APPLICATION OF DERIVATIVES

CHAPTER – 6

APPLICATION OF DERIVATIVES

RATE OF CHANGES OF QUANTITIES

Let y = f(x) be a function of x. Then, $\frac{dy}{dx}$ represents the rate of change of y with respect to x. Also, $\frac{dy}{dx}\big|_{x=x_0}$ represents the rate of change of y with respect to x at $x = x_0$

If two variables x and y are varying with respect to another variable t, i.e. x = f(t) and y = g(t), then

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, where $\frac{dx}{dt} \neq 0$ (by chain rule)

In other words, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to t.

Note

$\frac{dy}{dx}$ is positive, if y increases as x increases and it is negative, if y decreases as x increases.

Example

A stone is dropped into a quiet lake and waves move in the form of circles at a speed of 4 cm/sec. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Solution: We know that the area of a circle with radius "r" is given by $A = \pi r^2$.

Hence, the rate of change of area "A' with respect to the time "t" is given by:

$\frac{dA}{dt} = \frac{d}{dt}\pi r^2$

By using the chain rule, we get: $\frac{d}{dr}(\pi r^2).\frac{dr}{dt} = 2\pi r.\frac{dr}{dt}$

It is given that, $\frac{dr}{dt} = 4$ cm/sec

Therefore, when r = 10 cm, $\frac{dA}{dt} = 2\pi.(10).(4)$ $\frac{dA}{dt} = 80\pi$

Hence, when r = 10 cm, the enclosing area is increasing at a rate of 80π cm^2/sec.

MARGINAL COST: Marginal cost represents the instantaneous rate of change of the total cost at any level of output. If C(x) represents the cost function for x units produced, then marginal cost (MC) is given by MC = $\frac{d}{dx}\{C(x)\}$

Example

The total cost C(x) in Rupees, associated with the production of c units of an item is given by C(x) = $0.005x^3$ – $0.02x^2$ +30x +5000.Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Solution: Since marginal cost is the rate of change of total cost with respect to the output, we have cost (MC) = $\frac{dC}{dx}$ = 0.005 $(3x^2) - 0.02(2x) + 30$

When x = 3, MC = $0.015(3^2) - 0.04(3) + 30$

= 0.135 – 0.12 +30 = 30.015

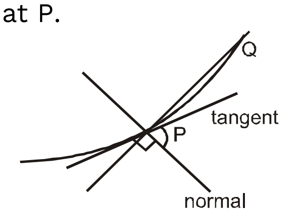
Hence, the required marginal cost Rs.30.02 (nearly)

MARGINAL REVENUE: Marginal revenue represents the rate of change of total revenue with respect to the number of items sold at an instant.

If R(x) is the revenue function for x units sold, then marginal revenue (MR) is given by MR = $\frac{d}{dx}\{R(x)\}$

TANGENTS AND NORMALS

Let y = f(x) be function with graph as shown in figure. Consider secant PQ. If Q tends to P along the curve passing through the points Q_1, Q_2, ...I.e. Q → P, secant PQ will become tangent at P. A line through P perpendicular to tangent is called normal at P.



GEOMETRICAL MEANING OF $\frac{dy}{dx}$

As Q → P, h → 0 and slope of chord PQ tends to slope of tangent at P (see figure).

Slope of chord PQ = $\frac{f(x+h)-f(x)}{h}$

$\lim_{Q\to P}$ slope of chord PQ = $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

⇒ slope of tangent at P = f′(x) = $\frac{dy}{dx}$

SLOPE OF LINE PASSING THROUGH TWO POINTS

Let two points be P(x_1, y_1) and Q(x_2, y_2) then the slope of line 'm' = $\frac{y_2-y_1}{x_2-x_1}$

Example

The slope of a passing through (2 , -1) and (3 , 4) is m = $\frac{4-(-1)}{3-2} = 5$

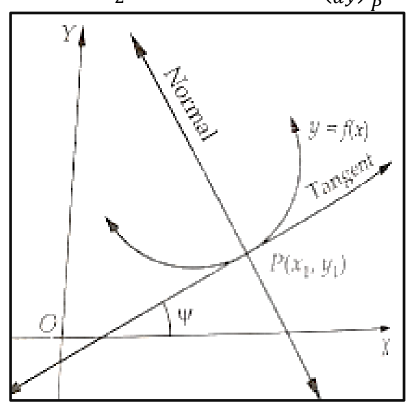
SLOPE OF TANGENT AND NORMAL

SLOPE OF TANGENT :

Let y = f(x) be a continuous curve , and let P(x_1, y_1) be a point n it. Then , $\left(\frac{dy}{dx}\right)_P$ is the slope of the tangent to the curve y = f(x) at point P i.e., $\left(\frac{dy}{dx}\right)_P$ = tan ψ = Slope of the tangent at P, where ψ is the angle which the tangent at P(x_1, y_1) makes with the positive direction of x – axis.

⇒ If the tangent at P is parallel to x -axis , then ψ = 0 ⇒ tan ψ = 0 ⇒ $\left(\frac{dy}{dx}\right)_P = 0$

⇒ If the tangent at P is perpendicular to x -axis or parallel to y -axis then , then ψ = $\frac{\pi}{2}$ ⇒ cot ψ = 0 ⇒ $\left(\frac{dx}{dy}\right)_P = 0$



SLOPE OF NORMAL: The normal to a curve at P(x_1, y_1) is a line perpendicular to the tangent at P and passing through P.

∴ Slope of the normal at p = $\frac{-1}{\text{Slope of the tangent at P}} = \frac{-1}{\left(\frac{dy}{dx}\right)_P} = -\left(\frac{dx}{dy}\right)_P$

Example

Find the slopes of the tangent and the normal to the curve $x^2 + 3y + y^2 = 5$ at (1 ,1)

Solution: The equation of the curve is $x^2 + 3y + y^2 = 5$

Differentiating with respect to x , we get

⇒ $2x + 3\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

⇒ $\frac{dy}{dx} = \frac{-2x}{2y+3}$

⇒ $\left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{2}{2+3} = -\frac{2}{5}$

And , slope of the normal at (1, 1) = $\frac{-1}{\left(\frac{dy}{dx}\right)_{(1,1)}} = \frac{-1}{\frac{-2}{5}} = \frac{5}{2}$

EQUATIONS OF TANGENT AND NORMAL

The equation of a line passing through a point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.

Since the normal at P(x_1, y_1) passes through P and has slope $= \frac{-1}{\left(\frac{dy}{dx}\right)_P}$. Therefore, the equation of the normal at P(x_1, y_1) to the curve y = f(x) is $y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_P}(x - x_1)$

Remark

1. If $\left(\frac{dx}{dy}\right)_P = \infty$, then t`he tangent at (x_1, y_1) is parallel to y – axis and its equation is x = x_1.

2 : If $\left(\frac{dx}{dy}\right)_P = 0$,then the normal at (x_1, y_1) is normal is parallel to y – axis and its equation is x = x_1.

Example

Find the equation of normal to the curve $y = 2x^2 + 3\sin x$ at x = 0

Solution: The equation of the given curve is $y = 2x^2 + 3\sin x$ ……..(i)

Putting x = 0 in equation (i), we get y = 0

So, the point of contact is (0 ,0).

Now , $y = 2x^2 + 3\sin x$

$\Rightarrow \frac{dy}{dx} = 4x + 3\cos x$

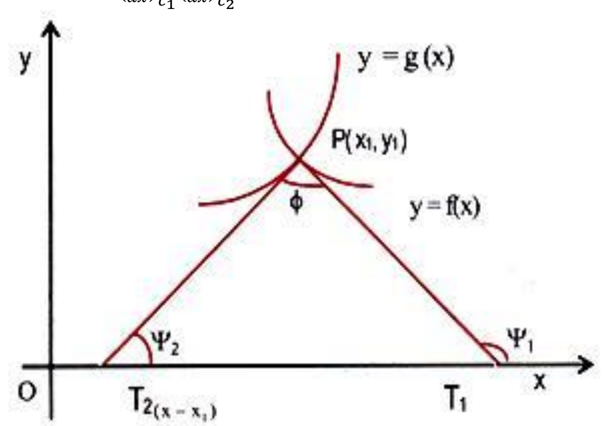
$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)} = 4 \times 0 + 3\cos 0 = 3$

So, the equation of the normal at (0 ,0) is $y - 0 = \frac{-1}{3}(x - 0)$ or $x + 3y = 0$

ANGLE OF INTERSECTION OF TWO CURVE

The angle of intersection of two curve is defined to be the angle between the tangents to the two curves at their point of intersection.

$\Rightarrow \tan\phi = \frac{\left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}}$



ORTHOGONAL CURVES

If the angle of intersection of two curves is a right angle, the two curves are said to intersect orthogonally and the curves are called orthogonal curves.

If the curve are orthogonal, then $\phi = \frac{\pi}{2}$

$\therefore\ m_1 . m_2 = -1$

$\Rightarrow \left(\frac{dy}{dx}\right)_{C_1} . \left(\frac{dy}{dx}\right)_{C_2} = -1$

Example

Find the angle of intersection od the curve $xy = 6$ and $x^2 y = 12$

Solution: The equations of the two curves are $xy = 6$ …….(i)

and $x^2 y = 12$ ….(ii)

From (i), we obtain y = $\frac{6}{x}$. Putting this value of y in (ii), we obtain $x^2\left(\frac{6}{y}\right) = 12 \Rightarrow 6x = 12 \Rightarrow x = 2$

Putting x = 2 in (i) or (ii), we get y = 3.Thus, the two curves intersect at P (2, 3).

Differentiating (i) w.r.t x, we get

$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x} \Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{-3}{2}$

Differentiating (ii) w.r.t x, we get

$x^2\frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} = \frac{-2y}{x} \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(2,3)} = -3$

Let θ be the angle of intersection of curve (i) and (ii) at point P, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\frac{3}{2} + 3}{1 + \left(-\frac{3}{2}\right)(-3)} = \frac{3}{11} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{11}\right)$$

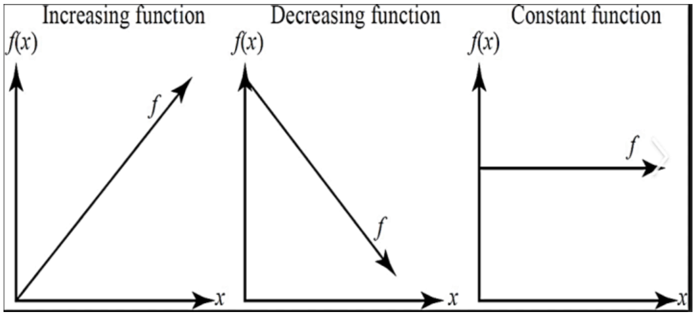
INCREASING AND DECREASING FUNCTION:

Let I be an open interval contained in the domain of a real valued function f. Then, f is said to be

- increasing on I, if $x_1 < x_2$ in I ⇒ $f(x_1) \le f(x_2)$, ∀ $x_1, x_2 \in$ I.
- strictly increasing on I, if $x_1 < x_2$ in I ⇒ $f(x_1) < f(x_2)$, ∀ $x_1, x_2 \in$ I.
- decreasing on I, if $x_1 < x_2$ in I ⇒ $f(x_1) \ge f(x_2)$, ∀ $x_1, x_2 \in$ I.
- strictly decreasing on I, if $x_1 < x_2$ in $f(x_1) > f(x_2)$, ∀ $x_1, x_2 \in$ I.
- Let x_0 be a point in the domain of definition of a real-valued function f, then f is said to be increasing, strictly increasing, decreasing or strictly decreasing at x_0, if there exists an open interval I containing x_0 such that f is increasing, strictly increasing, decreasing or strictly decreasing, respectively in I.

