Hyperbola

Chapter 30

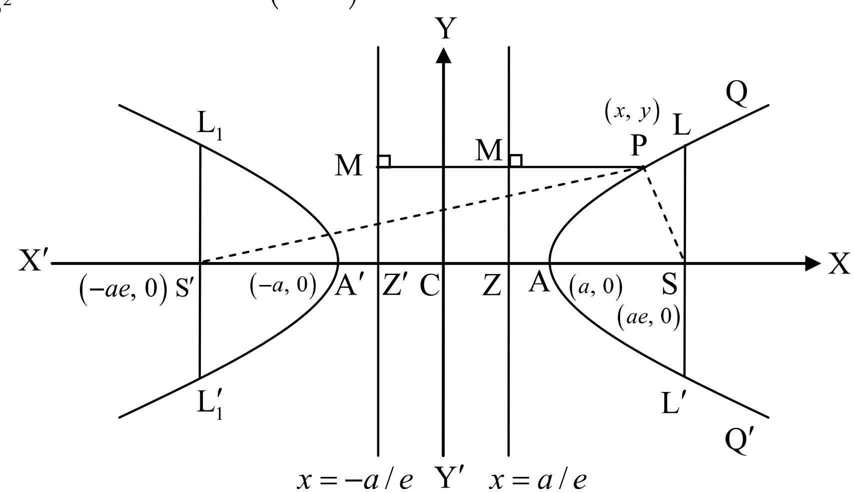
DEFINITION

A hyperbola is the locus of a point which moves in the plane in such a way that the ratio of its distance from a fixed point in the same plane to its distance from a fixed line is always constant which is greater than unity.

STANDARD EQUATION OF THE HYPERBOLA

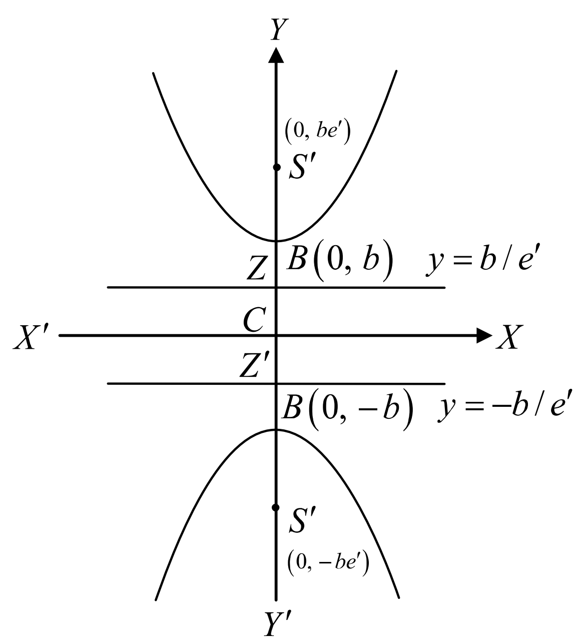
Let S be the focus, ZM be the directrix and e be the eccentricity of the hyperbola, then by definition,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \text{ where } b^2=a^2\left(e^2-1\right).$$



CONJUGATE HYPERBOLA

This hyperbola whose transverse and conjugate axis are respectively the conjugate and transverse axis of a given hyperbola is called conjugate hyperbola of the given hyperbola.



Equation of conjugate hyperbola of a given hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

Difference between both hyperbola will be clear from the following table :

Imp. terms \ Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	$(0, 0)$	$(0, 0)$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be')$
Equation of directrices	$x = \pm\, a/e$	$y = \pm\, b/e'$
Eccentricity	$e=\sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e'=\sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of latus rectum	$2b^2/a$	$2a^2/b$
Parametric co-ordinates	$(a\sec\phi, b\tan\phi)$ $0\le\phi<2\pi$	$(a\tan\phi, b\sec\phi)$ $0\le\phi<2\pi$
Focal radii	(a) If P lies on right branch $SP=ex_1-a$, $S'P=ex_1+a$ (b) If P lies on left branch $SP=-ex_1+a$, $S'P=-ex_1-a$	(a) If P lies on upper branch $SP=e'y_1-b$, $S'P=e'y_1+b$ (b) If P lies on lower branch $SP=-e'y_1+b$, $S'P=-e'y_1-b$
Difference of focal radii $\lvert S'P-SP\rvert$	$2a$	$2b$
Tangent at the vertices	$x=-a$, $x=a$	$y=-b$, $y=b$
Equation of the transverse axis	$y=0$	$x=0$
Equation of conjugate axis	$x=0$	$y=0$

Note :

(a) If e, e' are eccentricity of hyperbola & conjugate hyperbola then $\frac{1}{e^2}+\frac{1}{e'^2}=1$.

