18 Chapter

APPLICATION OF DERIVATIVES-1

(TANGENTS & NORMALS, MEAN VALUE THEOREM & RATE MEASUREMENTS)

A SINGLE CORRECT CHOICE TYPE

Each of these questions has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

1. If the tangent at the point (a, b) to the curve $x^3 + y^3 = c^3$ meets the curve again at the point (p, q), then

(a) $ap + bq + pq = 0$ (b) $aq + bp + ab = 0$

(c) $a^2 + b^2 = p^2 + q^2$ (d) None of these

2. The equation of the normal to the curve $x + y = x^y$, where it cuts x-axis is

(a) $y = x$ (b) $y = x + 1$

(c) $y = x - 1$ (d) $x + y = 1$

3. If the parabola $y = f(x)$, having axis parallel to the y-axis, touches the line $y = x$ at (1, 1), then

(a) $2f'(0) + f(0) = 1$ (b) $2f(0) + f'(0) = 1$

(c) $2f(0) - f'(0) = 1$ (d) $2f'(0) - f(0) = 1$

4. The sum of intercepts on the axes of coordinates by any tangent to the curve $\sqrt{x} + \sqrt{y} = 2$ is

(a) 2 (b) 4

(c) 8 (d) $2\sqrt{2}$

5. The angle between the tangents at any point P and the line joining P to the origin O, where P is a point on the curve $\ln(x^2 + y^2) = c\tan^{-1}\frac{y}{x}$, c is a constant, is

(a) Constant (b) varies as $\tan^{-1}(x)$

(c) varies as $\tan^{-1}(y)$ (d) equal to $\frac{\pi}{2}$

6. The general value of α such that the line $x\cos\alpha + y\sin\alpha = p$ is a normal to the curve $(x + a)y = c^2$ is

(a) $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right) \cup \left(2n\pi + \frac{3\pi}{2}, (2n+2)\pi\right)$

(b) $\left(2n\pi + \pi,\ 2n\pi + \frac{3\pi}{2}\right)$

(c) $\left(2n\pi + \frac{3\pi}{2}, (2n+2)\pi\right)$

(d) $\left(2n\pi,\ 2n\pi + \frac{\pi}{2},\right) \cup \left((2n+1)\pi,\ 2n\pi + \frac{3\pi}{2},\right)$

$(n \in I)$

7. The tangent at any point on the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ meets the axes in P and Q. The locus of the mid point of PQ is

(a) $x^{3/2} + y^{3/2} = a^{3/2}$ (b) $x^{2/3} + y^{2/3} = a^{2/3}$

(c) $4(x + y) = a$ (d) $4(x^2 + y^2) = a^2$

8. If the relation between subnormal SN and subtangent ST at any point S on the curve $by^2 = (x + a)^3$ is $p(SN) = q(ST)^2$, then $\frac{p}{q}$ is equal to

(a) $\frac{a}{27b}$ (b) $\frac{8a}{27b}$

(c) $\frac{8b}{27b}$ (d) $\frac{8b}{27}$

9. The value of n in the equation of curve $y = a^{1-n}x^n$, so that the sub-normal may be of constant length is

(a) 2 (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) 1

MARK YOUR RESPONSE					
	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ
	6. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ	

10. The equation of the tangent to the curve $y=\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^2}}$ at $x = 1$ is

(a) $\sqrt{2}y+1=x$ (b) $\sqrt{3}x+1=y$
(c) $\sqrt{3}x+1+\sqrt{3}=y$ (d) None of these

11. Given a function $f:[0, 4]\longrightarrow R$ is differentiable, then for some $a,b\in(0, 4)$ $[f(4)]^2-[f(0)]^2=$

(a) $8f'(b)f(a)$ (b) $4f'(b)f(a)$
(c) $2f'(b)f(a)$ (d) $f'(b)f(a)$

12. A function f is differentiable in the interval $0\le x\le 5$ such that $f(0) = 4$ and $f(5) = -1$. If $g(x)=\frac{f(x)}{x+1}$, then there exists some 'c' in $0 < c < 5$ such that $f'(c)$ equals

(a) $\frac{1}{6}$ (b) $-\frac{5}{6}$
(c) $-\frac{1}{6}$ (d) none of these

13. The distance covered by a particle moving in a straight line from a fixed point on the line is s. Where $s^2=at^2+2bt+c$, then acceleration is proportional to

(a) s^{-2} (b) s^{-3}
(c) $s^{-1/2}$ (d) None of these

14. A man is moving away from a tower 41.6m high at a rate of 2m/s. If the eye level of the man is 1.6m above the ground, then the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30 m from the foot of the tower is

(a) $-\frac{4}{125}\text{rad}-s^{-1}$ (b) $-\frac{2}{125}\text{rad}-s^{-1}$
(c) $-\frac{\pi}{300}\text{rad}-s^{-1}$ (d) None of these

15. At any point of the curve $2x^2y^2-x^4=c$, the mean proportional between the abscissa and the difference between the abscissa and the subnormal drawn to the curve at the same point is equal to

(a) ordinate (b) radius vector
(c) x-intercept of tangent (d) subtangent

16. Two men P and Q start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, the rate at which they are being separated is

