Chapter 17

DIFFERENTIATION

A SINGLE CORRECT CHOICE TYPE

Each of these questions has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

1. Consider the function : $f(x)=\begin{cases} x[x-1], & 0\le x<2 \\ [x](x-1), & 2\le x<3 \end{cases}$

$[x]$ denotes the greatest integer $\le x$. Then

(a) $f(x)$ is continuous every where

(b) $f(x)$ is not differentiable at infinite points

(c) $f(x)$ is not continuous at $x = 1$

(d) $f(x)$ is continuous at $x = 2$ but not differentiable

2. Let $f(x)=x-x^2$ and

$g(x)=\begin{cases} \max f(t), & 0\le t\le x,\ 0\le x\le 1 \\ \sin \pi x, & x>1 \end{cases}$, then in the interval $[0, \infty)$

(a) g (x) is every where continuous except at two points

(b) g (x) is every where differentiable except at two points

(c) g (x) is every where differentiable except at $x = 1$

(d) none of these

3. Given $f:[-2a,2a]\longrightarrow R$ is an odd function such that the left hand derivative at x = a is zero and $f(x)=f(2a-x)\ \forall\ x\in(a,2a)$, then its left had derivative at $x=-a$ is

(a) 0 (b) a

(c) $-a$ (d) does not exist

4. If $f:R\to R$ be a differentiable function such that $f(x+2y)=f(x)+f(2y)+4xy\ \forall\ x,y\in R$ then;

(a) $f'(1)=f'(0)+1$ (b) $f'(1)=f'(0)-1$

(c) $f'(0)=f'(1)+2$ (d) $f'(0)=f'(1)-2$

5. If $y=\tan^{-1}\sqrt{\dfrac{1-\sin x}{1+\sin x}}$, then the value of $\dfrac{dy}{dx}$ at $x=\dfrac{\pi}{6}$ is

(a) $-\dfrac{1}{2}$ (b) $\dfrac{1}{2}$

(c) 1 (d) does not exist

6. If the function $f(x)=\left[\dfrac{(x-2)^3}{a}\right]\sin(x-2)+a\cos(x-2)$,

[.] denotes the greatest integer function is continuous and differentiable in [4, 6], then

(a) $a\in[8, 64]$ (b) $a\in(0, 8]$

(c) $a\in[64,\infty]$ (d) none of these

7. Suppose that f is a differentiable function with the property that $f(x+y)=f(x)+f(y)+xy$ and $\lim_{h\to 0}\dfrac{1}{h}f(h)=3$, then

(a) f is a linear function (b) $f(x)=3x+x^2$

(c) $f(x)=3x+\dfrac{x^2}{2}$ (d) none of these

8. Suppose the function f satisfies the equation $f(x+y)=f(x)f(y)$ for all x and y and $f(x)=1+xg(x)$ where $\lim_{x\to 0} g(x)=\log a$ If $f^n(x)=Kf(x)$, then $K=$

(a) $\log a$ (b) $n\log a$

(c) $(\log a)^n$ (d) $n(\log a)^n$

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ
	6. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ		

9. A function $y = f(x)$ defined parametrically by $x = e^t \sin t$, $y = e^t \cos t$ satisfies the relation

(a) $y''(x+y)^2 = 2(xy' - y)$

(b) $y'(x+y) = 2(xy'' - y)^2$

(c) $y'(x+y'') = 2(x - y'')$

(d) none of these

10. If $y = \cos^{-1}(\cos x)$, then $\frac{dy}{dx}$ at $x = \frac{5\pi}{4}$ is

(a) 1 (b) –1

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{5\pi}{4}$

11. If $y = \frac{1}{x}$, then $\frac{dy}{\sqrt{1+y^4}} + \frac{dy}{\sqrt{1+x^4}} =$

(a) 0 (b) $-\frac{1}{x^2}$

(c) $\sqrt{\frac{1+y^4}{1+x^4}}$ (d) none of these

12. If $f(x) = \cos\left\{\frac{\pi}{2}[x] - x^3\right\}$, $-1 < x < 2$ and $[x]$ is the greatest integer less than or equal to x, then $f'\left(\sqrt[3]{\frac{\pi}{2}}\right)$ is equal to

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

13. If $\sqrt{1-x^{2n}} + \sqrt{1-y^{2n}} = a(x^n - y^n)$, then $\sqrt{\frac{1-x^{2n}}{1-y^{2n}}}\frac{dy}{dx} =$

(a) 1 (b) $\frac{x}{y}$

(c) $\frac{x^{n-1}}{y^{n-1}}$ (d) none of these

14. If $y = f\left(\frac{3x+4}{5x+6}\right)$ and $f'(x) = \tan x^2$, then $\frac{dy}{dx}$ is equal to

(a) $-2\tan\left(\frac{3x+4}{5x+6}\right)^2 \times \frac{1}{(5x+6)^2}$

(b) $\mathrm{f}\left(\frac{3\tan \mathrm{x}^2 + 3}{5\tan \mathrm{x}^2 + 6}\right)\tan \mathrm{x}^2$

(c) $\tan x^2$ (d) none of these

15. Let $g(x)$ be the inverse of an invertible function $f(x)$ which is differentiable for all real x, then $g''(f''(x))$ equals

(a) $-\frac{f''(x)}{(f'(x))^3}$

(b) $\frac{f'(x)f''(x) - (f'(x))^3}{f'(x)}$

(c) $\frac{f'(x)f''(x) - (f'(x))^2}{(f'(x))^2}$ (d) none of these.

16. Let $F(x) = f(x)g(x)h(x)$ for all real x, where $f(x), g(x)$ and $h(x)$ are differentiable functions. At some point $x_0, F'(x_0) = 21\,F(x_0),\ f'(x_0) = 4f(x_0), g'(x_0) = -7g(x_0)$ and $h'(x_0) = kh(x_0)$. Then the value of k can be equal to

(a) 1 (b) 7

(c) 24 (d) 5

17. If $F(x) = \frac{1}{x^2}\int_4^x (4t^2 - 2F'(t))dt$ then $F'(4)$ equals

(a) $\frac{32}{9}$ (b) $\frac{64}{3}$

(c) $\frac{64}{9}$ (d) none of these

18. If $f(x) = x + \tan x$ and f is inverse of g, then $g'(x)$ is equal to

(a) $\frac{1}{1+[g(x)-x]^2}$ (b) $\frac{1}{2-[g(x)-x]^2}$

(c) $\frac{1}{2+[g(x)-x]^2}$ (d) none of these

MARK YOUR RESPONSE					
	9. ⓐⓑⓒⓓ	10. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ	12. ⓐⓑⓒⓓ	13. ⓐⓑⓒⓓ
	14. ⓐⓑⓒⓓ	15. ⓐⓑⓒⓓ	16. ⓐⓑⓒⓓ	17. ⓐⓑⓒⓓ	18. ⓐⓑⓒⓓ

19. If $u = f(x^3)$, $v = g(x^2)$, $f'(x) = \cos x$ and $g'(x) = \sin x$, then $\frac{du}{dv} =$

(a) $\frac{1}{2}x\cos x^3 \text{cosec } x^2$ (b) $\frac{3}{2}x\cos x^3 \text{cosec } x^2$

(c) $\frac{1}{2}x\sec x^3 \sin x^2$ (d) $\frac{3}{2}x\sec x^3 \text{cosec } x^2$

20. If $y = x - x^2$ then the derivative of y^2 with respect to x^2 is

(a) $1-2x$ (b) $2-4x$

(c) $3x-2x^2$ (d) $1-3x+2x^2$

21. Let $f(x+y) = f(x)+f(y)+2xy-1 \ \forall\ x,\ y \in \mathbf{R}$. If $f(x)$ is differentiable and $f'(0) = \sin\phi$, then

(a) $f(x) > 0\ \forall\ x \in \mathbf{R}$ (b) $f(x) < 0\ \forall\ x \in \mathbf{R}$

(c) $f(x) = \sin\phi\ \forall\ x \in \mathbf{R}$ (d) none of these

22. If $f'(1) = 2$ and $g'(\sqrt{2}) = 4$, then the derivative of $f(\tan x)$ with respect to $g(\sec x)$ at $x = \frac{\pi}{4}$ is

(a) 1 (b) $\sqrt{2}$

(c) $\frac{\sqrt{2}}{2}$ (d) 2

23. If $2x = y^{1/5} + y^{-1/5}$, then $(x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} =$

(a) $5y$ (b) $25y$

(c) $25y^2$ (d) $y+25$

24. If $y = \sqrt{(a-x)(x-b)} - (a-b)\tan^{-1}\sqrt{\frac{a-x}{x-b}}$ where $0 < b < x < a$, then $\frac{dy}{dx}$ can be equal to

(a) 1 (b) $\sqrt{\frac{a-x}{x-b}}$

(c) $\sqrt{(a-x)(x-b)}$ (d) $\frac{1}{\sqrt{(a-x)(x-b)}}$

25. Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a_1, a_2, a_3 are in A.P. then $f'(a_1), f'(a_2), f'(a_3)$ are in

(a) A. P. (b) G.P.

(c) H. P. (d) none of these

26. If $g(x) = \frac{2h(x)+|h(x)|}{2h(x)-|h(x)|}$ where $h(x) = \sin x - \sin^n x$, $n \in R^+$, where R^+ is the the set of positive real numbers, and

$$f(x) = \begin{cases} [g(x)], & x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \\ 3, & x = \frac{\pi}{2} \end{cases}$$

Where $[x]$ denotes the greatest integer function, then

(a) $f(x)$ is continuous and differentiable at $x = \frac{\pi}{2}$, when $0 < n < 1$

(b) $f(x)$ is continuous and differentiable at $x = \frac{\pi}{2}$, when $n > 1$

(c) $f(x)$ is continuous but not differentiable at $x = \frac{\pi}{2}$, when $0 < n < 1$

(d) $f(x)$ is continuous but not differentiable at $x = \frac{\pi}{2}$, when $n > 1$

27. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\frac{dy}{dx} =$

(a) $\frac{2(1+x-x^2)}{x^2+1}\sin\left(\frac{2x-1}{x^2+1}\right)$

(b) $\frac{2(1+x-x^2)}{(x^2+1)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$

(c) $\frac{2(1+x-x^2)}{x^2+1}\sin\left(\frac{2x-1}{x^2+1}\right)^2$

(d) none of these

28. If $x = \text{cosec}\,\theta - \sin\theta$; $y = \text{cosec}^n\theta - \sin^n\theta$ then $(x^2+4)\left(\frac{dy}{dx}\right)^2 - n^2y^2 =$

(a) n^2 (b) $2n^2$

(c) $3n^2$ (d) $4n^2$

MARK YOUR RESPONSE					
	19. ⓐⓑⓒⓓ	20. ⓐⓑⓒⓓ	21. ⓐⓑⓒⓓ	22. ⓐⓑⓒⓓ	23. ⓐⓑⓒⓓ
	24. ⓐⓑⓒⓓ	25. ⓐⓑⓒⓓ	26. ⓐⓑⓒⓓ	27. ⓐⓑⓒⓓ	28. ⓐⓑⓒⓓ

29. If $f(x)=x^3+x^2 f'(1)+xf''(2)+f'''(3)$ for all $x\in\mathbf{R}$, then $f(x)=$

(a) x^3+2x^2-5x+6 (b) x^3+2x^2+6x-5

(c) x^3-5x^2+6x+2 (d) x^3-5x^2+2x+6

30. If $y=\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\tan^{-1}\frac{1}{x^2+5x+7}+....\text{to } n$ terms, then $\frac{dy}{dx}=$

(a) $\frac{1}{x^2+n^2}-\frac{1}{x^2+1}$ (b) $\frac{1}{(x+n)^2+1}-\frac{1}{x^2+1}$

(c) $\frac{1}{x^2+(n+1)^2}-\frac{1}{x^2+1}$ (d) none of these

31. If $f'(x)=\sin x+\sin 4x.\cos x$ then $f'\left(2x^2+\frac{\pi}{2}\right)=$

(a) $\cos 2x^2-\sin 8x^2\sin 2x^2$

(b) $4x(\cos 2x^2-\sin 8x^2\sin 2x^2)$

(c) $4x^2(\cos 2x^2-\sin 8x^2\sin 2x^2)$

(d) none of these

32. If $y=(1+x)(1+x^2)(1+x^4)..........(1+x^{2^n})$ then $\frac{dy}{dx}$ at $x=0$ is

(a) 0 (b) 1

(c) n (d) 2^n

33. If for all x, y a differentiable function f is defined by $f(x)+f(y)+f(x)f(y)=1$ and $f(x)>0$ then $f'(x)=$

(a) 0 (b) x

(c) $\sqrt{x}-1$ (d) $f'(x)\geq 0\ \forall\ x$

34. If $|x|<1$ then

$$\frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1-x^2+x^4}+\frac{4x^3-8x^7}{1-x^4+x^8}+......\infty=$$