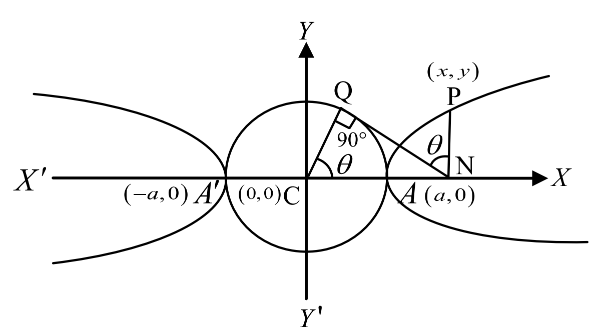
(b) Foci of a hyperbola & conjugate hyperbola are concyclic.
i.e. $ae = be'$

AUXILIARY CIRCLE OF HYPERBOLA

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the hyperbola, than equation of the auxiliary circle is

$$x^2+y^2=a^2.$$

Let $\angle QCN=\theta$. Here P and Q are the corresponding points on the hyperbola and the auxiliary circle $(0\le\theta<2\pi)$.



PARAMETRIC EQUATIONS OF HYPERBOLA

The equation $x = a\sec\theta$ and $y = b\tan\theta$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Thus $(a\sec\theta, b\tan\theta)$ lies on the hyperbola for all values of θ.

POSITION OF A POINT WITH RESPECT TO A HYPERBOLA

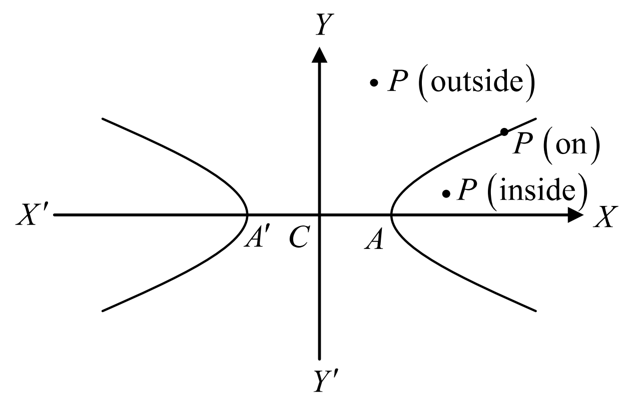
Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then $P(x_1, y_1)$ will lie inside, on or outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive, zero or negative.

i.e. $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0 \Rightarrow$ inside

$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 = 0 \Rightarrow$ on

$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0 \Rightarrow$ outside



INTERSECTION OF A LINE AND A HYPERBOLA

The straight line $y = mx + c$ will cut the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in two points may be real, coincident or imaginary according as $c^2 >, =, < a^2m^2 - b^2$.

Condition of tangency : If straight line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c^2 = a^2m^2 - b^2$.

EQUATIONS OF TANGENT IN DIFFERENT FORMS

1. **Point form** : The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

2. **Parametric form** : The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta, b\tan\theta)$ is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1.$$

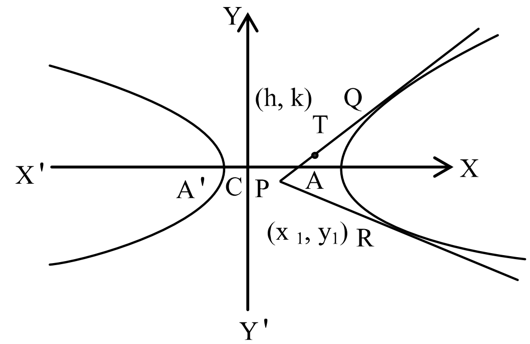
3. **Slope form :** The equations of tangents of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

The coordinates of points of contacts are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right).$$

EQUATION OF PAIR OF TANGENTS



If $P(x_1, y_1)$ be any point outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then a pair of tangents PQ, PR can be drawn to it from P. The equation of pair of tangents PQ and PR is $SS_1 = T^2$ where

$$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1, S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1, T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

DIRECTOR CIRCLE

The director circle is the locus of points from which perpendicular tangents are drawn to the given hyperbola. The equation of the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$x^2 + y^2 = a^2 - b^2.$$

EQUATIONS OF NORMAL IN DIFFERENT FORMS

1. **Point form : The equation of normal to the hyperbola**

   $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$.

