Polynomials

POLYNOMIAL

A function $p(x)$ of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

where $a_0, a_1, a_2, \ldots, a_n$ are real numbers, $a_n \neq 0$ and n is a non-negative integer is called a *polynomial* in x over reals.

The real number $a_0, a_1, \ldots, a_n$ are called the *coefficients of the polynomial*.

If $a_0, a_1, a_2, \ldots, a_n$ are all integers, we call it a *polynomial over integers*.

If they are rational numbers, we call it a *polynomial over rationals*.

Illustration 1

(*a*) $4x^2 + 7x - 8$ is a polynomial over integers.

(*b*) $\frac{7}{4}x^3 + \frac{2}{3}x^2 - \frac{8}{7}x + 5$ is a polynomial over rationals.

(*c*) $4x^2 - \sqrt{3}\,x + \sqrt{5}$ is a polynomial over reals.

Monomial

A polynomial having only one term is called a monomial. For example, $7, 2x, 8x^3$ are monomials.

Binomial

A polynomial having two terms is called a binomial.

For example, $2x + 3$, $7x^2 - 4x$, $x^2 + 8$ are binomials.

Trinomial

A polynomial having three terms is called a trinomial.

For example, $7x^2 - 3x + 8$ is a trinomial.

Degree of a Polynomial

The exponent in the term with the highest power is called the degree of the polynomial.

For example, in the polynomial $8x^6 - 4x^5 + 7x^3 - 8x^2 + 3$, the term with the highest power is x^6. Hence, the degree of the polynomial is 6.

A polynomial of degree 1 is called a *linear polynomial*.

It is of the form $ax + b$, $a \neq 0$.

A polynomial of degree 2 is called a *quadratic polynomial*.

It is of the form $ax^2 + bx + c$, $a \neq 0$.

Division of a Polynomial by a Polynomial

Let $p(x)$ and $f(x)$ be two polynomials and $f(x) \neq 0$. Then, if we can find polynomials $q(x)$ and $r(x)$, such that

$$p(x) = f(x).\, q(x) + r(x),$$

where degree $r(x) <$ degree $f(x)$, then we say that $p(x)$ divided by $f(x)$, gives $q(x)$ as *quotient* and $r(x)$ as *remainder*.

If the remainder $r(x)$ is zero, we say that *divisor* $f(x)$ is a factor of $p(x)$ and we have

$$p(x) = f(x).\, q(x).$$

Illustration 2 Divide $f(x) = 5x^3 - 70x^2 + 153x - 342$ by $g(x) = x^2 - 10x + 16$. Find the quotient and the remainder

Solution:

$$\begin{array}{r|l}
 & 5x - 20 \\ \hline
x^2 - 10x + 16 & 5x^3 - 70x^2 + 153x - 342 \\
 & 5x^3 - 50x^2 + 80x \\
 & -\quad +\qquad\quad - \\ \hline
 & -20x^2 + 73x - 342 \\
 & -20x^2 + 200x - 320 \\
 & +\qquad\quad -\quad + \\ \hline
 & -127x - 22
\end{array}$$

$\therefore$ Quotient $= 5x - 20$ and

Remainder $= -127x - 22$.

Illustration 3 Determine if $(x-1)$ is a factor of

$$p(x) = x^3 - 3x^2 + 4x + 2$$

$$
\begin{array}{r|l}
 & x^2 - 2x + 2 \\
\hline
x - 1 & x^3 - 3x^2 + 4x + 2 \\
 & x^3 - x^2 \\
 & - \quad + \\
\hline
 & -2x^2 + 4x \\
 & -2x^2 + 2x \\
 & \quad - \quad + \\
\hline
 & \quad 2x + 2 \\
 & \quad 2x - 2 \\
 & \quad - \quad + \\
\hline
 & \qquad 4
\end{array}
$$

Since the remainder is not zero, $(x-1)$ is not a factor of $p(x)$.

Some Basic Theorems

Factor Theorem

Let $p(x)$ be a polynomial of degree $n > 0$. If $p(a) = 0$ for a real number a, then $(x-a)$ is a factor of $p(x)$.

Conversely, if $(x-a)$ is a factor of $p(x)$, then $p(a) = 0$.

Illustration 4 Use factor theorem to determine if $(x-1)$ is a factor of $x^8 - x^7 + x^6 - x^5 + x^4 - x + 1$

Solution: Let $p(x) = x^8 - x^7 + x^6 - x^5 + x^4 - x + 1$

Then, $p(1) = (1)^8 - (1)^7 + (1)^6 - (1)^5 + (1)^4 - 1 + 1 = 1 \neq 0$

Hence, $(x-1)$ is not a factor of $p(x)$

Remainder Theorem

Let $p(x)$ be any polynomial of degree ≥ 1 and a any number.

If $p(x)$ is divided by $x - a$, the remainder is $p(a)$.