Note

If for a given interval I ⊆ R, function f increase for some values in I and decrease for other values in I, then we say function is neither increasing nor decreasing.



THEOREM

Let f be continuous on [a, b] and differentiable on the open interval (a, b). Then,

- f is increasing in [a, b] if f'(x) > 0 for each x ∈ (a, b).
- f is decreasing in [a, b] if f'(x) < 0 for each x ∈ (a, b).
- f is a constant function in [a, b], if f'(x) = 0 for each x ∈ (a, b).

NECESSARY AND SUFFICIENT CONDITIONS FOR MONOTONICITY

THEOREM 1: (Necessary condition) Let f(x) be continuous on [a , b] and differentiable on (a , b).

(i) If f(x) is strictly increasing on (a , b) , then $f'(x) > 0$ for all $x \in (a, b)$.

(ii) If f(x) is strictly decreasing on (a , b) , then $f'(x) < 0$ for all $x \in (a, b)$.

THEOREM 2: (sufficient condition) Let f be a differentiable real function defined on an open interval (a , b).

(i) If $f'(x) > 0$ for all $x \in (a , b)$, then f(x) is increasing on (a , b)

(ii) If $f'(x) < 0$ for all $x \in (a , b)$, then f(x) is decreasing on (a , b)

COROLLARY : Let f(x) be a function defined on (a , b).

(i) If $f'(x) > 0$ for all $x \in (a , b)$ except for a finite number of points , where $f'(x) = 0$ then f(x) is increasing on (a , b).

(ii) If $f'(x) < 0$ for all $x \in (a , b)$ except for a finite number of points , where $f'(x) = 0$,then f(x) is decreasing on (a , b).

Example

Show that the function f(x) = tan x – 4x is strictly decreasing on [-π/3, π/3]

Solution: Given that, f(x) = tan x – 4x
Then, the differentiation of the function is given by:
f'(x)= $\sec^2$x – 4
When -π/3 <x π/3, 1<sec x <2
Then, 1<$\sec^2$x <4
Hence, it becomes -3 < ($\sec^2$x-4) <0
Hence, for -π/3 <x π/3, f'(x)<0
Therefore, the function "f" is strictly decreasing on [-π/3, π/3].

Example

Prove that the function given by f(x) = cos x is increasing in $(\pi , 2\pi)$

Solution: We know that f'(x) = - sin x
Since for each x belongs to (π , 2π), sin x < 0 , we have f'(x)> 0 and so f is increasing in (π , 2π),

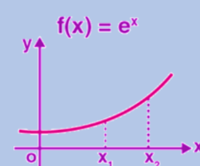
MONOTONIC FUNCTION

A function which is either increasing or decreasing in a given interval I, is called monotonic function.

A function f(x) is said to be increasing (decreasing) at a point x_0 if there is an interval $(x_0 - h , x_0 + h)$ containing x_0 such that f(x) is increasing (decreasing) on $(x_0 - h , x_0 + h)$.it is also increasing (decreasing) at x = a and x = b.

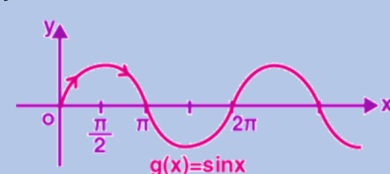
Example

Let f(x) = e^x is strictly increasing. Hence monotone function.



Example

Let f(x) = sin x is not a monotone function



APPROXIMATION: Let y = f(x) be any function of x. Let Δx be the small change in x and Δy be the corresponding change in y.i.e. Δy = f(x + Δx) – f(x).Then, dy = f'(x) dx or dy = $\frac{dy}{dx}$ Δx is a good approximation of Δy, when dx = Δx is relatively small and we denote it by dy ~ Δy.

Note

(i) The differential of the dependent variable is not equal to the increment of the variable whereas the differential of the independent variable is equal to the increment of the variable.

(ii) Absolute Error The change Δx in x is called absolute error in x.

Example

Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%

Solution: Note that V = x^3

dV = $\left(\frac{dV}{dx}\right)\Delta x = (3x^2)\Delta x$

= $(3x^2)(0.002x) = 0.06x^3\ m^3$ (as 2% of x is 0.02x)

Thus , the approximate change in volume is 0.06 $x^3\ m^3$.

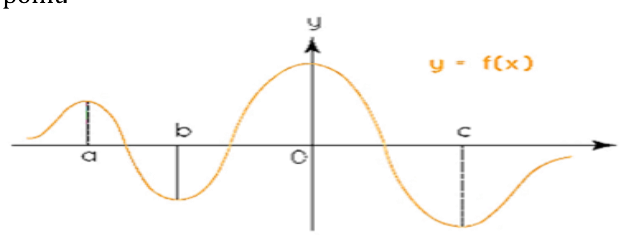
MAXIMUM AND MINIMUM VALUES OF A FUNCTION

Let f be a function defined on an interval I. Then,

(i) f is said to have a maximum value in I, if there exists a point c in I such that f(c) > f(x), ∀ x ∈ I. The number f(c) is called the maximum value of f in I and the point c is called a point of a maximum value of f in I.

(ii) f is said to have a minimum value in I, if there exists a point c in I such that f(c) < f(x), ∀ x ∈ I. The number f(c) is called the minimum value of f in I and the point c is called a point of minimum value of f in I.

(i) f is said to have an extreme value in I, if there exists a point c in I such that f(c) is either a maximum value or a minimum value of f in I. The number f(c) is called an extreme value off in I and the point c is called an extreme point.



Example

Find the maximum and the minimum value

(i) f(x) = $3x^2 + 6x + 8$, $x \in R$

Solution: (i) We have , f(x) = $3x^2 + 6x + 8$

Or , f(x) = $3\ (x^2 + 2x + 1) + 5 = 3(x+1)^2 + 5$

Clearly , $3(x+1)^2 \geq 0$ for all $x \in R$

$\Rightarrow f(x) \geq f(-1)$ for all $x \in R$

Thus , 5 is the minimum value of f(x) which it attains at x = -1.

Since f(x) can be made as large as we please. Therefore , the maximum value does not exist .

LOCAL MAXIMA AND LOCAL MINIMA

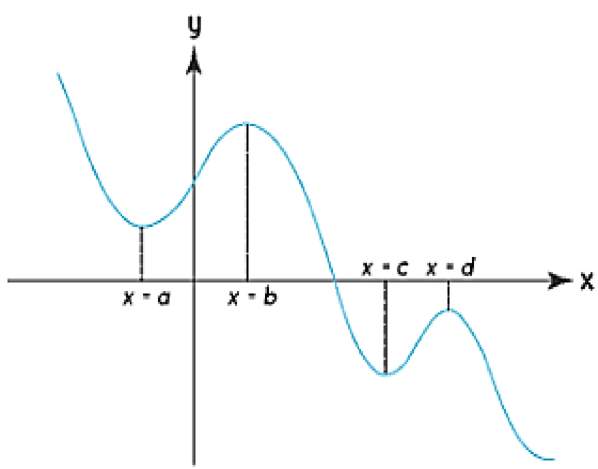
LOCAL MAXIMUM:

A function f(x) is said to have a local maximum value at point x = a, if there exists a neighbourhood (a − δ, a + δ) of a such that f(x) < f(a), ∀ x ∈ (a – δ, a + δ), x ≠ a. Here, f(a) is called the local maximum value of f(x) at the point x = a.

LOCAL MINIMUM:

A function f(x) is said to have a local minimum value at point x = a, if there exists a neighbourhood (a − δ, a + δ) of a such that f(x) > f(a), ∀ x ∈ (a – δ, a + δ), x ≠ a. Here, f(a) is called the local minimum value of f(x) at x = a.

- The points at which a function changes its nature from decreasing to increasing or vice-versa are called turning points. Note:
  - (i) Through the graphs, we can even find the maximum/minimum value of a function at a point at which it is not even differentiable.
  - (ii) Every monotonic function assumes its maximum/minimum value at the endpoints of the domain of definition of the function.
- Every continuous function on a closed interval has a maximum and a minimum value.
- Let f be a function defined on an open interval I. Suppose c is any point. If f has local maxima or local minima at x = c, then either f'(c) = 0 or f is not differentiable at c.



Example

Determine all the points of local maxima and local minima of the following function: f(x)= (-¾) $x^4 – 8x^3 – (45/2) x^2 + 105$

Solution: Given function: f(x) = (-¾) $x^4 – 8x^3 – (45/2) x^2 + 105$

differentiate the function with respect to x, we get

$f'(x) = –3x^3 – 24x^2 – 45x$

Now take, -3x as common:

$= – 3x (x^2 + 8x + 15)$

Factorize the expression inside the bracket, then we have:

$= – 3x (x +5) (x+3)$ $\qquad$ $f'(x) = 0$

$\Rightarrow x = –5, x = –3, x = 0$

Now, again differentiate the function:

$f''(x) = –9x^2 – 48x – 45$ $\qquad$ Take -3 outside,

$= –3 (3x^2 + 16x + 15)$

Now, substitute the value of x in the second derivative function.

$f''(0) = – 45 < 0$. Hence, x = 0 is point of local maxima $\qquad$ $f''(–3) = 18 > 0$. Hence, x = –3 is point of local minima

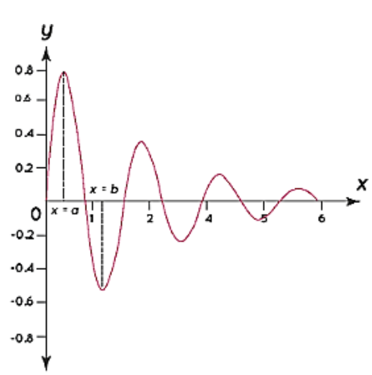
$f''(–5) = –30 < 0$. Hence, x = –5 is point of local maxima.

