2 INVERSE TRIGNOMETRIC FUNCTIONS

The domains and ranges (principal value branches) of inverse trigonometric functions are given in the following table:

Functions	Domains	Ranges
$y = \sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$y = \cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$y = \text{cosec}^{-1}x$	$R-(-1,1)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$
$y = \sec^{-1}x$	$R-(-1,1)$	$[0,\pi]-\{0\}$
$y = \tan^{-1}x$	R	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$y = \cot^{-1}x$	R	$[0,\pi]$

The value of an inverse trigonometric functions which lies in its principal value branch is called the principal value of that inverse trigonometric functions.

$y = \sin^{-1}x \Rightarrow x = \sin y$ $\quad \sin\left(\sin^{-1}x\right) = x$ $\quad \sin^{-1}(\sin x) = x$

$\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x$ $\quad \cos^{-1}(-x) = \pi - \cos^{-1}x$

$$\cos^{-1}\left(\frac{1}{x}\right) = \mathrm{cosec}^{-1}x$$

$$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$$

$$\cot^{-1}(-x) = \pi - \cot^{-1}x$$

$$\sec^{-1}(-x) = \pi - \sec^{-1}x$$

$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\mathrm{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\sin^{-1}(-x) = -\sin^{-1}x$$

$$\tan^{-1}(-x) = -\tan^{-1}x$$

$$\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}x$$

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

2 INVERSE TRIGNOMETRIC FUNCTIONS

Question :

Write the value of $\cos^{-1}\left(\frac{-1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

Solution :

$$\cos^{-1}\left(\frac{-1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$$

$$=\left[\pi-\cos^{-1}\left(\frac{1}{2}\right)\right]+2\sin^{-1}\left(\frac{1}{2}\right)$$

$$=\left[\pi-\cos^{-1}\left(\cos\frac{\pi}{3}\right)\right]+2\sin^{-1}\left(\sin\frac{\pi}{6}\right)=\left[\pi-\frac{\pi}{3}\right]+2\times\frac{\pi}{6}$$

$$=\frac{2\pi}{3}+\frac{2\pi}{6}=\pi$$

Question :

Write the value of $\cos^{-1}\left[\cos(680°)\right]$

Solution :

$$\cos^{-1}\left[\cos(680°)\right]$$

$$=\cos^{-1}\left[\cos(2\times360-40)°\right]$$

$$=\cos^{-1}(\cos 40°)=40°$$

Question :

Write the value of $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

Solution :

$$\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$$

$$=\cos\left[\cos^{-1}\left(\cos\frac{5\pi}{6}\right)+\frac{\pi}{6}\right]$$

$$=\cos\left[\frac{5\pi}{6}+\frac{\pi}{6}\right]=\cos\pi=\underline{\underline{-1}}$$

Question :

Write the value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$

Solution :

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$$

$$=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\left[\pi-\cos^{-1}\left(\frac{1}{2}\right)\right]$$

$$=\cos^{-1}\left(\cos\frac{\pi}{6}\right)+\left[\pi-\cos^{-1}\left(\cos\frac{\pi}{3}\right)\right]$$

$$=\frac{\pi}{6}+\pi-\frac{\pi}{3}=\frac{\pi}{6}+\frac{6\pi}{6}-\frac{2\pi}{6}$$

$$=\frac{5\pi}{6}$$

Question :

Write the value of $\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a-b}{a+b}\right)$

Solution :

$$\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a-b}{a+b}\right)$$

$$=\tan^{-1}\frac{\left(\frac{a}{b}-\left(\frac{a-b}{a+b}\right)\right)}{\left(1+\frac{a}{b}\left(\frac{a-b}{a+b}\right)\right)}$$

$$=\tan^{-1}\frac{\left(\frac{a(a+b)-b(a-b)}{b(a+b)}\right)}{\left(\frac{b(a+b)+a(a-b)}{b(a+b)}\right)}$$

$$=\tan^{-1}\left(\frac{a(a+b)-b(a-b)}{b(a+b)+a(a-b)}\right)$$

$$=\tan^{-1}\left(\frac{a^2+ab-ba+b^2}{ba+b^2+a^2-ab}\right)$$

$$=\tan^{-1}\frac{a^2+b^2}{a^2+b^2}$$

$$=\tan^{-1}1$$

$$=\frac{\pi}{4}$$

Question :