   The equation of normal at the point (x_1, y_1) to the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

   $$\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}$$

2. **Parametric form:** The equation of normal at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

   $$ax\cos\theta + by\cot\theta = a^2 + b^2$$

3. **Slope form:** The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of the slope m of normal is

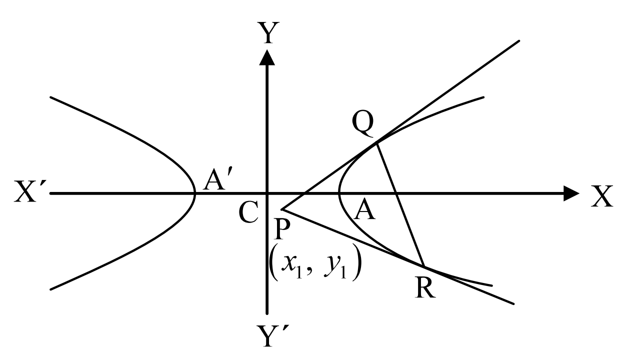
   $$y = mx \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2 m^2}}$$

4. **Condition for normality :** If $y = mx + c$ is the normal of

   $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c = \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - m^2 b^2}}$ or $c^2 = \frac{m^2 (a^2 + b^2)^2}{(a^2 - m^2 b^2)}$, which is condition of normality

Equation of chord of contact of tangents drawn from a point to a hyperbola

Let PQ and PR be tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ drawn from any external point $P(x_1, y_1)$. Then equation of chord of contact QR is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ or $T = 0$

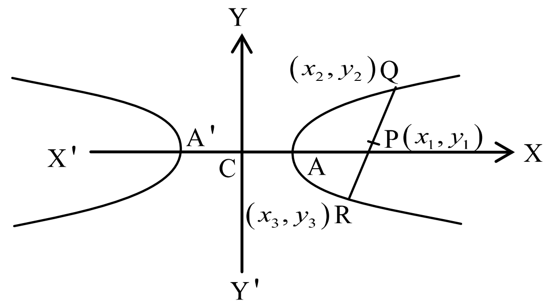


Equation of the chord of the hyperbola whose mid point (x_1, y_1) is given

Equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, bisected at the given point (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$

i.e. $T = S_1$



Equation of the chord joining two points on the hyperbola

The equation of the chord joining the points $P(a\sec\phi_1, b\tan\phi_1)$ and $Q(a\sec\phi_2, b\tan\phi_2)$ is

$$\frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

Point of intersection of tangents

Point of intersection of tangents at $P(a\sec\phi_1, b\tan\phi_1)$ and $Q(a\sec\phi_2, b\tan\phi_2)$ is

$$\left(a\frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)},\ b\frac{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}\right)$$

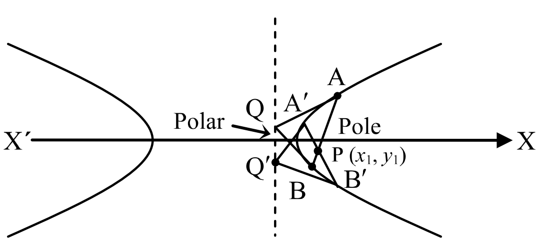
Point of intersection of normals

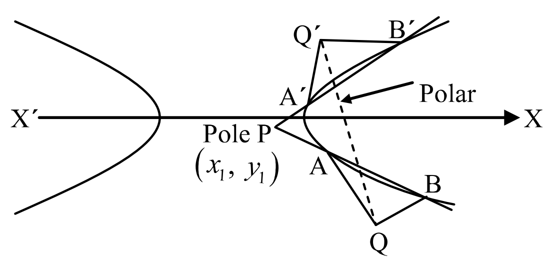
Coordinates of point of intersection of normals at $P(a\sec\phi_1, b\tan\phi_1)$ and $Q(a\sec\phi_2, b\tan\phi_2)$ is

$$\left(\frac{a^2+b^2}{a}\sec\phi_1\sec\phi_2\frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)},\ -\frac{a^2+b^2}{b}\tan\phi_1\tan\phi_2\frac{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}\right).$$