ABSOLUTE MAXIMA AND MINIMA

The highest point of a function within the entire domain is known as the absolute maxima of the function whereas the lowest point of the function within the entire domain of the function, is known as the absolute minima of the function. There can only be one absolute maximum of a function and one absolute minimum of the function over the entire domain. The absolute maxima and minima of the function can also be called the global maxima and global minima of the function.

- Absolute maxima: A point x = a is a point of global maximum for f(x) if f(x) ≤ f(a) for all x∈D (the domain of f(x)).
- Absolute minima: A point x = a is a point of global minimum for f(x) if f(x) ≥ f(a) for all x∈D (the domain of f(x)).

In the image given below, point x = a is the absolute maxima of the function and at x = b is the absolute minima of the function.



CRITICAL POINTS:

A point c in the domain of a function f at which either f'(c) = 0 or f is not differentiable, is called a critical point of f.

Example

Find the critical points of the function f (x) = x^2 ln x

Solution:

Take the derivative using the product rule:

$f'(x) = (x^2 \ln x)' = 2x \ln x + x^2 [1 / x] = 2x \ln x + x$

$= x (2 \ln x + 1)$.

Determine the points where the derivative is zero:

$f'(c) = 0$,

$\Rightarrow c (2 \ln c + 1) = 0$.

The first root $c_1 = 0$ is not a critical point because the function is defined only for x > 0.

Consider the second root:

$2 \ln c + 1 = 0, \Rightarrow \ln c = –1 / 2$,

$\Rightarrow c_2 = e^{-1/2} = 1 / \sqrt{e}$.

Hence, $c_2 = 1 / \sqrt{e}$ is a critical point of the given function.

FIRST DERIVATIVE TEST FOR LOCAL MAXIMA AND MINIMA

THEOREM 1: (First derivative test) Let f be a differentiable function defined on an interval I and let $a \in I$.

Then ,

(a) x = a is a point of local maximum value of f , if

(i) $f'(a) = 0$

(ii) $f'(x)$ change sign from positive to negative as x increase through a i.e., $f'(x) > 0$ at every point sufficiently close to and to the left of a , and $f'(x) > 0$ at every point sufficiently close to and to the right of a.

(b) x = a is a point of local minimum value of f , if

(i) $f'(a) = 0$

(ii) $f'(x)$ changes sign from negative to positive as x increase through a

(c) If $f'(a) = 0$ and $f'(x)$ does not change sign as increases through a i.e., $f'(x)$ has the same sign in the complete neighbourhood of a , then 'a' is neither a point of local maximum value nor a point of local minimum value . In fact , such 'a' point is called a point of inflexion.

Example

Find local maximum and local minimum values of the function f given by $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$ using the first derivative test.

Solution: Given,

$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$

Step 1: Evaluate the first derivative of f(x), i.e. f'(x)

$f'(x) = (d/dx)\ [3x^4 + 4x^3 - 12x^2 + 12]$

$= 3(4x^3) + 4(3x^2) - 12(2x) + 0$

$= 12x^3 + 12x^2 - 24x$

$= 12\,x\,(x^2 + x - 2)$

$= 12\,x\,(x^2 + 2x - x - 2)$

$= 12\,x\,[x\,(x + 2) - 1(x + 2)]$

$= 12\,x\,(x - 1)\,(x + 2)$

Thus, $f'(x) = 12x\,(x - 1)\,(x + 2)$

Step 2: Identify the critical points, i.e. value(s) of c by assuming f'(x) = 0

i.e. $12x\,(x - 1)\,(x + 2) = 0$

$x = 0,\ x - 1 = 0$ or $x + 2 = 0$

That means, f'(x) = 0 at x = 0, x = 1 and x = -2.

Therefore, the critical points are -2, 0, and 1.

SECOND DERIVATIVE TEST:

Let f(x) be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then,

(i) x = c is a point of local maxima, if $f'(c) = 0$ and $f''(c) < 0$.

(ii) x = c is a point of local minima, if $f'(c) = 0$ and $f''(c) > 0$.

(iii) the test fails, if $f'(c) = 0$ and $f''(c) = 0$.

Note

(i) If the test fails, then we go back to the first derivative test and find whether a is a point of local maxima, local minima or a point of inflexion.

(ii) If we say that f is twice differentiable at o, then it means second order derivative exists at a.

Example

Find the local maxima and minima of the function $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$.

Solution: For stationary points f'(x) = 0.

$f'(x)= 12x^3 + 12x^2 - 24x = 0$

$\Rightarrow 12x\,(x^2 + x - 2) = 0$

$\Rightarrow 12x\,(x - 1)\,(x + 2) = 0$

=> Hence, x = 0, x = 1 and x = -2

By second derivative test:

$f''(x) = 36x^2 + 24x - 24$

$f''(x) = 12(3x^2 + 2x - 2)$

At x = -2

$f''(-2) = 12(3(-2)^2 + 2(-2) - 2) = 12\,(12 - 4 - 2)$

$= 12(6) = 72 > 0$

At x = 0

$f''(0) = 12(3(0)^2 + 2(0) - 2) = 12(-2) = -24 < 0$

At x = 1

$f''(1) = 12(3(1)^2 + 2(1) - 2) = 12(3+2-2) = 12(3)$

$= 36 > 0$

Therefore, by the second derivative test x=0 is the point of local maxima while x = -2 and x=1 are the points of local minima.

Example

For the given curve: $y = 5x - 2x^3$, when x increases at the rate of 2 units/sec, then how fast is the slope of curve changes when x = 3?

Solution: Given that, $y = 5x - 2x^3$

Then, the slope of the curve, $\frac{dy}{dx} = 5-6x^2$

$\Rightarrow \frac{d}{dt}\left[\frac{dy}{dx}\right] = -12x.\frac{dx}{dt}$

$= -12(3)(2)$

$= -72$ units per second

Hence, the slope of the curve is decreasing at the rate of 72 units per second when x is increasing at the rate of 2 units per second.

POINT OF INFLECTION

An arc of a curve y = f(x) is called concave upward if , at each of its points , the arc lies above the tangent at the point.

If y = f(x) is a concave upward curve , the as x increases , $f'(x)$ either is of the same sign and increasing or changes sign

QUESTIONS

MCQ

Q1. Find the rate of change of the area of a circle per second with respect to its radius r =10 cm
(a) 20π (b) 30π
(c) 10π (d) 40π

Q2. A stone is dropped into a quiet lake and waves move in circles at a speed Of 4cm per second. How fast is the enclosed area increasing when radius is 10cm?"
(a) 50π (b) 30π
(c) 80π (d) 70π

Q3. The total cost C(x) in Rupees, associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$ .Find the marginal cost when 5 units are produced(approx.) .
(a) 25 (b) 29
(c) 28 (d) 30

Q4. The function $f(x) = x^3 - 3x^2 + 4x, x \in \mathbf{R}$ is
(a) Increasing (b) Decreasing
(c) Neither increasing and decreasing (d) None

Q5. the function given by $f(x) = \cos x \; in \; (\pi, 2\pi)$ is
(a) Increasing (b) Decreasing
(c) Neither increasing and decreasing (d) None

Q6. The intervals in which the function f given by f(x)=$x^2 - 4x + 6$ is decreasing
(a) $(2, \infty)$ (b) $(-2, \infty)$
(c) $(-\infty, 2)$ (d) $(-\infty, -2)$

Q7. Find the intervals in which the function f given by $f(x) = \sin x + \cos x\,, 0 \leq x \leq 2\pi$ is decreasing.
(a) $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$
(b) $\left(-\frac{\pi}{4}, \frac{5\pi}{4}\right)$
(c) $\left(-\frac{5\pi}{4}, \frac{-\pi}{4}\right)$
(d) $\left(\frac{-5\pi}{4}, \frac{\pi}{4}\right)$

Q8. Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$.
(a) 12 (b) -11
(c) 11 (d) -12

Q9. Find the point at which the tangent to the curve y=$\sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.
(a) (2,3) (b) (5,2)
(c) (2,5) (d) (3,2)

Q10. Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis.
(a) $20y - x + 7 = 0$ (b) $20y + x + 7 = 0$
(c) $20y - x - 7 = 0$ (d) $-20y - x + 7 = 0$

Q11. Find the equation of tangent to the curve given by $x = a\sin^3 t, y = b\cos^3 t$ at a point where $t = \frac{\pi}{2}$.
(a) $x = 0$ (b) $y = 0$
(c) $x = y$ (d) $x = -y$

Q12. Find the approximate change in the volume V of a cube of side x meters
caused by increasing the side by 2%.
(a) 0.04 (b) 0.06
(c) 0.02 (d) 0.05

Q13. A balloon which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$ Find the rate of change of its volume with respect to x.
(a) $\frac{27\pi}{8}(2x+1)^2$ (b) $-\frac{27\pi}{8}(2x+1)^2$
(c) $\frac{27\pi}{8}(2x-1)^2$ (d) $\frac{27\pi}{8}(2x+1)$

Q14. The total cost C(x) in rupees associated with the production of x units of an item Given by C(x)=$0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when " 17" units are produced."
(a) 30.65
(b) 19.20
(c) 20.97
(d) 25.5

Q15. Find the maximum and minimum values of f , if any, of the function given by f(x) = |x| , x belongs to **R**
(a) Maximum value = 1, Minimum value = 0
(b) No maximum value, Minimum value = 0
(c) Maximum value = 1, Minimum value = 1
(d) Maximum value = 1, Minimum value = -1

Q16. The function given by $f(x) = 3x + 17$ is
(a) Decreasing
(b) Increasing
(c) Neither increasing nor decreasing
(d) None

Q17. The function given by $f(x) = e^{2x}$ is
(a) Decreasing
(b) Increasing
(c) Neither increasing nor decreasing
(d) None

Q18. Which of the following function is monotonic function
(a) F(x) = cos x (b) F(x) = sin x
(c) F(x) = e^x (d) None of these

Q19. Find the approximate value of f (2·01), where f(x)=$4x^2 + 5x + 2$
(a) 25.21 (b) 24.21
(c) 27.21 (d) 28.21

Q20. The interval in which the function $(x+1)^3(x-3)^3$ is strictly increasing

(a) $(2,\infty)$ (b) $[2,\infty)$

(c) $(1,\infty)$ (d) $[1,\infty)$

Q21. Which of the following is true about $y=\frac{4\sin\theta}{(2+\cos\theta)}-\theta$,when θ is in $\left[0,\frac{\pi}{2}\right]$

(a) The function is increasing

(b) The function is decreasing

(c) The function is neither increasing nor decreasing

(d) None

Q22. The interval in which $y=x^2e^{-x}$ is increasing in:

(a) $(-\infty,\infty)$ (b) $(-2,0)$

(c) $(2,\infty)$ (d) $(0,2)$

Q23. Find the slope of tangent to the curve $y=x^3-x+1$ at the given point whose x −coordinate is 2.

