIPLE be arranged:

(i) Without changing the order of the vowels

(ii) Keeping the position of each vowel fixed and without changing the relative order/position or vowels & consonants.

Q.27 Let N denotes the number of all 9 digits numbers if

(a) The digit of each number are all from the set {5, 6, 7, 8, 9} and

(b) Any digit that appears in the number, repeats at least three times. Find the value of N/5.

Q.28 How many integers between 1000 and 9999 have exactly one pair of equal digit such as 4049 or 9902 but not 4449 or 4040?

Q.29 How many 6 digits odd numbers greater than 60,000 can be formed from the digits 5, 6, 7, 8, 9, 0 if

(i) Repetitions are not allowed

(ii) Repetitions are allowed.

Exercise 2

Single Correct Choice Type

Q.1 An eight digit number divisible by 9 is to be formed by using 8 digits out of the digits 0, 1, 2, 3, 4, 5, 6, 7 8, 9 without replacement. The number of ways in which this can be done is

(A) 9! (B) 2(7!) (C) 4(7!) (D) (36) (7!)

Q.2 Number of 4 digit numbers of the form N = abcd which satisfy following three conditions

(i) $4000 \le N < 6000$

(ii) N is a multiple of 5

(iii) $3 \le b < c \le 6$ is equal to

(A) 12 (B) 18 (C) 24 (D) 48

Q.3 5 Indian & 5 American couples meet at a party and shake hands. If no wife shakes hands with her own husband and no Indian wife shakes hands with a male then the number of handshakes that takes place in the party is

(A) 95 (B) 110 (C) 135 (D) 150

Q. 4 The 9 horizontal and 9 vertical lines on an 8 × 8 chessboard form 'r' rectangles and 's' squares, The ratio s/r in its lowest terms is

(A) $\frac{1}{6}$ (B) $\frac{17}{108}$ (C) $\frac{4}{27}$ (D) None

Q.5 Number of different natural numbers which are smaller than two hundred million and use only the digits 1 or 2 is

(A) $(3) . 2^8 - 2$ (B) $(3) . 2^8 - 1$

(C) $2 (2^9 - 1)$ (D) None

Q.6 There are counters available in x different colors. The counters are all alike except for the color. The total number of arrangements consisting of y counters, assuming sufficient number of counters of each color, if no arrangement consists of all counters of the same color is:

(A) $x^y - x$ (B) $x^y - y$ (C) $y^x - x$ (D) $y^x - y$

Q.7 If m denotes the number of 5 digit numbers of each successive digits are in their descending order magnitude and n is the corresponding figure, when the digits are in their ascending order of magnitude then (m – n) has the value

(A) ${}^{10}C_4$ (B) ${}^{9}C_5$ (C) ${}^{10}C_3$ (D) ${}^{9}C_3$

Q.8 There are m points on straight line AB & n points on the line AC none of them being the point A. Triangles are formed with these points as vertices, when

(i) A is excluded

(ii) A is included. The ration of number of triangles in the two cases is:

(A) $\frac{m+n-2}{m+n}$ (B) $\frac{m+n-2}{m+n-1}$

(C) $\frac{m+n-2}{m+n+2}$ (D $\frac{n(n-1)}{(m+1)(n+1)}$

Q.9 The number of 5 digit numbers such that the sum of their digits is even is

(A) 50000 (B) 45000 (C) 60000 (D) None

Q.10 Number of ways in which 8 people can be arranged in a line if A and B must be next each other and C must be somewhere behind D, is equal to

(A) 10080 (B) 5040 (C) 5050 (D) 10100

Q.11 Seven different coins are to be divided amongst three persons. If no two of the persons receive the same number of coins but each receives at least one coin & none is left over, then the number of ways in which the division may be made is

(A) 420 (B) 630 (C) 710 (D) None

Q.12 Let there be 9 fixed point on the circumference of a circle. Each of these points is joined to every one of the remaining 8 points by a straight line and the points are so positioned on the circumference that at most 2 straight lines meet in any interior point of the circle. The number of such interior intersection points is:

(A) 126 (B) 351 (C) 756 (D) None

Q.13 The number of ways in which 8 distinguishable apples can be distributed among 3 boys such that every boy should get at least 1 apple & at most 4 apples is K. ${}^7P_3$ where K has the value equal to

(A) 14 (B) 66 (C) 44 (D) 22

Q.14 There are five different peaches and three different apples. Number of ways they can be divided into two packs of four fruits if each pack must contain at least one apple, is

(A) 95 (B) 65 (C) 60 (D) 30

Q.15 Let P_n denote the number of ways in which three people can be selected out of 'n' people sitting in a row, if no two of them are consecutive. If $P_{n+1} - P_n = 15$ then the value of 'n' is

(A) 7 (B) 8 (C) 9 (D) 10

Q.16 The number of positive integers not greater than 100, which are not divisible by 2, 3 or 5 is

(A) 26 (B) 18 (C) 31 (D) None

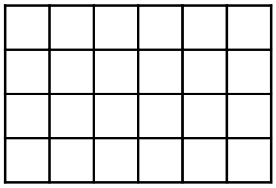
Q.17 There are six periods in each working day of a school. Number of ways in which 5 subjects can be arranged if each subject is allotted at least one period and no period remains vacant is

(A) 210 (B) 1800 (C) 360 (D) 3600

Q.18 An old man while dialing a 7 digit telephone number remembers that the first four digits consists of one 1's, one 2's and two 3's. He also remembers that the fifth digit is either a 4 or 5 while has no memorizing of the sixth digit, he remembers that the seventh digit is 9 minus the sixth digit. Maximum number of distinct trials he has to try to make sure that he dials the correct telephone number, is

(A) 360 (B) 240 (C) 216 (D) None

Q.19 Number of rectangles in the grid shown which are not squares is



(A) 160 (B) 162 (C) 170 (D) 185

Q.20 All the five digit numbers in which each successive digit exceeds its predecessor are arranged in the increasing order of their magnitude. The 97th number in the list does not contains the digit

(A) 4 (B) 5 (C) 7 (D) 8

Q.21 There are n identical red balls & m identical green balls. The number of different linear arrangements consisting of n red ball but not necessarily all the green balls' is ${}^xC_y$ then

(A) x = m + n, y = m

(B) x = m + n + 1, y = ,m

(C) x = m + n + 1, y = m + 1

(D) x = m + n, y = n

Q.22 A gentleman invites a party of m + n ($m \neq n$) friends to a dinner & places m at one table T_1 and n at another table T_2, the table being round, if not all people shall have the same neighbor in any two arrangement, then the number of ways in which he can arrange the guests, is

(A) $\frac{(m+n)!}{4mn}$ (B) $\frac{1}{2}\frac{(m+n)!}{mn}$

(C) $2\frac{(m+n)!}{mn}$ (D) None

Q.23 Consider a determinant of order 3 all whose entries are either 0 or 1. Five of these entries are 1 and four of them are '0'. Also $a_{ij} = a_{ji}\ \forall\ 1 \le i, j \le 3$. The number of such determinants, is equal to

(A) 6 (B) 8 (C) 9 (D) 12

Q.24 A team of 8 students goes on an excursion, in two cars, of which one can seat 5 and the other only 4. If internal arrangement inside the car does not matter then the number of ways in which they can travel, is

(A) 91 (B) 182 (C) 126 (D) 3920

Q.25 One hundred management students who read at least one of the three business magazines are surveyed to study the readership pattern. It is found that 80 read Business India, 50 read Business world, and 30 read Business Today. Five students read all the three magazines. How many read exactly two magazines?

(A) 50 (B) 10 (C) 95 (D) 45

Q.26 Six people are going to sit in a row on a bench. A and B are adjacent, C does not want to sit adjacent to D. E and F can sit anywhere. Number of ways in which these six people can be seated, is

(A) 200 (B) 144 (C) 120 (D) 56

Q.27 Number of cyphers at the end of ${}^{2002}C_{1001}$ is

(A) 0 (B) 1 (C) 2 (D) 200

Q.28 Three vertices of a convex n sided polygon are selected. If the number of triangles that can be constructed such that none of the sides of the triangle is also the side of the polygon is 30, then the polygon is a

(A) Heptagon (B) Octagon

(C) Nonagon (D) Decagon

Q.29 Given 11 points, of which 5 lie on one circle, other than these 5, no 4 lie on one circle. Then the maximum number of circles that can be drawn so that each contains at least three of the given points is:

(A) 216 (B) 156 (C) 172 (D) None

Q.30 Number of 5 digit numbers divisible by 25 that can be formed using only the digits 1, 2, 3, 4, 5, & 0 taken five at a time is

(A) 2 (B) 32 (C) 42 (D) 52

Q.31 Let P_n denotes the number of ways of selecting 3 people out of 'n' sitting in a row, if no two of them are consecutive and Q_n is the corresponding figure when they are in a circle. If $P_n - Q_n = 6$, then 'n' is equal to

(A) 8 (B) 9 (C) 10 (D) 12

Q.32 Let m denote the number of ways in which 4 different books are distributed among 10 persons, each receiving none or one only and let n denote the number of ways of distribution if the books are all alike. Then:

(A) m = 4n (B) n = 4m (C) m = 24 n (D) none

Q.33 The number of ways of choosing a committee of 2 women & 6 men, if Mr. A refuses to serve on the committee if Mr. B is a member & Mr. B can only serve, if Miss C is the member of the committee, is

(A) 60 (B) 84 (C) 124 (D) None

Q.34 Six person A, B, C, D, E and F are to be seated at a circular table. The number of ways this can be done if A must have either B or C on his right and B must have either C or D on his right is:

(A) 36 (B) 12 (C) 24 (D) 18

Q.35 There are 100 different books in a shelf. Number of ways in which 3 books can be selected so that no two of which are adjacent is