POLE AND POLAR

The locus of the point of intersection of the tangents at the end of a variable chord drawn from a fixed point P on the hyperbola is called the polar of the given point P with respect to the hyperbola and the point P is called the pole of the polar. The equation of the required polar with (x_1, y_1) as its pole is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.





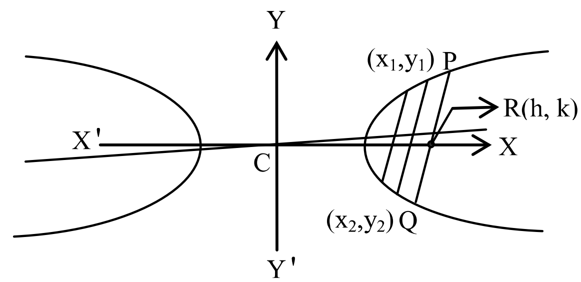
PROPERTIES OF POLE AND POLAR

1. If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be **conjugate points**.

2. If the pole of a line $lx+my+n=0$ lies on the another line $l_1x+m_1y+n_1=0$ then pole of the second line will lie on the first and such lines are said to be **conjugate lines**.
3. Pole of a given line is same as point of intersection of tangents as its extremities.

DIAMETER OF THE HYPERBOLA

The locus of the middle point of a system of parallel chords of a hyperbola is called a diameter and the point where the diameter intersects the hyperbola is called the vertex of the diameter.



Let $y=mx+c$ system of parallel chords to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for different chords then the equation of diameter of the hyperbola is $y=\frac{b^2x}{a^2m}$, which is passing through (0, 0).

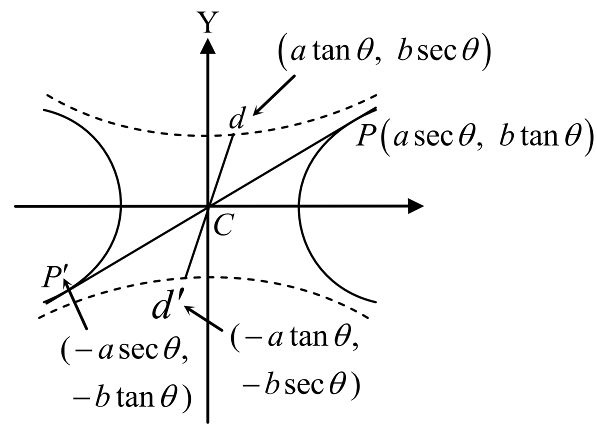
Conjugate diameter : Two diameter are said to be conjugate when each bisects all chords parallel to the other.

If $y=m_1x,\ y=m_2x$ be conjugates, diameters, then $m_1m_2=\frac{b^2}{a^2}$.

Note : If the two extremities of a diameter lie in the first and third quadrants, the extremities of the conjugate diameter also lie in the first and third quadrants.

The coordinates of the four extremities of two conjugate diameters are shown in the adjoining figure.

Caution : The extremities d and d' of the conjugate diameter do not lie on the hyperbola.



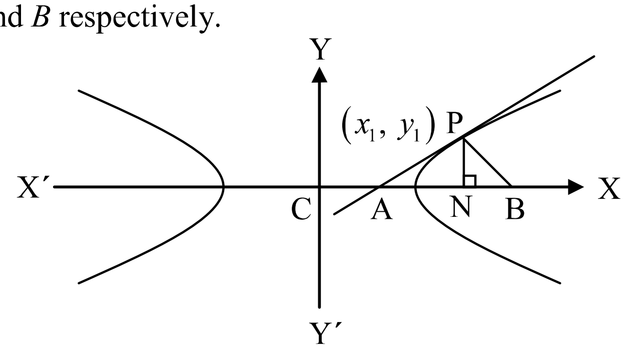
Subtangent and Subnormal of the hyperbola

Let the tangent and normal at $P(x_1,\ y_1)$ meet the x-axis at *A* and *B* respectively.