(a) 10 (b) 13

(c) 11 (d) 14

Q24. The slope of the normal to the curve $y=2x^2+3\sin x$ at $x=0$ is:

(a) 3 (b) 1/3

(c) -3 (d) -1/3

Q25. If the radius of a sphere is measured as 7m with an error of 0.02m, then find the approximate error in calculating its volume.

(a) 12.32 (b) 13.41

(c) 15.87 (d) 11.52

Q26. Find the maximum and minimum values, of the function " $g(x)=-|x+1|+3$

(a) Maximum=1 Minimum =-1

(b) Maximum doesn't exist Minimum =-1

(c) Maximum doesn't exist Minimum =-3

(d) Minimum doesn't exist Maximum=3

Q27. Which of the following is true for the function

(a) f is said to have a maximum value of f in I ,if there exists a point c in I such that f(c)> $f(x), for\ all\ x \in \mathrm{R}$

(b) f is said to have a minimum value of f in I ,if there exists a point c in I such that f(c)> $f(x), for\ all\ x \in \mathrm{R}$

(c) f is said to have a maximum value of f in I ,if there exists a point c in I such that f(c)< $f(x), for\ all\ x \in \mathrm{R}$

(d) None of these

Q28. Find local minimum value of the function f given by f(x) = 3 +|x|, $x \in \mathrm{R}$

(a) $f(0)=3$ (b) $f(3)=0$

(c) $f(1)=0$ (d) None of these

Q29. Find the absolute maximum value of function f given by f(x) = $2x^3-15x^2+36x+1$ on the interval [1, 5]

(a) 34 (b) 56

(c) 49 (d) 23

Q30. Find two positive integers whose sum is 16 and sum of whose cubes is minimum.

(a) 7,7 (b) 8,8

(c) 6,6 (d) 5,5

SUBJECTIVE QUESTIONS

Q1. Find the equation of tangent and normal to the curve $x=\frac{2at^2}{1+t^2}$, $y=\frac{2at^3}{1+t^2}$ at the point $t=\frac{1}{2}$.

Q2. Find value of 'c' such that line joining (0, 4) and (5, –1) become tangent to curve $y=\frac{c}{x+1}$.

Q3. Find the length of tangent for the curve $y = x^3 + 3x^2 + 4x - 1$ at point x = 0.

Q4. The volume of a cube is increasing at a rate of 7 cm^3/sec. How fast is the surface area increasing when the length of an edge is 4 cm?

Q5. f(x) = [x] is a step up function. Is it a strictly increasing function for $x \in R$.

NUMERICAL TYPE QUESTIONS

Q1. If $f(x) = x^3 + ax^2 + bx + c$ has extreme values at x = – 1 and x = 3. Find the value of product of a and b_____.

Q2. Find the greatest value of $f(x) = x^3 - 12x$, $x \in [-1, 3]$______.

Q3. Find points of local minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$ ________.

Q4. Find the sum of two positive numbers x and y such that x + y = 60 and xy^3 is maximum_______.

Q5. The number of tangents drawn to the curve xy = 4 from point (0, 1) is________.

TRUE AND FALSE

Q1. Find the intervals of monotonicity of the following functions , $f(x) = x^2 (x - 2)^2$ Decreasing in [0, 1] and in [2, ∞)

Q2. points of maxima or minima of f(x) = $x^2 (x - 2)^2$ is x = 1 or x = 0, 2 respectively.

Q3. f(x) = ($x^3 - 6x^2 + 12x - 8$) does not have any point of local maxima or minima.

Q4. Find equation of tangent to y = e^x at x = 0 is y = x - 1

Q5. The approximate value of $25^{1/3}$ is 2.926

ASSERTION AND REASONING

Directions (Q. No. 1 – 5) Each of these questions contains two statements , one is Assertion (A) and other is Reason (R).Each of these questions also has four alternative choice ,only one of which is the correct answer.

(a) Both A and R are individually true and R is the correct explanation of A.
(b) Both A and R are individually true but R is not the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true.

Q1. **Assertion(A):** Curve $y = xe^x$ is minimum at the point x = -1

Reason (R): $\frac{dy}{dx} < 0$ at x = -1.

Q2. Assertion(A): $y = x^2e^{-x}$ increases on the interval (0 ,2)

Reason (R): Since , $\frac{dy}{dx} > 0$ on the interval (0 ,2).

Q3. Assertion(A): The tangent to the curve $y = x^3 - x^2 - x + 2$ at (1 ,1) is parallel to the x - axis

Reason (R): The slope of the tangent to the curve at (1,1) is zero.

Q4. Assertion(A): The interval in which the function f(x) = $x^{\frac{1}{x}}$ is increasing in (-∞ , e)

Reason (R): Let f(x) be differentiable. If f′(a) > 0 then f(x) is increasing at x = a.

Q5. Assertion(A): A particle moves in a straight line in such a way that its velocity at any point is given by $v^2 = 2 - 3x$, where x is measured from a fixed point. Then acceleration is $\frac{-3}{2}$

Reason (R): If x and v denotes the displacement and velocity of a particle at any instant t , then acceleration is given by a = $\frac{d^2x}{dt^2}$

HOMEWORK

MCQ

Q1. The angle between the curves y = sin x and y = cos x is

(a) $\tan^{-1}(2\sqrt{2})$ (b) $\tan^{-1}(3\sqrt{2})$
(c) $\tan^{-1}(3\sqrt{3})$ (d) $\tan^{-1}(5\sqrt{2})$

Q2. f(x) = x + 1/x, x ≠ 0 is increasing when -
(a) |x| < 1
(b) |x| > 1
(c) |x| < 2
(d) |x| > 2

Q3. If f(x) = $2x^3 - 9x^2 + 12x - 6$, then in which interval f(x) is monotonically increasing
(a) (1, 2)
(b) (−∞, 1)
(c) (2, ∞)
(d) $(-\infty, 1) \cup (2, \infty)$

Q4. The function f(x) = a sin x + $\frac{1}{3}$ sin 3x has a maximum at x = π/3, then a equals-
(a) – 2 (b) 2
(c) –1 (d) 1

Q5. f (c) is a maximum value of f(x) if-
(a) f′ (c) = 0, f″ (c) > 0
(b) f′ (c) = 0, f″ (c) < 0
(c) f′ (c) ≠ 0, f″ (c) = 0
(d) f′ (c) < 0, f″ (c) > 0

Q6. The ratio between the height of a right circular cone of maximum volume inscribed in a given sphere and the diameter of the sphere is-
(a) 2 : 3 (b) 3 : 4
(c) 1 : 3 (d) 1 : 4

Q7. The function f(x) = $\frac{x}{2} + \frac{2}{x}$ has a local minimum at –
(a) x = – 2 (b) x = 0
(c) x = 1 (d) x = 2

Q8. The maximum value $x^3 - 3x$ in the interval [0, 2] is
(a)-2 (b)0
(c) 2 (d)1

Q9. If $f(x) = k x^3 - 9 x^2 + 9 x + 3$ is increasing on R, then
(a) $k < 3$ (b) $k > 3$
(c) $k \leq 3$ (d) None of these

Q10. The function $f(x) = x^2 e^{-x}$ increases in the interval
(a) (0, 2) (b)(2, 3)
(c) (3, 4) (d) (4, 5)

SUBJECTIVE QUESTIONS

Q1. If a right circular cylinder is inscribed in a given cone. Find the dimensions of the cylinder such that its volume is maximum.

Q2. Find the points of local maxima or minima for f(x) = sin2x – x, x ∈ (0, π).

Q3. Find the critical points of the function f(x) = $4x^3 - 6x^2 - 24x + 9$ if
(i) x ∈ [0, 3]
(ii) x ∈ [–3, 3]
(iii) x ∈ [– 1, 2].

Q4. Let f(x) = $x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$. If f(x) has positive point of maxima, then find possible values of 'a'.

Q5. Find possible values of 'a' such that f(x) = $e^{2x} - (a + 1) e^x + 2x$ is monotonically increasing for x ∈ R

NUMERICAL TYPE QUESTIONS

Q1. Determine 'p' such that the length of the sub-tangent and sub-normal is equal for the curve y = e^{px} + p x at the point (0, 1)______.

Q2. Find the least value of k for which the function $x^2 + kx + 1$ is an increasing function in the interval 1 < x < 2._________.

Q3. A point moves in a straight line during the time t = 0 to t =3 according to the laws s = $15t - 2t^2$. The average velocity of the point is _______.

Q4. The maximum value of f(x) = $2x^3 - 24x + 107$ in the interval [1 ,3] is ________.

Q5. If sum of two number is 3 .The maximum value of the product of first and the square of second is _____.

TRUE AND FALSE

Q1. Let f be a differentiable a\real function defined on an open interval (a , b) . If f'(x) > 0 for all $x \in (a, b)$,then f(x) is increasing on (a , b).

Q2. The function $y = \tan^{-1} x - x$ is always increasing.

Q3. The value of x for which the polynomial $2x^3 - 9x^2 + 12x + 4$ is a decreasing function of x , is 1 < x< 2.

Q4. The local maximum value of $x(1-x)^2$, 0 ≤ x ≤ 2 is $\frac{4}{27}$

Q5. Angle between the tangents to the curve y = $x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is $\frac{\pi}{2}$

ASSERTION AND REASONING

Directions (Q.No. 1 – 5) Each of these questions contains two statements , one is Assertion (A) and other is Reason (R).Each of these questions also has four alternative choice ,only one of which is the correct answer.
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(b) Both A and R are individually true but R is not the correct explanation of A (c) A is true but R is false
(d) A is false but R is true.

Q1. **Assertion(A):** f(x) = ln x is an decreasing function on (0 , ∞).
Reason (R): f(x) = $e^x - x \ln x$ is an increasing function on (1 , ∞).

Q2. Consider the function f(x) = $\frac{x^2-1}{x^2+1}$, where $x \in R$.
Assertion (A): f(x) attain minimum value at x = 0
Reason (R): The minimum value of f(x) is 1

Q3. **Assertion(A):** $f(x) = x^2 - 5x + 6$ is decreasing function in the interval (-∞ , 2).
Reason (R): . If f'(x) < 0 for all $x \in (a, b)$,then f(x) is decreasing on (a , b).