Illustration 5 Let $p(x) = x^5 + 5x^4 - 3x + 7$ be divided by $(x-1)$. Find the remainder

Solution: Remainder $= p(1) = (1)^5 + 5(1)^4 - 3(1) + 7$

$= 10$

Some Useful Results and Formulae

1. $(A+B)^2 = A^2 + B^2 + 2AB$
2. $(A-B)^2 = A^2 + B^2 - 2AB = (A+B)^2 - 4AB$
3. $(A+B)(A-B) = A^2 - B^2$
4. $(A+B)^2 + (A-B)^2 = 2(A^2 + B^2)$
5. $(A+B)^2 - (A-B)^2 = 4AB$
6. $(A+B)^3 = A^3 + B^3 + 3AB(A+B)$
7. $(A-B)^3 = A^3 - B^3 - 3AB(A-B)$
8. $A^2 + B^2 = (A+B)^2 - 2AB$
9. $A^3 + B^3 = (A+B)(A^2 + B^2 - AB)$
10. $A^3 - B^3 = (A-B)(A^2 + B^2 + AB)$
11. $(A+B+C)^2 = A^2 + B^2 + C^2 + 2(AB + BC + CA)$
12. $(A^3 + B^3 + C^3 - 3ABC$
    $= (A+B+C)(A^2 + B^2 + C^2 - AB - CA - BC)$
13. $A + B + C = 0 \Rightarrow A^3 + B^3 + C^3 = 3ABC.$
14. $A^n - B^n$ is divisible by $(A-B)$ for all values of n.
15. $A^n - B^n$ is divisible by $(A+B)$ only for even values of n.
16. $A^n + B^n$ is never divisible by $(A-B)$.
17. $A^n + B^n$ is divisible by $(A+B)$ only when n is odd.

A Useful Short-cut Method

When a polynomial $f(x)$ is divided by $x - a$ and $x - b$, the respective remainders are A and B. Then, if the same polynomial is divided by $(x-a)(x-b)$, the remainder will be

$$\frac{A-B}{a-b}x + \frac{Ba-Ab}{a-b}.$$

Illustration 6 When a polynomial $f(x)$ is divided by $(x-1)$ and $(x-2)$, the respective remainders are 15 and 9. What is the remainder when it is divided by

$$(x-1)(x-2)?$$

Solution: Remainder $= \dfrac{A-B}{a-b}x + \dfrac{Ba-Ab}{a-b}$

$= \dfrac{15-9}{1-2}x + \dfrac{9(1)-15(2)}{1-2}$

$= (-x + 21)$

Practice Exercises

Difficulty Level-1
(Based on Memory)

1. If $(a + b + 2c + 3d)(a - b - 2c + 3d)$
$= (a - b + 2c - 3d) \times (a + b - 2c - 3d)$,
then $2bc$ is equal to:

(*a*) $3\,ad$ (*b*) $\frac{3}{2}$

(*c*) a^2d^2 (*d*) $\frac{3a}{2d}$

[Based on MAT, 2003]

2. What is the value of the following expression?
$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 - x)$

(*a*) $1 + x^{16}$ (*b*) $1 - x^{16}$

(*c*) $x^{16} - 1$ (*d*) $x^8 + 1$

[Based on MAT, 2000]

3. If $x^2 - 6x + a$ is divisible by $x - 2$, then a is equal to:

(*a*) 8 (*b*) 6

(*c*) 0 (*d*) None of these

4. If $(x - 2)$ is a factor of the polynomial $x^3 - 2ax^2 + ax - 1$, find the value of a.

(*a*) 5/6 (*b*) 7/6

(*c*) 11/6 (*d*) None of these

5. Divide the polynomial $4y^3 - 3y^2 + 2y - 4$ by $y + 2$ and find the quotient and remainder.

(*a*) $4y^2 - 11y + 24, - 52$ (*b*) $6y^2 - 13y + 36, - 64$

(*c*) $4y^2 + 13y - 24, + 52$ (*d*) None of these

6. Resolve into factors: $16(x - y)^2 - 9(x + y)^2$.

(*a*) $(x - 5y)(5x - y)$ (*b*) $(x + 7y)(7x + y)$

(*c*) $(x - 7y)(7x - y)$ (*d*) None of these

7. Resolve into factors: $4x^2 + 12xy + 9y^2 - 8x - 12y$.

(*a*) $(3x + 2y)(4x + 2y - 3)$

(*b*) $(2x + 3y)(2x + 3y - 4)$

(*c*) $(2x - 3y)(2x + 3y + 4)$

(*d*) None of these

8. Resolve into factors:
$16x^2 - 72xy + 81y^2 - 12x + 27y$

(*a*) $(6x - 7y)(6x - 7y - 5)$ (*b*) $(4x - 9y)(4x - 9y - 3)$

(*c*) $(4x + 9y)(4x + 9y + 3)$ (*d*) None of these

9. Resolve into factors: $(a + b)^2 - 14c(a + b) + 49c^2$.

(*a*) $(a - b - 9c)^3$ (*b*) $(a + b - 7c)^2$

(*c*) $(a + b + 9c)^2$ (*d*) None of these

10. Resolve into factors: $81x^2y^2 + 108xyz + 36z^2$.

(*a*) $(6xy + 9z)^2$ (*b*) $(9xy - 7z)^2$

(*c*) $(9xy + 6z)^2$ (*d*) None of these

11. If $a + b + c = 0$, then the value of $a^2(b + c) + b^2(c + a) + c^2(a + b)$ is:

(*a*) abc (*b*) $3\,abc$

(*c*) $-3\,abc$ (*d*) 0

[Based on MAT, 1999]