(a) $\frac{1}{1+x+x^2}$ (b) $\frac{1+2x}{1+x+x^2}$

(c) $\frac{1-x+x^2}{1+x+x^2}$ (d) 1

35. Let $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ and $f'(0)=a$ and $f(0)=b$ then $f''(x)=$

(a) 0 (b) a

(c) b (d) $a+b$

36. If the function $f(x)=|x^2+a|x|+|b||$ has exactly three points of non-derivability, then

(a) $b=0, a<0$ (b) $b<0,\ a\in R$

(c) $b>0,\ a\in R$ (d) None of these

37. Let $f(x)=\begin{cases}\cot^{-1}x, & |x|\geq 1\\ \frac{1}{2}|x|+\frac{\pi}{4}-\frac{1}{2}, & |x|<1\end{cases}$, then number of points which domain of $f'(x)$ does not contain is

(a) one (b) two

(c) three (d) none of these

38. Which of the following functions is not differentiable at $x=1$?

(a) $f(x)=(x^2-1)|(x-1)(x-2)|$

(b) $f(x)=\sin(|x-1|)-|x-1|$

(c) $f(x)=\tan(|x-1|)+|x-1|$

(d) none of these

MARK YOUR RESPONSE					
	29. ⓐⓑⓒⓓ	30. ⓐⓑⓒⓓ	31. ⓐⓑⓒⓓ	32. ⓐⓑⓒⓓ	33. ⓐⓑⓒⓓ
	34. ⓐⓑⓒⓓ	35. ⓐⓑⓒⓓ	36. ⓐⓑⓒⓓ	37. ⓐⓑⓒⓓ	38. ⓐⓑⓒⓓ

39. Let

$$f(x)=(x^2-3x+2)\,|(x^3-6x^2+11x-6)|+\left|\sin\left(x+\frac{\pi}{4}\right)\right|.$$

The set of points at which the function $f(x)$ is not differentiable in $[0, 2\pi]$ is

(a) $\left\{1,2,3,\frac{3\pi}{4},\frac{7\pi}{4}\right\}$ (b) $\{1,2,3\}$

(c) $\left\{3,\frac{3\pi}{4},\frac{7\pi}{4}\right\}$ (d) $\left\{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4}\right\}$

40. Let $f(x)$ be a periodic function with period 3 and $f\left(-\frac{2}{3}\right)=7$ and $g(x)=\int_0^x f(t+n)dt$, where $n=3k, k\in N$. Then $g'\left(\frac{7}{3}\right)$ is equal to

(a) $-\frac{2}{3}$ (b) 7

(c) –7 (d) $\frac{7}{3}$

41. Let $f(x)=[r+p\sin x], x\in(0,\pi), r\in I$ and p is prime number ([.] denotes greatest integer function). The number of points at which $f(x)$ is non-differentiable is

(a) p (b) $p-1$
(c) $2p+1$ (d) $2p-1$

42. Let $f(x)$ be a four times differentiable function such that $f(2x^2-1)=2xf(x)\,\forall x\in R,$ then $f^{iv}(0)$ is equal to (where $f^{iv}(0)$ represents fourth derivative of $f(x)$ at $x=0$)

(a) 0 (b) 1
(c) –1 (d) data insufficient

MARK YOUR RESPONSE	39. ⓐⓑⓒⓓ	40. ⓐⓑⓒⓓ	41. ⓐⓑⓒⓓ	42. ⓐⓑⓒⓓ	

B COMPREHENSION TYPE

This section contains groups of questions. Each group is followed by some multiple choice questions based on a paragraph. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

PASSAGE-1

Consider a series of the form $\frac{f_1'(x)}{f_1(x)}+\frac{f_2'(x)}{f_2(x)}+\frac{f_3'(x)}{f_3(x)}+........$

The sum of such a series can be obtained in the following way :

Step 1 : Obtain the product $f_1(x)f_2(x).....f_n(x)$.
Let $f_1(x)f_2(x).......f_n(x)=g(x)$

Step 2 : Take log of both the sides, you get
$\log f_1(x)+\log f_2(x)+.....+\log f_n(x)=\log g(x)$

Step 3 : Differentiating both sides with respect to x, you get
$$\frac{f_1'(x)}{f_1(x)}+\frac{f_2'(x)}{f_2(x)}+.......+\frac{f_n'(x)}{f_n(x)}=\frac{g'(x)}{g(x)}$$

The above method can be extended for the sum of infinite terms of the series provided the series is convergent.

1. The sum of n terms of the series

$$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+.....\text{ is}$$

(a) $\frac{1}{1-x}-\frac{2^n x^{2^n-1}}{1-x^{2^n}}$

(b) $\frac{1}{1-x}+\frac{2^n x^{2^n-1}}{1+x^{2^n}}$

(c) $-\frac{1}{1-x}-\frac{2^n x^{2^n-1}}{1-x^{2^n}}$

(d) $-2^n x^{2^{n-1}}\left[\frac{1}{1-x^{2^n}}+\frac{1}{1+x^{2^n}}\right]$

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ				

2. If $|x|<1$ then the sum to infinite terms of the series given in (1) is

(a) $\frac{1}{1-x}$ (b) $\frac{1}{1+x}$

(c) $\frac{1}{x-1}$ (d) 1

3. The sum of the series

$\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\frac{1}{2^3}\tan\frac{x}{2^3}+......+\frac{1}{2^n}\tan\frac{x}{2^n}$ is

(a) $\cot x$ (b) $-\cot x+\frac{1}{2^n}\cot\frac{x}{2^n}$

(c) $\cot x-\cot\frac{x}{2^n}$ (d) $-\tan x$

PASSAGE-2

If u be a function of two variables x and y given by the relation $u=f(x,y)$ then $\underset{\partial x\to 0}{Lt}\frac{f(x+\partial x,y)-f(x,y)}{\partial x}$

provided this limit exists is called **Partial differential coefficient** of $f(x,y)$ i.e., u w. r. t. x and is denoted by the symbol $\frac{\partial f}{\partial x}$ or $\frac{\partial u}{\partial x}$ or f_x.

Similarly, $\underset{\delta y\to 0}{Lt}\frac{f(x,y+\delta y)-f(x,y)}{\delta y}$

If it exists is called partial differential coefficient of $f(x,y)$ or of u w.r.t y and is denoted by the symbol $\frac{\partial f}{\partial y}$ or $\frac{\partial u}{\partial y}$ or f_y.

Hence in order to find $\frac{\partial f}{\partial x}$ we have to differentiate $f(x,y)$ w.r.t x treating y as constant and $\frac{\partial f}{\partial y}$ is obtained by differentiating the given function w. r. t. y treating x as constant.

4. If $u=\tan^{-1}\frac{x}{y}$ then

(a) $\frac{\partial u}{\partial x}=\frac{cx}{x^2+y^2}$ (b) $\frac{\partial u}{\partial y}=\frac{cx}{x^2+y^2}$

(c) $\frac{\partial u}{\partial y}=\frac{cy}{x^2+y^2}$ (d) $\frac{\partial u}{\partial x}=\frac{cy}{x^2+y^2}$

5. If $u=(x^2+y^2+z^2)^{-1/2}$ then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=$

(a) 0 (b) u

(c) $-u$ (d) u^2

6. If $u=\tan^{-1}\frac{x^2+y^2}{x+y}$, then $\frac{\partial u}{\partial x}:\frac{\partial u}{\partial y}=$

(a) $\frac{x^2+2xy-y^2}{-x^2+2xy+y^2}$ (b) $\frac{x^2-y^2}{x^2+y^2}$

(c) $\frac{(x+y)^2}{x^2+y^2}$ (d) $\frac{x^2-xy+y^2}{x^2+xy+y^2}$

PASSAGE-3

If $y=f(x)$ be such that it possesses a differential coefficient $f'(x)$ which is called the first derivative of $f(x)$, then $f'(x)$ being itself a function of x is capable of being differentiated further. Now if $f'(x)$ is differentiable i.e. it $\underset{\delta x\to 0}{Lt}\frac{f'(x+\delta x)-f'(x)}{\delta x}$ exists finitely, then this limit is called the first derivative of $f'(x)$ or second derivative of $f(x)$ and is denoted by $f''(x)$. Similarly the first derivative of $f''(x)$ is called the second derivative of $f'(x)$ and third derivative of $f(x)$ and is denoted by $f'''(x)$. The nth derivative of y is also denoted as $y_n, \frac{d^n y}{dx^n}, D^n y, y^n, f^n(x)$.

LEIBNITZ THEOREM

The nth derivative of product of two functions of *x*.

Suppose u and v are two functions of x and all their derivatives exist; then by Leibnitz's theorem.

$$D^n(uv)=(D^n u)v+{}^nC_1(D^{n-1}u)(Dv)+{}^nC_2(D^{n-2}u)(D^n v)+......$$
$$+{}^nC_r(D^{n-r}u)(D^2 v)+.........+u(D^n v).$$

Note : When one of the two functions in the above theorem is of the form x^m, then we should choose it as v and the other as u because x^m shall have only m differential coefficients and not more, which may be simplified.

MARK YOUR RESPONSE	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ	6. ⓐⓑⓒⓓ

7. If $y=\dfrac{x^2+1}{(x-1)(x-2)(x-3)}$ then $y_n=$

(a) $\dfrac{(-1)^n.n!}{(x-1)^{n+1}(x-2)^{n+1}(x-3)^{n+1}}$

(b) $(-1)^n\,n!\left[\dfrac{1}{(x-1)^{n+1}}+\dfrac{1}{(x-2)^{n+1}}+\dfrac{1}{(x-3)^{n+1}}\right]$

(c) $(-1)^n\,n!\left[\dfrac{1}{(x-1)^{n+1}}+\dfrac{2}{(x-2)^{n+1}}+\dfrac{3}{(x-3)^{n+1}}\right]$

(d) None of these

8. If $x^2y_2+xy_1+y=0$ then

$x^2y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=$

(a) 0 (b) y_n

(c) $-\,y_n$ (d) $n\,y_n$

9. Differentiating x^{2n} successively n times by two different ways, the sum of the series

$1+\dfrac{n^2}{1^2}+\dfrac{n^2(n-1)^2}{1^2.2^2}+\dfrac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2}+$............ can be obtained, which is

(a) $(2n)!$ (b) $\dfrac{(2n)!}{n!}$

(c) $\dfrac{(2n)!}{(n!)^2}$ (d) $(n!)^2$

PASSAGE-4

A function $f:R\to R$ satisfies the following conditions :

(1) $f(x)\neq 0$ for any $x\in R$

(2) $f(x+y)=f(x)\,.\,f(y)$ for all x, y in R

(3) $f(x)$ is differentiable

(4) $f'(0)=2$

10. The derivative of $f(x)$, i.e., f' satisfies the equation

(a) $f'(x+y)=f'(x)+f'(y)$ (b) $f'(x+y)=f'(x)f'(y)$

(c) $f'(x+y)=f'(x)f(y)$ (d) $f'(x+y)=f'(x)+f(y)$

11. The ratio $\dfrac{f'(x)}{f(x)}$ for all x, equals to

(a) 1 (b) 2

(c) x (d) $2x$

12. $\lim\limits_{x\to 0}\dfrac{f(x)-f(-x)}{x}=$

(a) 1 (b) 2

(c) 3 (d) 4

PASSAGE-5

Let $f(x)$ be a real valued function not identically zero, which satisfies the following conditions.

I. $f(x+y^{2n+1})=f(x)+\{f(y)\}^{2n+1}$, $n\in N$ x, y are any real numbers

II. $f'(0)\geq 0$

III. $f(1)\geq 0$

13. The value of $f(5)$ is

(a) 1 (b) 4

(c) 5 (d) $5n$

14. The value of $f'(10)$ is

(a) 10 (b) 0

(c) $2n+1$ (d) 1

15. The function $f(x)$ is

(a) Continuous and differentiable everywhere

(b) Continuous everywhere but not differentiable at some points

(c) Discontinuous at exactly one point

(d) Discontinuous at infinitely many points

MARK YOUR RESPONSE					
	7. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ	10. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ
	12. ⓐⓑⓒⓓ	13. ⓐⓑⓒⓓ	14. ⓐⓑⓒⓓ	15. ⓐⓑⓒⓓ	

C REASONING TYPE

In the following questions two Statements (1 and 2) are provided. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct. Mark your responses from the following options :

(a) Both Statement-1 and Statement-2 are true and Statement-2 is the correct explanation of Statement-1.
(b) Both Statement-1 and Statement-2 are true and Statement-2 is not the correct explanation of Statement-1.
(c) Statement-1 is true but Statement-2 is false.
(d) Statement-1 is false but Statement-2 is true.

1. **Statement-1** : $f(x) = \cos^2 x + \cos^2\left(x + \frac{\pi}{3}\right) - \cos x \cos\left(x + \frac{\pi}{3}\right)$

then $f'(x) = 0$

Statement-2 : Derivative of constant function is zero.

2. **Statement-1** : $f(x) = x^n \sin\left(\frac{1}{x}\right)$ is differentiable for all real values of x $(n \geq 2)$.