Q4. **Assertion(A):** The shortest distance between the line y – x = 1 and the curve x = y^2 is $\frac{3\sqrt{2}}{8}$
Reason (R): The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is $y - 3 = 0$.

Q5. **Assertion(A):** Critical point of f(x) = $x^3 - 3x^2 + 3x - 100$ is 1
Reason (R): A point c in the domain of a function f at which either f'(c) = 0 or f is not differentiable, is called a critical point of f.

SOLUTIONS

MCQ

S1. (a): The area A of a circle with radius r is given by $\mathrm{A} = \pi r^2$.
The rate of change of the area A w.r.t r is $\frac{d\mathrm{A}}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$.
When $r = 10\mathrm{cm}$, $\frac{d\mathrm{A}}{dr} = 20\pi$.
Thus, the correct answer is (a).

S2. (c): The area A of a circle with radius r is given by $\mathrm{A} = \pi r^2$. Therefore, the rate of change of area A with respect to time t is
$\frac{d\mathrm{A}}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt} = 2\pi r \frac{dr}{dt}$
Given,
$\frac{dr}{dt} = 4\mathrm{cm/s}$
$\frac{d\mathrm{A}}{dt} = 2\pi(10)(4) = 80\pi [\because r = 10]$
Thus, the correct answer is (c).

S3. (d): $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$
$\frac{dC}{dx} = 0.005(3x^2) - 0.02(2x) + 30$
$x = 5, \text{Marginal Cost} = 0.015(5^2) - 0.04(5) + 30 = 30.175$
Thus, the correct answer is (d).

S4. (a): $f'(x) \& = 3x^2 - 6x + 4$
$= 3(x^2 - 2x + 1) + 1$
$= 3(x-1)^2 + 1 > 0$
"Therefore, the function " f" is increasing on " **R**"."
Thus, the correct answer is (a).

S5. (a): Since for each $x \in (\pi, 2\pi)$, $\sin x < 0$, we have $f'(x) > 0$ and so f is increasing in $(\pi, 2\pi)$.
Thus, the correct answer is (a).

S6. (c): Given,
$f(x) = x^2 - 4x + 6$
$f'(x) = 2x - 4$
$\Rightarrow f'(x) = 0 \; gives \; x = 2$
In the interval $(-\infty, 2)$, $f'(x) = 2x - 4 < 0$
$\therefore f$ is decreasing in this interval.
Thus, the correct answer is (c).

S7. (a): $f(x) = \sin x + \cos x$
$f'(x) = \cos x - \sin x$
Now $f'(x) = 0$ gives $\sin x = \cos x \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$ as $0 \leq x \leq 2\pi$
$Also, f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$
$\Rightarrow f$ is decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$
Thus, the correct answer is (a).

S8. (c): $\left.\frac{dy}{dx}\right]_{x=2} = 3x^2 - 1\Big]_{x=2} = 11$
Thus, the correct answer is (c).

S9. (d): Slope of tangent to the given curve at (x, y) is $\frac{dy}{dx} = \frac{1}{2}(4x-3)^{\frac{-1}{2}} 4 = \frac{2}{\sqrt{4x-3}}$
The slope is given to be $\frac{2}{3}$.
$\Rightarrow \frac{2}{\sqrt{4x-3}} \& = \frac{2}{3}$

$\Rightarrow 4x - 3\& = 9$
$\Rightarrow x\& = 3$
Now $y = \sqrt{4x-3} - 1$. So when $x = 3, y = \sqrt{4(3)-3} - 1 = 2$.
Therefore, the required point is $(3,2)$.
Thus, the correct answer is (d).

S10. (a): Given,
$y = \frac{x-7}{(x-2)(x-3)} \dots (i)$
Now when the curve cuts the x-axis the y-coordinate will be 0.
From (i), we get that if y=0 then x=7.
Thus, the curve cuts the x-axis at $(7,0)$.
Differentiating (i) w.r.t x
$\frac{dy}{dx} = \frac{1-y(2x-5)}{(x-2)(x-3)}$
$\left.\frac{dy}{dx}\right]_{(7,0)} = \frac{1-0}{(5)(4)} = \frac{1}{20}$
Therefore, the slope of the tangent at $(7,0)$ is $\frac{1}{20}$.
Hence, the equation of the tangent at $(7,0)$ is
$y - 0 = \frac{1}{20}(x-7) \Rightarrow 20y - x + 7 = 0$.
Thus, the correct answer is (a).

S11. (b): Given,
$x = a\sin^3 t, y = b\cos^3 t$
Differentiating w.r.t t, we get
$\frac{dx}{dt} = 3a\sin^2 t\cos t$ and $\frac{dy}{dt} = -3b\cos^2 t\sin t$
$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3b\cos^2 t\sin t}{3a\sin^2 t\cos t} = \frac{-b}{a}\frac{\cos t}{\sin t}$
Therefore, slope of the tangent at $t = \frac{\pi}{2}$ is
$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{2}} = \frac{-b\cos\frac{\pi}{2}}{a\sin\frac{\pi}{2}} = 0$
Also, when $t = \frac{\pi}{2}, x = a$ and $y = 0$. Hence, the equation of tangent is
$y - 0 = 0(x-a), \Rightarrow y = 0$
Thus, the correct answer is (b).

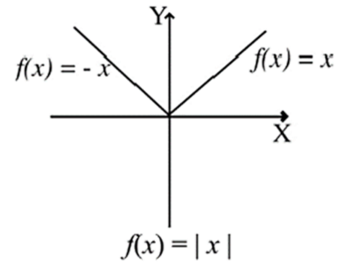
S12. (b): We know,
$V\& = x^3$
$dV\& = \left(\frac{dV}{dx}\right)\Delta x = (3x^2)\Delta x$
As 2% of x is $0.02x$
$= (3x^2)(0.02x) = 0.06x^3 \text{m}^3$
Thus, the correct answer is (b).

S13. (a): Given: Diameter of the balloon $= \frac{3}{2}(2x+1)$
$\Rightarrow$ Radius of the balloon $= \frac{3}{4}(2x+1)$
So, Volume of the balloon $= \frac{4}{3}\pi\left(\frac{3}{4}(2x+1)\right)^3 = = \frac{9\pi}{16}(2x+1)^3$
Now, Rate of change of volume w. r .t. $x = \frac{dV}{dx}$
$= \frac{9\pi}{16} \cdot 3(2x+1)^2 \cdot \frac{d}{dx}(2x+1)$
$= \frac{27\pi}{16}(2x+1)^2 \times 2$
$= \frac{27\pi}{8}(2x+1)^2$
Thus, the correct answer is (a).

S14. (c): Marginal cost is given by $= \frac{dC}{dx}$
$= \frac{d}{dx}(0.007x^3 - 0.003x^2 + 15x + 4000)$
$= 0.021x^2 - 0.006x + 15$

At $x = 17$
$= 0.021(17)^2 - 0.006 \times 17 + 15$
$= 6.069 - 0.102 + 15 = 20.967$
Thus, the correct answer is (c).

S15. (b):



From the graph of the given function, note that $f(x) \geq 0$, for all x belongs to **R** and f(X) = 0 if x = 0.
Therefore, the function f has a minimum value 0 and the point of minimum value of f is x = 0. Also, the graph clearly show that f has no maximum value in **R** and hence no point of maximum value in **R**

S16. (b): Given : $f(x) = 3x + 17$
Differentiate w.r.t x , we get
$f'(x) = 3(1) + 0 = 3 > 0$ that is, positive for all $x \in$ R
$\therefore, f(x)$ is strictly increasing on R.
Thus, the correct answer is (b).

S17. (b): Given : $f(x) = e^{2x}$
$f'(x) = e^{2x} \frac{d}{dx} 2x = e^{2x}(2) = 2e^{2x} > 0$ that is, positive for all x ∈ R
Therefore, $f(x)$ is strictly increasing on R.
Thus, the correct answer is (b).

S18. (c): By a monotonic function f in an interval I, we mean that f is either increasing in I or decreasing in I.
So, f(x) = sin x and f(x) = cos x are non-monotonic function but f(x) = e^x is strictly increasing function hence, f(X) = e^x is monotonic function.

S19. (d): Given $f(x) = 4x^2 + 5x + 2$
Differentiating w. r. t. x, we get $f'(x) = 8x + 5$
We know that $f(x + \delta x) = f(x) + f'(x)\delta x$
Let $x = 2$ and $x + \delta x = 2 \cdot 01$ so that $\delta x = 0 \cdot 01$
$f(2 \cdot 01) = f(2) + f'(2) \times 0 \cdot 01$
$= 4 \times 2^2 + 5 \times 2 + 2 + (8 \times 2 + 5) \times 0 \cdot 01$
$= 28 + 21 \times 0 \cdot 01 = 28 + 0 \cdot 21 = 28.21$
Thus, the correct answer is (d).

S20. (c): Given : $f(x) = (x + 1)^3(x - 3)^3$
$f'(x) = (x + 1)^3 \cdot 3(x - 3)^2 + (x - 3)^3 \cdot 3(x + 1)^2$
$f'(x) = 3(x + 1)^2(x - 3)^2(x + 1 + x - 3)$
$f'(x) = 3(x + 1)^2(x - 3)^2(2x - 2)$
$f'(x) = 6(x + 1)^2(x - 3)^2(x - 1)$
Here, factors $(x + 1)^2$ and $(x - 3)^2$ are non-negative for all x.
Therefore, f(x) is strictly increasing if $f'(x) > 0$
$x - 1 > 0$
$\Rightarrow x > 1$
So, f is strictly increasing in $(1, \infty)$.
Thus, the correct answer is (c).

S21. (a): Given, $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$
Differentiating y w.r.t θ
$\frac{dy}{d\theta} = \frac{(2+\cos\theta)\cdot 4\cos\theta - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$
$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$
$= \frac{dy}{d\theta} = \frac{8\cos\theta + 4(\cos^2\theta + \sin^2\theta) - (2+\cos\theta)^2}{(2+\cos\theta)^2}$
$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$
$\Rightarrow \frac{dy}{d\theta} = \frac{(8\cos\theta + 4) - (4 + 4\cos\theta + \cos^2\theta)}{(2+\cos\theta)^2}$

$= \frac{\cos\,\theta(4-\cos\,\theta)}{(2+\cos\,\theta)^2}$

Since $0 \leq \theta \leq \frac{\pi}{2}$ and we have $0 \leq \cos\,\theta \leq 1$, therefore $4 - \cos\,\theta > 0$.