12. If $(x + 1)$ is a factor of $2x^3 - ax^2 - (2a - 3)x + 2$, then the value of '$a$' is:

(*a*) 3 (*b*) 2

(*c*) 3/2 (*d*) 1/2

[Based on MAT, 1999]

13. If $P = \frac{x^2 - 36}{x^2 - 49}$ and $Q = \frac{x + 6}{x + 7}$, then the value of P/Q is:

(*a*) $\frac{x - 6}{x - 7}$ (*b*) $\frac{x - 6}{x + 7}$

(*c*) $\frac{x - 6}{x + 6}$ (*d*) $\frac{x + 6}{x - 7}$

[Based on MAT, 1999]

14. What is the value of the following expression?
$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 - x)$

(*a*) $1 + x^{16}$ (*b*) $1 - x^{16}$

(*c*) $x^{16} - 1$ (*d*) $x^8 + 1$

[Based on MAT, 2000]

15. Resolve into factors:
$9(3x + 5y)^2 - 12(3x + 5y)(2x + 3y) + 4(2x + 3y)^2$

(*a*) $(7x + 9y)^2$ (*b*) $(5x + 9y)^2$

(*c*) $(5x - 9y)^2$ (*d*) None of these

16. If $x + 1/x = 3$, then $x^3 + 1/x^3$ is equal to:

(*a*) 9 (*b*) 18

(*c*) 27 (*d*) 6

[Based on IIFT, 2005]

17. Factorize: $45a^3b + 5ab^3 - 30a^2b^2$

(*a*) $5ab(5a - b)^2$ (*b*) $7ab(5a - b)^2$

(*c*) $5ab(3a - b)^2$ (*d*) None of these

18. Find the factors of $(a - b)^3 + (b - c)^3 + (c - a)^3$

(*a*) $3(a + b)(b + c)(c + a)$

(*b*) $5(a - b)(b - c)(c - a)$

(*c*) $3(a - b)(b - c)(c - a)$

(*d*) None of these

19. Factorize $a^2 + \frac{1}{a^2} + 3 - 2a - \frac{2}{a}$

(a) $\left(a+\frac{1}{a}-1\right)\left(a-\frac{1}{a}+1\right)$

(b) $\left(a+\frac{1}{a}-1\right)\left(a+\frac{1}{a}+1\right)$

(c) $\left(a+\frac{1}{a}+1\right)\left(a+\frac{1}{a}+1\right)$

(d) $\left(a+\frac{1}{a}-1\right)\left(a+\frac{1}{a}-1\right)$

20. If $x + \frac{1}{x} = 2$, find the value of $x^4 + \frac{1}{x^4}$.

(a) 2 (b) 4
(c) 6 (d) 8

21. If $x + \frac{1}{x} = 2$, then $x^3 + \frac{1}{x^3}$ is equal to:

(a) 64 (b) 14
(c) 8 (d) 2

22. If $\sqrt{x} + \frac{1}{\sqrt{x}} = 5$, what will be the value of $x^2 + \frac{1}{x^2}$?

(a) 927 (b) 727
(c) 527 (d) 627

23. If $x + \frac{1}{x} = 3$, the value of $x^6 + \frac{1}{x^6}$ is:

(a) 927 (b) 414
(c) 364 (d) 322

24. If $\left(x^3 + \frac{1}{x^3}\right) = 52$, the value of $x + \frac{1}{x}$ is:

(a) 4 (b) 3
(c) 6 (d) 13

25. If $x + \frac{1}{y} = 1$ and $y + \frac{1}{z} = 1$, find the value of $z + \frac{1}{x}$

(a) 2 (b) 1
(c) 0 (d) 3

Difficulty Level-2
(Based on Memory)

1. $(x^n - a^n)$ is divisible by $(x - a)$
(a) For all values of n
(b) Only for even values of n
(c) Only for odd values of n
(d) Only for prime values of n

2. Which of the following expressions are exactly equal in value?
(a) $(3x - y)^2 - (5x^2 - 2xy)$
(b) $(2x - y)^2$
(c) $(2x + y)^2 - 2xy$
(d) $(2x + 3y)^2 - 8y(2x + y)$

(a) (a) and (b) only (b) (a), (b) and (c) only
(c) (b) and (d) only (d) (a), (b) and (d) only

[Based on IRMA, 2002]

3. Find the values of m and n in the polynomial $2x^3 + mx^2 + nx - 14$ such that $(x - 1)$ and $(x + 2)$ are its factors.
(a) $m = 4, n = 5$ (b) $m = 9, n = 3$
(c) $m = 6, n = 7$ (d) None of these

4. Factorize $(a - b + c)^2 + (b - c + a)^2 + 2(a - b + c)(b + c - a)$.
(a) $4a^2$ (b) $6a^2$
(c) $8a^2$ (d) None of these

5. Value of k for which $(x - 1)$ is a factor $(x^3 - k)$, is:
(a) – 1 (b) 1
(c) 8 (d) – 8

[Based on FMS, 2005]

6. If $x^{1/3} + y^{1/3} + x^{1/3} = 0$, then:
(a) $x + y + z = 0$ (b) $(x + y + z)^3 = 27\,xyz$
(c) $x + y + z = 3\,xyz$ (d) $x^3 + y^3 + z^3 = 0$