Statement-2 : For $n \geq 2$, $\lim_{x \to 0} f(x) = 0$

3. **Statment-1** : $f(x) = \sin^{-1}(\cos x)$ is differentiable for all x

Statment-2 : $f'(x) = -1, \ 0 < x < \pi$

4. **Statement-1** : The function $f(x) = |x|^3$ is not differentiable at $x = 0$

Statement-1 : The function $f(x) = |x|$ is not differentiable at $x = 0$

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	

D MULTIPLE CORRECT CHOICE TYPE

Each of these questions has 4 choices (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

1. Let $f(x) = \begin{cases} \int_0^x \{1 + |1-t|\} \, dt & \text{If } x > 2 \\ 5x - 7 & \text{If } x \leq 2 \end{cases}$, then

(a) f is not continuous at $x = 2$
(b) f is continuous but not differentiable at $x = 2$
(c) f is differentiable every where
(d) $f'(2^+) = 2$

2. Let $f(x) = \begin{cases} 2x^2 + 12x + 16 & -4 \leq x \leq -2 \\ 2 - |x| & -2 < x \leq 1 \\ 4x - x^2 - 2 & 1 < x \leq 3 \end{cases}$

then $[f(x)]$ (where [ ] represents the greatest integer function) is not differentiable at $x =$

(a) $-4, -3, 2$
(b) $-3 \pm \frac{1}{\sqrt{2}}$
(c) $-2, -1, 0$
(d) $0, 1, 3$

3. If $F(x) = f(x)\, g(x)$ and $f'(x)\, g'(x) = c$, then

(a) $F' = c\left[\frac{f}{f'} + \frac{g}{g'}\right]$
(b) $\frac{F''}{F} = \frac{f''}{f} + \frac{g''}{g} + \frac{2c}{fg}$
(c) $\frac{F'''}{F} = \frac{f'''}{f} + \frac{g'''}{g}$
(d) $\frac{F'''}{F''} = \frac{f'''}{f''} + \frac{g'''}{g''}$

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ		

4. If $f : R \to R$ is a function such that

$f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3), x \in \mathbf{R}$,

then $f(1) - f(0)$ is equal to

(a) –2 (b) $f(2)$

(c) $f(3)$ (d) $f'(2)$

5. Let $f(x) = \begin{cases} x^2 + a, & 0 \le x < 1 \\ 2x + b, & 1 \le x \le 2 \end{cases}$

and $g(x) = \begin{cases} 3x + b, & 0 \le x < 1 \\ x^3, & 1 \le x \le 2 \end{cases}$

If $\frac{df}{dg}$ exists then

(a) $a = -1$ (b) $b = -2$

(c) $\frac{df}{dg} = \frac{2}{3}$ (d) none of these

6. If $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$ then

(Given $\alpha \ne \beta \ne \gamma$)

(a) $2f(\beta) = f(\alpha) + f(\gamma)$

(b) $2f(\alpha) + f(\beta) = 3f(\gamma)$

(c) $f'(\alpha) = 0$

(d) $f(\alpha) = 0$

7. If 1 is a twice repeated root of the equation $ax^3 + bx^2 + bx + d = 0$ then

(a) $a = b = d$ (b) $a + b = 0$

(c) $b + d = 0$ (d) $a = d$

8. The function

$f(x) = ||\tan x + \cot x| - |\tan x - \cot x||, \ 0 < x < \frac{\pi}{2}$ is

(a) continuous every where

(b) differentiable everywhere

(c) not continuous at some points

(d) not differentiable at exactly one point

9. Let $f(x) = \begin{cases} \min\{x, x^2\} & x \ge 0. \\ \max\{2x, x^2 - 1\} & x < 0 \end{cases}$. Then

(a) $f(x)$ is continuous at $x = 0$

(b) $f(x)$ is differentiable at $x = 1$

(c) $f(x)$ is not differentiable at exactly three point

(d) $f(x)$ is every where continuous

10. Let $f(x) = x^3 + 3x^2 - 33x - 33$ for $x > 0$ and 'g' be its inverse, then the value of 'k' such that $kg'(2) = 1$ is equal to

(a) -36 (b) 51

(c) 72 (d) 42

11. Let $f(x) = |x-1| + |x-2|$ and

$g(x) = \begin{cases} \min\{f(t)\}; 0 \le t \le x, 0 \le x \le 3 \\ x - 2, \quad x > 3 \end{cases}$, then $g(x)$ is not differentiable at

(a) $x = 1$ (b) $x = 2$

(c) $x = 3$ (d) $x = \frac{3}{2}$

12. $f(x)$ is defined for $x \ge 0$ and has a continuous derivative. It satisfies $f(0) = 1, f'(0) = 0$ and $(1 + f(x)) f''(x) = 1 + x$. The values $f(1)$ can't take is (are)

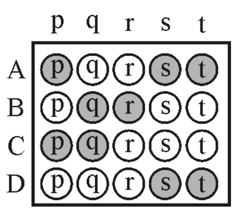
(a) 2 (b) 1.75

(c) 1.50 (d) 1.35

MARK YOUR RESPONSE					
	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ	6. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ
	9. ⓐⓑⓒⓓ	10. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ	12. ⓐⓑⓒⓓ	

E MATRIX-MATCH TYPE

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labeled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column -I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example:
If the correct matches are A–p, s and t; B–q and r; C–p and q; and D–s and t; then the correct darkening of bubbles will look like the given.



1. **Observe the following Columns :**

Column-I	Column-II
(A) $\frac{d}{dx}\left(\sin^2 \cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$, where $-1 < x < 1$, is	p. 1
(B) If $h(x) = e^{e^x}$ then $\frac{h'(x)}{h(x)\log(h(x))}$ is	q. 0
(C) If $f(x) = \sqrt{ax} + \frac{a^2}{\sqrt{ax}}$ then $f'(a)$, where $a > 0, x > 0$, is	r. $-\frac{1}{2}$
(D) If $xy = (x+y)^n$ and $\frac{dy}{dx} = \frac{y}{x}$ then n is	s. 2

2. **Observe the following Columns :**

Column-I	Column-II
(A) Let $f(x) = \begin{cases} \tan^{-1} x, & \lvert x \rvert \geq 1 \\ \frac{x^2-1}{4}, & \lvert x \rvert < 1 \end{cases}$, then $f(x)$ is not differentiable at x equal to	p. –1
(B) $f(x)=(x^2-4)\left\lvert x^2-5x+6\right\rvert + \cos\lvert x \rvert$ is non derivable at x equal to	q. 1
(C) If $\sin(x+y) = e^{x+y} - 2$, then $\frac{dy}{dx}$ is equal to	r. 2
(D) Let $f : R \to R$ is defined by the equation $f(x+y) = f(x)f(y)\ \forall\ x, y \in R, f(0) \neq 0$ and $f'(0) = 2$, then $\frac{f'(x)}{f(x)}$ is equal to	s. 3
	t. 0

3. Let $f : R \to R$ satisfies $\lvert f(x) \rvert \leq x^2 \forall x \in R$ and $g : R \to R$ satisfies $g(x+y) = g(x) - g(y) + 2xy - 1$ and $g'(0) = \sqrt{3+a+a^2}$. Now match the entries from the following two columns :

Column-I	Column-II
(A) At $x = 0$, $f(x)$ is necessarily	p. continuous
(B) At $x = 0$, $g(x)$ is necessarily	q. differentiable
(C) The number of roots of the equation $g(x) = f'(0)$ is	r. 0
(D) If $f(t)$ can be a non zero root of the equation $g(x) = 0$ then the least integral value of t can be	s. 1
	t. 2

MARK YOUR RESPONSE

1.
	p	q	r	s
A	(p)	(q)	(r)	(s)
B	(p)	(q)	(r)	(s)
C	(p)	(q)	(r)	(s)
D	(p)	(q)	(r)	(s)

2.
	p	q	r	s	t
A	(p)	(q)	(r)	(s)	(t)
B	(p)	(q)	(r)	(s)	(t)
C	(p)	(q)	(r)	(s)	(t)
D	(p)	(q)	(r)	(s)	(t)

3.
	p	q	r	s	t
A	(p)	(q)	(r)	(s)	(t)
B	(p)	(q)	(r)	(s)	(t)
C	(p)	(q)	(r)	(s)	(t)
D	(p)	(q)	(r)	(s)	(t)

F

NUMERIC/INTEGER ANSWER TYPE

The answer to each of the questions is either numeric (eg. 304, 40, 3010 etc.) or a single-digit integer, ranging from 0 to 9.

The appropriate bubbles below the respective question numbers in the response grid have to be darkened.

For example, if the correct answers to a question is 6092, then the correct darkening of bubbles will look like the given.

For single digit integer answer darken the extreme right bubble only.

0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9

1. Let $f(x) = x^2 + xg'(1) + g''(2)$ and

$g(x) = f(1)x^2 + xf'(x) + f''(x)$ then $f(g(1))$ is equal to

2. If $y^3 - y = 2x$ then $y\left[\left(x^2 - \frac{1}{27}\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx}\right]^{-1}$ is equal to

3. The number of point where $f(x) = \begin{cases} [\cos x], & x \le 1 \\ |2x-3|[x-2] & x > 1 \end{cases}$, (where [.] denotes the greatest integer function) in [0, 2] is non-differentiable is

4. Let $f(x) = [x] + |1-x|, -1 \le x < 3$, (here [.] denotes greatest integer function). The number of points, where $f(x)$ is non-differentiable is

MARK YOUR RESPONSE

1.				2.				3.				4.			
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9

Answerkey

A SINGLE CORRECT CHOICE TYPE

1	(c)	**6**	(c)	**11**	(a)	**16**	(c)	**21**	(a)	**26**	(b)	**31**	(b)	**36**	(a)	**41**	(d)
2	(c)	**7**	(c)	**12**	(a)	**17**	(a)	**22**	(c)	**27**	(b)	**32**	(b)	**37**	(c)	**42**	(a)
3	(a)	**8**	(c)	**13**	(c)	**18**	(c)	**23**	(b)	**28**	(d)	**33**	(a)	**38**	(c)		
4	(d)	**9**	(a)	**14**	(a)	**19**	(b)	**24**	(b)	**29**	(d)	**34**	(b)	**39**	(c)		
5	(a)	**10**	(b)	**15**	(a)	**20**	(d)	**25**	(a)	**30**	(b)	**35**	(a)	**40**	(b)		

B COMPREHENSION TYPE

1	(a)	**4**	(d)	**7**	(d)	**10**	(c)	**13**	(c)
2	(a)	**5**	(c)	**8**	(c)	**11**	(b)	**14**	(d)
3	(b)	**6**	(a)	**9**	(c)	**12**	(d)	**15**	(a)

C REASONING TYPE

1	(a)	**2**	(d)	**3**	(d)	**4**	(d)

D MULTIPLE CORRECT CHOICE TYPE

1	(b, d)	**3**	(a,b,c)	**5**	(a, b, c)	**7**	(b,c,d)	**9**	(a, c, d)	**11**	(a)
2	(a,b,c)	**4**	(a,b)	**6**	(a, b, c)	**8**	(a,d)	**10**	(a, b, c)	**12**	(a,b,c,d)

E MATRIX-MATCH TYPE

1. **A-r; B-p; C-q; D-s**

2. **A-p, q, B-s, C-q, D-r**

3. **A-p, q, B-p, q, C-t, D-s**

F NUMERIC/INTEGER ANSWER TYPE

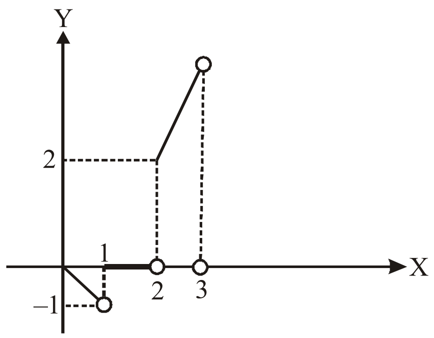
1	4	**2**	9	**3**	5	**4**	3

Solutions

A SINGLE CORRECT CHOICE TYPE

1. **(c)** Let $0 \le x < 1 \Rightarrow -1 \le x-1 < 0 \Rightarrow [x-1] = -1$

Let $1 \le x < 2 \Rightarrow 0 \le x-1 < 1 \Rightarrow [x-1] = 0$



Let $2 \le x < 3 \Rightarrow [x] = 2$

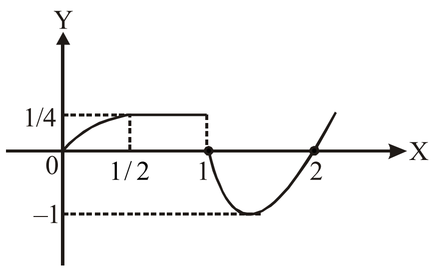
$$\therefore \; f(x) = \begin{cases} -x, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \\ 2(x-1), & 2 \le x < 3 \end{cases}$$

Form the graph it is evident that
$f(x)$ is not continuous at $x = 1$ and 2 and hence not differentiable also.