Length of subtangent $AN=CN-CA=x_1-\frac{a^2}{x}$.

Length of subnormal $BN=CB-CN=\frac{(a^2+b^2)}{a^2}x_1-x_1$

$=\frac{b^2}{a^2}x_1=(e^2-1)x_1$.



ASYMPTOTES OF A HYPERBOLA

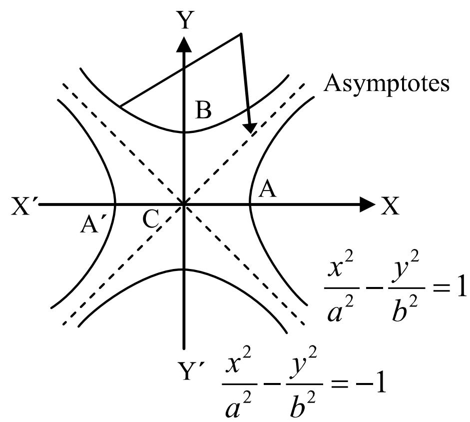
If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called an **asymptote of the hyperbola**.

The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $y=\pm\frac{b}{a}x$ or $\frac{x}{a}\pm\frac{y}{b}=0$.

SOME IMPORTANT POINTS ABOUT ASYMPTOTES

1. The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.

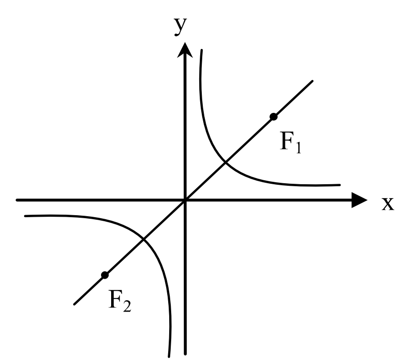
2. When $b = a$ i.e., the asymptote of rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$, which are at right angles.
3. A hyperbola and its conjugate hyperbola have the same asymptotes.



4. The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only i.e., Hyperbola – Asymptotes = Asymptotes – Conjugate hyperbola or
$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right) - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} + 1\right)$$
5. The asymptotes pass through the centre of the hyperbola.
6. The bisector of the angles between the asymptotes are the coordinate axes.
7. The angle between the asymptotes of the hyperbola $S = 0$ i.e., $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\frac{b}{a}$ or $2\sec^{-1} e$.
8. Asymptotes are equally inclined to the axes of the hyperbola.

RECTANGULAR OR EQUILATERAL HYPERBOLA

1. **Definition :** A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola. The eccentricity of rectangular hyperbola is always $\sqrt{2}$.
   The general equation of second degree represents a rectangular hyperbola if $\Delta \neq 0, h^2 > ab$ and coefficient of $x^2 + \text{coefficiant of } y^2 = 0$
2. Rotating the axes by an angle $-\pi/4$ about the same origin the equation of rectangular hyperbola $x^2 - y^2 = a^2$ reduces to $xy = c^2 \left(= \frac{a^2}{2}\right)$.
3. **Parametric co-ordinates of a point on the hyperbola xy = c² If t is non zero variable , the co ordinates of any point on the rectangular** hyperbola xy = c^2 can be written as (ct, c/t). The point (ct, c/t) on the hyperbola $xy = c^2$ is generally referred as the point 't'.
   For rectangular hyperbola $xy = c^2$, the coordinates of foci are $(c\sqrt{2}, c\sqrt{2})$ and $(-c\sqrt{2}, -c\sqrt{2})$ directrices are $x + y = \pm c\sqrt{2}$.



4. **Equation of the chord joining points t_1 and t_2 :** The equation of the chord joining two points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy = c^2 \Rightarrow x + y t_1 t_2 = c(t_1 + t_2)$.