(A) ${}^{100}C_3 - 98$ (B) ${}^{97}C_3$ (C) ${}^{96}C_3$ (D) ${}^{98}C_3$

Q.36 Number of ways in which four different toys and five indistinguishable marbles can be distributed between Amar, Akbar and Anthony, if each child receives at least one toy and one marble, is

(A) 42 (B) 100 (C) 150 (D) 216

Q.37 A 3 digit palindrome is a 3 digit number (not starting with zero) which reads the same backwards as forwards. For example 171. The sum of all even 3 digit palindromes, is

(A) 22380 (B) 25700 (C) 22000 (D) 22400

Q.38 Two classrooms A and B having capacity of 25 and (n–25) seats respectively A_n denotes the number of possible seating arrangements of room 'A', when 'n' students are to be seated in these rooms, starting from room 'A' which is to be filled up full to its capacity. If $A_n - A_{n-1} = 25!\,({}^{49}C_{25})$ then 'n' equals

(A) 50 (B) 48 (C) 49 (D) 51

Q.39 Number of positive integral solution satisfying the equation $(x_1 + x_2 + x_3)(y_1 + y_2) = 77$, is

(A) 150 (B) 270 (C) 420 (D) 1024

Q.40 There are counters available in 3 different colors (at least four of each color). Counters are all alike except for the color. If 'm' denotes the number of arrangements of four counters if no arrangement consists of counters of same color and 'n' denotes the corresponding figure when every arrangement consists of counters of each color, then:

(A) m = 2 n (B) 6 m = 13 n

(C) 3 m = 5 n (D) 5 m = 3 n

Q.41 Three digit numbers in which the middle one is a perfect square are formed using the digits 1 to 9. Their sum is:

(A) 134055 (B) 270540

(C) 170055 (D) none

Q.42 A guardian with 6 wards wishes every one of them to study either Law of Medicine or Engineering. Number of ways in which he can make up his mind with regard to the education of his wards if every one of them be fit for any of the branches to study, and at least one child is to be sent in each discipline is:

(A) 120 (B) 216 (C) 729 (D) 540

Q.43 There are (p + q) different books on different topics in Mathematics ($p \neq q$)

If L = The number of ways in which these books are distributed between two students X and Y such that X get p books and Y gets q books.

M = The number of ways in which these books are distributed between two students X and Y such that one of them gets p books and another gets q books.

N = The number of ways in which these books are divided into two groups of p books and q books then,

(A) L = M = N (B) L = 2M = 2N

(C) 2L = M = 2N (D) L = M = 2N

Q.44 Number of ways in which 5A' and 6B's can be arranged in a row which reads the same backwards and forwards, is

(A) 6 (B) 8 (C) 10 (D) 12

Q.45 Coefficient of $x^2 y^3 z^4$ in the expansion of $(x + y + z)^9$ is equal to

(A) The number of ways in which 9 objects of which 2 alike of one kind, 3 alike of 2nd kind, and 4 alike of 3rd kind can be arranged.

(B) The number of ways in which 9 identical objects can be distributed in 3 persons each receiving at least two objects.

(C) The number of ways in which 9 identical objects can be distributed in 3 persons each receiving none one or more.

(D) The number of ways in which 9 different books can be tied up in to three bundles one containing 2, other 3 and third containing 4 books.

Multiple Correct Choice Type

Q.46 The combinatorial coefficient C(n, r) is equal to

(A) number of possible subsets of r members from a set of n distinct members.

(B) number of possible binary messages of length n with exactly r l's.

(C) number of non-decreasing 2-D paths from the lattice point (0, 0) to (r, n).

(D) number of ways of selecting r objects out of n different objects when a particular object is always included plus the number of ways of selecting 'r' objects out of n, when a particular object out of n, when a particular object is always excluded.

Q.47 There are 10 questions, each question is either True or False. Number of different sequences of incorrect answers is also equal to

(A) Number of ways in which a normal coin tossed 10 times would fall in a definite order if both Heads and Tails are present.

(B) Number of ways in which a multiple choice question containing 10 alternatives with one or more than one correct alternatives, can be answered.

(C) Number of ways in which it is possible to draw a sum of money with 10 coins of different denominations taken some or all at a time.

(D) Number of different selection of 10 indistinguishable objects taken some or all at a time.

Q.48 The number of ways in which five different books can be distributed among 3 persons so that each person gets at least one book, is equal to the number of ways in which

(A) 5 persons are allotted 3 different residential flats so that and each person is allotted at most one flat and no two persons are allotted the same flat.

(B) Number of parallelograms (some of which may be overlapping) formed by one set of 6 parallel lines and other set of 5 parallel lines that goes in other direction.

(C) 5 different toys are to be distributed among 3 children, so that each child gets at least one toy.

(D) 3 mathematics professors are assigned five different lecturers to be delivered, so that each professor gets at least one lecturer.

Q.49 If k is odd then ${}^kC_r$ is maximum for r equal to

(A) $r = \frac{1}{2}(k-1)$ (B) $r = \frac{1}{2}(k+1)$

(C) k – 1 (D) k

Q.50 Which of the following statements are correct?

(A) Number of words that can be formed with 6 only of the letters of the word "CENTRIFUGAL' if each word must contain all the vowels is 3 . 7!

(B) There are 15 balls of which some are white and the rest black. If the number of ways in which the balls con be arranged in a row, is maximum then the number of white balls must be equal to 7 or 8. Assume balls of the same color to be alike.

(C) There are 12 objects, 4 alike of one kind, 5 alike and of another kind and the rest are all different. The total number of combinations in 240

(D) Number of selections that can be made of 6 letters from the word "COMMITTEE" is 35.

Q.51 Number of ways in which the letters of the word 'B U L B U L' can be arranged in a line is a definite order is also equal to the

(A) Number of ways in which 2 alike Apples and 4 alike Mangoes can be distributed in 3 children so that each child receives any number of fruits.

(B) Number of ways in which 6 different books can be tied up into 3 bundles, if each bundle is to have equal number of books.

(C) Coefficient of $x^2y^2z^2$ in the expansion of $(x + y + z)^6$.

(D) Number of ways in which 6 different prizes can be distributed equally in three children.

Comprehension Type

Paragraph 1: Consider the word W = MISSISSIPPI

Q.52 If N denotes the number of different selections of 5 letters from the word W = MISSISSIPPI then N belongs to the set

(A) {15, 16, 17, 18, 19} (B) {20, 21, 22, 23, 24}

(C) {25, 26, 27, 28, 29} (D) {30, 31, 32, 33, 34}

Q.53 Number of ways in which the letters of the word W can be arranged if at least one vowel is separated from rest of the vowels

(A) $\frac{8!.161}{4!.4!.2!}$ (B) $\frac{8!.161}{4.4!.2!}$ (C) $\frac{8!.161}{4!.2!}$ (D) $\frac{8!}{4!.2!}.\frac{165}{4!}$

Q.54 If the number of arrangements of the letters of the word W if all the S's and P's are separated is (k) $\left(\frac{10!}{4!.4!}\right)$ then k equals

(A) $\frac{6}{5}$ (B) 1 (C) $\frac{4}{3}$ (D) $\frac{3}{2}$

Paragraph 2: 16 players P_1, P_2, P_3, ….. P_{16} take part in a tennis tournament. Lower suffix player is better than any higher suffix player. These players are to be divided into 4 groups each comprising of 4 players and the best from each group is selected for semifinals.

Q.55 Number of ways in which 16 players can be divided into four equal groups, is

(A) $\frac{35}{27}\prod_{r=1}^{8}(2r-1)$ (B) $\frac{35}{24}\prod_{r=1}^{8}(2r-1)$

(C) $\frac{35}{52}\prod_{r=1}^{8}(2r-1)$ (D) $\frac{35}{6}\prod_{r=1}^{8}(2r-1)$

Q.56 Number of ways in which they can be divided into 4 equal groups if the players P_1, P_2, P_3 and P_4 are in different groups, is:

(A) $\frac{(11)!}{36}$ (B) $\frac{(11)!}{72}$ (C) $\frac{(11)!}{108}$ (D) $\frac{(11)!}{216}$

Match the Columns

Q.57

Column-I	Column-II
(A) Number of increasing permutations of m symbols are there from the n set numbers $\{a_1, a_2, \ldots, a_n\}$ where the order among the number is given by $a_1 < a_2 < a_3 < \ldots a_{n-1} < a_n$ is	(p) n^m
(B) There are m men and n monkeys. Number of ways in which every monkey has a master, if a man can have any number of monkeys	(q) ${}^mC_n$
(C) Number of ways in which n red balls are (m – 1) green balls can be arranged in a line, so that no two red balls are together, is (balls of the same color are alike)	(r) ${}^nC_m$
(D) Number of ways in which 'm' different toys can be distributed in 'n' children if every child may receive any number of toys, is	(s) m^n

Q.58

Column-I	Column-II
(A) Four different movies are running in a town. Ten students go to watch these four movies. The number of ways in which every movie is watched by at least one student, is (Assume each way differs only by number of students watching a movies)	(p) 11
(B) Consider 8 vertices of a regular octagon and its center. If T denotes the number of triangles and S denotes the number of straight lines that can be formed with these 9 points then the value of (T – S) equals	(q) 36
(C) In an examination, 5 children were found to have their mobiles in their pocket. The Invigilator fired them and took their mobiles in his possession. Towards the end of the test, Invigilator randomly returned their mobiles. The number of ways in which at most two children did not get their own mobiles is	(r) 52
(D) The product of the digits of 3214 is 24. The number of 4 digit natural numbers such that the product of their digits is 12, is	(s) 60
(E) The number of ways in which a mixed double tennis game can be arranged from amongst 5 married couple if no husband & wife plays in the same game, is	(t) 84

Previous Years' Questions

Q.1 Five balls of different colors are to be placed in three boxes of different sizes. Each box can hold all five. In how many different ways can we place the balls so that no box remains empty? ***(1981)***

Q.2 7 relatives of a man comprises 4 ladies and 3 gentlemen, his wife has also 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relative and 3 of the wife's relatives? ***(1985)***

Q.3 A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box, if at least one black ball is to be included in the draw? ***(1986)***

Q.4 Eighteen guests have to be seated half on each side of a long table. Four particular guests desire to sit one particular side and three other on the other side. Determine the number of ways in which the sitting arrangements can be made. ***(1991)***

Q.5 A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be included in a committee? In how many of these committees

(a) the women are in majority?