[Based on FMS, 2005]

7. If $3x^3 - 9x^2 + kx - 12$ is divisible by $x - 3$, then it is also divisible by:
(a) $3x^2 - 4$ (b) $3x^2 + 4$
(c) $3x - 4$ (d) $3x + 4$

[Based on FMS, 2010]

8. If the expression $ax^2 + bx + c$ is equal to 4 when $x = 0$, leaves a remainder 4 when divided by $x + 1$ and a remainder 6 when divided by $x + 2$, then the values of a, b and c are respectively:
(a) 1, 1, 4 (b) 2, 2, 4
(c) 3, 3, 4 (d) 4, 4, 4

[Based on XAT, 2006]

9. The condition that $x^5 + 10x^4 - 7x^3 + 10ax + 5a^2$ will contain $x + 1$ as a factor is:

(a) $a = \sqrt{-169}$ (b) $a = -2$

(c) $5a^2 - 10a + 16 = 0$ (d) $5a^2 - 10a - 16 = 0$

[Based on XAT, 2006]

10. If $x^3 + 2x^2 + ax + b$ is exactly divisible by $x^2 - 1$, then the values of a and b are respectively:

(a) 1 and 2 (b) 1 and 0

(c) –1 and –2 (d) 0 and 1

[Based on XAT, 2006]

11. If the polynomial $x^3 + px + q$ has three distinct roots, then which of the following is a possible value of p?

(a) –1 (b) 0

(c) 1 (d) 2

[Based on XAT, 2007]

12. If $\frac{x}{y} + \frac{y}{x} = 6$, find the value of $\frac{x^3}{y^3} + \frac{y^3}{x^3}$.

(a) 176 (b) 198

(c) 184 (d) None of these

13. If $x + y + 2 = 0$, what will be the value of $\frac{x^2 + y^2 + z^2}{x^2 - yz}$?

(a) 4 (b) 6

(c) 2 (d) 8

14. Which of the following must be equal to zero for all real numbers x?

I. $x^3 - x^2$ II. x^0 III. x^1

(a) II only (b) I only

(c) I and II only (d) None of these

[Based on NMAT, 2005]

15. Factorize: $(2x + 3y)^2 + 2(2x + 3y)(2x - 3y) + (2x - 3y)^2$

(a) $16x^2$ (b) $18x^2$

(c) $12x^2$ (d) None of these

16. Factors of $a^2 + \frac{1}{4} + a$ will be:

(a) $\left(a+\frac{1}{2}\right)\left(a-\frac{1}{2}\right)$ (b) $\left(a+\frac{1}{2}\right)^2$

(c) $\left(a+\frac{1}{2}\right)^3$ (d) $\left(a+\frac{1}{2}\right) \times a$

17. If $a + b + c = 0$, the value of $\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}\right)$ is:

(a) 1 (b) 0

(c) –1 (d) 3

18. If $x + y + z = 9$ and $xy + yz + zx = 23$, the value of $x^3 + y^3 + z^3 - 3xyz$ is:

(a) 108 (b) 207

(c) 669 (d) 729

19. When $(x^3 - 2x^2 + px - q)$ is divided by $x^2 - 2x - 3$ the remainder is $(x - 6)$. The values of p and q are:

(a) $p = -2, q = -6$ (b) $p = 2, q = -6$

(c) $p = -2, q = 6$ (d) $p = 2, q = 6$

20. Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants. If $f(x)$ is divided by $ax - b$, the remainder is:

(a) $f\left(\frac{b}{a}\right)$ (b) $f\left(\frac{-b}{a}\right)$

(c) $f\left(\frac{a}{b}\right)$ (d) $f\left(\frac{-a}{b}\right)$

21. If $(x^{3/2} - xy^{1/2} + x^{1/2}y - y^{3/2})$ is divided by $(x^{1/2} - y^{1/2})$, the quotient is:

(a) $x + y$ (b) $x - y$

(c) $x^{1/2} + y^{1/2}$ (d) $x^2 - y^2$

22. When $4x^3 - ax^2 + bx - 4$ is divided by $x - 2$ and $x + 1$, the respective remainders are 20 and –13. Find the values of a and b.

(a) $a = 3, b = 2$ (b) $a = 5, b = 4$

(c) $a = 7, b = 6$ (d) $a = 9, b = 8$

23. When a polynomial $f(x)$ is divided by $x - 3$ and $x + 6$, the respective remainders are 7 and 22. What is the remainder when $f(x)$ is divided by $(x - 3)(x + 6)$?