Write the principal value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$

Solution :

$$\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$$

$$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\cos^{-1}\left(-\cos\frac{\pi}{3}\right)$$

$$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\cos^{-1}\left(\cos\left(\pi-\frac{\pi}{3}\right)\right)$$

$$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)=\frac{\pi}{4}+\frac{2\pi}{3}=\frac{11\pi}{12}$$

Question :

Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$

Solution :

$$\tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$$

$$=\tan^{-1}\left[2\sin\left(\cos^{-1}\left(2.\frac{3}{4}-1\right)\right)\right]$$

$\because 2\cos^{-1}x=\cos^{-1}\left(2x^2-1\right)$

$$=\tan^{-1}\left[2\sin\left(\cos^{-1}\left(\frac{3}{2}-1\right)\right)\right]$$

$$=\tan^{-1}\left[2\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right)\right]$$

$$=\tan^{-1}\left[2\sin\left(\cos^{-1}\left(\cos\frac{\pi}{3}\right)\right)\right]$$

$$=\tan^{-1}\left[2\sin\frac{\pi}{3}\right]$$

$$=\tan^{-1}\left[2.\frac{\sqrt{3}}{2}\right]$$

$$=\tan^{-1}\left[\tan\frac{\pi}{3}\right]=\frac{\pi}{3}$$

Question :

Solve for x, $\tan^{-1}x+2\cot^{-1}x=\frac{2\pi}{3}$

Solution :

$$\tan^{-1}x+2\tan^{-1}\frac{1}{x}=\frac{2\pi}{3}$$

$$\Rightarrow\ \tan^{-1}x+\tan^{-1}\left(\frac{2\times\frac{1}{x}}{1-\frac{1}{x^2}}\right)=\frac{2\pi}{3}\ \Rightarrow\ \tan^{-1}x+\tan^{-1}\left(\frac{\frac{2x}{x^2}}{\frac{x^2-1}{x^2}}\right)=\frac{2\pi}{3}$$

$$\Rightarrow\ \tan^{-1}x+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}\ \Rightarrow\ \tan^{-1}\left(\frac{x+\frac{2x}{x^2-1}}{1-\frac{2x^2}{x^2-1}}\right)=\frac{2\pi}{3}$$

$$\Rightarrow\ \frac{x+\frac{2x}{x^2-1}}{1-\frac{2x^2}{x^2-1}}=\tan\frac{2\pi}{3}\ \Rightarrow\ \frac{\frac{x\left(x^2-1\right)+2x}{x^2-1}}{\frac{1\left(x^2-1\right)-2x^2}{x^2-1}}=\tan\frac{2\pi}{3}$$

$$\Rightarrow \quad \frac{x^3 - x + 2x}{x^2 - 1 - 2x^2} = \tan\left(\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow \quad \frac{x^3 + x}{-x^2 - 1} = -\tan\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad \frac{x\left(x^2 + 1\right)}{-\left(x^2 + 1\right)} = -\sqrt{3}$$

$$\Rightarrow \quad x = \sqrt{3}$$

Question :

Prove that, $\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right) = 7$ or $\frac{\pi}{4} - 2\cot^{-1}3 = \cot^{-1}7$

Solution :

$$\cot^{-1}7 + 2\cot^{-1}3 = \frac{\pi}{4}$$

$$\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = \frac{\pi}{4}$$

$$\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{2.\frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2} = \frac{\pi}{4}$$

$$\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{\frac{2}{3}}{\frac{8}{9}} = \frac{\pi}{4}$$