2. **(c)** $f'(x) = 1 - 2x > 0$ If $x < \frac{1}{2}$ and < 0 if $x > \frac{1}{2}$

$\therefore f(x)$ is increasing in $0 \le x \le \frac{1}{2}$

$\Rightarrow$ max $f(t) = f(x)$



Also maximum of

$f(x) = f\left(\frac{1}{2}\right) = \frac{1}{4}$ $\qquad (0 \le x \le 1)$

$\therefore$ Max $f(t) = f\left(\frac{1}{2}\right) = \frac{1}{4}$ $\qquad$ If $\frac{1}{2} \le x \le 1$

$$\text{so } g(x) = \begin{cases} x - x^2, & 0 \le x \le \frac{1}{2} \\ \frac{1}{4}, & \frac{1}{2} \le x \le 1 \\ \sin \pi x, & x > 1 \end{cases}$$

It is clear from the graph that
$g(x)$ is continuous every where except at $x = 1$,
hence not differentiable at $x = 1$

At $x = \frac{1}{2}$, $f(x)$ is continuous as well as differentiable.

3. **(a)** Given $f'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = 0$

Now $f'(-a^-) = \lim_{h \to 0} \frac{f(-a-h) - f(-a)}{-h}$

$= \lim_{h \to 0} \frac{-f(a+h) + f(a)}{-h}$ $\qquad [\because f(x)$ is odd function$]$

$= \lim_{h \to 0} \frac{-f(a-h) + f(a)}{-h}$

$[\because f(2a - x) = f(x) \Rightarrow f(a+x) = f(a-x)]$

$= \lim_{h \to 0} \frac{f(a-h) - f(a)}{h} = 0$ [From (1)]

4. **(d)** Let $x = y = 0 \Rightarrow f(0) = 0$
Now we have

$$\frac{f(x+2y) - f(x)}{2y} = 2x + \frac{f(2y)}{2y} = 2x + \frac{f(2y) - f(0)}{2y}$$

Taking limit on both sides as $2y \to 0$, we get

$f'(x) = 2x + f'(0)$ as

$f(0) = 0 \Rightarrow f'(0) = f'(1) - 2$

5. **(a)** $y = \tan^{-1} \sqrt{\left(\frac{\sin \frac{x}{2} - \cos \frac{x}{2}}{\sin \frac{x}{2} + \cos \frac{x}{2}} \right)^2} = \tan^{-1} \left| \frac{\sin \frac{x}{2} - \cos \frac{x}{2}}{\sin \frac{x}{2} + \cos \frac{x}{2}} \right|$

.... (1)

In the neighbourhood of $x = \frac{\pi}{6}, \sin\frac{x}{2} - \cos\frac{x}{2} < 0$

$$\therefore \quad \left|\frac{\sin\frac{x}{2} - \cos\frac{x}{2}}{\sin\frac{x}{2} + \cos\frac{x}{2}}\right| = -\frac{\sin\frac{x}{2} - \cos\frac{x}{2}}{\sin\frac{x}{2} + \cos\frac{x}{2}}$$

$$= \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}} = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

From (1) $\qquad y = \frac{\pi}{4} - \frac{x}{2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}$

Note : At (1) if you write

$$\sqrt{\left(\frac{\sin\frac{x}{2} - \cos\frac{x}{2}}{\sin\frac{x}{2} + \cos\frac{x}{2}}\right)^2} = \frac{\sin\frac{x}{2} - \cos\frac{x}{2}}{\sin\frac{x}{2} + \cos\frac{x}{2}}$$

You get $\frac{dy}{dx} = \frac{1}{2}$, which is wrong. Actually in this function $\frac{dy}{dx} = -\frac{1}{2}$ or $\frac{1}{2}$ depending on the value of x.

6. **(c)** Since $[x^3]$ is not continuous and differentiable at integral points. So $f(x)$ is continuous and differentiable in

$[4, 6]$ if $\left[\frac{(x-2)^3}{a}\right] = 0 \Rightarrow a \geq 64$

7. **(c)** $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x) + f(h) + xh - f(x)}{h}$$

$$= \lim_{h\to} \frac{1}{h} f(h) + x = 3 + x$$

Integrating we get $f(x) = 3x + \frac{x^2}{2} + c$

Putting $x = y = 0$ in the given equation, we get

$f(0) = 0 \Rightarrow c = 0 \qquad \therefore f(x) = 3x + \frac{x^2}{2}$

8. **(c)** $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x)\{f(h) - 1\}}{h} = \lim_{h\to 0} \frac{f(x)(1 + hg(h) - 1)}{h}$$

$$= \lim_{h\to 0} f(x)g(h) = \log a\ f(x)$$

Hence $f''(x) = (\log a)(f'(x)) = (\log a)^2 f(x)$

Thus $f^n(x) = (\log a)^n f(x)$ so $K = (\log a)^n$

9. **(a)** Here $x = e^t \sin t$

On differentiating with respect to t we get,

$$\frac{dx}{dt} = e^t \cos t + e^t \sin t = e^t(\cos t + \sin t)$$

and $y = e^t \cos t$

On differentiating with respect to t we get,

$$\frac{dy}{dt} = e^t(-\sin t) + e^t(\cos t) = e^t(\cos t - \sin t)$$

So, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t(\cos t - \sin t)}{e^t(\cos t + \sin t)} = \frac{y-x}{y+x}$

or $\frac{dy}{dx} = \frac{y-x}{y+x}$

Differentiating both sides w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{(y+x).\left(\frac{dy}{dx} - 1\right) - (y-x)\left(\frac{dy}{dx} + 1\right)}{(y+x)^2}$$

or $y'' = \frac{(y+x).(y'-1) - (y-x)(y'+1)}{(y+x)^2}$

$$\Rightarrow (x+y)^2 y'' = 2(xy' - y)$$

10. **(b)** $y = \cos^{-1}(\cos x) = \cos^{-1}\{\cos(2\pi - x)\} = 2\pi - x$

$[\because\ 0 \leq \cos^{-1} x \leq \pi \Rightarrow$ in the neighbourhood of $x = \frac{5\pi}{4}$ we have $0 < 2\pi - x < \pi$]

$$\therefore \quad \frac{dy}{dx} = -1$$

11. **(a)** Here $\sqrt{(1+y^4)} = \sqrt{\left(1 + \frac{1}{x^4}\right)} = \frac{\sqrt{1+x^4}}{x^2} \quad \left(\because y = \frac{1}{x}\right)$

$$\Rightarrow \frac{\sqrt{1+y^4}}{\sqrt{1+x^4}} = \frac{1}{x^2} \qquad ....(1)$$

But $y = \frac{1}{x} \qquad \therefore \frac{dy}{dx} = -\frac{1}{x^2} \qquad ....(2)$

Form (1) and (2) $\frac{\sqrt{1+y^4}}{\sqrt{1+x^4}} = -\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{\sqrt{(1+y^4)}} + \frac{dx}{\sqrt{(1+x^4)}} = 0$$

12. **(a)** As we know

$1 < \sqrt[3]{\frac{\pi}{2}} < 2 \qquad \therefore$ If $x = \sqrt[3]{\frac{\pi}{2}} \Rightarrow [x] = 1$

So, $f(x) = \cos\left\{\frac{\pi}{2} - x^3\right\} = \sin x^3$

$\Rightarrow f'(x) = \cos x^3 . 3x^2$

$\therefore \quad f'\left(\sqrt[3]{\frac{\pi}{2}}\right) = 3\left(\frac{\pi}{2}\right)^{2/3} . \cos\frac{\pi}{2} = 0$

$\Rightarrow f'\left(\sqrt[3]{\frac{\pi}{2}}\right) = 0$

13. **(c)** We have $\sqrt{(1-x^{2n})} + \sqrt{(1-y^{2n})} = a(x^n - y^n)$ ....(1)

Putting $x^n = \sin\theta \Rightarrow \theta = \sin^{-1} x^n$ and $y^n = \sin\phi$

$\Rightarrow \phi = \sin^{-1} y^n$ ......(2)

Then (1), becomes $\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$\Rightarrow 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) = a2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$

$\Rightarrow 2\cot\left(\frac{\theta-\phi}{2}\right) = a \Rightarrow \theta - \phi = 2\cot^{-1} a$

$\Rightarrow \sin^{-1} x^n - \sin^{-1} y^n = 2\cot^{-1} a$

Differentiating both w.r.t. x we get

$$\frac{1}{\sqrt{(1-x^{2n})}} . nx^{n-1} - \frac{1}{\sqrt{(1-y^{2n})}} ny^{n-1}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^{n-1}}{y^{n-1}}\sqrt{\left(\frac{1-y^{2n}}{1-x^{2n}}\right)}$$

14. **(a)** $\frac{dy}{dx} = f'(u)\frac{du}{dx}$, where $u = \frac{3x+4}{5x+6}$

But $\frac{du}{dx} = \frac{3(5x+6)-(3x+4)5}{(5x+6)^2} = -\frac{2}{(5x+6)^2}$

Thus $\frac{dy}{dx} = -2\tan\left(\frac{3x+4}{5x+6}\right)^2 \frac{1}{(5x+6)^2}$

15. **(a)** Given that $g^{-1}(x) = f(x)$

$\Rightarrow x = g(f(x))$ or $g'(f(x))f'(x) = 1$

$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$

$\Rightarrow g''(f(x)).f'(x) = -\frac{f''(x)}{[f'(x)]^2}$

$\Rightarrow g''(f(x)) = -\frac{f''(x)}{[f'(x)]^3}$

16. **(c)** $F(x) = f(x)g(x)h(x)$

$\Rightarrow F'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

$\therefore \quad F'(x_0) = f'(x_0)g(x_0)h(x_0) + f(x_0)g'(x_0)h(x_0) + f(x_0)g(x_0)h'(x_0)$

$\Rightarrow 21F(x_0) = 4f(x_0)g(x_0)h(x_0) + f(x_0)(-7)g(x_0)h(x_0) + f(x_0)g(x_0)kh(x_0)$

$\Rightarrow 21f(x_0)g(x_0)h(x_0) = (4-7+k)f(x_0)g(x_0)h(x_0) \Rightarrow k = 24$

17. **(a)** $F(x) = \frac{1}{x^2}\int_4^x (4t^2 - 2F'(t))dt$

$\Rightarrow x^2 F(x) = \int_4^x [4t^2 - 2F'(t)]dt$ ......(1)

$\Rightarrow 2xF(x) + x^2 F'(x) = 4x^2 - 2F'(x)$

From (1), $F(4) = 0$

$\therefore \quad 16F'(4) = 64 - 2F'(4)$

$18F'(4) = 64 \Rightarrow F'(4) = \frac{64}{18} = \frac{32}{9}$

18. **(c)** $f(x) = x + \tan x$

$f(f^{-1}(y)) = f^{-1}(y) + \tan(f^{-1}(y))$

$\Rightarrow y = g(y) + \tan g(y)$ or $x = g(x) + \tan g(x)$

Differentiating $1 = g'(x) + \sec^2 g(x).g'(x)$

$$g'(x) = \frac{1}{1+\sec^2 g(x)} = \frac{1}{2+\tan^2(g(x))}$$

$$= \frac{1}{2+\{g(x)-x\}^2}$$

19. **(b)** Here, $u = f(x^3)$

$\Rightarrow \frac{du}{dx} = f'(x^3).\frac{d}{dx}(x^3)$

$= (\cos(x^3)).3x^2 = 3x^2.\cos x^3$ and $v = g(x^2)$

$\Rightarrow \frac{dv}{dx} = g'(x^2).\frac{d}{dx}(x^2) = (\sin x^2).(2x) = 2x\sin x^2$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} \frac{3x^2.\cos x^3}{2x.\sin x^2}$$

$\Rightarrow \frac{du}{dv} = \frac{3}{2}x.\cos x^3.\text{cosec } x^2$

20. **(d)** Let $u = y^2$ and $v = x^2$

$\therefore \quad \frac{du}{dx} = \frac{d}{dx}y^2 = \frac{d}{dy}y^2.\frac{dy}{dx} =$

$2y.(1-2x) = 2(x-x^2)(1-2x) = 2x(1-x)(1-2x)$ ....(1)

and $\frac{dv}{dx} = 2x$ .....(2)

Hence, $\frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)} = \frac{2x(1-x)(1-2x)}{2x}$

(from (1) and (2))

$= (1-x)(1-2x) = 1-3x+2x^2$

21. **(a)** $f(x+y) = f(x)+f(y)+2xy-1$

Put $x = y = 0 \Rightarrow f(0) = 2f(0) - 1 \Rightarrow f(0) = 1$

Also, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{f(x)+f(h)+2xh-1-f(x)}{h}$

$= 2x + \lim_{h\to 0}\frac{f(h)-1}{h} = 2x + f'(0) = 2x + \sin\phi$

Integrating, we get $f(x) = x^2 + x\sin\phi + C$

$f(0) = 1 \Rightarrow 1 = C$

$\therefore f(x) = x^2 + x\sin\phi + 1 > 0 \forall x \in R \; [\because \Delta < 0]$

22. **(c)** Let $u = f(\tan x) \Rightarrow \frac{du}{dx} = f'(\tan x).\sec^2 x$

Let $v = g(\sec x) \Rightarrow \frac{dv}{dx} = g'(\sec x).\sec x\tan x$

$\therefore \quad \frac{du}{dv} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x} = \frac{f'(\tan x)}{g'(\sec x)}\text{cosec } x$

$\therefore \quad \left.\frac{du}{dv}\right|_{x=\frac{\pi}{4}} = \frac{f'(1)}{g'(\sqrt{2})}.(\sqrt{2}) = \frac{2}{4}\sqrt{2} = \frac{\sqrt{2}}{2}$