(a) $\frac{-5}{3}x + 12$ (b) $\frac{-7}{3}x + 14$

(c) $\frac{-5}{3}x + 16$ (d) $\frac{-7}{3}x + 12$

24. If $(x - 1)$ is a factor of $Ax^3 + Bx^2 - 36x + 22$ and $2^B = 64^A$, find A and B.

(a) $A = 4, B = 16$ (b) $A = 6, B = 24$

(c) $A = 2, B = 12$ (c) $A = 8, B = 16$

25. Find the remainder when $a^3 - 5a^2 + 7a - 9$ is divided by $a^2 + a - 6$.

(a) $19a - 31$ (b) $19a - 38$

(c) $19a - 49$ (d) $19a - 45$

[Based on CAT, 2009]

Answer Keys

Difficulty Level-1

1. (*a*)	**2.** (*b*)	**3.** (*a*)	**4.** (*b*)	**5.** (*a*)	**6.** (*c*)	**7.** (*b*)	**8.** (*b*)	**9.** (*b*)	**10.** (*c*)	**11.** (*c*)	**12.** (*a*)	**13.** (*a*)
14. (*b*)	**15.** (*b*)	**16.** (*b*)	**17.** (*c*)	**18.** (*c*)	**19.** (*d*)	**20.** (*a*)	**21.** (*d*)	**22.** (*c*)	**23.** (*d*)	**24.** (*a*)	**25.** (*b*)	

Difficulty Level-2

1. (*a*)	**2.** (*d*)	**3.** (*b*)	**4.** (*a*)	**5.** (*b*)	**6.** (*b*)	**7.** (*b*)	**8.** (*a*)	**9.** (*c*)	**10.** (*c*)	**11.** (*a*)	**12.** (*b*)	**13.** (*c*)
14. (*d*)	**15.** (*a*)	**16.** (*b*)	**17.** (*d*)	**18.** (*a*)	**19.** (*c*)	**20.** (*a*)	**21.** (*a*)	**22.** (*a*)	**23.** (*a*)	**24.** (*c*)	**25.** (*d*)	

Explanatory Answers

Difficulty Level-1

1. (*a*) Given expression

$\Rightarrow (a+b)(a-b) - (a+b)(2c-3d) + (2c+3d)(a-b) - (2c+3d)(2c-3d)$

$= (a-b)(a+b) - (a-b)(2c+3d) + (2c-3d)(a+b) - (2c-3d)(2c+3d)$

$\Rightarrow \quad (a+b)(2c-3d) = (a-b)(2c+3d)$

$\Rightarrow 2ac - 3ad + 2bc - 3bd = 2ac + 3ad - 2bc - 3bd$

$\Rightarrow \quad 4bc = 6ad$

$\Rightarrow \quad 2bc = 3ad.$

2. (*b*) Given expression

$= (1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)$

$= (1-x^2)(1+x^2)(1+x^4)(1+x^8)$

$= (1-x^4)(1+x^4)(1+x^8)$

$= (1-x^8)(1+x^8) = 1 - x^{16}.$

3. (*a*)

4. (*b*) Let, $\quad p(x) = x^3 - 2ax^2 + ax - 1$

Since $x - 2$ is a factor of $p(x)$, we must have $p(2) = 0$

$\therefore \quad (2)^3 - 2a(2)^2 + 2a - 1 = 0$

$\Rightarrow 8 - 8a + 2a - 1 = 0$

$\Rightarrow \quad -6a = -7 \Rightarrow a = \frac{7}{6}.$

5. (*a*)

$$\begin{array}{r|l} & 4y^2 - 11y + 24 \\ \hline y+2 & 4y^3 - 3y^2 + 2y - 4 \\ & 4y^3 + 8y^2 \\ & - \quad - \\ \hline & -11y^2 + 2y - 4 \\ & -11y^2 - 22y \\ & + \quad + \\ \hline & 24y - 4 \\ & 24y + 48 \\ & - \quad - \\ \hline & -52 \end{array}$$

$\therefore$ Quotient $= 4y^2 - 11y + 24$

Remainder $= -52.$

6. (*c*) $16(x-y)^2 - 9(x+y)^2$

$= [4(x-y)]^2 - [3(x+y)]^2$

$= [4(x-y) - 3(x+y)][4(x-y) + 3(x+y)]$

$= (4x - 4y - 3x - 3y)(4x - 4y + 3x + 3y)$

$= (x - 7y)(7x - y).$

7. (*b*) $4x^2 + 12xy + 9y^2 - 8x - 12y$

$= [(2x)^2 + 2(2x)(3y) + (3y)^2] - 4(2x + 3y)$

$= (2x + 3y)^2 - 4(2x + 3y)$

$= (2x + 3y)(2x + 3y - 4).$

8. (*b*) $16x^2 - 72xy + 81y^2 - 12x + 27y$

$= (4x)^2 - 2(4x)(9y) + (9y)^2 - 3(4x - 9y)$

$= (4x - 9y)^2 - 3(4x - 9y)$

$= (4x - 9y)(4x - 9y - 3).$

9. (*b*) $(a + b)^2 - 14c(a + b) + 49c^2$

$= (a + b)^2 - 2(a + b)\cdot(7c) + (7c)^2$

$= (a + b - 7c)^2.$

10. (*c*) $81x^2y^2 + 108xyz + 36z^2$

$= (9xy)^2 + 2(9xy)(6z) + (6z)^2$

$= (9xy + 6z)^2.$

11. (*c*) If $a + b + c = 0$, then

$a^3 + b^3 + c^3 = 3abc$

$\therefore\ a^2(b + c) + b^2(c + a) + c^2(a + b)$

$= a^2(-a) + b^2(-b) + c^2(-c)$

$= -a^3 - b^3 - c^3$

$= -(a^3 + b^3 + c^3)$

$= -3abc.$

12. (*a*) $x = -1$ satisfies the equation

$2x^3 - ax^2 - (2a - 3)x + 2 = 0$

$\Rightarrow\quad a = 3$

13. (*a*) $\dfrac{P}{Q} = \dfrac{x^2 - 36}{x^2 - 49} \times \dfrac{x + 7}{x + 6} = \dfrac{x - 6}{x - 7}$