$\frac{dy}{d\theta} \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$

So, y is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

S22. (d): Given function

$y = x^2 e^{-x}$

$\Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx} e^{-x} + e^{-x} \frac{d}{dx} x^2$

$= \&x^2 e^{-x}(-1) + e^{-x}(2x)$

$\Rightarrow \& \frac{dy}{dx} = -x^2 e^{-x} + 2xe^{-x}$

$= \&xe^{-x}(-x+2)$

$\frac{dy}{dx} = \frac{x(2-x)}{e^x}$

In $(0,2), \frac{dy}{dx} > 0\ for\ all\ x.$

S23. (c): Equation of the curve $y = x^3 - x + 1$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 1$

Slope of the tangent at point $x = 2$ to the curve

$= 3(2)^2 - 1 = 11$

S24. (d): Equation of the curve $y = 2x^2 + 3\sin\,x$

Slope of the tangent at point (x, y) is $\frac{dy}{dx} = 4x + 3\cos\,x$

Slope of the tangent at $x = 0{,}4(0) + 3\cos\,0 = 3 = m$

Slope of the normal $= \frac{-1}{m} = \frac{-1}{3}$

Thus, the correct answer is (d).

S25. (a): Volume of sphere $(\mathrm{V}) = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{d\mathrm{V}}{dr} = \frac{4}{3}\pi \cdot 3r^2$

Approximate error in calculating the volume $=$ Approximate value of $\Delta\mathrm{V}$

$d\mathrm{V} = \frac{d\mathrm{V}}{dr}(dr)$

$= \left(\frac{4}{3}\pi.3r^2\right)dr$

$= 4\pi(7)^2(0.02)$

$= 3.92 \times \frac{22}{7}$

$= 12.32\mathrm{m}^3$

Thus, the correct answer is (a).

S26. (d): Given $g(x) = -|x+1| + 3$

As $|x+1| \geq 0$ for all $x \in R$

$\Rightarrow -|x+1| + 3 \leq 3$

$\Rightarrow g(x) \leq 3$

Maximum value of $g(x)$ is 3 which is obtained when $x + 1 = 0$ or $x = -1$.

minimum value of $g(x) \rightarrow -\infty$, does not exist.

Thus, the correct answer is (d).

S27. (a): f is said to have a maximum value of f in I, if there exists a point c in I such that

$f(c) > f(x), for\ all\ x \in \mathrm{R}$

S28. (a): Note that the given function is not differentiable at x = 0.So, second derivative test fails .Let us try first derivative test. Note that 0 is a critical point of f. Now to the left of 0, f(X) = 3 -x and so f'(x) = -1$<$ 0. Also to the right of 0 , f(x) = 3 +x and so f'(x) =1$>$ 0.$Therefore$, by first derivative test , x = 0 is appoint of local minima of f and local minimum value of f is f(0)=3.

S29. **(b):** We have f(x) = $2x^3 - 15x^2 + 36x + 1$

$f'(x) = 6x^2 - 30x + 36 = 6(x-3)(x-2)$

Note that $f'(x) = 0$ gives x = 2 and x = 3

We shall now evaluate the value of f at these points and at the end points of the interval [1, 5] ,i.e., at x =1, x = 2, x = 3 and at x = 5. So, f(1) = 2(1) – 15 (1) +36 +1 = 24

f(2) = 2 (8) – 15(4) +36(2) +1 =29

f(3) = 2(27)-15(9) +36(3) +1 = 28

f(5) = 2(125) -15(25) +36(5)+1 = 56

Thus, we conclude that absolute maximum value of f on [1,5] is 56, occurring at x =5 .

S30. **(b):** Consider the two positive numbers are x and y.

$x + y = 16$

$\Rightarrow y = 16 - x$

Consider $z = x^3 + y^3$

$z = x^3 + (16-x)^3$

$z = x^3 + (16)^3 - x^3 - 48x(16-x)$

$= (16)^3 - 768x + 48x^2$

$\frac{dz}{dx} = -768 + 96x$ and $\frac{d^2z}{dx^2} = 96$

$If \frac{dz}{dx} = 0$

$-768 + 96x = 0 \Rightarrow x = 8$

$At x = 8 \quad \frac{d^2z}{dx^2} = 96 > 0$

$x = 8$ is a point of local minima and z is minimum when $x = 8$.

$\Rightarrow y = 16 - 8 = 8$

"the required numbers are " 8" and " 8"."

SUBJECTIVE QUESTIONS

S1. Given that

x = $\frac{2at^2}{1+t^2}$ y = $\frac{2at^3}{1+t^2}$

at t =$\frac{1}{2}$, x = $\frac{2a}{5}$, y =$\frac{a}{5}$

also $\frac{dx}{dt} = \frac{4at}{(1+t^2)^2}$ and $\frac{dy}{dt} = \frac{2at^2(3+t^2)}{(1+t^2)^2}$

∴ $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2}$ t (3 + t^2)

when t = $\frac{1}{2}$, $\frac{dy}{dx} = \frac{1}{2}\frac{1}{2}\left(3+\frac{1}{4}\right) = \frac{13}{16}$

∴ The equation of the tangent when t = $\frac{1}{2}$ is

$\Rightarrow$ y − $\frac{a}{5} = \left(\frac{13}{16}\right)\left(x - \frac{2a}{5}\right)$ $\Rightarrow$ 13x − 16y = 2a

and the equation of the normal is $\left(y - \frac{a}{5}\right)\left(\frac{13}{16}\right)$ + x − $\frac{2a}{5}$= 0

$\Rightarrow$ 16x + 13y = 9a

S2. Equation of line joining A & B is x + y = 4

Solving this line and curve we get

4 − x = $\frac{c}{x+1}$

$\Rightarrow$ x^2 − 3x + (c − 4) = 0 ......(i)

For tangency, roots of this equation must be equal.

Hence discriminant of quadratic equation = 0

$\Rightarrow$ 9 = 4(c − 4) $\Rightarrow$ $\frac{9}{4}$ = c − 4

$\Rightarrow$ c = $\frac{9}{4}$ + 4 $\Rightarrow$ c = $\frac{25}{4}$

$\Rightarrow$ x^2 − 3x + $\frac{9}{4}$ = 0 $\Rightarrow$ x^2 − 2. $\frac{3}{2}$x + $\frac{9}{4}$ −$\frac{9}{4}$ + $\frac{9}{4}$ = 0

$\Rightarrow \left(x - \frac{3}{2}\right)^2 = 0 \qquad \Rightarrow \qquad x = \frac{3}{2}$

Hence point of contact becomes $\left(\frac{3}{2}, \frac{5}{2}\right)$

S3. Here, $m = \left.\frac{dy}{dx}\right|_{x=0}$

$\frac{dy}{dx} = 3x^2 + 6x + 4$

$\Rightarrow m = 4$

and, $k = y(0)$

$\Rightarrow k = -1$

$\ell = |k|\sqrt{1 + \frac{1}{m^2}}$

$\Rightarrow \ell = |(-1)|\sqrt{1 + \frac{1}{16}} = \frac{\sqrt{17}}{4}$

S4. Let at some time t, the length of edge is x cm.

$v = x^3$

$\Rightarrow \frac{dv}{dt} = 3x^2 \frac{dx}{dt} \qquad \left(\text{but } \frac{dv}{dt} = 7\right)$

$\Rightarrow \frac{dx}{dt} = \frac{7}{3x^2}$ cm/sec.

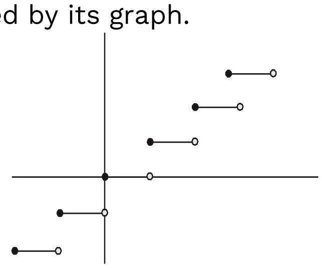
Now, $S = 6x^2$

$\frac{dS}{dt} = 12x \frac{dx}{dt}$

$\Rightarrow \frac{dS}{dt} = 12x . \frac{7}{3x^2} = \frac{28}{x}$

when x = 4 cm, $\frac{dS}{dt} = 7$ cm^2/sec.

S5. No, f(x) = [x] is increasing (monotonically increasing) (non-decreasing), but not strictly increasing function as illustrated by its graph.



NUMERICAL TYPE QUESTIONS

S1. (27) Extreme values basically mean maximum or minimum values, since f(x) is differentiable function so

$f'(-1) = 0 = f'(c)$ $\qquad$ $f'(x) = 3x^2 + 2ax + b$

$f'(c) = 27 + 6a + b = 0$ $\qquad$ $f'(-1) = 3 - 2a + b = 0$

$\Rightarrow a = -3, b = -9, c \in R$

$\Rightarrow$ Product of a and b = 27

S2. (11) The possible points of maxima/minima are critical points and the boundary points.

For $x \in [-1, 3]$ and $f(x) = x^3 - 12x$

x = 2 is the only critical point.

Examining the value of f(x) at points x = –1, 2, 3. We can find greatest and least values.

$\therefore$ Minimum f(x) = – 16 & Maximum f(x) = 11.

x	f(x)
–1	11
2	–16
3	–9

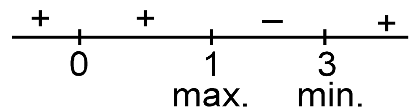
S3. (3) $f'(x) = 5x^2 (x - 1)(x - 3)$

$\because$ function increases from $-\infty$ to 1 and decreases from 1 to 3

hence, $x = 1$ will be point of local maximum

$\therefore$ function decreases from 1 to 3 and increases from 3 to ∞

hence, $x = 3$ will be point of local minima



S4. (60) $x + y = 60$

$\Rightarrow x = 60 - y$

$\Rightarrow xy^3 = (60 - y)y^3$

Let $f(y) = (60 - y)\, y^3$; $y \in (0, 60)$

for maximizing f(y) let us find critical points

$f'(y) = 3y^2 (60 - y) - y^3 = 0$

$f'(y) = y^2 (180 - 4y) = 0$

$\Rightarrow y = 45$

$f'(45^+) < 0$ and $f'(45^-) > 0$. Hence local maxima at $y = 45$.

So $x = 15$ and $y = 45$.

$\therefore x + y = 15 + 45 = 60$

S5. (0) For tangent , we have to find $\frac{dy}{dx}$.

So , $xy = 4$

We have to differentiate above equation , we get

$\Rightarrow x\frac{dy}{dx} + y = 0$

So , $\frac{dy}{dx} = \frac{-y}{x}$

Here , x can not be zero .So, for point (0 , 1) tangent can't be drawn.

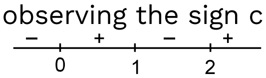
So, the number of tangents is zero .