$$\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{4} = \frac{\pi}{4}$$

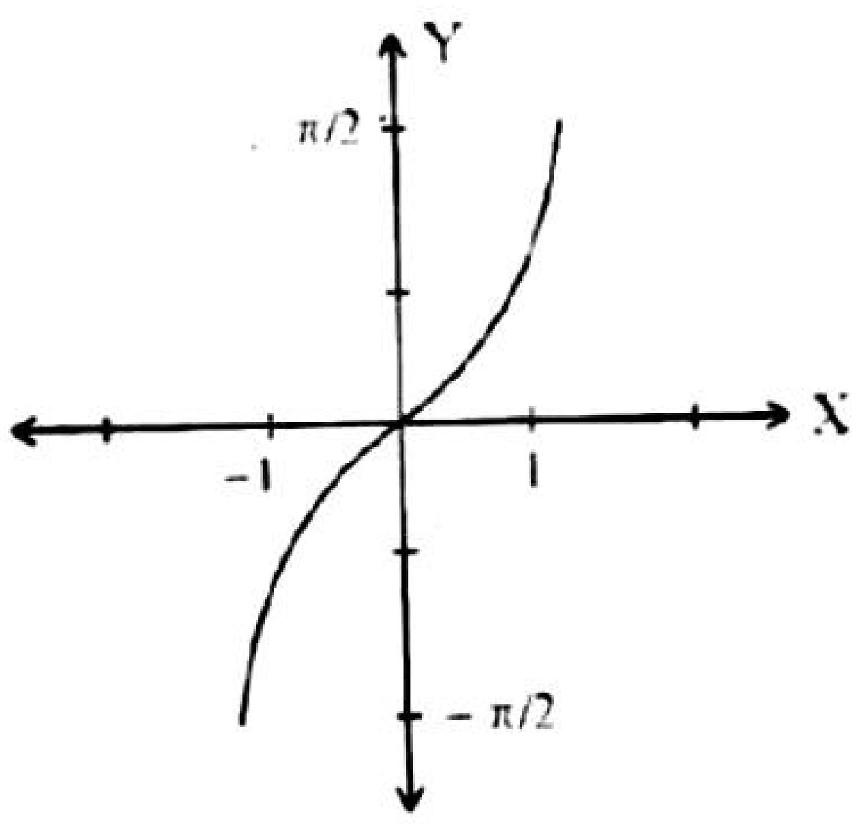
$$\tan^{-1}\frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7}.\frac{3}{4}} = \frac{\pi}{4}$$

$$\tan^{-1}\frac{\frac{25}{28}}{\frac{25}{28}} = \tan^{-1}1 = \frac{\pi}{4}$$

$$\frac{\pi}{4} = \frac{\pi}{4}$$

HOME WORK QUESTIONS

Question : (March 2018)



(a) Identify the function from the above graph.

(i) $\tan^{-1}x$ (ii) $\sin^{-1}x$ (iii) $\cos^{-1}x$ (iv) $\operatorname{cosec}^{-1}x$

(b) Find the domain and range of the function represented in above graph.

(c) Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{2}{11}=\tan^{-1}\frac{3}{4}$.

Hint or Answer :

(a) $\sin^{-1}x$

(b) Domain $[-1,1]$ and Range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(c) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{2}{11}=\tan^{-1}\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}}=\tan^{-1}\frac{15}{20}=\tan^{-1}\frac{3}{4}$.

Question : (Sept 2017)

(a) $\sin\left(\tan^{-1}1\right)$ is equal to $\frac{1}{\sqrt{2}}$, 1 , $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$

(b) If $x\in\left(0,\frac{\pi}{2}\right)$, show that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2}$

Hint or Answer :

(a) $\frac{1}{\sqrt{2}}$

(b) $1+\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}=\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$

$$\text{LHS}=\cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right)=\cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)=\frac{x}{2}$$

Question : (March 2017)

(a) Principle value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is ... $\left(\frac{\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}\right)$

(b) Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$

Hint or Answer :

(a) $\frac{2\pi}{3}$

(b) $$\tan^{-1}\left(\frac{\left(\frac{x-1}{x-2}\right)+\left(\frac{x+1}{x+2}\right)}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x^2-4}{-3}\right)=\frac{\pi}{4}$$

$$\Rightarrow \frac{2x^2-4}{-3}=1 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Question : (Imp 2016)

(a) Principle value of $\tan^{-1}\left(-\sqrt{3}\right)$ is ..... $\left(\frac{\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{4}, -\frac{\pi}{6}\right)$

(b) Show that $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{2}{11}\right)=\tan^{-1}\left(\frac{3}{4}\right)$

Hint or Answer : (a) $-\frac{\pi}{3}$

Question : (March 2016)

(a) If $xy < 1$, $\tan^{-1}x+\tan^{-1}y=$ ....

(b) Show that $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{31}{17}\right)$

Hint or Answer :

a) $\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

b) $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{2\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$=\tan^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right)$

$=\tan^{-1}\left(\frac{31\times 21}{21\times 17}\right)=\tan^{-1}\left(\frac{31}{17}\right)$