23. **(b)** $(y^{1/5}+y^{-1/5})^2 = (y^{1/5}-y^{-1/5})^2 + 4$

Then, $(y^{1/5}-y^{-1/5}) = 4x^2 - 4$

[using $2x = y^{1/5}+y^{-1/5}$]

$\therefore \quad y^{1/5}-y^{-1/5} = \pm 2\sqrt{x^2-1}$ ......(i)

and $y^{1/5}+y^{-1/5} = 2x$ ......(ii) [given]

Adding (1) and (2), we get

$2y^{1/5} = 2x \pm 2\sqrt{x^2-1}$

$\therefore \quad y^{1/5} = x \pm \sqrt{x^2-1}$

or $y = (x\pm\sqrt{x^2-1})^5$ ......(iii)

On differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = 5(x\pm\sqrt{x^2-1})^4.\left\{1 \pm \frac{1.2x}{2\sqrt{x^2-1}}\right\}$

Using (+) sign $\frac{dy}{dx} = \frac{5(x+\sqrt{x^2-1})^5}{\sqrt{x^2-1}} = \frac{5y}{\sqrt{x^2-1}}$

or $(x^2-1)\left(\frac{dy}{dx}\right)^2 = 25y^2$

Using (–) sign $\frac{dy}{dx} = -\frac{5(x-\sqrt{x^2-1})^5}{\sqrt{x^2-1}} = -\frac{5y}{\sqrt{x^2-1}}$

$\Rightarrow (x^2-1)\left(\frac{dy}{dx}\right)^2 = 25y^2$ ......(iv)

Again differentiating both sides w.r.t. x get

$2x\left(\frac{dy}{dx}\right)^2 + (x^2-1)2\frac{dy}{dx}.\frac{d^2y}{dx^2} = 50y\frac{dy}{dx}$, dividing

by 2 $\frac{dy}{dx}$ on both sides,

$x\frac{dy}{dx} + (x^2-1)\frac{d^2y}{dx^2} = 25y \; \left\{\because \frac{dy}{dx} \neq 0\right\}$

24. **(b)** Let $x = a\cos^2\theta + b\sin^2\theta$

$\therefore \quad a - x = a - a\cos^2\theta - b\sin^2\theta = (a-b)\sin^2\theta$

and $x - b = a\cos^2\theta + b\sin^2\theta - b = (a-b)\cos^2\theta$

$\therefore \quad y = (a-b)\sin\theta\cos\theta - (a-b)\tan^{-1}\tan\theta$

$= \frac{a-b}{2}\sin 2\theta - (a-b)\theta$

$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(a-b)\cos 2\theta - (a-b)}{(b-a)\sin 2\theta}$

$= \frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta = \sqrt{\frac{a-x}{x-b}}$

25. **(a)** Let $f(x) = lx^2 + mx + n \Rightarrow f'(x) = 2\ell x + m$

Now, $f(1) = f(-1) \Rightarrow \ell + m + n = \ell - m + n$

$\Rightarrow m = 0 \qquad \therefore f'(x) = 2lx$

$\therefore \quad f'(a_1) = 2\ell a, f'(a_2) = 2\ell a_2, f'(a_3) = 2\ell a_3$

As a_1, a_2, a_3 are in A.P.

$\therefore\ f'(a_1), f'(a_2), f'(a_3)$ are in A. P.

26. **(b)** $g(x) = \dfrac{2(\sin x - \sin^n x) + |\sin x - \sin^n x|}{2(\sin x - \sin^n x) - |\sin x - \sin^n x|}$

for $0 < n < 1$, $\sin x < \sin^n x$, so $g(x) = \dfrac{1}{3}$ and for

$n > 1$, $\sin x > \sin^n x$, so $g(x) = 3$

$\therefore$ for $n > 1$, $f(x) = 3,\ x \in (0, \pi)$

$\therefore$ f (x) is continuous and differentiable at $x = \dfrac{\pi}{2}$

and for $0 < n < 1$

$$f(x) = \begin{cases} \left[\frac{1}{3}\right] = 0 & x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \\ 3, & = \frac{\pi}{2} \end{cases}$$

$\therefore f(x)$ is not continuous at $x = \dfrac{\pi}{2}$. Hence $f(x)$ is also not differentiable at $x = \dfrac{\pi}{2}$

27. **(b)** Here $\dfrac{dy}{dx} = \dfrac{d}{dx} f\left(\dfrac{2x-1}{x^2+1}\right)$

$$= f'\left(\frac{2x-1}{x^2+1}\right)\left\{\frac{(x^2+1).(2) - (2x-1)(2x)}{(x^2+1)^2}\right\}$$

$$= \sin\left(\frac{2x-1}{x^2+1}\right)^2 . \frac{2(x^2+1-2x^2+x)}{(x^2+1)^2}$$

$$= \sin\left(\frac{2x-1}{x^2+1}\right)^2 . \frac{2(1+x-x^2)}{(x^2+1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2(1+x-x^2)}{(x^2+1)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$$

28. **(d)** $\because\ x = \text{cosec}\theta - \sin\theta$

$\Rightarrow\ x^2 + 4 = (\text{cosec}\theta - \sin\theta)^2 + 4$

$= (\text{cosec}\theta + \sin\theta)^2$ .... (1)

and $y^2 + 4 = (\text{cosec}^n\theta - \sin^n\theta)^2 + 4$

$= (\text{cosec}^n\theta + \sin^n\theta)^2$ ....(2)

Now, $\dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$

$$= \frac{n(\text{cosec}^{n-1}\theta)(-\text{cosec}\theta\cot\theta) - n\sin^{n-1}\theta\cos\theta}{-\text{cosec}\theta\cot\theta - \cos\theta}$$

$$= \frac{n(\text{cosec}^n\theta\cot\theta + \sin^{n-1}\theta\cos\theta}{(\text{cosec}\theta\cot\theta + \cos\theta)}$$

$$= \frac{n\cot\theta(\text{cosec}^n\theta + \sin^n\theta)}{\cot\theta(\text{cosec}\theta + \sin\theta)}$$

$$= \frac{n(\text{cosec}^n\theta + \sin^n\theta)}{(\text{cosec}\theta + \sin\theta)} = \frac{n\sqrt{y^2+4}}{\sqrt{x^2+4}}$$

[From (1) and (2)]

Squaring both side, we get $\left(\dfrac{dy}{dx}\right)^2 = \dfrac{n^2(y^2+4)}{(x^2+4)}$

or $(x^2+4)\left(\dfrac{dy}{dx}\right)^2 = n^2(y^2+4)$

29. **(d)** Given $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$

Let $f'(1) = a, f''(2) = b, f'''(3) = c$ ...... (i)

$\therefore\ f(x) = x^3 + ax^2 + bx + c$

$\Rightarrow\ f'(x) = 3x^2 + 2ax + b$

or $f'(1) = 3 + 2a + b$ ...... (ii)

$\Rightarrow\ f''(x) = 6x + 2a$

or $f'(2) = 12 + 2a$ ....(iii)

$\Rightarrow\ f'''(x) = 6$ or $f'''(3) = 6$ ....(iv)

From (i) and (iv), $c = 6$

From (i), (ii), and (iii) we have, $a = -5, b = 2$

$\therefore\ f(x) = x^3 - 5x^2 + 2x + 6$

30. **(b)** Given, $y = \tan^{-1}\dfrac{1}{x^2+x+1}$

$$+\tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \text{..... to } n \text{ terms}$$

$$= \tan^{-1}\left\{\frac{1}{1+x(x+1)}\right\} + \tan^{-1}\left\{\frac{1}{1+(x+1)(x+2)}\right\}$$

$$+ \tan^{-1}\left\{\frac{1}{1+(x+2)(x+3)}\right\}$$

$$+ \text{.........} + \tan^{-1}\left\{\frac{1}{1+(x+(n-1))(x+n)}\right\}$$

$$= \tan^{-1}\left\{\frac{(x+1)-x}{1+(x+1)x}\right\} + \tan^{-1}\left\{\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}\right\}$$

$$+ \tan^{-1}\left\{\frac{(x+3)-(x+2)}{1+(x+3)(x+2)}\right\}$$

$$+ \text{.........} + \tan^{-1}\left(\frac{(x+n)-(x+n-1)}{1+(x+n)(x+n-1)}\right)$$

$$\therefore\ y = \{\tan^{-1}(x+1) - \tan^{-1}(x)$$

$$+\{\tan^{-1}(x+2) - \tan^{-1}(x+1)\}$$

$$+\{\tan^{-1}(x+3) - \tan^{-1}(x+2)$$

$$+ \text{........} + \{\tan^{-1}(x+n) - \tan^{-1}(x+(n-1)\}$$

So, $y = \tan^{-1}(x+n) - \tan^{-1}(x)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

31. **(b)** Since, $f'(x) = \sin x + \sin 4x.\cos x$

$$\therefore\ f'\left(2x^2+\frac{\pi}{2}\right) = \frac{d\left\{f\left(2x^2+\frac{\pi}{2}\right)\right\}}{dx}$$

$$= \frac{d\left\{f\left(2x^2+\frac{\pi}{2}\right)\right\}}{d\left(2x^2+\frac{\pi}{2}\right)} \cdot \frac{d\left(2x^2+\frac{\pi}{2}\right)}{dx}$$

$$= \left\{\sin\left(2x^2+\frac{\pi}{2}\right) + \sin(8x^2+2\pi).\cos\left(2x^2+\frac{\pi}{2}\right)\right\}4x$$

$$= \{\cos 2x^2 + \sin 8x^2.(-\sin 2x^2)\}4x$$

$$= \{\cos 2x^2 - \sin 8x^2 \sin 2x^2)4x$$

32. **(b)** Here, $y = (1+x)(1+x^2)(1+x^4)\text{.......}(1+x^{2^n})$

$$= \frac{1}{1-x}\{(1-x)(1+x)(1+x^2)(1+x^4)\text{......}(1+x^{2^n})\}$$

$$= \frac{1}{1-x}\ \{(1-x^2)(1+x^2)(1+x^4)\text{.......}(1+x^{2^n})\}$$

$$= \frac{1}{1-x}\ \{(1-x^4)(1+x^4)\text{..........}(1+x^{2^n})\}$$

$$\therefore\ y = \frac{(1-x^{2^{n+1}})}{(1-x)}$$

$$\therefore\ \frac{dy}{dx} = \frac{-2^{n+1}.(x^{2^{n+1}-1})\ \ (1-x)-(1-x^{2^{n+1}}).(-1)}{(1-x)^2}$$

Now, $\frac{dy}{dx}$ at $x=0$;

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{-2^{n+1}.(0).(1)+(1-0)}{(1-0)^2} = 1.$$

33. **(a)** Here $f(x)+f(y)+f(x).f(y)=1$

Put $x=y=0$, we get; $2f(0)+f(0)^2 = 1$

$\Rightarrow\ \{f(0)\}^2 + 2f(0) - 1 = 0$

$$f(0) = \frac{-2\pm\sqrt{4+4}}{2} = -1-\sqrt{2} \text{ or, } -1+\sqrt{2}.$$

as $f(0)>0 \Rightarrow f(0) = \sqrt{2}-1$

Again, putting $y=x$ in (i);

$$2f(x)+\{f(x)\}^2 = 1$$

On differentiating with respect to x

$$2f'(x)+2f(x)f'(x) = 0 \Rightarrow 2f'(x)(1+f(x)) = 0$$

$\Rightarrow\ f'(x)=0;$ because $f(x)>0$

Thus $f'(x)=0$ when $f(x)>0$.

34. **(b)** We have

$$(1+x+x^2)(1-x+x^2) = (1-x^2)^2 - x^2 = 1+x^2+x^4$$

Now, $(1+x+x^2)(1-x+x^2)(1-x^2+x^4)$

$$= (1+x^2+x^4)(1-x^2+x^4) = 1+x^4+x^8$$

Continuing in this way, we have

$$(1+x+x^2)\ (1-x+x^2)\ (1-x^2+x^4)$$

$$(1-x^4+x^8)\text{........}(1-x^{2^{n-1}}+x^{2^n}) = (1+x^{2^n}+x^{2^{n+1}})$$

Now, for $|x|<1, x^\infty = 0$, Taking Limit $n\to\infty$ in (1), we get

$$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)$$

$$(1-x^4+x^8)\text{.....} = 1$$

Taking logarithm of both sides, we get

$$\Rightarrow\ \ln\ (1+x+x^2) + \ln\ (1-x+x^2) + \ln$$

$$(1-x^2+x^4) + \ln\ (1-x^4+x^8) + \text{..........} = 0$$

Differentiating both sides w.r.t. x, we get

$$\Rightarrow \quad \frac{1+2x}{1+x+x^2}+\frac{-1+2x}{1-x+x^2}+\frac{-2x+4x^3}{1-x^2+x^4}$$

$$+\frac{-4x^3+8x^7}{1-x^4+x^8}+ \ldots\ldots\ldots\ldots = 0$$

Hence,

$$\frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1-x^2+x^4}+\frac{4x^3 8x^7}{1-x^4+x^8}+\ldots.. = \frac{1+2x}{1+x+x^2}$$

35. (a) $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(x)}{2}$, this holds for any real x, y and y is independent of x i.e., $\frac{dy}{dx}=0$.