(b) the men are in majority? ***(1994)***

Q.6 Match the conditions/expressions in column I with statement in column II.

Consider all possible permutations of the letters of the word ENDEANOEL. ***(2008)***

Column I	Column II
(A) The number of permutations containing the word ENDEA. is	(p) 5!
(B) The number of permutations in which the letter E occurs in the first and the last positions, is	(q) $2 \times 5!$
(C) The number of permutations in which none of the letters D, L, N occurs in the last five positions, is	(r) $7 \times 5!$
(D) The number of permutations in which the letters A, E, O occur only in odd positions, is	(s) $21\times5!$

Questions

JEE Main/Boards

Exercise 1

Q.6 Q.10 Q.13 Q.16 Q.22
Q.26 Q.27

Exercise 2

Q.3 Q.13 Q.15 Q.18 Q.25
Q.33 Q.43 Q.46 Q.47 Q.50
Q.52

Previous Years' Questions

Q.4 Q.5 Q.7

JEE Advanced/Boards

Exercise 1

Q.5 Q.6 Q.12 Q.13 Q.16
Q.20 Q.22 Q.28 Q.30

Exercise 2

Q.1 Q.3 Q.8 Q.13 Q.20
Q.26 Q.32 Q.39 Q.42 Q.43
Q.49 Q.55 Q.58

Previous Years' Questions

Q.1 Q.3 Q.4 Q.6

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $2 + 6 + 24 = 32$

Q.2 $3 \times 4 \times 3 \times 2 \times 1 = 72$

Q.3 $35 + 15 = 50$

Q.5 $n = 6$

Q.6 180

Q.7 365

Q.8 1332

Q.9 1023

Q.10 (i) $2^1 + 2^2 + 2^3 + 2^4 = 30$ (ii) $= 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 2 + 4 + 8 + 16 + 32 = 62$

Q.11 $34650 - 840 = 33810$

Q.12 4! 3!

Q.13 369600

Q.14 302399, 30240

Q.15 3600

Q.16 126

Q.17 $3 + 4\frac{1}{4} + 4 = 11\frac{1}{2}$ Hours.

Q.18 $r = 3$

Q.20 ${}^{52}C_4 = 270725$

Q.22 817190

Q.23 (i) 116 (ii) 40 (iii) 185

Q.24 31

Q.25 $\frac{50!}{(12!)^3.(7!)^2 3!}$

Q.26 26

Q.27 150

Q.28 2454

Exercise 2

Single Correct Choice Type

Q.1 C	**Q.2** C	**Q.3** D	**Q.4** A	**Q.5** B	**Q.6** C
Q.7 A	**Q.8** D	**Q.9** A	**Q.10** C	**Q.11** D	**Q.12** B
Q.13 D	**Q.14** B	**Q.15** A	**Q.16** A	**Q.17** A	**Q.18** D
Q.19 D	**Q.20** A	**Q.21** B	**Q.22** C	**Q.23** C	**Q.24** C
Q.25 D	**Q.26** A	**Q.27** A	**Q.28** C	**Q.29** B	**Q.30** D
Q.31 C	**Q.32** B	**Q.33** C	**Q.34** D	**Q.35** C	**Q.36** D
Q.37 A	**Q.38** B	**Q.39** A	**Q.40** D	**Q.41** C	**Q.42** C
Q.43 A	**Q.44** C	**Q.45** C	**Q.46** D	**Q.47** A	**Q.48** B
Q.49 D	**Q.50** D	**Q.51** C	**Q.52** B	**Q.53** B	**Q.54** B
Q.55 E	**Q.56** D				

Previous Years' Questions

Q.1 C	**Q.2** D	**Q.3** A	**Q.4** A	**Q.5** C	**Q.6** C
Q.7 C					

JEE Advanced/Boards

Exercise 1

Q.1 162 **Q.2** 7560 **Q.3** 2500 **Q.4** 826 **Q.5** 510 **Q.6** 10, 3

Q.7 60 **Q.8** 64 **Q.9** 244 **Q.10** 24 **Q.11** (i)15, (ii) 126, (iii) 60 (iv) 105

Q.12 (a) 72; 78120; (b) 23 (c) 32; (d) 435 **Q.13** 2252 **Q.14** 30980 **Q.15** 276

Q.16 440 **Q.17** 121 **Q.18** (a) 128; (b) 54 **Q.19** 48 **Q.20** 42

Q.21 3119976 **Q.22** (a) 1680; (b) 1140 **Q.23** 40 **Q.24** 974 **Q.25** 186

Q.26 (i) 3359; (ii) 59; (iii) 359 **Q.27** 4201 **Q.28** 3888 **Q.29** (i) 240, (ii) 15552

Exercise 2

Single Correct Choice Type

Q.1 D	**Q.2** C	**Q.3** C	**Q.4** B	**Q.5** A	**Q.6** A
Q.7 B	**Q.8** A	**Q.9** B	**Q.10** B	**Q.11** B	**Q.12** A
Q.13 D	**Q.14** D	**Q.15** D	**Q.16** A	**Q.17** B	**Q.18** B
Q.19 A	**Q.20** B	**Q.21** B	**Q.22** A	**Q.23** D	**Q.24** C
Q.25 A	**Q.26** B	**Q.27** B	**Q.28** C	**Q.29** B	**Q.30** C
Q.31 C	**Q.32** C	**Q.33** C	**Q.34** D	**Q.35** D	**Q.36** D
Q.37 C	**Q.38** A	**Q.39** C	**Q.40** B	**Q.41** A	**Q.42** D
Q.43 C	**Q.44** C	**Q.45** D			

Multiple Correct Choice Type

Q.46 A, B, D **Q.47** B, C **Q.48** B, C, D **Q.49** A, B **Q.50** B, D **Q.51** C, D

Comprehension Type

Q.52 C **Q.53** B **Q.54** B **Q.55** A **Q.56** C

Matric Match Type

Q.57 A → r; B → s; C → q; D → p **Q.58** A → t; B → r; C → p; D → q; E → s

Previous Years' Questions

Q.1 300 **Q.2** 485 **Q.3** 64 **Q.4** ${}^9P_4 \times {}^9P_3 \times (11)!$

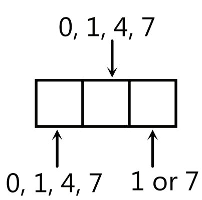
Q.5 6062, (a) 2702 (b) 1008 **Q.6** A → p; B → s; C → q; D → q.

Solutions

JEE Main/Boards

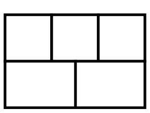
Exercise 1

Sol 1:



Total numbers = 4 × 4 × 2 = 32

Sol 2:



First arrange those 3 persons in rear seats

Then remaining in front.

Total ways to seat = 5!

Two particular people cannot seat on the driver self

So for this case total ways ⇒ 4! + 4!

Therefore, required number of ways = 5! – 4! – 4! = 4! (5 – 2) = 3 × 4 × 3 × 2 × 1 = 72

Sol 3: ${}^7C_1 \times {}^5C_1 + {}^3C_1 \times {}^5C_1 = 35 + 15 = 50$

Sol 4: $(2n + 1)! = 1 \times 2 \times 3........ (2n + 1)$

$= (2 \times 4 \times 6......2n)(1 \times 3 \times 5...... (2n + 1))$

$=2^n (1 \times 2 \times 3.....n)(1 \times 3 \times 5......(2n +1))$

$=2^n.n! (1 \times 3 \times 5........ (2n + 1))$

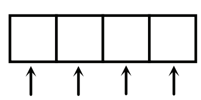
$\frac{(2n+1)}{n!} = 2^n. (1 \times 3 \times 5........ (2n + 1))$

Sol 5: ${}^nP_4 = 360$; $\frac{n!}{(n-4)!} = 360$

$n(n – 1)(n – 2)(n – 3) = 360 \Rightarrow 6 \times 5 \times 4 \times 3 = 360$

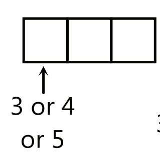
$n = 6$

Sol 6: Case-I: 4 digits



$2 \times {}^5C_3 \times 3! = 120$

Case-II: 3 digits

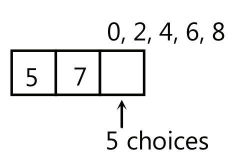


$3 \times {}^5C_2 \times 2! = 60$

Total number of numbers = 60 + 120 = 180

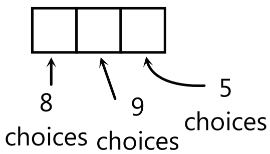
Sol 7: 5 can be there only in the thousand's digit

Case-I: 5 is there



Therefore, total number for case I = 5

Case-II: 5 is not there



Therefore, total number for case II = 8 × 9 × 5 = 360

Total ways = 5 + 360 = 365

Sol 8: 123+132+213+231+312+321 = 1332

Sol 9: $4^5 - 1 = 1024 - 1 = 1023$

One case is excluded when all arms are at rest.