14. (*b*) Given expression

$= (1 - x)(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$

$= (1 - x^2)(1 + x^2)(1 + x^4)(1 + x^8)$

$= (1 - x^4)(1 + x^4)(1 + x^8)$

$= (1 - x^8)(1 + x^8) = 1 - x^{16}.$

15. (*b*) $9(3x + 5y)^2 - 12(3x + 5y)(2x + 3y) + 4(2x + 3y)^2$

$= [3(3x + 5y)]^2 - 2[3(3x + 5y)][2(2x + 3y)] + [2(2x + 3y)]^2$

$= [3(3x + 5y) - 2(2x + 3y)]^2$

$= (9x + 15y - 4x - 6y)^2$

$= (5x + 9y)^2.$

16. (*b*) $\left(x + \dfrac{1}{x}\right)^3 = x^3 + \dfrac{1}{x^3} + 3\cdot x\cdot\dfrac{1}{x}\left(x + \dfrac{1}{x}\right)$

$\Rightarrow\quad 27 - 9 = x^3 + \dfrac{1}{x^3}$

$\therefore\quad x^3 + \dfrac{1}{x^3} = 18.$

17. (*c*) $45a^3b + 5ab^3 - 30a^2b^2$

$= 5ab[9a^2 + b^2 - 6ab]$

$= 5ab[9a^2 - 6ab + b^2]$

$= 5ab[(3a)^2 - 2(3a)(b) + (b)^2]$

$= 5ab[3a - b]^2.$

18. (*c*) Suppose, $a - b = x,\ b - c = y,\ c - a = z$

$\therefore\ (a - b) + (b - c) + (c - a) = x + y + z$

$\Rightarrow\quad 0 = x + y + z$

$\therefore\quad x + y = -z \qquad (1)$

$\therefore\quad (x + y)^3 = (-z)^3$

or, $x^3 + y^3 + 3xy(x + y) = -z^3$

or, $x^3 + z^3 + z^3 + 3xy(-z) = -z^3$

[On substituting $x + y = -z$ from eq. (1)]

or, $x^3 + y^3 - 3xyz = -z^3$

or, $x^3 + y^3 + z^3 = 3xyz$

$\therefore\ (a - b)^3 + (b - c)^3 + (c - a)^3$

$= 3(a - b)(b - c)(c - a)$

19. (*d*) $a^2 + \dfrac{1}{a^2} + 3 - 2a - \dfrac{2}{a}$

$= \left(a^2 + \dfrac{1}{a^2} + 2\right) - 2a - \dfrac{2}{a} + 1$

$= \left(a + \dfrac{1}{a}\right)^2 - 2\left(a + \dfrac{1}{a}\right) + 1$

$= x^2 - 2x + 1 \quad \left[\text{suppose } a + \dfrac{1}{a} = x\right]$

$= (x - 1)^2$

$= \left(a + \dfrac{1}{a} - 1\right)^2.$

20. (*a*) $x + \dfrac{1}{x} = 2 \Rightarrow \left(x + \dfrac{1}{x}\right)^2 = (2)^2$

$\therefore\quad x^2 + \dfrac{1}{x^2} + 2x.\dfrac{1}{x} = 4$

$\Rightarrow\quad x^2 + \dfrac{1}{x^2} + 2 = 4$

$\Rightarrow\quad x^2 + \dfrac{1}{x^2} = 2$

$\therefore\quad \left(x^2 + \dfrac{1}{x^2}\right)^2 = (2)^2$

$\Rightarrow\quad x^4 + \dfrac{1}{x^4} + 2x^2.\dfrac{1}{x^2} = 4$

$\Rightarrow\quad x^4 + \dfrac{1}{x^4} + 2 = 4$

$\therefore\quad x^4 + \dfrac{1}{x^4} = 2.$

21. (*d*) $$x + \frac{1}{x} = 2$$

$$\Rightarrow \left(x+\frac{1}{x}\right)^3 = 23$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3\left(x+\frac{1}{x}\right) = 8$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3 \times 2 = 8$$

$$\Rightarrow x^3 + \frac{1}{x^3} = 2.$$

22. (*c*) $$\sqrt{x} + \frac{1}{\sqrt{x}} = 5$$

$$\Rightarrow \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 = (5)^2$$

$$\therefore x + 2\sqrt{x}\ \frac{1}{\sqrt{x}} + \frac{1}{x} = 25$$

$$\therefore 2 + x + \frac{1}{x} = 25$$

$$\Rightarrow x + \frac{1}{x} = 23$$

$$\therefore \left(x+\frac{1}{x}\right)^2 = (23)^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} + 2 = 529$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 527.$$