On differentiating w.r.t. x, we get

$$f'\left(\frac{x+y}{2}\right)\frac{1}{2}\left(1+\frac{dy}{dx}\right) = \frac{1}{2}\left\{f'(x)+f'(y).\frac{dy}{dx}\right\}$$

$$\therefore \quad \frac{1}{2}f'\left(\frac{x+y}{2}\right) = \frac{1}{2}f'(x) \qquad \left\{\text{as } \frac{dy}{dx}=0\right\}$$

or $f'\left(\frac{x+y}{2}=f'(x)\right)$

Taking $x=0$ and $y=x$ in (i), we get

$$f'\left(\frac{0+x}{2}\right)=f'(0) \Rightarrow f'\left(\frac{x}{2}\right)=a$$

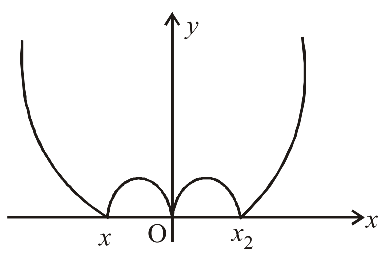
On integrating we get

$$f\left(\frac{x}{2}\right)=a\frac{x}{2}+c \text{ or } f(x)=ax+c$$

If $x=0, f(0)=b=c$

$\therefore f(x)=ax+b \Rightarrow f'(x)=a$ and $f''(x)=0$.

36. (a) For exactly three points of non-differentiability of $f(x)$, the graph should be as shown in figure.



That is the equation $x^2+ax+b=0$ has one root zero and other positive root.

$\Rightarrow b=0$ and $a<0$.

37. (c) $f(x)=\begin{cases}\cot^{-1}x, & |x|\geq 1\\ -\frac{1}{2}x+\frac{\pi}{4}-\frac{1}{2}, & -1<x<0\\ \frac{1}{2}x+\frac{\pi}{4}-\frac{1}{2}, & 0\leq x<1\end{cases}$

$f(1-0)=\frac{1}{2}+\frac{\pi}{4}-\frac{1}{2}=\frac{\pi}{4}$ and $f(1+0)=\cot^{-1}(1)=\frac{\pi}{4}$

$f(0-0)=f(0+0)=\frac{\pi}{4}-\frac{1}{2}$

$f(-1-0)=\cot^{-1}(-1)=\frac{3\pi}{4}$ and $f(-1+0)=\frac{\pi}{4}$

$\therefore$ $f(x)$ is continuous at $x=1$ and 0 but discontinuous at $x=-1$

Now $f'(x)=\begin{cases}-\frac{1}{1+x^2}, & |x|>1\\ -\frac{1}{2}, & -1<x<0\\ \frac{1}{2}, & 0<x<1\end{cases}$

$f'(1-0)=\frac{1}{2}$ and $f'(1+0)=-\frac{1}{2}$

$f'(0-0)=-\frac{1}{2}$ and $f'(0+0)=\frac{1}{2}$

Thus, $f(x)$ is not differentiable at $x=0$ and 1

Further $f(x)$ is discontinuous at $x=-1$, so not differentiable.

38. (c) $f(x)=(x^2-1)\,|(x-1)(x-2)|$

$=(x+1)(x-1)\,|x-1||x-2|$

Clearly $f'(1-0)=f'(1+0)=0$

$\Rightarrow f(x)$ is differentiable

$f(x)=\sin(|x-1|)-|x-1|$

$=\begin{cases}-\sin(x-1)+(x-1) & \text{if } x<1\\ \sin(x-1)-(x-1) & \text{if } x\geq 1\end{cases}$

$\therefore \quad f'(x)=\begin{cases}-\cos(x-1)+1 & \text{if } x<1\\ \cos(x-1)-1 & \text{if } x>1\end{cases}$

Clearly $f'(1-0)=f'(1+0)=0 \Rightarrow f(x)$ is differentiable at $x=1$

$f(x)=\tan(|x-1|)+|x-1|$

$=\begin{cases}-\tan(x-1)-x+1 & \text{if } x<1\\ \tan(x-1)+x-1 & \text{if } x\geq 1\end{cases}$

$$\therefore \quad f'(x)=\begin{cases}-\sec^2(x-1)-1, & \text{if } x<1\\ \sec^2(x-1)+1, & \text{if } x>1\end{cases}$$

Thus $f'(1-0)=-2$ and $f'(1+0)=2 \Rightarrow f(x)$ is not differentiable at $x=1$

39. **(c)** $f(x)=(x-1)(x-2)\,|(x-1)(x-2)(x-3)|$

$$+\left|\sin\left(x+\frac{\pi}{4}\right)\right|.$$

Clearly $f(x)$ is not differentiable at $x=3, \frac{3\pi}{4}, \frac{7\pi}{4}.$

40. **(b)** $g'(x)=f(x+n)=f(x+(n-3)+3)$

$=f(x+n-3)$

$=$.......................

$=f(x+3)=f(x) \Rightarrow g'(x)=f(x)$

$$\Rightarrow g'\left(\frac{7}{3}\right)=f\left(\frac{7}{3}\right)=f\left(-\frac{2}{3}+3\right)=f\left(-\frac{2}{3}\right)=7.$$

41. **(d)** $f(x)$ is non- differentiable at only those points where p sin x acquires integral value. Now $\sin x=\frac{r}{p}$ will have two solution in $(0, \pi)$ for $r \in \{1, 2..., p-1\}$ and $\sin x = 1$ will have only one solution.

$\Rightarrow$ total number of points of non-differentiability $=2(p-1)+1=2p-1.$

42. **(a)** $f(2x^2-1)=2x\,f(x)$ ......(1)

Replace x by $-x$ then

$f(2x^2-1)=-2x\,f(-x)$ ......(2)

From (1) and (2)

$2x\left[f(x)+f(-x)\right]=0 \Rightarrow f(x)+f(-x)=0$

$\therefore$ $f(x)$ is an odd function.

$\Rightarrow$ $f^{iv}(x)$ is also odd $\Rightarrow f^{iv}(0)=0$

B COMPREHENSION TYPE

1. **(a)** We have $(1+x)(1+x^2)(1+x^4).........(1+x^{2^{n-1}})$

$$=\frac{1}{1-x}(1-x^2)(1+x^2)(1+x^4)......(1+x^{2^{n-1}})$$

$$=\frac{1}{1-x}(1-x^4)(1+x^4).....(1+x^{2^{n-1}})$$

$$=.......=\frac{(1-x^{2^{n-1}})(1+x^{2^{n-1}})}{1-x}=\frac{1-x^{2^n}}{1-x}$$

Taking log of product obtained above, we get

$$\log(1+x)+\log(1+x^2)+\log(1+x^4)+.....+\log(1+x^{2^{n-1}})$$

$$=\log(1-x^{2^n})-\log(1-x)$$

Differentiating both the sides, we get

$$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+.....+\frac{2^{n-1}.x^{2^{n-1}-1}}{1+x^{2^{n-1}}}=\frac{1}{1-x}-\frac{2^n x^{2^n-1}}{1-x^{2^n}}$$

2. **(a)** If $|x|<1$, then $\lim\limits_{n\to\infty} x^{2^n}=\lim\limits_{n\to\infty} x^{2^{n-1}}=0$

So, the sum of the series to infinite terms $=\frac{1}{1-x}$

3. **(b)** $$\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}........\cos\frac{x}{2^n}=\frac{1}{2\sin\frac{x}{2^n}}$$

$$\left(2\sin\frac{x}{2^n}\cos\frac{x}{2^n}\right)\cos\frac{x}{2^{n-1}}.....\cos\frac{x}{2}$$

$$=\frac{1}{2\sin\frac{x}{2^n}}\sin\frac{x}{2^{n-1}}\cos\frac{x}{2^{n-1}}.......\cos\frac{x}{2}$$

$$=\frac{1}{2^2\sin\frac{x}{2^n}}\sin\frac{x}{2^{n-2}}......\cos\frac{x}{2}$$

$$=\frac{1}{2^{n-1}\sin\frac{x}{2^n}}\sin\frac{x}{2}\cos\frac{x}{2}=\frac{\sin x}{2^n\sin\frac{x}{2^n}}$$

Taking log and differentiating both sides of the product obtained above, we get

$$-\frac{1}{2}\tan\frac{x}{2}-\frac{1}{2^2}\tan\frac{x}{2^2}-......-\frac{1}{2^n}\tan\frac{x}{2^n}$$

$$=\cot x-\frac{1}{2^n}\cot\frac{x}{2^n}$$

$$\Rightarrow \frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+......+\frac{1}{2^n}\tan\frac{x}{2^n}$$

$$= -\cot x+\frac{1}{2^n}\cot\frac{x}{2^n}$$

4. **(d)** $\frac{\partial u}{\partial x}=c.\frac{1}{1+\frac{x^2}{y^2}}.\frac{1}{y}=\frac{cy}{x^2+y^2}.$ (y constant)

Again, $\frac{\partial u}{\partial y}=c.\frac{1}{1+\frac{x^2}{y^2}}.\frac{-x}{y^2}=\frac{-cx}{(x^2+y^2)}$ (x constant)

5. **(c)** $u=\frac{1}{(x^2+y^2+z^2)^{1/2}}$

$$x\frac{\partial u}{\partial x}=x\left[-\frac{1}{2(x^2+y^2+z^2)^{3/2}}.2x\right]-\frac{x^2}{(x^2+y^2+z^2)^{3/2}}$$

Similarly, we can find $y\frac{\partial u}{\partial y}$ and $z\frac{\partial u}{\partial z}$.

$$\therefore \sum x\frac{\partial u}{\partial x}=-\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{-1}{\sqrt{(x^2+y^2+z^2)}}=-u$$

6. **(a)** $\frac{\partial u}{\partial x}=\frac{1}{1+\left(\frac{x^2+y^2}{x+y}\right)^2}\left\{\frac{(x+y).2x-(x^2+y^2)}{(x+y)^2}\right\}$

$$=\frac{x^2+2xy-y^2}{1+(x^2+y^2)^2}$$

$\frac{\partial u}{\partial y}=\frac{-x^2+2xy+y^2}{1+(x^2+y^2)^2}$ (By symmetry)

$$\therefore \frac{\partial u}{\partial x}:\frac{\partial u}{\partial y}=\frac{x^2+2xy-y^2}{-x^2+2xy+y^2}$$

7. **(d)** $y=\frac{x^2+1}{(x-1)(x-2)(x-3)}$

Breaking into partial fractions ,

$$\frac{x^2+1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

Then, $A=\frac{1^2+1}{(1-2)(1-3)}=1$

$$B=\frac{2^2+1}{(2-1)(2-3)}=-5$$

$C=\frac{3^2+1}{(3-1)(3-2)}=5$ [Also, see partial fractions]

Thus, $y=\frac{x^2+1}{(x-1)(x-2)(x-3)}$

$$=\frac{1}{x-1}-\frac{5}{x-2}+\frac{5}{x-3}$$

$$\therefore \quad y_n=(-1)^n\, n!\frac{1}{(x-1)^{n+1}}$$

$$-5(-1)^n\, n!\frac{1}{(x-2)^{n+1}}+5(-1)^n\, n!\frac{1}{(x-3)^{n+1}}=(-1)^n\, n!$$

$$\left[\frac{1}{(x-1)^{n+1}}-\frac{5}{(x-2)^{n+1}}+\frac{5}{(x-3)^{n+1}}\right]$$

8. **(c)** We have,

$$D^n(x^2y_2)=x^2y_{n+2}+n.(2x)y_{n+1}+\frac{n(n-1)}{2!}.2y_n$$

$$D^n(xy_1)=xy_{n+1}+n.y_n$$

$$D^n(y)=y_n$$

Adding, we get

$$0=x^2y_{n+2}+(2n+1)\,xy_{n+1}+(n^2+1)\,y_n$$

$$\therefore\ x^2y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=-y_n$$

9. **(c)** If $y=x^{2n}$, then

$$y_n=\frac{(2n)!}{(2n-n)!}x^{2n-n}=\frac{(2n)!}{n!}x^n \quad ....(i)$$

Again regard x^{2n} as $x^n.\ x^n$.

$$\therefore\ D^n\ (x^{2n})=D^n(x^n.x^n)$$

$$=D^n(x^n).x^n+{}^nC_1D^{n-1}(x^n).Dx^n$$

$$+{}^nC_2D^{n-2}(x^n).D^2x^n+...........x^n.\ D^n\ (x^n)$$

$$=\quad n!\ x^n+n.\ \frac{n!}{1!}x.nx^{n-1}$$

$$+\frac{n(n-1)}{2!}\frac{n!}{2!}x^2\times n\,(n-1)\ x^{n-2}+..........x^n\ .n!$$

$$= x^n \cdot n! \left[1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{2!.2!}+..........\right]$$

$$= \frac{(2n)!}{n!} x^n \qquad [\text{by (i)}]$$

$$\therefore\ 1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2.2^2}+..........=\frac{(2n)!}{(n!)^2}$$