Sol 10: (i) $2^1 + 2^2 + 2^3 + 2^4 = 30$

(ii) (a) $2^5 = 32$ (b) $2^1 + 2^2 + 2^3 \dots\dots\dots 2^5 = 62$

Sol 11: M I S S I S S I P P I

M (1)

S (4)

I (4)

P (2)

Ways = Total permutations – Permutations with 4 I's together

$$= \frac{11!}{4!\ 4!\ 2!} - \frac{8!}{4!\ 2!} = 33810$$

Sol 12: ↓ ↓ ↓ ↓ B B B B

| G | G | G |

× × ×

first arrange girls → 3!

4 seats remains in which 4 boys will be

seated → 4!

Total ways = 3! × 4! = 144

Sol 13: $\frac{12!}{(3!)^4} = 369600$

Sol 14: BENEVOLENT

No. of letters = 10

B (1), E (3), N (2), V (1), O (1), L (1), T (1)

Total rearrangements = $\frac{10!}{3!\ 2!} - 1 = 302400 - 1 = 302399$

Total rearrangements with L in the end = $\frac{9!}{3!\ 2!} = 30240$

Sol 15: PATALI PUTRA

Vowels = 5

Consonants = 6

Vowels: A(3) I(1) U(1)

Consonants: P(2) T(2)L(1)R(1)

Total no. of words = $\frac{5!}{3!} \times \frac{6!}{2!\ 2!} = 3600$

Sol 16: $\frac{9!}{4!\ 5!} = 126$

Sol 17: 3! ways

A → III

B → I

C → II

Time = $4 + 4\frac{1}{4} + 3 = 11\frac{1}{4}$ hours

Sol 18: ${}^{15}C_{3r} = {}^{15}C_{r+3}$

(1) $3r = r + 3 \Rightarrow r = \frac{3}{2}$

(2) $3r + r + 3 = 15 \Rightarrow 4r = 12 \Rightarrow r = 3$

$r = \frac{3}{2}$ Not possible

r = 3 Possible

Sol 19: ${}^{n}C_{r} \times {}^{r}C_{s} = {}^{n}C_{s} \times {}^{n-s}C_{r-s}$

LHS: ${}^{n}C_{r} + {}^{r}C_{s} = \frac{n!}{r!(n-r)!}, \frac{r!}{s!(r-s)!}$

$$= \frac{n!}{s!} \times \frac{(n-s)!}{(n-s)!} \times \frac{1}{(n-r)!.(r-s)!} = \frac{n!}{s!(n-s)!} \times \frac{(n-s)!}{(n-r)!(r-s)!}$$

$= {}^{n}C_{s} \times {}^{n-s}C_{r-s} = \text{RHS}$

Sol 20: ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$

$= \underbrace{{}^{47}C_4 + ({}^{47}C_3}\ + {}^{48}C_3 \dots\dots {}^{51}C_3 = {}^{48}C_4 + ({}^{48}C_3 + \dots\dots + {}^{51}C_3)$

$= {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4 = 270725$

Sol 21: Product = (n + 1)(n + 2) ...... (n + r)

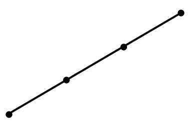
$$= \frac{(n+r)!}{n!} = r! \times \frac{(n+r)!}{n!\ r!} = r! \times {}^{n+r}C_r$$

${}^{n+r}C_r$ will be integer

Hence product is divisible by r!.

Sol 22: ${}^{22}C_{10} + {}^{3}C_{3}\ {}^{22}C_{7} = 817190$

Sol 23:



(i) No. of triangles = ${}^{10}C_3 - {}^{4}C_3 = 116$

(ii) No. of straight lines = ${}^{6}C_2 + 6 \times 4 + 1$

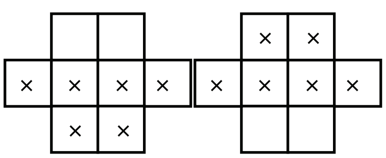
= 15 + 24 + 1 = 40

(iii) No. of quadrilaterals = ${}^{10}C_4 - {}^{4}C_3 \times 6 - {}^{4}C_4 = 185$

Sol 24: $2^5 - 1 = 31$

Sol 25: $\dfrac{50!}{(12!)^3\, 3!\ (7!)^2\, 2!}$

Sol 26: Cases not allowed



${}^{8}C_6 - 2 = 26$

Sol 27: Possible groups

1 1 3

1 2 2

No. of ways

$$= 3! \times \left(\frac{5!}{1!\ 1!\ 3!\ (2!)} + \frac{5!}{1!\ 2!\ 2!\ (2!)}\right) = 60 + 90 = 150$$

Sol 28: EXAMINATION

4 DIFF

${}^{8}C_4 \times 4! = 1680$

2 diff. 2 alike

${}^{3}C_1 \times {}^{7}C_2 \times \dfrac{4!}{2!} = 756$

2A, 2A

${}^{3}C_2 \times \dfrac{4!}{2!\ 2!} = 18$

Total = 2454

Exercise 2

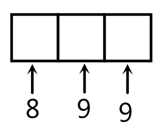
Single Correct Choice Type

Sol 1: (C)

A N R U V
to
N A R U V   4!
to
R A N U V   4!
to
U A N R V   4!
to
V A N R U   4!   96
V A N R U 97
V A N U R 98
V A R N U 99
V A R U N 100

Sol 2: (C) Numbers = Total – Numbers with no digit 7

Total = 900



Number of numbers with at least

One digit 7 = 900 – 8 × 9 × 9 = 252

Sol 3: (D)

A E H M S
to
A S M H E   24 — 24
E A H M S
to
E S M H A   24 — 48
H A E M S
to
H S M E A   24 — 72
M A E H S — 73
to
M A S H E   6 — 78
M E A H S
to
M E S H A   6 — 84
M H A E S
M H A S E — 86

Sol 4: (A)

4	3	3	3	3	3

Y, G, B, R

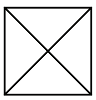
$4 \times 3^5 = 12 \times 81$

Sol 5: (B) Prime number in 0 – 9

2, 3, 5, 7

	1		1		1		1		1
4		4		4		4		4	

$2 \times 4^5 = 2048$

Sol 6: (C) 

${}^5C_3 - 2 = 10 - 2 = 8$

Sol 7: (A)

5	5	5

No. 5 with no even digit

$900 - 5^3 = 775$

Sol 8: (D) The sum of the 5-digits used must be divisible by 3.

Only possible combinations are:

1, 2, 3, 4, 5&0, 1, 2, 4, 5

↓↓

5!　　　　4.4!

Total = 9.4!

= 9 × 24 = 216

Sol 9: (A) ${}^7C_2 \times 2^5 = 672$

Sol 10: (C) ${}^7C_4 \times {}^4C_2 \times 6! = 151200$

Sol 11: (D) 5 can be there only in the thousand's digit

Case-I: 5 is there

		0, 2, 4, 6, 8
5	7	
		5 choices

5

Case-II: 5 is not there

8 choices	9 choices	5 choices

$8 \times 9 \times 5 = 360$

Total ways = 5 + 360 = 365

Sol 12: (B)

				5
8	8	7	6	

5	4	3	2	0

${}^4C_1 \times 8 \times 7 \times 6$

Total ways = $8 \times 8 \times 7 \times 6 + {}^4C_1 \times 8 \times 7 \times 6 = 168 \times 4!$

Sol 13: (D) 4 odd, 3 even

Arrangements: 4 + 2 3 + 3

Total numbers = ${}^3C_2 \times 2 \times 4! + {}^4C_3 \times {}^3C_2 \times 2 \times 4! = 720$

Sol 14: (B) $9000 - 9 \times 9 \times 8 \times 7 = 4464$

Sol 15: (A) It can be a digit number with digits 1 to 9 or a 10 digit number with digits 0 to 9

9! + 9.9! = 10!

Sol 16: (A)

n		
	n	n

$$\sum_{n=1}^{9} n^2 = \frac{9 \times 10 \times 19}{6} = 285$$

Sol 17: (A) Consider 1 & 2, 3 & 4, 5 & 6 to be identical

Permutations = $\dfrac{9!}{(2!)^3} = 9.7!$

Sol 18: (D)

Last digit 9

			9
7	8	7	

Last digit not 9

6	8	7	4

Total = $7 \times 8 \times 7 + 6 \times 8 \times 7 \times 4 = 1736$

Sol 19: (D)

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

× × × × × × × × × ×

$\frac{10 \text{ persons}}{{}^{10}C_5 \ . \ 5!}$

Choose 5 persons from 10 getting books & distribute books to then 5! ways .

Sol 20: (A) ${}^5C_3 \times {}^8C_7 + {}^5C_4 \times {}^8C_6 + {}^5C_5 \times {}^8C_5$

$= 80 + 5 \times 28 + 56 = 276$

Sol 21: (B) $\underbrace{\boxed{}\boxed{}}_{9}\underset{9}{\boxed{\uparrow}}$ $\quad$ $\underset{9}{\boxed{\uparrow}}\underbrace{\boxed{}\boxed{}}_{9}$

Total = 162

Sol 22: (C) $\frac{(n-2)}{n} \times 180 = 150$

$6n - 12 = 5n \Rightarrow n = 12$

Diagonals $= {}^{12}C_2 - 12 = 54$

Sol 23: (C) $2^n - 2 = 510$

$2^n = 512$

$n = 9$

Sol 24: (C) $\boxed{7}\underset{9}{\boxed{\uparrow}}\underset{8}{\boxed{\uparrow}}\underset{7}{\boxed{\uparrow}}$ $\quad$ $\underset{8}{\boxed{\uparrow}}\underset{8}{\boxed{\uparrow}}\boxed{7}\underset{7}{\boxed{\uparrow}}$

$9 \times 8 \times 7 + 8 \times 8 \times 7 \times 3 = 33 \times 8 \times 7 = 1848$

Sol 25: (D) Total appointment = 11

$11 \rightarrow \boxed{2}\ \boxed{3}\ \boxed{6}$

total ways to disposed = (2! 3! 6!) × 3 = 8640 × 3= 25920

Sol 26: (A) Illegal ways $= 2 \times {}^7C_4 + {}^7C_3 - 2 \times {}^5C_2 = 85$

No. of possible ways $= {}^9C_5 - 85 = 126 - 85 = 41$

Sol 27: (A) 1 1 3

1 2 2

$3 \times ({}^4C_1 \times {}^4C_1 \times {}^4C_3 + {}^4C_1 \times {}^4C_2 \times {}^4C_2)$

$= 3 \times (64 + 144) = 624$

Sol 28: (C) Combine those 2 green bottles

× × × × × × × ×

9 spaces

$6 \times {}^9C_6 = 84 \times 6 = 504$

Sol 29: (B) ${}^{25}C_5 - {}^{24}C_4 = {}^{24}C_5$

Sol 30: (D) ${}^5C_4 \times 2^4 = 80$

Select 4 pairs out of 5 different pairs. Now in each pair you can choose 2 different shoes.