23. (*d*) $$\left(x+\frac{1}{x}\right)^2 = 3^2$$

$$\Rightarrow x2 + \frac{1}{x^2} = 7$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^3 = 7^3$$

$$\therefore x^6 + \frac{1}{x^6} + 3\left(x^2+\frac{1}{x^2}\right) = 343$$

$$\Rightarrow x^6 + \frac{1}{x^6} + 3 \times 7 = 343$$

$$\therefore x^6 + \frac{1}{x^6} = 343 - 21 = 322.$$

24. (*a*) $$\left(x+\frac{1}{x}\right)^3 = \left(x^3+\frac{1}{x^3}\right) + 3\left(x+\frac{1}{x}\right)$$

$$\therefore \left(x+\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{x}\right) = x^3 + \frac{1}{x^3} = 52$$

$$\Rightarrow y^3 - 3y = 52$$

where, $y = x + \frac{1}{x}$

i.e., $y^3 - 3y - 52 = 0$

Clearly, $y = 4$, satisfies $y^3 - 3y - 52 = 0$

$$\therefore x + \frac{1}{x} = 4.$$

25. (*b*) $$x + \frac{1}{y} = 1 \Rightarrow x = 1 - \frac{1}{y} = \frac{y-1}{y}$$

$$\Rightarrow \frac{1}{x} = \frac{y}{y-1}$$

and $$y + \frac{1}{z} = 1 \Rightarrow \frac{1}{z} = 1-y \Rightarrow z = \frac{1}{1-y}$$

$$\therefore z + \frac{1}{x} = \frac{1}{1-y} + \frac{y}{y-1} = \frac{1}{1-y} - \frac{y}{1-y}$$

$$= \frac{1-y}{1-y} = 1.$$

Difficulty Level-2

1. (*a*)

2. (*d*) $(a) = (3x - y)^2 - (5x^2 - 2xy)$

$= 9x^2 + y^2 - 6xy - 5x^2 + 2xy$

$= 4x^2 + y^2 - 4xy$

$= (2x - y)^2$

$(b) = (2x - y)^2$

$(c) = (2x + y)^2 - 2xy$

$= 4x^2 + y^2 + 2xy$

$(d) = (2x + 3y)^2 - 8y\,(2x + y)$

$= 4x^2 + 9y^2 + 12xy - 16xy - 8y^2$

$= 4x^2 + y^2 - 4xy = (2x - y)^2.$

3. (*b*) Let, $f(x) = 2x^3 + mx^2 + nx - 14$

Since, $x - 1$ is a factor of $f(x)$

$\therefore \quad f(1) = 0$ [By factor theorem]

$\Rightarrow \quad 2(1)^3 + m(1)^2 + n(1) - 14 = 0$

$\Rightarrow \quad 2 + m + n - 14 = 0$

$\Rightarrow \quad m + n = 12 \quad (1)$

Since, $x + 2$, i.e., $x - (-2)$ is a factor of $f(x)$

$\therefore \quad f(-2) = 0$ [By factor theorem]

$\Rightarrow 2(-2)^3 + m(-2)^2 + n(-2) - 14 = 0$

$\Rightarrow \quad -16 + 4m - 2n - 24 = 0$

$\Rightarrow \quad 4m - 2n - 30 = 0$

$\Rightarrow \quad 2m - n = 15 \quad (2)$

Adding Eqs. (1) and (2), we get

$$3m = 27 \Rightarrow m = 9$$

Put, $m = 9$ in Eq. (1),

we get $9 + n = 12$

$\Rightarrow \quad n = 3$

4. (*a*) $(a - b + c)^2 + (b - c + a)^2 + 2(a - b + c)(b + c - a)$

$= (a - b + c)^2 + 2(a - b + c)(b + c - a) + (b - c + a)^2$ [rearranging]

$= [(a - b + c) + (b - c + a)]^2 = (2a)^2 = 4a^2$

5. (*b*) Put $x - 1 = 0 \Rightarrow x = 1$

Putting the value in

$x^3 - k = 0$

$\therefore \quad k = 1.$

6. (*b*) If $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, then

$$(x + y + z) = 3x^{\frac{1}{3}} \times y^{\frac{1}{3}} \times x^{\frac{1}{3}}$$

(If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$)

$\Rightarrow (x + y + z)^3 = 27\,xyz.$

7. (*b*) $3x^3 - 9x^2 + kx - 12$ is divisible by $x - 3$

$\therefore \quad f(3) = 0$

$\Rightarrow \quad 3 \times 3^3 - 9 \times 3^2 + 3k - 12 = 0$

$\Rightarrow \quad 81 - 81 + 3k - 12 = 0$

$\Rightarrow \quad 3k = 12$

$\Rightarrow \quad k = 4$

So, the equation will be

$$3x^3 - 9x^2 + 4x - 12 = 0$$

$\therefore \quad (x - 3)(3x^2 + 4) = 0$

So, the given equation is also divisible by $(3x^2 + 4)$.

8. (*a*) Applying factor theorem and Remainder theorem

$f(0) = 4$

$\Rightarrow \quad c = 4$

$f(-1) = 4$

$\Rightarrow \quad a - b + c = 4$

and, $\quad f(-2) = 6$

$\Rightarrow 4a - 2b + c = 6$

On solving, we get

$$a = 1, b = 1, c = 4.$$

9. (*c*) Using factor theorem

$$x + 1 = 0 \Rightarrow x = -1$$

Putting $f(-1) = 0$, we get

$$5a^2 - 10a + 16 = 0$$

10. (*c*) Using factor theorem

$$f(-1) = 0 \text{ and } f(1) = 0$$

we get $a = -1$ and $b = -2$.