10. **(c)** Given $f(x+y)=f(x)f(y)$ ....(1)
Put $x=y=0 \Rightarrow f(0)=\{f(0)\}^2 \Rightarrow f(0)=1$
[since $f(0)\neq 0$]
Differentiating both the sides of (1), we get

$$f'(x+y)\left\{1+\frac{dy}{dx}\right\}=f(x)\ f'(y)\frac{dy}{dx}+f'(x)f(y)$$

Above holds for all x, y, so $\frac{dy}{dx}=0$

$\therefore f'(x+y)=f'(x)f(y)$

11. **(b)** Put $x=0$, we get $f'(y)=f'(0)f(y)$

$\therefore \frac{f'(y)}{f(y)}=f'(0)=2\ \forall\ y$

12. **(d)** $\frac{f'(x)}{f(x)}=2$. Integrating we get $\log_e|f(x)|=2x+C$.
Putting $x=0$, we get $C=0$

$\therefore f(x)=\pm e^{2x} \qquad \because f'(0)=2$

$\Rightarrow f(x)=e^{2x}$

$$\lim_{x\to 0}\frac{f(x)-f(-x)}{x}=\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{x}=2-(-2)=4$$

13. **(c)** Given $f(x+y^{2n+1})=f(x)+\{f(y)\}^{2n+1}$ ...(1)
Put $x=y=0$

$\Rightarrow f(0)=f(0)+\{f(0)\}^{2n+1}\Rightarrow f(0)=0$

Next put $x=0, y=1$, we get,

$f(0+1)=f(0)+\{f(1)\}^{2n+1}\Rightarrow f(1)[\{f(1)\}^{2n}-1]=0$

$f(1)\neq 0 \Rightarrow f(1)=1$. If $f(1)=0$, then $f(x)=0\ \forall\ x$ identically.
Put $y=1$ in the functional equation we get

$f(x+1)=f(x)+\{f(1)\}^{2n+1}\Rightarrow f(x+1)=f(x)+1$

$\therefore f(2)=2,\ f(3)=3,\ f(4)=4$ *and* $f(5)=5$

14. **(d)** Differentiating (1), we get $f'(x+y^{2n+1})=f'(x)$

[x and y are independent so, $\frac{dy}{dx}=0$]

$\Rightarrow f'(x)$ is constant, say $f'(x)=k$
Integrating we get $f(x)=kx+c$
Now $f(0)=0 \Rightarrow c=0$ and $f(1)=1 \Rightarrow k=1$

$\therefore f(x)=x \Rightarrow f'(x)=1\ \forall\ x$

15. **(a)** As $f(x)=x$, so it is everywhere continuous and differentiable.

C REASONING TYPE

1. **(a)** $f(x)=\cos^2 x+\cos^2\left(x+\frac{\pi}{3}\right)-\cos x\cos\left(x+\frac{\pi}{3}\right)$

$$=\frac{1+\cos 2x}{2}+\frac{1+\cos\left(2x+\frac{2\pi}{3}\right)}{2}-\frac{1}{2}[2\cos x\cos\left(x+\frac{\pi}{3}\right)$$

$$=1+\cos\left(2x+\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)-\frac{1}{2}\left[\cos\left(2x+\frac{\pi}{3}\right)+\cos\frac{\pi}{3}\right]$$

$$=1+\frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right)-\frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right)-\frac{1}{4}=\frac{3}{4}$$

$f'(x)=0$
Derivative of constant function is zero.

2. **(d)** $\lim_{x\to 0} f(x)=\lim_{x\to 0} x^n\sin\left(\frac{1}{x}\right)=0$ for positive integer n
Now f (0) does not exist, hence function is not continuous at $x=0$

3. **(d)** $f'(x)=\frac{-\sin x}{\sqrt{1-\cos^2 x}}=\frac{-\sin x}{|\sin x|}$

$$=\begin{cases}1 & \text{if } \sin x<0\\ -1 & \text{if } \sin x>0\\ \text{does not exist} & \text{if sinx}=0\end{cases}$$

4. **(d)** $f(x)=|x|^3=\begin{cases}-x^3 & \text{if } x<0\\ x^3 & \text{if } x\geq 0\end{cases}$

Clearly $f(x)$ is continuous at $x=0$

$$f'(x)=\begin{cases}-3x^2, x<0\\ 3x^2,\ x\geq 0\end{cases}$$

Thus $f'(0^+)=f'(0^-)=0$

$\therefore f(x)$ is differentiable at $x=0$
Statement 2 is clearly true.

D MULTIPLE CORRECT CHOICE TYPE

1. **(b,d)** For $x > 2, \int_0^x \{1 + |1-t|\} dt$

$$= \int_0^1 (2-t)dt + \int_1^x t\ dt = 1 + \frac{x^2}{2}$$

Thus, $f(x) = \begin{cases} 1 + \frac{x^2}{2}, & x > 2 \\ 5x - 7, & x \le 2 \end{cases}$

$$\lim_{x \to 2+} f(x) = 1 + \frac{4}{2} = 3 = f(2) = \lim_{x \to 2-} f(x)$$

$$f'(2+) = \lim_{x \to 0+} \frac{1 + (1/2)(2+h)^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{(1/2)h^2 + 2h}{h} = 2$$

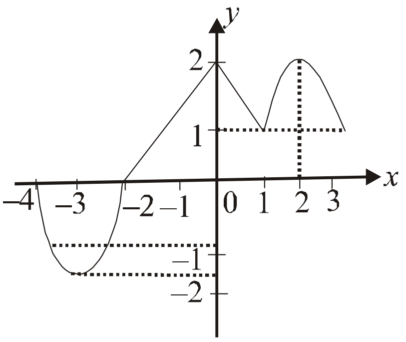
$$f'(2-) = \lim_{x \to 0-} \frac{5(2+h) - 7 - 3}{h} = 5$$

Hence f is continuous but not differentiable at $x =$ 2.

2. **(a,b,c)** $f(x) = \begin{cases} 2x^2 + 12x + 16, & -4 \le x \le -2 \\ 2 - |x|, & -2 < x \le 1 \\ 4x - x^2 - 2, & 1 < x \le 3 \end{cases}$

$$= \begin{cases} 2(x+3)^2 - 2, & -4 \le x \le -2 \\ 2 - |x|, & -2 < x \le 1 \\ 2 - (x-2)^2, & 1 < x \le 3 \end{cases}$$

The graph of $y = f(x)$ is shown



Solving $2x^2 + 12x + 16 = -1$

$$\Rightarrow 2x^2 + 12x + 17 = 0,$$

we get $x = -3 - \frac{1}{\sqrt{2}}$ and $x = -3 + \frac{1}{\sqrt{2}}$

Thus we get

$$[f(x)] = \begin{cases} 0, & x = -4 \\ -1, & -4 < x \le -3 - \frac{1}{\sqrt{2}} \\ -2, & -3 - \frac{1}{\sqrt{2}} < x < -3 + \frac{1}{\sqrt{2}} \\ -1, & -3 + \frac{1}{\sqrt{2}} \le x < -2 \\ 0 & -2 \le x < -1 \\ 1, & -1 \le x < 0 \\ 2, & x = 0 \\ 1, & 0 < x < 2 \\ 2, & x = 2 \\ 1, & 2 < x \le 3 \end{cases}$$

$\therefore$ $[f(x)]$ is discontinuous at

$$x = -4, -3 - \frac{1}{\sqrt{2}}, -3 + \frac{1}{\sqrt{2}}, -2, -1, 0, 2.$$

3. **(a,b,c)** Given $F(x) = f(x) \cdot g(x)$....(1)

Differentiating both sides w.r.t., x we get

$$F'(x) = f'(x).g(x) + g'(x).f(x)$$

$$\Rightarrow F'(x) = f'(x)g'(x)\left[\frac{f(x)}{f'(x)} + \frac{g(x)}{g'(x)}\right]$$

$$\Rightarrow F' = c\left[\frac{f}{f'} + \frac{g}{g'}\right] \Rightarrow \text{(a) is correct}$$

Again differentiating both sides w.r.t., x we get

$$F''(x) = f''(x).g(x) + g''(x).f(x) + 2f'(x).g'(x)$$

$$\Rightarrow F''(x) = f''(x).g(x) + g''(x).f(x) + 2c \text{ .....(2)}$$

Dividing both sides by $F(x) = f(x) \,.\, g(x)$

$\{\because f'(x).g'(x) = c\}$

then $\dfrac{F''(x)}{F(x)} = \dfrac{f''(x)}{f(x)} + \dfrac{g''(x)}{g(x)} + \dfrac{2c}{f(x)g(x)}$

or $\dfrac{F''}{F} = \dfrac{f''}{f} + \dfrac{g''}{g} + \dfrac{2c}{fg} \Rightarrow$ (b) is correct.

Again given $f'(x)\, g'(x) = c$

Differentiating both sides w.r.t., x we get

$f'(x)g''(x) + g'(x)f''(x) = 0$

From (2), $F''(x) = f''(x) \,.\, g(x) + g''(x) \,.\, f(x) + 2c$

Differentiating both sides w.r.t., x we get

$F'''(x) = f''(x).g'(x) + f'''(x).g(x) + g''(x).f'(x) + f(x).g'''(x) + 0$

$= f'''(x) \,.\, g(x) + g'''(x) \,.\, f(x) + 0$ [from (3)]

Now dividing both sides by $F(x) = f(x)\, g(x)$

Then $\dfrac{F'''(x)}{F(x)} = \dfrac{f'''(x)}{f(x)} + \dfrac{g'''(x)}{g(x)}$

or $\dfrac{F'''}{F} = \dfrac{f'''}{f} + \dfrac{g'''}{g}$

4. **(a,b)** Given that $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ ....(1)

Putting $x = 0$ and $x = 1$ in (1) we get

$f(0) = f'''(3)$ and

$f(1) = 1 + f'(1) + f''(2) + f'''(3)$

$\therefore\ f(1) - f(0) = 1 + f'(1) + f''(2)$ .....(2)

Differentiating both sides of (1) w.r.t. x we get

$f'(x) = 3x^2 + 2xf'(1) + f''(2)$ ......(3)

and $f''(x) = 6x + 2f'(1)$ ....(4)

also $f'''(x) = 6$ ......(5)

Putting $x = 1, 2, 3$ in (3), (4), (5) respectively, we get

$f'(1) = 3 + 2f'(1) + f''(2)$

or $f'(1) + f''(2) = -3$ ......(6)

$\Rightarrow\ f''(2) = 12 + 2f'(1)$ or $2f'(1) - f''(2) = -12$ ......(7)

and $f'''(3) = 6$ ......(8)

Solving (6) and (7), we get

$f'(1) = -5$ and $f''(2) = 2$

Hence $f(1) - f(0) = 1 - 5 + 2 = -2$ ......(9)

Also from (1), $f(2) = 8 + 4f'(1) + 2f''(2) + f'''(3)$

$= 8 - 20 + 4 + 6 = -2$ .....(10)

Hence from (9) and (10), we get $f(2) = f(1) - f(0)$.

5. **(a, b, c)** $f(x)$ is differentiable at $x = 1$ if $1 + a = 2 + b$ .....(1)

$g(x)$ is differentiable at $x = 1$ if $3 + b = 1$ ......(2)

From (1) and (2), $b = -2$ and $a = -1$

Now $\dfrac{df}{dx} = 2$ and $\dfrac{dg}{dx} = 3 \Rightarrow \dfrac{df}{dg} = \dfrac{2}{3}$

6. **(a, b, c)** $f'(x) = \begin{vmatrix} -\sin(x+\alpha) & -\sin(x+\beta) & -\sin(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$

$+ \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$

$= 0 + 0 \Rightarrow$ f (x) is a constant function.

Thus $f(\alpha) = f(\beta) = f(\gamma)$

7. **(b,c,d)** Let $f(x) = ax^3 + bx^2 + bx + d$

The $f(1) = 0$ and $f'(1) = 0$

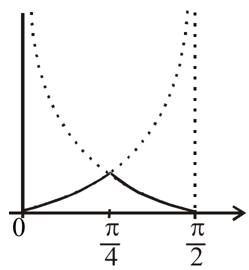
$\Rightarrow\ a + 2b + d = 0$ ....(1) and

$3a + 2b + b = 0$ ....(2)

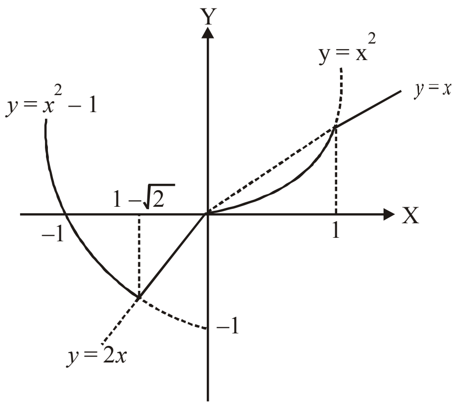
From (1) and (2) $a = d = -b$

8. **(a,d)** $f(x) = \min\{|\tan x|, |\cot x|\}$

From the graph it is clear that $f(x)$ is everywhere continuous but not derivable when $x = \dfrac{\pi}{4}$



9. **(a, c, d)** All the options can be checked easily from the graph of the function. We find that $f(x)$ is every where continuous but not differentiable at $x = 1-\sqrt{2}, 0, 1$.