Sol 31: (C) 2 2 2 3

$\frac{3!\ 9!}{3!\ (2!)^3\ 3!} = \frac{9!}{3!\ (2!)^3}$

Sol 32: (B) $\underbrace{\times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times}_{\text{10 red balls}}$

11 spaces

choose 9 spaces to fill green balls. 10! ways to arrange red balls.

${}^{11}C_9 \times 10!$

Sol 33: (C) $7! \times 2^3$

7! ways to arrange

2 order (ascending/descending)

Sol 34: (D) $\frac{6!}{2!\ 2!\ 2!}$

Consider man and wife to be identical and arrange them.

Similar concept as used in Q. 18

Sol 35: (C) $\frac{7!}{7!\ 0!} + \frac{7!}{6!\ 1!} + \frac{7!}{5!\ 2!} \ldots\ldots \frac{7!}{2!\ 5!}$

${}^7C_0 + {}^7C_1 + {}^7C_2 \ldots\ldots {}^7C_5 = 2^7 - {}^7C_6 - {}^7C_7 = 128 - 7 - 1 = 120$

Sol 36: (D) ↓ × ↓ × ↓ × ↓ × ↓

EE AA

EAEA

AEAE

EAAE

$4! \times ({}^5C_2 \times (2! + 2! + 1 + 1)) = 1440$

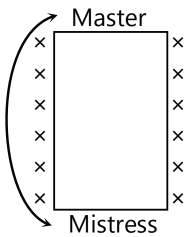
Sol 37: (A) The last match has to be won by India

$1 + {}^{5}C_1 + {}^{6}C_2 + {}^{7}C_3 + {}^{8}C_4 = 126$

Sol 38: (B) $\underbrace{\boxed{}\boxed{}\boxed{}}_{635/674}\boxed{}\boxed{}\boxed{}\boxed{}$

$2(9^3 \times 1 + 3 \times 4 \times 9^2) = 2(1701) = 3402$

Sol 39: (A)



$2 \times 10 \times 10! = $ Sol.20 10!

Sol 40: (D) $S_1\ S_3\ S_2$

$S_3S_1S_2$

Consider $S_1\ S_2\ S_3$ identical & arrange

$2 \times \frac{10!}{3!} = \frac{10!}{3}$

Sol 41: (C) $3^{10} - 1$

3 choices question its alternative & no question.

Sol 42: (C) 9 9 9 9 9 9 7

9 9 9 9 9 8 8

$\frac{7!}{6!} + \frac{7!}{5!\ 2!}$ f= 7 + 21 = 2s

Sol 43: (A) $\frac{9!}{2!\ 3!} - 4!\ 3! = 6(7! - 4!)$

Sol 44: (C) $1000 = 2^3 \times 5^3$

The product of even divisors of 1000 will be

$$= \left(2\times 2^2\times 2^3\right)\times\left(2\times 5\right)\times\left(2\times 5^2\right)\times\left(2\times 5^3\right)\times\left(2^2\times 5\right)$$

$$\times\left(2^2\times 5^2\right)\times\left(2^2\times 5^3\right)\times\left(2^3\times 5\right)\times\left(2^3\times 5^2\right)\times\left(2^3\times 5^3\right)$$

$= (2^6)^4 \times (5 \times 5^2 \times 5^3) = (2^6)^4 \times (5^6)^3 = 64 \times 10^{18}$

Sol 45: (C) $x_1 + x_2 + \ldots\ldots\ldots x_{10} = 7$

7 people are distributed to 10 floor

Total ways $= {}^{7+10-1}C_{10-1} = {}^{16}C_9 = {}^{16}C_7$

Sol 46: (D) Total ways $= 3^n + {}^{n}C_2\,.3^{n-2} + {}^{n}C_4 . 3^{n-4} + \ldots\ldots$

$= \frac{(1+3)^n}{2} + \frac{2^n}{2} = 2^{2n-1} + 2^{n-1} = 2^{n-1}(2^n + 1)$

Sol 47: (A) Each digit will be present at unit's ten's and hundred's place 6 times.

$(1 + 2 + 3 + 4) \times 6 = 60$

$$\text{Sum of digits} = \begin{array}{r} 60 \\ 60\times \\ \underline{60\times\times} \\ 6660 \end{array}$$

Sol 48: (B) ${}^{8}C_1 + {}^{8}C_1 \times {}^{7}C_1 = 64$

Sol 49: (D) No. of books of algebra = No of books of calculus

Sol 50: (D) No. of ways = Total ways – No letter is in wrong envelope = 5! – 1 = 119

Sol 51: (C) $P_1\ P_2\ P_3\ P_4$

$\frac{4!}{(2!)^2 2!} = 3$

Total matches $= {}^{8}C_4 \times 3$

Sol 52: (B) ${}^{9}C_6 \times 5 \times 3! = 2520$

Sol 53: (B) ${}^{6}C_1 \times 2^1 \times 2^5 + {}^{6}C_3 \times 2^3 \times 2^3 + {}_6C^5$

$\times\, 2^5 \times 2^1 = 2^6\,({}^{6}C_1 + {}^{6}C_3 + {}^{6}C_5)$

$= 2^6 \times \frac{2^6}{2} = 2^{11}$

Sol 54: (B) 2.6.10 .... (4n – 2)

$= 2^n(1.3.5 \ldots\ldots (2n-1)) \times \frac{n!}{n!} = \frac{(2n)!}{n!} = {}^{2n}P_n$

Sol 55: (E) No. of permutations

$= {}^{2n}C_n = \frac{(2n)!}{n!\ n!} = \frac{2^n\ .\ n![1.3.5\ldots\ldots(2n-3)(2n-1)]}{n!}$

Sol 56: (D) n is odd

$1, 2, 3 \ldots\ldots\ldots \frac{n+1}{2} \ldots\ldots n$

Total A.P.S. = 0 + 1.....+ $\frac{n-1}{2}$ +....+1+ 0 = $\frac{(n-1)^2}{4}$

n is even

1, 2, ... $\frac{n}{2}$, $\frac{n}{2}$ + 1, ... n

Total A.P.S

= 2 × $\left(0+1+.......\frac{n}{2}-1\right)$ = $\frac{n(n-2)}{4}$

Previous Years' Questions

Sol 1: (C) Here, $^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$

= $^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{47}C_3$

= $({}^{47}C_4 + {}^{47}C_3) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

(using $^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$)

= $({}^{48}C_4 + {}^{48}C_3) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

= $({}^{49}C_4 + {}^{49}C_3) + {}^{50}C_{43} + {}^{51}C_3$

= $({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3 = {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4$

Sol 2: (D) Since, the first 2 women select the chairs amongst 1 to 4 in $^{4}P_2$ ways

Now, from the remaining 6 chairs, three men could be arranged in $^{6}P_3$.

∴ Total number of arrangements=$^{4}P_2 \times {}^{6}P_3$.

Sol 3: (A) Since, a five digits number is formed using the digits {0, 1, 2, 3, 4 and 5} divisible by 3 ie, only possible when sum of the digits is multiple of three.

Case I : Using digits 0, 1, 2, 4, 5

Number of ways = 4 × 4 × 3 × 2 × 1 = 96

Case II : Using digits 1, 2, 3, 4, 5

Number of ways = 5×4×3×2×1 = 120

∴ Total number formed =120+96=216

Sol 4: (A) Since, 240 = 2^4 × 3 × 5

∴ Total number of divisors = (4 +1) (2) (2) = 20

Out of these 2, 6, 10, and 30 are of the fomr 4n + 2.

Therefore, (a) is the answer.

Sol 5: (C) Since, r, s, t are prime numbers.

∴ Selection of p and q are as under

Pq Number of ways

R^0r^2 1 way

$R^1r^2$1 way

R2r^0, r^1, $r^2$3 ways

∴ Total number of ways to select r = 5.

Selection of s as under

$s^0s^4$1 way

$s^1s^4$1 way

$s^2s^4$1 way

$s^3s^4$1 way

$s^4s^4$1 way

∴ Total number of ways to select s = 9.

Similarly, the number of ways to select t = 5.

∴ Total number of ways = 5 × 9 × 5 = 2Sol.25

Sol 6: (C) Arrange the letters of the word COCHIN as in the order of dictionary CCHINO.

Consider the words starting from C.

There are 5! Such words. Number of words with the two C' s occupying first and second place = 4!.