TRUE AND FALSE

S1. **(False)** $f(x) = x^2 (x - 2)^2$

$\Rightarrow f'(x) = 4x (x - 1)(x - 2)$

observing the sign change of $f'(x)$



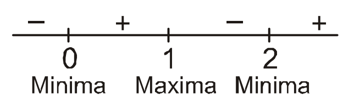
Hence increasing in [0, 1] and in $[2, \infty)$ and decreasing for $x \in (-\infty, 0]$ and [1, 2]

S2. **(True)** $f(x) = x^2 (x - 2)^2$

$f'(x) = 4x (x - 1)(x - 2)$

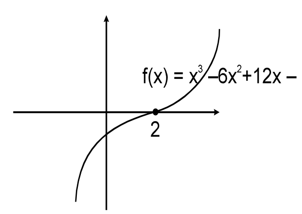
$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 2$

examining the sign change of $f'(x)$



Hence $x = 1$ is point of maxima, $x = 0, 2$ are points of minima.

S3. **(True)** $f(x) = x^3 - 6x^2 + 12x - 8$



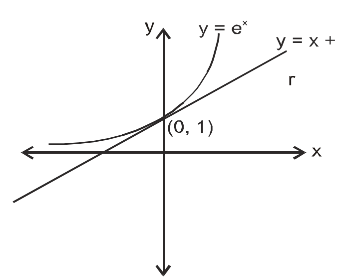
$f'(x) = 3(x^2 - 4x + 4) \qquad f'(x) = 3(x - 2)^2$

$f'(x) = 0 \qquad \Rightarrow x = 2$

But clearly $f'(x)$ does not change sign about $x = 2$. $f'(2^+) > 0$ and $f'(2^-) > 0$.

So f(x) has no point of maxima or minima. In fact f(x) is a monotonically increasing function for $x \in R$.

S4. **(False)** At $x = 0$



$\Rightarrow y = e^0 = 1$

$\Rightarrow \frac{dy}{dx} = e^x$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 1$

Hence equation of tangent is $1(x - 0) = (y - 1)$

$\Rightarrow$ y = x + 1

S5. (True) Let $y = x^{1/3}$

Let x = 27 and $\Delta x = -2$

Now $\Delta y = (x + \Delta x)^{1/3} - x^{1/3} = (25)^{1/3} - 3$

$\frac{dy}{dx} \Delta x = 25^{1/3} - 3$

At x = 27, $25^{1/3} = 3 - 0.074 = 2.926$

ASSERTION AND REASONING

S1. (c): Assertion : $y = xe^x$

$\therefore \frac{dy}{dx} = xe^x + e^x$

Put $\frac{dy}{dx} = 0$, for maxima or minima

$\Rightarrow xe^x + e^x = 0$

$\Rightarrow x = -1$

Now , $\frac{d^2y}{dx^2} = 2e^x + xe^x$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=-1} = e^{-1}(2-1) > 0$

Hence , at x = -1 , y is minimum.

Thus , A is true but R is false.

S2. (a): Since $y = x^2e^{-x}$

$\Rightarrow \frac{dy}{dx} = 2xe^{-x} - x^2e^{-x}$

Function will be increasing , if $\frac{dy}{dx} > 0$

$\therefore 2xe^{-x} - x^2e^{-x} > 0$

$\Rightarrow e^{-x}(-x^2 + 2x) > 0$

$\Rightarrow -x^2 + 2x > 0$

$\Rightarrow x \in (0,2)$

Both A and R are individually true and R is the correct explanation of A.

S3. (a): Given that , the tangent to the curve , curve $y = x^3 - x^2 - x + 2$ at (1 ,1) is parallel to the x – axis .

$\therefore \frac{dy}{dx} = 3x^2 - 2x - 1$

$\left(\frac{dy}{dx}\right)_{(1,1)} = 3 - 2 - 1 = 0$

$\therefore$ The equation of tangent at (1,1) is given by $(y-1) = \left(\frac{dy}{dx}\right)_{(1,1)}(x-1)$

$\Rightarrow y - 1 = 0$

$\Rightarrow y = 1$

Which is parallel to x- axis and $\left(\frac{dy}{dx}\right)_{(1,1)} = 0$

$\therefore$ Both A and R are true and R is the correct explanation of A.

S4. (a): Given that , f(x) = $x^{\frac{1}{x}}$

$\Rightarrow f'(x) = \frac{1}{x^2}(1 - \log x).x^{\frac{1}{x}}$

$\Rightarrow f'(x) > 0$, if 1 – log x>0

$\Rightarrow \log x < 1$

$\Rightarrow x < e$

$\therefore$ f(x) is increasing in the interval (-∞ , e).

$\therefore$ Both A and R are true and R is the correct explanation of A.

S5. (a): We have , $v^2 = 2 - 3x$

On differentiating w. r. t. t . we get

$\Rightarrow 2v\frac{dv}{dt} = 0 - 3\frac{dx}{dt}$

$\Rightarrow 2v\frac{dv}{dt} = -3v$

$\Rightarrow \frac{dv}{dt} = \frac{-3}{2}$

$\therefore$ Required acceleration = $\frac{-3}{2}$

$\therefore$ Both A and R are true and R is the correct explanation of A.

HOMEWORK

MCQ

S1. **(a):** The point of intersection is $(\pi / 4, 1 / \sqrt{2})$
$y = \sin x$
$dy / dx = \cos x$
$m_1 = (dy / dx)_{(\pi / 4, 1 / \sqrt{2})} = 1 / \sqrt{2}$
$y = \cos x$
$dy / dx = -\sin x$
$m_2 = (dy / dx)_{(\pi / 4, 1 / \sqrt{2})} = -1 / \sqrt{2}$
$\tan \theta = |m_2 - m_1| / [1 + m_1 m_2]$
$= 2\sqrt{2}$
$\theta = \tan^{-1} (2\sqrt{2})$

S2. **(b):** $f(x) = x + \frac{1}{x}$
$\Rightarrow f'(x) = 1 - \frac{1}{x^2}$
Now , $f'(x) > 0$
Gives $1 > \frac{1}{x^2}$ $or\ x^2 > 1$
Or $|x| > 1$ for f(x) to be an increasing function ,
Or $x \in (-\infty, -1) \cup (1, \infty)$

S3. **(d):** f(x) = $2x^3 - 9x^2 + 12x - 6$
$f'(x) = 6x^2 - 18x + 12$
$f'(x) = 6(x^2 - 3x + 2)$
$f'(x) = 6(x-2)(x-1)$
For critical point
$f'(x) = 0$
$\Rightarrow 6\,(x-1)(x-2) = 0$
$\Rightarrow x = 1, 2$
$\Rightarrow f'(x) > 0$ for all $x \in (-\infty, 1) \cup (2, \infty)$
So, f(x) monotonic increasing in $(-\infty, 1) \cup (2, \infty)$

S4. **(b):** Given function f(x) = a sin x + $\frac{1}{3}\sin 3x$
Then , $f'(x) = a\cos x + 3 \times \frac{1}{3}(\cos 3x)$
$f'(x) = a\cos x + \cos 3x$
We know that the maximum value or global maxima of any function occurs at the critical point which is given as $f'(x) = 0$.Also , it is given that the maximum value of f(x) = a sin x + $\frac{1}{3}\sin 3x$ occur at x = $\frac{\pi}{3}$. So , $f'\left(\frac{\pi}{3}\right) = 0$
$\Rightarrow a\cos\left(\frac{\pi}{3}\right) + \cos\left(3 . \frac{\pi}{3}\right) = 0$
$\Rightarrow \frac{a}{2} - 1 = 0$
$\Rightarrow a = 2$

S5. **(b):** If $f''(c) < 0$ then x = c is a point of maxima

S6. **(a):** In triangle AOB , $AO^2 + AB^2 = OB^2$
$\Rightarrow\ (h-k)^2 + r^2 = R^2$
$\Rightarrow\ r^2 = R^2 - (h-R)^2$.......(1)
Volume of cone = $\frac{1}{3}\pi r^2 h$
$\Rightarrow V = \frac{1}{3}\pi[R^2 - (h-R)^2]h$ ....(2)
For maximum value of V
$\frac{dV}{dh} = 0$

$\Rightarrow \frac{dV}{dh} = \frac{1}{3}\pi[-2(h-R)h + R^2 - (h-R)^2] = 0$
$\Rightarrow -3h + 4R = 0$
$\Rightarrow \frac{h}{R} = \frac{4}{3}$
$\therefore \frac{h}{2R} = \frac{2}{3}$

S7. (d): Given , f(x) = $\frac{x}{2} + \frac{2}{x}$
For extremum $f'(x) = 0$
$f'(x) = \frac{1}{2} - \frac{2}{x^2}$
$\Rightarrow \frac{1}{2} - \frac{2}{x^2} = 0$
$\Rightarrow x = \pm 2$
For local minima $f''(x) > 0$
$f''(x) = \frac{4}{x^3}$
$\Rightarrow f''(2) > 0$, at x = 2

S8. (c): f(x) = $x^3 - 3x$
$f'(x) = 3\,(x^2 - 1)$
For maxima or minima , $f'(x) = 0$
$\Rightarrow x = -1, 1$
$\Rightarrow x = 1$ (since -1 does not belongs in the given interval)
$\Rightarrow f''(x) = 6x$
$\Rightarrow f''(1) = 6 > 0$, so f(x) has a minimum at x = 1.
Now we will find the value of function at end points
f(0) =0 , f(2) = 8-6 =2
So, maximum value is 2.

S9. (b): f(x) = k$x^3 - 9x^2 + 9x + 3$ is monotonically increasing for all $x \in R$
$\Rightarrow f'(x) = 3kx^2 - 18k + 9$
$\Rightarrow f'(x) = 3(kx^2 - 6x + 3) > 0$
For the equation
$ax^2 + bx + c$ to be +ve $\Rightarrow$ a > 0 and D <0
$\therefore$ k >0 and $b^2 - 4ac < 0$
12k > 36
$\therefore$ k > 3 for f(x) to be monotonically increasing in R.