5. **Equation of the tangents in different forms**

(i) Point form: The equation of tangent at (x_1, y_1) to the hyperbola

$$xy = c^2 \text{ is } xy_1 + yx_1 = 2c^2 \text{ or } \frac{x}{x_1} + \frac{y}{y_1} = 2.$$

(ii) Parametric form: The equation of the tangents at $\left(ct, \frac{c}{t}\right)$ to the hyperbola

$$xy = c^2 \text{ is } \frac{x}{t} + yt = 2c \;\Rightarrow\; x + t^2 y = 2ct$$

On replacing x_1 by ct and y_1 by $\frac{c}{t}$ in the equation of the tangents at (x_1, y_1)

i.e. $xy_1 + yx_1 = 2c^2$ we get $\frac{x}{t} + yt = 2c$.

Point of intersection of tangents at 't_1' and 't_2' is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.

6. **Equation of the normal in different forms:**

(i) Point form: The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(ii) Parametric form: The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola

$xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$

This equation is a fourth degree in t. So, in general four normals can be drawn from a point to the hyperbola $xy = c^2$, and point of intersection of normals at t_1 and t_2 is

$$\left(\frac{c\left\{t_1t_2\left(t_1^2 + t_1t_2 + t_2^2\right) + 1\right\}}{t_1t_2(t_1 + t_2)}, \frac{c\left\{t_1^3t_2^3 + \left(t_1^2 + t_1t_2 + t_2^2\right)\right\}}{t_1t_2(t_1 + t_2)}\right).$$

(iii) If the normal at $P\left(ct, \frac{c}{t}\right)$ cuts the rectangular hyperbola $xy = c^2$ at $Q\left(ct', \frac{c}{t'}\right)$ then $t' = -\frac{1}{t^3}$.

7. Equation of diameter of rectangular hyperbola $xy = c^2$ is $y + mx = 0$ (m is the slope of the chord joining two points lies on the rectangular hyperbola)

Two diameters $y + m_1x = 0$ and $y + m_2x = 0$ are conjugate diameter if $m_1 + m_2 = 0$.

PROPERTIES OF HYPERBOLA $x^2/a^2 - y^2/b^2 = 1$

1. If *PN* be the ordinate of a point *P* on the hyperbola and the tangent at *P* meets the transverse axis in *T*, then $ON.OT = a^2$, O being the origin.
2. If PM be drawn perpendiculars to the conjugate axis from a point *p* on the hyperbola and the tangent at P meets the conjugate axis in T, then $OM.OT = -b^2$; O, being the origin.
3. If the normal at *P* on the hyperbola meets the transverse axis in *G*, then $SG = eSP$; S being a foci and e the eccentricity of the hyperbola.
4. The tangent and normal at any point of a hyperbola bisect the angle between the focal radii to that point.
5. The locus of the feet of the perpendiculars from the foci on a tangent to a hyperbola is the auxiliary circle.
6. The product of the length of the perpendicular drawn from foci on any tangent to hyperbola is b^2.

7. From any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ perpendicular are drawn to the asymptotes then product is

$$\frac{a^2b^2}{a^2+b^2} \text{ and for rectangular hyperbola } =\frac{a^2}{2}$$

8. If a circle cuts the rectangular hyperbola $xy=1$ in $(x_r,\ y_r)$ (four points) $r=1, 2, 3, 4$ then

$$\frac{1}{x_1x_2x_3x_4}=y_1y_2y_3y_4=1.$$

9. A rectangular hyperbola with centre at C is cut by any circle of radius R in four points L, M, N, P then the value of $CL^2+CM^2+CN^2+CP^2=4R^2$.
10. If a triangle is inscribed in a rectangular hyperbola then the orthocenter of triangle lies on the rectangular hyperbola.
11. The portion of tangent intercepted between the asymptotes at any point of the hyperbola is bisected by the point of contact.
12. Whenever any circle and any hyperbola cut each other at four points the mean position of these four points is the mid point of the line segment joining centre of hyperbola and centre of circle.
13. The harmonic mean of focal radi for any focal chord $=\frac{b^2}{a}$
14. Tangent drawn at the ends of any focal chord meet on the directrix.
15. Local of point of intersection two perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a circle called director circle whose equation is $x^2+y^2=a^2-b^2$
    and if $a<b$ then there is no real point from where we can draw two perpendicular tangents to the hyperbola.
16. The portion of tangent between point of contact and the point where it cuts the directrix subtend 90° angle at the focus.