Sol 7: (C) There are two possible cases

Case-I Five 1's , one 2's, one 3's

Number of numbers = $\frac{7!}{5!}$ = 42

Case-II Four 1's three 2's

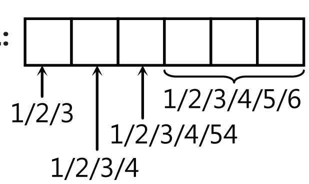
Number of numbers = $\frac{7!}{4!\,3!}$ = 35

Total number of numbers = 42 + 35 = 77

JEE Advanced/Boards

Exercise 1

Sol 1:



$\underbrace{3\times3\times3\times3}_{\text{Filling first 4 places}} \times \underbrace{2}_{\text{last two places}} = 81 \times 2 = 162$

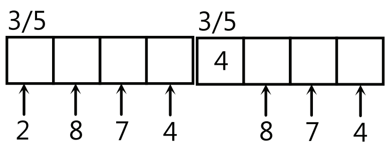
Sol 2: ${}^9C_1 \times {}^8C_2 \times \dfrac{5!}{2!\ 2!} = 9 \times 28 \times \dfrac{120}{84} = 7560$

Sol 3: $3^1\ 3^2\ 3^3$ .... 3^{51} .... $3^{98}\ 3^{99}\ 3^{100}\ 3^{101}$

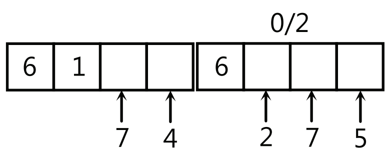
Fix the middle element of G.P. the find the number of G.P.S possible

$0 + 1 + 2 + \ldots\ldots + 49 + 50 + 49 + \ldots\ldots + 1 + 0 = 2500$

Sol 4:



$2 \times 8 \times 7 \times 4 = 448\ 8 \times 7 \times 5 = 280$



$7 \times 4 = 282 \times 7 \times 5 = 70$

Total = 826

Sol 5: Case-I: 2 2 2

$\dfrac{6!}{(2!)^3} = 90$

Case-II: 1 2 3

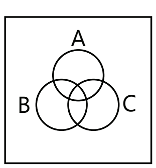
$\dfrac{6!}{1!\ 2!\ 3!} \times 3! = 360$

Case-III: 0 3 3

$\dfrac{6!}{(3!)^2\ 2!} \times 3! = 60$

Total = 90 + 360 + 60 = 510

Sol 6:



$n(A) = 8! = n(B) = n(C)$

$n(A \cap B) = n(B \cap C) = n(A \cap C) = 7!$

$n(A \cap B \cap C) = 6!$

Sol 7: $3 \times n(A \cap B) - 3n(A \cap B \cap C)$

$3 \times 7! - 3 \times 6!$

$= (63 - 3)6! = 60(6!)$

Sol 8: $n(A) - 2n(A \cap B) + 2n(A \cap B \cap C)$

$= 8! - 2 \times 7! + 6! = (56 + 14 + 1$

$6! = 43\ 6!$

Sol 9: $9! - n(A \cup B \cup C)$

Sol 10: $\dfrac{360}{k}$ Should be an integer.

$\therefore$ k is a factor of 360

$360 = 3^2 \times 2^3 \times 5$

Total factors = 4 × 3 × 2= 24

Sol 11: (i) ${}^6C_4 = 15$

(ii) All, 4 diff. ${}^6C_4 = 15$

2diff, 2Alike ${}^6C_2 \times {}^4C_1 = 60$

2 alike, 2alike${}^6C_2 = 15$

1 diff, 3alike $2 \times {}^6C_2 = 30$

4 alike ${}^6C_1 = 6$

126

(iii)3 different flavours = 60

(iv) 2 or 3 different flavours =60 +15 + 30 = 105

Sol 12: (a) x = 21600

$x = 6^3 \times 100 = 2^5\ 3^3\ 5^2$

No. of divisors = 6 × 4 × 3 = 72

Sum = $(2^0 + 2^1 + \ldots + 2^5)$

$(3^0 + 3^1 + \ldots 3^3)(5^0 + 5^1 + 5^2) = 60 \times 40 \times 31 = 78120$

(b) x = 7056

$x = 2^4 \times 3^2 \times 7^2$

Total factors = 5 × 3 × 3 = 45

Answer = $\dfrac{45+1}{2} = 23$

(c) $300300 = 7 \times 11 \times 13 \times 10^2 \times 3$

$= 2^2 \times 3 \times 5^2 \times 7 \times 11 \times 13$

$\dfrac{2^6}{2} = 32$

(d) $10^{10}\ 15^7\ 18^{11}$

$2^{10} \times 5^{10},\ 3^7 \times 5^7,\ 2^{11} \times 3^{22}$

HCF of 10^{10}, 15^7 & $18^{11} = 1$

HCF of 10^{10} & $15^7 = 5^7$

HCF of 10^{10} & $18^{11} = 2^{10}$

HCF of 15^7 & 18^{11} = 3^7

Total divisors = $(11\times11+8\times8+12\times23)-(8+11+8)+1=435$

Sol 13: $\sum_{x=0}^{5} ({}^5C_x)^3 = 1 + 5^3 + 10^3 + 10^3 + 5^3 + 1 = 2252$

Sol 14: ${}^{18}C_9 - {}^{10}C_5 \times {}^8C_4 = 48620 - 252 \times 70 = 30980$

Sol 15: $5^2 - 1 = 24 \quad \therefore {}^{24}C_2$ triangles

Sol 16: $x + y + 2 + w = 13$

$x \le 5y \ge 2$

${}^{10}C_2 + {}^{11}C_2 + ... {}^{15}C_2 = 45 + 55 + 66 + 78 + 91 + 105 = 440$

Sol 17: 2 digits should be odd or

2 digits should be even for average to be integer

111 222 .... 999 each repeats 3 times

$${}^3C_2 \times \underbrace{{}^5C_1 \times {}^5C_1}_{\text{odd}} + {}^3C_2 \times \underbrace{{}^4C_1 \times {}^4C_1}_{\text{even/0}} + \underbrace{{}^2C_1 \times 4 \times 3}_{\text{zero}} - \underbrace{2 \times 9}_{\text{repetition}} = 121$$

Sol 18: (a) 1 element ${}^{12}C_1 = 12$

2 elements ${}^{10}C_2 = 45$

3 elements ${}^8C_3 = 56$

4 elements ${}^6C_4 = 15$

5 elements 0

Total = 128

(b) Select 3 elements

Each is this element is either present in S, T or both S and T

$\therefore$ Total = ${}^6C_3 \times 3^3 = 540$

$\lambda = 54$

Sol 19: Sum at first places can be 6 at max.

4 3 1 2 5 not allowed

	First 2 digits	Last 2 digits	Permutations
Sum 6:	15	43	4
	24	53	4
Sum5:	14	53/52	8
	23	54/51	8
Sum4:	13	54/52/42	12
Sum3:	12	54/53	12
		Total	48

Sol 20: Total wins = Total matches = $10 \times {}^8C_2$

Wins of champion = A + 7d

$\frac{8}{2}[2A + 7d] = 280$

$A = \frac{70 - 7d}{2}$

Now $d \neq 0$ d = 2

A = 28

Wins of champion = $28 + 7 \times 2 = 42$

Sol 21:

S_1	S_2	S_3	S_4	S_5

$S_1 = (1 + 2 + 4 + 5) \times 4! = 288$

$S_2 = S_3 = S_4 = S_5 = (1 + 2 + 4 + 5) \times 3 \times 3! = 216$

Sum = $S_1 \times 10^4 + S_2(10^3 + 10^2 + 10 + 1)$

$= 288 \times 10^4 + 216(1111) = 3119976$

Sol 22: (a) ${}^8C_3 \times {}^5C_3 \times 3! = 1680$

(b) ${}^3C_1 \frac{5!}{3!\ 2!}\ 2 + {}^3C_2 \times 3 \times 2 \times \frac{5!}{1!\ 2!\ 2!} \times 2!$

$= 60 + 1080 = 1140$

a, b, c distinct

Sol 23: $\{1, x, x\}\ ({}^5C_1 + {}^5C_2) = 15$

$\{x, x, x\}\ \frac{5!}{1!\ 1!\ 3!\ 2!} + \frac{5!}{1!\ 2!\ 2!\ 2!} = 25$

Total = 15 + 25 = 40

Sol 24: No. of integers

2 digit no. 89, 98 2

3 digit no.

Zero included 890, 809 4

980, 908

Zero excludes

$2\times {}^3C_2 + {}^7C_1 \times 3! 48$

4 digit no.

Tow zeroes ${}^3C_2 \times 26$

One zero ${}^3C_1 \times ({}^7C_1 \times 3! + 6) 144$

No zero $2 \times {}^4C_2 \times 7 \times 7 + 2 \times {}^4C_2 \times 2 \times 7\ 770 + 2 \times {}^4C_3 + {}^4C_2 = 974$

Sol 25: A – 1B – 9 1

A – 2 B – $8{}^8C_1$

A – 3B – $7{}^8C_2$

A – 4B – $6{}^8C_3$

A – 5B – 50

N = $2(1 + {}^8C_1 + {}^8C_2 + {}^8C_3)$

N = 186

Sol 26: (i) $M^U\ LT^I\ PL^E$

$$\frac{8!}{3!\ 2!} - 1 = 3359$$

(ii) $\frac{5!}{2!} - 1 = 59$

(iii)$3! \times \frac{5!}{2!} - 1 = 359$

Sol 27: 3 digits $^5C_3 \times \frac{9!}{3!\ 3!\ 3!} = 16,800$

2 digits $2 \times {}^5C_2 \times \left(\frac{9!}{3!\ 6!} + \frac{9!}{4!\ 5!}\right) = 4,200$

1 digit $^5C_1 = 5$

Total = 21005

$\frac{N}{5} = 4201$

Sol 28: $^3C_2 \times 9 \times 8 + 9 \times {}^3C_2 \times 8 \times 8 + 9 \times 3 \times 9 \times 8 = 3888$

Sol 29: (i) $4 \times 4! + 2 \times 3 \times 4! = 240$

(ii) $4 \times 6^4 \times 3 = 15552$

Exercise 2

Single Correct Choice Type

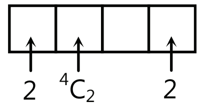
Sol 1: (D) We have 0 + 1 + 2 + 3 ... + 8 + 9 = 45

To obtain an eight digit number exactly divisible by 9, we must not use either (0, 9) or (2, 7) or (3, 6) or (4, 5). [Sum of the remaining eight digits is 36 which is exactly divisible by 9.]