11. (*a*) Since, coefficient of $x^2 = 0$. Sum of roots $\alpha + \beta + \gamma = 0$.

This means at least one of the roots must be negative.

12. (*b*)

$$\frac{x}{y} + \frac{y}{x} = 6$$

$\Rightarrow \quad \left(\frac{x}{y} + \frac{y}{x}\right)^3 = (6)^3$

$\therefore \quad \frac{x^3}{y^3} + \frac{y^3}{x^3} + 3\left(\frac{x}{y} + \frac{y}{x}\right) = 216$

$\therefore \quad \frac{x^3}{y^3} + \frac{y^3}{x^3} + 3 \times 6 = 216$

$\therefore \quad \frac{x^3}{y^3} + \frac{y^3}{x^3} = 216 - 18 = 198.$

13. (*c*) $\because \quad x + y + z = 0 \Rightarrow (x + y + z)^2 = 0$

$\therefore\ x^2 + y^2 + z^2 + 2\,(xy + yz + zx) = 0$

$\therefore\ x^2 + y^2 + z^2 = -2\,(xy + yz + zx)$

$= -2\,[x\,(y + z) + yz]$

$= -2\,(x \times -x + yz)$

$(\because x + y + z = 0)$

$= 2\,(x^2 - yz)$

$$\therefore \quad \frac{x^2 + y^2 + z^2}{x^2 - yz} = 2$$

14. (*d*) None of the given expression will be zero by hit and trial method.

15. (*a*) $(2x + 3y)^2 + 2\,(2x + 3y)\,(2x - 3y) + (2x - 3y)^2$

$= [2x + 3y) + (2x - 3y)]^2 = (4x)^2 = 16x^2$

16. (*b*) $$a^2 + \frac{1}{4} + a = a^2 + \left(\frac{1}{2}\right)^2 + 2.a\left(\frac{1}{2}\right)$$

$$= \left(a + \frac{1}{2}\right)^2.$$

17. (*d*) $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

$$\therefore \quad \frac{a^3}{abc} + \frac{b^3}{abc} + \frac{c^3}{abc} = 3 \text{ or } \frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} = 3.$$

18. (*a*) $x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$

$= (x + y + z)\,[(x + y + z)^2 - 3\,(xy + yz + zx)]$

$= 9\,[(9)^2 - 3\,(23)] = 9\,[81 - 69]$

$= 9 \times 12 = 108.$

19. (*c*) On actual division, remainder is $(p + 3)\,x - q$

$\therefore\ (p + 3)\,x - q = x - 6$

$\Rightarrow \quad p + 3 = 1$ and $q = 6$

$\Rightarrow \quad p = -2,\ q = 6$

20. (*a*) $$ax - b = 0 \Rightarrow x = \frac{b}{a}$$

So, remainder $= f\left(\dfrac{b}{a}\right)$.

21. (*a*) $x^{3/2} - xy^{1/2} + x^{1/2}y - y^{3/2}$

$= x\,(x^{1/2} - y^{1/2}) + y\,(x^{1/2} - y^{1/2})$

$= (x + y)\,(x^{1/2} - y^{1/2})$

$$\therefore \quad \frac{x^{3/2} - xy^{1/2} + x^{1/2}y - y^{3/2}}{x^{1/2} - y^{1/2}} = (x + y)$$

22. (*a*) Let $f(x) = 4x^3 - ax^2 + bx - 4$. When the expression $f(x)$ is divided by $x - 2$, the remainder is

$f(2) = 4\,(2)^3 - a\,(2)^2 + b\,(2) - 4 = 20$ (given)

$\Rightarrow 2b - 4a + 28 = 20$

$\Rightarrow \quad 2a - b = 4$ (1)

Similarly, when the expression $f(x)$ is divided by $x - (-1)$, the remainder is

$f(-1) = 4 \times (-1)^3 - a\,(-1) + b\,(-1) - 4 = -13$ (given)

$\Rightarrow -4 - a - b - 4 = -13$

$\Rightarrow \quad a + b = 5$ (2)

Solving Eqs. (1) and (2), we get

$a = 3,\ b = 2.$

23. (*a*) The function $f(x)$ is not known

Here, $a = 3,\ b = -6$

$A = 7,\ B = 22.$

Required remainder

$$= \frac{A - B}{a - b}\,x + \frac{Ba - Ab}{a - b}$$

$$= \frac{7 - 22}{3 - (-6)}\,x + \frac{22 \times 3 - 7 \times (-6)}{3 - (-6)}$$

$$= -\frac{5}{3}x + 12.$$

24. (*c*) Since $x - 1$ is a factor of $Ax^3 + Bx^2 - 36x + 22$

$\therefore\ A\,(1)^3 + B\,(1)^2 - 36\,(1) + 22 = 0$

$\Rightarrow \quad A + B = 14$

and, $\quad 2^B = (2^6)^A$

$\Rightarrow \quad B = 6A$

$\therefore \quad A = 2,\ B = 12.$

25. (*d*) By direct division method,

$(a^2 + a - 6)a^3 - 5a^2 + 7a - 9(a - 6)$

$(-)a^3 + a^2 - 6a$

$-6a^2 + 13a - 9$

$(-) - 6a^2 - 6a + 36$

$19a - 45$

$\therefore$ Remainder $= 19a - 45.$