Notice that $x = 1-\sqrt{2}$ is the point of intersection of $y = 2x, y = x^2 - 1\ (x < 0)$.

10. **(a, b, c)** $g(f(x)) = x \Rightarrow g'(f(x))f'(x) = 1$

$$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

Now, $f(x) = 2 \Rightarrow x^3 + 3x^2 - 33x - 33 = 2$

$$\Rightarrow x^3 + 3x^2 - 33x - 35 = 0$$

$$\Rightarrow x^3 - 5x^2 + 8x^2 - 40x + 7x - 35 = 0 \text{ or}$$

$$\Rightarrow (x-5)(x^2 + 8x + 7) = 0$$

$$\Rightarrow (x-5)(x+1)(x+7) = 0$$

$$\therefore x = -7, -1, 5$$

Thus we have,

$$k = f'(-1) = 3(-1)^2 + 6(-1) - 33$$

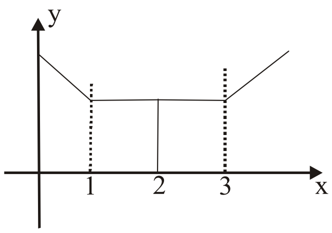
$$= 3 - 6 - 33 = -36$$

$$k = f'(-7) = 3(-7)^2 + 6(-7) - 33$$

$$= 147 - 63 - 33 = 51$$

$$k = f'(5) = 3.5^2 + 6.5 - 33 = 75 + 30 - 33 = 72$$

11. **(a)** $f(x) = \begin{cases} 3-2x, & x < 1 \\ 1, & 1 \le x < 2 \\ 2x-3, & 2 \le x \le 3 \end{cases}$



Thus $g(x) = \begin{cases} f(x), & 0 \le x < 1 \\ 1, & 1 \le x \le 3 \\ x-2, & x > 3 \end{cases}$

$\Rightarrow g(x) = \begin{cases} 3-2x, & 0 \le x < 1 \\ 1, & 1 \le x \le 3 \\ x-2, & x > 3 \end{cases}$

$\Rightarrow$ Thus $g(x)$ is not differentiable at $x = 1$

12. **(a,b,c,d)** $1 + x$ is never zero, so $1 + f(x)$ is never zero. It is 1 for $x = 0$, so it is always positive.

Hence $f''(x)$ is always positive. $f'(0) = 0$, so $f'(0) > 0$ for all $x > 0$ and hence f is strictly increasing.

So, in particular, $1 + f(x) \ge 2$ for all x.

We have $f''(x) \le \frac{(1+x)}{2}$.

Integrating , $f'(x) \le f'(0) + \frac{x}{2} + \frac{x^2}{4} = \frac{x}{2} + \frac{x^2}{4}$.

Integrating , again, $f(x) \le f(0) + \frac{x^2}{4} + \frac{x^3}{12}$. Hence

$$f(1) \le 1 + \frac{1}{4} + \frac{1}{12} = \frac{4}{3}.$$

E MATRIX-MATCH TYPE

1. **A-r; B-p; C-q; D-s**

(A) Put $x = \cos 2\theta$

$$\Rightarrow \frac{d}{dx}\Sigma(\sin^2 .\cot^{-1}(\cot\theta))$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1+\cos 2\theta}{2}\right) = \frac{d}{dx}\left(\frac{1-x}{2}\right) = \frac{-1}{2}$$

(B) $\log(h(x)) = e^x \Rightarrow \frac{1}{h(x)}h'(x) = e^x = \log(h(x))$

(C) $f'(x) = \sqrt{a}.\frac{1}{2\sqrt{x}} + a\sqrt{a}.\left(\frac{-1}{2}\right)(x)^{\frac{-3}{2}}$;

$$f'(a) = \frac{1}{2} - \frac{1}{2}a\sqrt{a}.\frac{1}{a\sqrt{a}} = 0$$

(D) Take log on both sides, $\log x + \log y = n\log(x+y)$

Diff. w.r.t. 'x' we get

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = n\left(\frac{1}{x+y}\right)\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}.\frac{y}{x} = n\left(\frac{1}{x+y}\right)\left(1+\frac{y}{x}\right)$$

$$\Rightarrow \frac{2}{x} = n\left(\frac{1}{x+y}\right)\left(\frac{x+y}{x}\right) \Rightarrow n = 2$$

2. **A-p, q, B-s, C-q, D-r**

(A) $f(x) = \begin{cases} \tan^{-1}x, & |x| \geq 1 \\ \frac{x^2-1}{4}, & |x| < 1 \end{cases}$

Clearly $f(1-0) = 0$ and $f(1+0) = \frac{\pi}{4}$

$f(-1-0) = -\frac{\pi}{4}$ and $f(-1+0) = 0$

$\therefore$ $f(x)$ is discontinuous at $x = -1$ and 1, hence not differentiable also.

Again $f'(x) = \begin{cases} \frac{1}{1+x^2}, & |x| > 1 \\ \frac{2x}{4}, & |x| < 1 \end{cases}$

$f'(x)$ exist for all x except $x = \pm 1$

(B) $f(x) = \begin{cases} (x^2-4)(x^2-5x+6)+\cos x, & x \leq 2 \text{ or } x \geq 3 \\ (4-x^2)(x^2-5x+6)+\cos x, & 2 < x < 3 \end{cases}$

Clearly $f(x)$ is everywhere continuous. So,

$$f'(x) = \begin{cases} (x^2-4)(2x-5)+2x(x^2-5x+6)-\sin x, & x < 2 \text{ or } x > 3 \\ (4-x^2)(2x-5)-(x^2-5x+6)-\sin x, & 2 < x < 3 \end{cases}$$

Now $\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^+} f'(x) = -\sin 2$;

$\lim_{x\to 3^-} f'(x) = -5-\sin 3$ and $\lim_{x\to 3^-} f'(x) = -5-\sin 3$

Thus $f(x)$ is differentiable at $x = 2$ but not at $x = 3$.

(C) Differentiating both sides we get,

$$\cos(x+y)\left\{1+\frac{dy}{dx}\right\} = e^{x+y}\left\{1+\frac{dy}{dx}\right\} \Rightarrow \frac{dy}{dx} = -1$$

(D) $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)[f(h)-f(0)]}{h} = f(x)f'(0) \quad [\because f(0)=1]$$

$$\therefore \frac{f'(x)}{f(x)} = 2$$

3. **A-p, q, B-p, q, C-t, D-s**

(A) $|f(0)| \leq 0 \Rightarrow f(0) = 0$

Also, $f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{f(x)}{x}$ and

$$\left|\frac{f(x)}{x}\right| \leq |x| \Rightarrow \lim_{x\to 0}\left|\frac{f(x)}{x}\right| \leq 0 \Rightarrow f'(0) = 0$$

$\therefore f(x)$ is continuous and differentiable at $x = 0$.

(B) Put $x = y = 0$, we get $g(0) = -1$

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{g(x)-g(h)+2xh-1-g(x)}{h}$$

$$= 2x - \lim_{x\to 0}\frac{g(h)-g(0)}{h} = 2x - g'(0)$$

$\therefore g(x) = x^2 - \sqrt{3+a+a^2}\,x + C$ and

$g(0) = -1 \Rightarrow C = -1$

$\therefore g(x) = x^2 - \sqrt{3+a+a^2}\,x - 1$

So, $g(x)$ is continuous and differentiable at $x = 0$.

(C) $g(x) = f'(0) \Rightarrow x^2 - \sqrt{3+a+a^2}\,x - 1 = 0$ which clearly has two distinct roots.

(D) $g(-1)g(1) = -(3+a+a^2) \le 0 \qquad (\because 3+a+a^2 \ge 0)$

Thus $g(x) = 0$ has at least one root in $[-1, 1]$.

Also $-1 \le f(1) \le 1$. So, $f(1)$ can be a root of $g(x) = 0$. [But not necessarily]

F NUMERIC/INTEGER ANSWER TYPE

1. Ans : 4

Let $g'(1) = a$, $g''(2) = b$ ......(i)

Then, $f(x) = x^2 + ax + b$, $f(1) = 1 + a + b$

$f'(x) = 2x + a$, $f''(x) = 2$

$\therefore \quad g(x) = (1+a+b)x^2 + (2x+a).x + 2$

$= x^2(3+a+b) + ax + 2$

$\Rightarrow \quad g'(x) = 2x(3+a+b) + a$ and

$g''(x) = 2(3+a+b)$

Hence, $g'(1) = 2(3+a+b) + a$ ......(ii)

$g''(2) = 2(3+a+b)$ ......(iii)

From (i), (ii) and (iii), we have,

$a = 2(3+a+b) + a$ and $b = 2(3+a+b)$

i.e., $3+a+b = 0$ and $b + 2a + 6 = 0$

Hence, $b = 0$ and $a = -3$

So, $f(x) = x^2 - 3x$ and $g(x) = -3x + 2$

$\Rightarrow f(g(x)) = (-3x+2)^2 - 3(-3x+2) = 9x^2 - 3x - 2$

2. Ans : 9

Given $y^3 - y = 2x$

Differentiate both sides with respect to x, we get

$$(3y^2 - 1)\frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} = \frac{2}{(3y^2-1)} \quad .....(1)$$

Again differentiating both sides with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{-2.6y\dfrac{dy}{dx}}{(3y^2-1)^2} \qquad \text{Using (1) we get}$$

$$\frac{d^2y}{dx^2} = \frac{-24y}{(3y^2-1)^3} \quad .....(2)$$

Now, $\left(x^2 - \frac{1}{27}\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$

$$= \left(x^2 - \frac{1}{27}\right)\left(\frac{-24y}{(3y^2-1)^3}\right) + \frac{2x}{(3y^2-1)} \quad \text{[From (1) and (2)]}$$

$$= \left(\frac{y^2(y^2-1)^2}{4} - \frac{1}{27}\right)\left(\frac{-24y}{(3y^2-1)^3}\right) + \frac{y(y^2-1)}{(3y^2-1)}$$

$(\because y^3 - y = 2x)$

$$= \frac{\{27y^2(y^2-1)^2 - 4\}}{108}\frac{(-24y)}{(3y^2-1)^3} + \frac{y(y^2-1)}{(3y^2-1)}$$

$$= \frac{y}{9}\left\{\frac{-54y^2(y^2-1)^2 + 8}{(3y^2-1)^3} + \frac{9(y^2-1)}{(3y^2-1)}\right\}$$

$$= \frac{y}{9}\left\{\frac{-2(1+\alpha)(\alpha-2)^2 + 8}{\alpha^3} + \frac{3(\alpha-2)}{\alpha}\right\} = \frac{y}{9} \quad (\alpha = 3y^2 - 1)$$

3. Ans : 5

$$f(x) = \begin{cases} [\cos \pi x], & x \le 1 \\ |2x-3|[x-2], & x > 1 \end{cases}$$

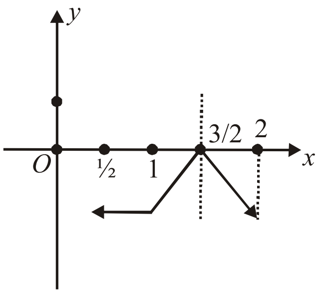
$$\text{i.e } f(x) = \begin{cases} [\cos \pi x], & 0 \le x \le 1 \\ -|2x-3| & 1 < x < 2 \\ 0, & x = 2 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } 0 < x \le \frac{1}{2} \\ -1, & \text{if } \frac{1}{2} < x \le 1 \\ 2x-3, & \text{if } 1 \le x < \frac{3}{2} \\ 3-2x, & \text{if } \frac{3}{2} \le x < 2 \\ 0, & \text{if } x = 2 \end{cases}$$

From graph,

$f(x)$ is discontinuous at $x = 0, \frac{1}{2}, 2$

$f(x)$ is non differentiable at $x = 0, \frac{1}{2}, 1, \frac{3}{2}, 2.$



4. Ans: 3

$y = [x] + |1 - x|$ for $-1 \le x < 3$

(i) For $-1 \le x < 1, y = -1 + 1 - x = -x$

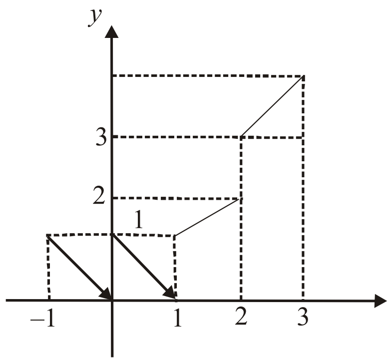
(ii) For $0 \le x < 1, \quad y = 0 + 1 - x = 1 - x$

(iii) For $1 \le x < 2, \quad y = 1 + x - 1 = x$

(iv) For $2 \le x < 3, \quad y = 2 + x - 1 = x + 1.$

Given function is discontinuous at $x = 0, 1, 2,$

Given function is not differentiable at $x = 0, 1, 2.$



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