When, we do not use (0, 9), then the number of required 8 digit number is 8!.

When, one of (1, 8) or (2, 7) or (3, 6) or (4, 5) is not used, the remaining digits can be arranged in 8! – 7! ways as 0 cannot be at extreme left.

Hence, there are 8! + 4(8! – 7!) = (36) (7!) numbers in the desired category.

Sol 2: (C) 

$2 \times {}^4C_2 \times 2 = 24$

Sol 3: (C) $^{20}C_2 - 10 - 5 \times 9 = 190 - 10 - 45 = 135$

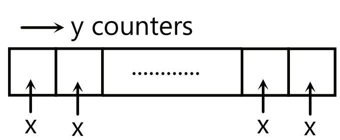
Sol 4: (B) $r = {}^9C_2 \times {}^9C_2 = 36^2$

$s = 8_2 + 7^2 + 6^2 + \ldots\ldots\ 1^2 = \frac{8 \times 9 \times 17}{6}$

$\frac{s}{r} = \frac{17}{108}$

Sol 5: (A) 200,000,000

$2^8 + 2^8 + 2^7 + 2^6 + \ldots\ldots\ 2^1 = 2^8 + 2.\left(\frac{2^8 - 1}{2 - 1}\right) = 3.\ (2)^8 - 2$

Sol 6: (A) 

$x^y - y$

Sol 7: (B) $M - n = {}^{10}C_5 - {}^9C_5 = {}^9C_5$

Sol 8: (A) $N_1 = {}^{m+n}C_3 - {}^mC_3 - {}^nC_3$

$N_2 = {}^{m+n+1}C_3 - {}^{m+1}C_3 - {}^{n+1}C_3$

$\frac{N_1}{N_2} = \frac{m+n-2}{m+n}$

Sol 9: (B) 10000 to 99999

Number of numbers = $\frac{90000}{2} = 45000$

Sol 10: (B) $\boxed{AB}$ D C E F G H

$2 \times \frac{7!}{2!} = 5040$

Sol 11: (B) $\frac{7!}{1!\ 2!\ 4!} \times 3! = 630$

Sol 12: (A) $^9C_4 = 126$

Sol 13: (D) Case-I: 1 3 4

$$\frac{8!}{1!\ 3!\ 4!} \times 3!$$

Case-II: 2 3 3

$$\frac{8!}{2!\ 3!\ 3!\ 2!} \times 3!$$

Case-III: 2 2 4

$$\frac{8!}{2!\ 2!\ 4!\ 2!} \times 3!$$

Total = 4620 = 22 × $^{7}P_{3}$

Sol 14: (D) Pack 1 Pack 2

Peaches Apples Peaches Apples

3 1 2 2

$^{5}C_{3} \times {}^{3}C_{1} = 30$

Sol 15: (D) $\underbrace{\downarrow \times \downarrow \times \downarrow \times \downarrow \times \downarrow}_{n-3}$

(n – 2) choices. select 3

$P_n = {}^{n-2}C_3$

$P_{n+1} - P_n = {}^{n-1}C_3 - {}^{n-2}C_3 = {}^{n-2}C_2 = 15$

$n - 2 = 6 \Rightarrow n = 8$

Sol 16: (A) $n(2) = 50 \quad n(2 \cap 3) = 16$

$n(3) = 33 \quad n(3 \cap 5) = 6 \quad n(2 \cap 3 \cap 5) = 5$

$n(5) = 20 \quad n(2 \cap 5) = 10$

$n(2 \cup 3 \cup 5) = (50 + 33 + 20) - (16 + 6 + 10) + 3$

$n(\overline{2 \cup 3 \cup 5}) = 100 - 74 = 26$

Sol 17: (B) $^{5}C_{1} \times \frac{6!}{2!} = 1800$

Choose subject with two periods and then arrange.

Sol 18: (B) $\underbrace{\boxed{\bullet}\boxed{\bullet}\boxed{\bullet}\boxed{\bullet}}_{^{4}C_{2} \times 2} \times \underbrace{\boxed{\ }}_{2} \times \underbrace{\boxed{\ }\boxed{\ }}_{10}$

240

Sol 19: (A) Rectangles = $^{7}C_{2} \times {}^{5}C_{2} = 21 \times 10 = 210$

Squares = 6 × 4 + 5 × 3 + 4 × 2 + 3 ×1

= 24 + 15 + 8 + 3 = 50

Rectangles which are not squares = 210-50 = 160

Sol 20: (B)

$$\underline{1}\_\_\_\_\_\_ \begin{pmatrix} \{15+10+6+3+1\}+ \\ \{10+6+3+1\} \\ +\{6+3+1\}+\{3+1\}+1 \end{pmatrix} = 70$$

$2\underline{3}\_\_\_\_\_ \ 10+6+3+1 = 20+70 = 90$

$2\ 4\ 5\_\_\_\_ \ 3+2+1 = 6+90 = 96$

$2\ 4\ 6\ 7\ 8 = 97^{th}$

Sol 21: (B) $^{n}C_{n} + {}^{n+1}C_{n} + {}^{n+2}C_{n} + \ldots\ldots\ {}^{n+m}C_{n}$

$= {}^{n}C_{0} + {}^{n+1}C_{1} + {}^{n+2}C_{2} + \ldots {}^{n+m}C_{m} - {}^{n+1}C_{0} + {}^{n+1}C_{0} = {}^{n+m+1}C_{m}$

Sol 22: (A) $\frac{1}{2} \times \frac{1}{2} \times {}^{m+n}C_{m}(m-1)(n-1)!$

$$= \frac{1}{4} \times \frac{(m+n)!}{mn}$$

Sol 23: (D) $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \ {}^{3}C_{1} = 3$

$\begin{vmatrix} 1 & x & x \\ x & 0 & x \\ x & x & 0 \end{vmatrix} \ {}^{3}C_{1} \times {}^{3}C_{2} = 9$

Total = 12

Sol 24: (C) 5 3 or 4 4

$^{8}C_{5} + {}^{8}C_{4} = 126$

Sol 25: (A) 100 = (80 + 50 + 30) – n + 5

n = 65

People reading exactly 2 magazines

= 65 – 3 × 5 = 50

Sol 26: (B) 2(5! – 4! × 2) = 144

Sol 27: (B)

Degree of 5 in prime factorization of $^{2002}C_{1001} = 1$

$^{2002}C_{1001}$ is clearly divisible by 2

Sol 28: (C) No. of triangles = $^nC_3 - [(n(n-4)) + n]$

$= \frac{n(n-1)(n-2)}{3!} - [(n^2 - 3n)] = 30 \Rightarrow n = 9$

Sol 29: (B) $^{11}C_3 - {^5C_3} + 1 = 156$

Sol 30: (C)

			2	5

			5	10

$3! + 3 \times 2 \times 2!\ {^4C_3} \times 3! = 18 = 24$

Total no.s = 18 + 24 = 42

Sol 31: (C) $P_n - Q_n$

The difference in P_n & Q_n is the number of ways in which first and last person of the row is selected

$P_n - Q_n = n - 4 = 6$

$n = 10$

Sol 32: (C) $m = {^{10}C_4} \times 4!$

$n = {^{10}C_4}$; $m = 24\ n$

Sol 33: (C) No. of ways of selecting a committee of 2W and 3M from 5W and 6M

$= {^4C_1} \times {^4C_2} + {^5C_2} \times {^5C_2} = 124$

Sol 34: (D) ABC, ACD, ABD

$3! + 3! + 3! = 18$

Sol 35: (D) Ways in which 2 are neighbours

$= 2 \times 97 + 97 \times 96 = 98 \times 97$

ways in which all 3 are neighbours = 98

Required ways = $^{100}C_3 - 98^2 = 152096 = {^{98}C_3}$

Sol 36: (D) Make 3 groups of boys 1, 1, 2

$\frac{4!}{2!\ (11)^2\ \ 2!} \times 3!$ ways to distribute

Identical marbles distribution

1 2 2– 3 ways

1 1 3 – 3 ways

Total ways = $\frac{4!}{2!\ 2!} \times 3! \times 6 = 216$

Sol 37: (C) S_1 S_2 S_3

$S_1 = S_3 = (2 + 4 + 6 + 8) \times 10 = 200$

$S_2 = (0 + 1 + 2 + \ldots\ldots 9) \times 4 = 180$

Sum = $200 + 180 \times 10 + 200 \times 100 = 22000$

Sol 38: (A) $A_n = {^nC_{25}} \times 25!$

$A_n - A_{n-1} = 25!\ ({^nC_{25}} - {^{n-1}C_{25}}) = 25!({^{n-1}C_{24}})$

$n = 50$

Sol 39: (C) $(x_1 + x_2 + x_3)(y_1 + y_2) = 77 = 7 \times 11$

$x_1 + x_2 + x_3 = 4$ & $x_1 + x_2 + x_3 = 8$

$y_1 + y_2 = 9 y_1 + y_2 = 5$

No. of possible solutions

$= {^6C_2} \times {^{10}C_1} + {^{10}C_2} \times {^6C_1} = 15 \times 10 + 45 \times 6$

$= 150 + 270 = 420$

Sol 40: (B) $m = 34 - 3 = 78$

$n = 3^4 - [3(2^4 - 2) + 3] = 81 - 45 = 36$

Hence, $\frac{m}{n} = \frac{78}{36} = \frac{13}{6}$

Sol 41: (A) Sum of digits at units place

$= (1 + 2 + \ldots\ldots + 9) \times 3 \times 9 = 45 \times 27 = 1215$

Sum of digits at ten's place

$= (1 + 4 + 9) \times 9 \times 9 = 14 \times 81 = 1134$

Sum of digits at hundred's place

$= (1 + 2 + \ldots\ldots 9) \times 3 \times 9 = 45 \times 27 = 1215$

$$\text{Sum} = \begin{array}{r} 1215 \\ 1134\times \\ \underline{1215\times\times} \\ 134055 \end{array}$$

Sol 42: (D) 3 possible distribution of wards for each subject.