S10. (a): The function $f(x) = x^2e^{-x}$ increases in the interval
$f'(x) = 0$
$\Rightarrow 2xe^{-x} - x^2e^{-x} = 0$
$\Rightarrow e^{-x}(2x - x^2) = 0$
$\Rightarrow x = 0, x = 2$
$\therefore$ f is increasing in the interval (0 , 2)

SUBJECTIVE QUESTIONS

S1. Let x be the radius of cylinder and y be its height v = $\pi x^2 y$
x, y can be related by using similar triangles
(as shown in figure).
$\frac{y}{r - x} = \frac{h}{r}$
$\Rightarrow y = \frac{h}{r}(r - x)$
$\Rightarrow v(x) = \pi x^2 \frac{h}{r}(r - x)$
$x \in (0, r)$

$\Rightarrow v(x) = \frac{\pi h}{r}(rx^2 - x^3)$

$v'(x) = \frac{\pi h}{r} x\,(2r - 3x)$

$v'\left(\frac{2r}{3}\right) = 0 \qquad \text{and } v'\left(\frac{2r}{3}\right) < 0$

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

S2. $f(x) = \sin 2x - x$

$f'(x) = 2\cos 2x - 1$

$f'(x) = 0$

$\Rightarrow \cos 2x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$f''(x) = -4 \sin 2x$

$f''\left(\frac{\pi}{6}\right) < 0$

$\Rightarrow$ Maxima at $x = \frac{\pi}{6}$

$f''\left(\frac{5\pi}{6}\right) > 0$

$\Rightarrow$ Minima at $x = \frac{5\pi}{6}3.$

S3. $f'(x) = 12(x^2 - x - 2) = 12(x - 2)(x + 1)$

$f'(x) = 0$

$\Rightarrow x = -1$ or 2

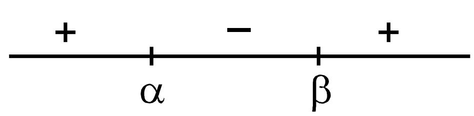
(i) if $x \in [0, 3]$, $x = 2$ is critical point.

(ii) if $x \in [-3, 3]$, then we have two critical points $x = -1, 2$.

(iii)If $x \in [-1, 2]$, then no critical point as both $x = 1$ and $x = 2$ become boundary points.

S4. $f'(x) = 3\,[x^2 + 2(a - 7)x + (a^2 - 9)]$

Let α, β be roots of $f'(x) = 0$ and let α be the smaller root. Examining sign change of $f'(x)$.



Maxima occurs at smaller root α which has to be positive. This basically implies that both roots of $f'(x) = 0$ must be positive and distinct.

(i) $D > 0 \qquad \Rightarrow a < \frac{29}{7}$

(ii) $-\frac{b}{2a} > 0 \qquad \Rightarrow a < 7$

(iii) $f'(0) > 0 \qquad \Rightarrow a \in (-\infty, -3) \cup (3, \infty)$

from (i), (ii) and (iii)

$\Rightarrow a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

S5. $f(x) = e^{2x} - (a + 1)e^x + 2x$

$f'(x) = 2e^{2x} - (a + 1)e^x + 2$

Now, $2e^{2x} - (a + 1)e^x + 2 \geq 0 \qquad$ for all $x \in R$

$\Rightarrow 2\left(e^x + \frac{1}{e^x}\right) - (a+1) \geq 0$ for all $x \in R$

$\Rightarrow (a+1) \leq 2\left(e^x + \frac{1}{e^x}\right)$ for all $x \in R$

$\Rightarrow a + 1 \leq 4 \left(\because e^x + \frac{1}{e^x} \text{ has minimum value } 2\right)$

$\Rightarrow a \leq 3$

NUMERICAL TYPE QUESTIONS

S1. **($\pm\frac{1}{2}$)** $\frac{dy}{dx} = pe^{px} + p$ at point (0, 1) = 2p

subnormal = $\left|y\frac{dy}{dx}\right|$

subtangent = $\left|y\frac{dx}{dy}\right|$

$\frac{dy}{dx} = \pm 1$

$\Rightarrow 2p = \pm 1$

$\Rightarrow p = \pm \frac{1}{2}$

S2. **(-2)** $f(x) = x^2 + kx + 1$

for f(x) to be increasing, $f'(x) > 0$

$\Rightarrow \frac{d}{dx}(x^2 + kx + 1) > 0$

$\Rightarrow 2x + k > 0$

$\Rightarrow k > -2x$

for $x \in (1, 2)$ the least value of k is -2

S3. **(9)** We have , $s = 15t - 2t^2$

On differentiating w.r.t t , we get

Velocity = $\frac{ds}{dt} = 15 - 4t$

$\Rightarrow \left(\frac{ds}{dt}\right)_{(t=0)} = 15$

And $\left(\frac{ds}{dt}\right)_{(t=3)} = 15 - 12 = 3$

$\therefore$ Average velocity = $\frac{15+3}{2} = 9$

S4. **(89)** Given , $f(x) = 2x^3 - 24x + 107$

$f'(x) = 6x^2 - 24$

For maximum or minimum , put $f'(x) = 0$

$\Rightarrow 6x^2 - 24 = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

But x = -2 $\notin [1,3]$

$\therefore$ x = 2 is the stationary point.

Now , f(1) = 2 – 24 +107 = 85

$f(2) = 2(2)^3 - 24(2) + 107 = 75$

$f(3) = 2(3)^3 - 24(3) + 107 = 89$

Hence , the maximum value of f(x) is 89 at x = 3.

S5. **(4)** Let the first number be 3 -x , then the second number will be x .

According to given condition,

Let f(x) = (3-x)x^2
$\Rightarrow f'(x) = 6x - 3x^2$
For maximum or minimum , put f'(x) = 0
$\Rightarrow x = 0 , 2$
Also , $f''(x) = 6 - 6x$
At x = 2,
$f''(x) = -6 < 0$
∴ f(2) = (3-2) $(2)^2 = 4$

TRUE AND FALSE

S1. **(True)** Let f be a differentiable a\real function defined on an open interval (a , b) . If f'(x) > 0 for all $x \in (a , b)$,then f(x) is increasing on (a , b).

S2. **(False)** Given , $y = \tan^{-1} x - x$
On differentiating w.r.t x , we get
$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2} - 1 = \frac{-x^2}{1+x^2}$
$\Rightarrow \frac{dy}{dx} < 0 , for\ all\ x \in R$
Hence , function is always decreasing.

S3. **(True)** Since , f(x) = $2x^3 - 9x^2 + 12x + 4$
$\Rightarrow f'(x) = 6x^2 - 18x + 12$
For function to be increasing , f'(x) <0
$\Rightarrow 6(x^2 - 3x + 2) < 0$
$\Rightarrow (x-2)(x-1) < 0$
$\Rightarrow 1 < x < 2$

S4. **(True)** Let $y = x(1-x)^2$, then for maximum / minimum
$\frac{dy}{dx} = 1.(1-x)^2 - 2x(1-x) = 0$
$\Rightarrow (1-x)(1-x-2x) = 0$
$\Rightarrow (1-x)(1-3x) = 0$
$\Rightarrow x = 1 , x = \frac{1}{3}$
Now $\frac{d^2y}{dx^2} = (-1)(1-3x) + (1-x)(-3) = (-1)(1-3) + (1-1)(-3) = 2$ at x = 1
$\frac{d^2y}{dx^2} = 2 > 0$
∴ At x = 1 , y is minimum
$\frac{d^2y}{dx^2} = (-1)\left[1 - 3 \times \frac{1}{3}\right] + \left(1 - \frac{1}{3}\right)(-3) = -2$
Maximum at x = $\frac{1}{3}$, $\frac{d^2y}{dx^2} < 0$
∴ y is maximum
∴ y = $\frac{1}{3}\left(1 - \frac{1}{3}\right)^2 = \frac{4}{27}$

S5. **(True)** Given equation $y = x^2 - 5x + 6$
Differentiating the equation w.r.t x
$\frac{dy}{dx} = 2x - 5$
Finding slope from given point.
Slope of tangent to the curve at (2 , 0) is
$\left(\frac{dy}{dx}\right)_{(2,0)} = 2(2) - 5 = -1$
$m_1 = -1$
Slope of tangent to the curve at (3 , 0) is
$\left(\frac{dy}{dx}\right)_{(3,0)} = 6 - 5 = 1$
$m_2 = 1$

Therefore angle between the tangents to the curve at (2, 0) and (3 , 0) is $\frac{\pi}{2}$

ASSERTION AND REASONING

S1. (d): Assertion (A)

$f(x) = \ln x$ $\quad\therefore f'(x) = \frac{1}{x}$

Here , $f'(x) > 0$ in the interval $(0 , \infty)$.

$\therefore$ f(x) is increasing function in $(0 , \infty)$

A is not correct

Reason : $f(x) = e^x - x\ln x$

$f'(x) = e^x - \ln x - x.\frac{1}{x} = e^x - \ln x - 1$

Here , $f'(x) > 0$ in the interval $(1, \infty)$.

$\therefore$ f(x) is decreasing function.

R is correct.

S2. (c): Assertion(A)

We have , f(x) = $\frac{x^2-1}{x^2+1}$

$\Rightarrow f'(x) = \frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$

$\Rightarrow \quad f'(x) = \frac{4x}{(x^2+1)^2}$

For critical points , put $f'(x) = 0$

$\Rightarrow \frac{4x}{(x^2+1)^2} = 0$

$\Rightarrow 4x = 0$

$\Rightarrow x = 0$

A is true

Reason (R)

f(0)= $\frac{0-1}{0+1} = -1$

R is false.

S3. (a): Assertion(A)

We have , f(x) = $x^2 - 5x + 6$

$f'(x) = 2x - 5$

For decreasing $f'(x) < 0$

2x -5 < 0

$\Rightarrow x < \frac{5}{2}$

$\Rightarrow x \in (-\infty , 2.5)\ or\ x \in (-\infty , 2)$

A is true ,R is true and correct explanation of A.

S4. (b): Assertion (A)

$y - x = 1$

$y^2 = x$

Differentiating w. r. t x

$2y\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{2y} = 1$

$y = \frac{1}{2} , x = \frac{1}{4}$

Tangent at $\left(\frac{1}{4} , \frac{1}{2}\right)$

$\frac{1}{2}y = \frac{1}{2}\left(x + \frac{1}{4}\right)$

$y = x + \frac{1}{4}$

Distance = $\left|\frac{1-\frac{1}{4}}{\sqrt{2}}\right| = \frac{3\sqrt{2}}{8}$

A is true.

Reason (R)
$y = x + \frac{4}{x^2}$
$\Rightarrow \frac{dy}{dx} = 1 - \frac{8}{x^3}$
Since , tangent is parallel to x- axis , i.e., $\frac{dy}{dx} = 0$
0 = 1 - $\frac{8}{x^3}$
$x^3 = 8$
$\Rightarrow x = 2$
Substitute in the equation of the curve
$y = 2 + \frac{4}{2^2} = 2 + 1 = 3$
Equation of tangent is given as $y - 3 = 0\ (\ x - 2)$
$\Rightarrow y - 3 = 0$
R is true ,but R is not correct explanation of A.

S5. (a): Assertion (A):
Critical point of f(x) = $x^3 - 3x^2 + 3x - 100$
$\Rightarrow f'(x) = \ 3x^2 - 6x + 3 = 3(x-1)^2$
$\Rightarrow f'(x) = 0$
$\Rightarrow 3(x-1)^2 = 0$
$\Rightarrow x = 1$
Reason (R): Definition of critical points.
Thus, both A and R are true and R is correct explanation of A.