1 1 4

1 2 3

2 2 2

Total ways = $3!\left(\frac{6!}{4!\ (1!)^2\ \ 2!} + \frac{6!}{1!\ 2!\ 3!} + \frac{6!}{(2!)^3 3!}\right) = 540$

Sol 43: (C) M = L × 2!

L = N $\quad \therefore$ 2L = M = 2N

Sol 44: (C) There are 11 position

At the 6^{th} position A should be present. In the 5 positions left to 6^{th} positions 2positions will have A.

${}^5C_2$ ways

Sol 45: (D) ${}^9C_2 \times {}^7C_3 \times {}^4C_4$

(A) $\dfrac{9!}{2!\ 3!\ 4!}$

(D) $\dfrac{9!}{2!\ 3!\ 4!}$

Multiple Correct Choice Type

Sol 46: (A, B, D) ${}^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r$

Sol 47: (B, C) $2^{10} - 1$

Sol 48: (B, C, D) 2 2 1

$$\frac{5!}{(2!)^2\ 1!\ 2!} \times 3! = \frac{120}{8} \times 6 = 90$$

Sol 49: (A, B) Let k = 2n + 1, then ${}^{2n+1}C_r$ is maximum when r = n. Also ${}^{2n+1}C_n = {}^{2n+1}C_{n+1}$. Thus, ${}^kC_r$ is maximum

when $r = \frac{1}{2}(k-1)$ or $r = \frac{1}{2}(k+1)$

Sol 50: (B, D) (A) 4 vowels, 7 consonants

${}^7C_2 \times 6! = 3.7!$

(B) $\dfrac{15}{n!(15-x)!}$ n = no of white balls

(C) $\dfrac{12!}{4!\ 5!}$

(D) 35

Sol 51: (C, D) BULBUL

$$\frac{6!}{2!\ 2!\ 2!} = 90$$

(A) ${}^4C_2 \times {}^6C_2 = 6 \times 15 = 90$

(B) $\dfrac{6!}{(2!)^3\ 3!} = 15$

(C) $\dfrac{6!}{(2!)^3} = 90$

(D) $\dfrac{6!}{(2!)^3\ 3!} \times 3! = 90$

Comprehension Type

Sol 52: (C) M I(4) S(4)P(2)

(i) ${}^2C_1 \times {}^3C_1 = 6$

(ii) ${}^2C_1 \times {}^2C_1 = 4$

(iii) ${}^2C_1 \times {}^3C_1 = 6$

(iv) ${}^3C_2 \times 2 = 6$

(v) ${}^3C_1 = 3$

Adding all these, we get = 25

Sol 53: (B) Total ways in which all vowels are together

$$= \frac{11!}{4!\ 4!\ 2!} - \frac{8!}{4!\ 2!} = \frac{8!}{4!\ 2!}\left(\frac{165}{4} - 1\right) = \frac{8!.161}{4\ 4!\ 2!}$$

Sol 54: (B) ${}^6C_2 \times {}^8C_4 \times 5 = 1 \times \left(\dfrac{10!}{4!\ 4!}\right)$

Sol 55: (A) No of ways $= \dfrac{16!}{(4!)^5}$

$$= \frac{(1\times3\times5........\times15)\times2^8\times8^1}{(4!)^5}$$

$$= \prod_{r=1}^{8}(2r-1) \times \frac{2^8\times8\times7\times6\times5\times4!}{(4!)^5}$$

$$=35\prod_{r=1}^{8}(2r-1) \times \frac{2^8\times48}{(24)^4} = \frac{35}{27}\prod_{r=1}^{8}(2r-1)$$

Sol 56: (C) $\dfrac{12!}{(3!)^4\ 4!} \times 4! = \dfrac{12\times11!}{6^4} = \dfrac{11!}{108}$

Match the Columns

Sol 57: A → r; B → s; C → q; D → p

(A) ${}^nC_m$ (B) m^n (C) ${}^mC_n$ (D) n^m

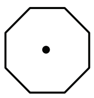
Sol 58: A → t; B → r; C → p; D → q; E → s

(A) x_1, x_2, x_3 & x_4 are the students watching a particular movie

$x_2 + x_2 + x_3 + x_4 = 10$

$x_i \geq 1$

$^9C_3 = \frac{9\times8\times7}{6} 84$



(B) $T = {}^9C_3 - 4 = 80$

$S = {}^8C_2 = 28$

$T - S = 52$

(C) $1 + {}^5C_2 = 11$

(D) $12 = 1 \times 2 \times 2 \times 3 = 1 \times 1 \times 4 \times 3 = 1 \times 1 \times 2 \times 6$

$\frac{4!}{2!} + \frac{4!}{2!} + \frac{4!}{2!} = 36$

(E) $^5C_2 \times {}^3C_2 \times 2 = 10 \times 3 \times 2 = 60$

$M_1M_2\ M_1\ M_2$

$F_1F_2\ F_2F_1$

Previous Years' Questions

Sol 1: Since, each box can hold five balls.

$\therefore$ Number of ways in which balls could be distributed so that none is empty are (2, 21) or (3, 1, 1).

ie, $({}^5C_2\ {}^3C_2\ {}^1C_1 + {}^5C_3\ {}^2C_1\ {}^1C_1) \times 3!$

$= (30 + 20) \times 6 = 300$

Sol 2: The possible cases are

Case-I: A man invites 3 ladies and women invites 3 gentlemen

Number of ways = $^4C_3 \, . \, {}^4C_3 = 16$

Case-II: A man invites (2 ladies, 1 gentlemen) and women invites (2 gentlemen, 1 lady).

Number of ways

$= ({}^4C_2.{}^3C_1)\ ({}^3C_1.{}^4C_1) = 324$

Case-III: A man invites (1 lady, 2 gentlemen) and women invites (2 ladies, 1 gentleman).

Number of ways

$= ({}^4C_1.{}^3C_2).({}^3C_2.{}^4C_1) = 144$

Case-IV: A man invites (3 gentlemen) and women invites (3 ladies).

Number of ways = $^3C_3.{}^3C_3 = 1$

$\therefore$ Total number of ways = 16 + 324 + 144 + 1 =485

Sol 3: Case-I: When one black and two others ball S are drawn

$\Rightarrow$ number of ways = $^3C_1 \, . \, {}^6C_2 = 45$

Case-II: When two black and one other balls are drawn

$\Rightarrow$ Number of ways = $^3C_2.{}^6C_1 = 18$

Case-III : When all three black balls are drawn

$\Rightarrow$ Number of ways = $^3C_3 = 1$

$\therefore$ Total number of ways = 45 + 18 + 1 = 64

Sol 4: Let the two sides be A and B. Assume that four particular guests wish to sit on side A. Four guests who wish to sit on side A can be accommodated on nine chairs in $^9P_4$ ways and there guests who wish to sit on side B and be accommodated in $^9P_3$ ways.

Now, the remaining guests are left who can sit on 11 chairs on both the sides of the table in (11!) ways.

Hence, the total number of ways in which 18 persons can be seated = $^9P_4 \times {}^9P_3 \times (11)!$.

Sol 5: There are 9 women and 8 men. A committee of 12, consisting of at least 5 women, can be formed by choosing:

(i) 5 women and 7 men

(ii) 6 women and 6 men

(iii) 7 women and 5 men

(iv) 8 women and 4 men

(v) 9 women and 3 men

$\therefore$ Total number of ways forming the committee

$= {}^9C_5 \times {}^8C_7 + {}^9C_6 \times {}^8C_6 + {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3$

$= 126 \times 8 + 84 \times 28 + 36 \times 56 + 9 \times 70 + 1 \times 56 = 6062$

(i) Clearly, women are in majority in (iii), (iv) and (v) cases as discussed above.

(ii) So, total number of committees in which women are in majority

$= {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3$

$= 36 \times 56 + 9 \times 70 + 1 \times 56 = 2702$

Clearly, men are in majority in only (i) case as discussed above.

So, total number of committees in which men are in majority

$= {}^9C_5 \times {}^8C_7 = 126 \times 8 = 1008$

Sol 6: A $\rightarrow$ p; B $\rightarrow$ s; C $\rightarrow$ q; D $\rightarrow$ q

(A) If ENDEA is fixed word, then assume this as a single letter.

Total number of letters = 5

Total number of arrangements = 5!

(B) If E is at first and last places, then total number of permutation of $\frac{7!}{2!} = 21 \times 5!$

(C) If D, L, N are not in last five positions

$\leftarrow$ D, I, N, N $\rightarrow$ $\leftarrow$ E, E, E, A, O $\rightarrow$

Total number of permutation

$= \frac{4!}{2!} \times \frac{5!}{3!} = 2 \times 5!$

(D) Total number of odd position = 5

Permutation of AEEEO are $\frac{5!}{3!}$

Total number of even positions = 4

Number of permutations of N, N, D, L = $\frac{4!}{2!}$

Hence, total number of permutation = $\frac{5!}{3!} \times \frac{4!}{2!} = 2 \times 5!$