(a) $v\sqrt{2}$ (b) $v\sqrt{2+\sqrt{2}}$
(c) $v\sqrt{2-\sqrt{2}}$ (d) $v/\sqrt{2}$

17. x and y are the sides of two squares such that $y = x - x^2$. The rate of change of the area of the second square with respect to the area of the first square is

(a) x^2-x+1 (b) $2x^2+2x-1$
(c) $2x^2-3x+1$ (d) x^2+x-1

18. With the usual meaning for a, b, c and s if Δ be the area of a triangle, then the error in Δ, $(\delta\Delta)$ resulting from a small error in the measurment of c (δc) is given by

(a) $\delta\Delta=\frac{\delta c}{4s}$ (b) $\delta\Delta=\frac{\Delta.\delta c}{4s}$
(c) $\Delta=\frac{(abc)sc}{4\,s^2}$ (d) None of these

19. If the sides and angles of a plane triangle vary in such a way that its circumradius remains constant, then the value of $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}$ is (where da, db, dc, are small increments in the sides a, b, c respectively)

(a) 0
(b) $a + b + c$
(c) $a \sin A + b \sin B + c \sin C$
(d) None of these

20. If $f(x)$ and $g(x)$ are differentiable functions for $0\le x\le 1$ such that $f(0) = 2, g(0) = 0, f(1) = 6, g(1) = 2$, then there exists $c\in(0,1)$, for which $f'(c) =$

(a) $g'(c)$ (b) $2g'(c)$
(c) $g'(c)-g'(0)$ (d) None of these

21. The equation $3x^2+4ax+b=0$ has at least one root in $(0, 1)$ if

(a) $4a+b+3=0$ (b) $2a+b+1=0$
(c) $b=0.\ a=-\frac{4}{3}$ (d) None of these

22. The real value of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots in (0, 1) is

(a) $k\in R$ (b) $0<k<1$
(c) $-1<k<\infty$ (d) $k\in\phi$

MARK YOUR RESPONSE	10. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ	12. ⓐⓑⓒⓓ	13. ⓐⓑⓒⓓ	14. ⓐⓑⓒⓓ
	15. ⓐⓑⓒⓓ	16. ⓐⓑⓒⓓ	17. ⓐⓑⓒⓓ	18. ⓐⓑⓒⓓ	19. ⓐⓑⓒⓓ
	20. ⓐⓑⓒⓓ	21. ⓐⓑⓒⓓ	22. ⓐⓑⓒⓓ		

23. If α and β are any two roots of equation $e^x \cos x = 1$, then the equation $e^x \sin x - 1 = 0$ has

(a) exactly one roots in (α, β)

(b) exactly two roots in (α, β)

(c) at least one root in (α, β)

(d) no root in (α, β)

24. If the normal to the curve $y = f(x)$ at $x = 0$ be given by the equation $3x - y + 3 = 0$, then the value of

$\lim\limits_{x\to 0} x^2\{f(x^2) - 5f(4x^2) + 4f(7x^2)\}^{-1} =$

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$

(c) $-\frac{1}{5}$ (d) $\frac{1}{4}$

25. Let P be a point on the hyperbola $x^2 - y^2 = a^2$, where a is a parameter, such that P is nearest to the line $y = 2x$. Then the locus of P is

(a) $y = 2x$ (b) $y = x$

(c) $2y = x$ (d) $x + y = 0$

26. For the curve $ax^2 + by^2 = c$, $y^3\left(\frac{d^2y}{dx^2}\right) + 4 = 0$, then the equation $ax^2 + bx + c = 0$ will have

(a) imaginary roots (b) real and unequal roots

(c) real and equal roots (d) both roots infinite

27. Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 1$, such that $f(0) = 2, g(0) = 0, f(1) = 6$. Let there exists a real number c in $[0, 1]$ such that $f'(c) = 2g'(c)$, then the value of $g(1)$ must be

(a) 1 (b) 2

(c) -2 (d) -1

28. If f' and g' exist for all $x \in [a,b]$ and if $g'(x) \ne 0\ \forall\, x \in (a,b)$ then for some $c \in (a,b)$,

$\frac{f(c)-f(a)}{g(b)-g(c)} =$

(a) $f'(c)g'(c)$ (b) $\frac{f'(c)}{g'(c)}$

(c) $f'(c) - \frac{1}{g'(c)}$ (d) $\frac{f(a)f'(c)}{g(b)g'(c)}$

29. If the equation $ax^2 + bx + c = 0$ has two distinct positive roots, then the equation $ax^2 + (b + 6a)x + (c + 3b) = 0$ has

(a) two positive roots

(b) Exactly one positive root

(c) at least one positive root

(d) no positive root

30. Let $f(x) = \ln x$ and $g(x) = x^2$. If $c \in (4, 5)$ then $c \ln\left(\frac{4^{25}}{5^{16}}\right)$ equals to

(a) $c \ln 5 - 8$ (b) $2(c^2 \ln 4 - 8)$

(c) $2(c^2 \ln 5 - 8)$ (d) $c \ln 4 - 8$

31. A rocket of length h is fired vertically upwards with velocity $v(t) = (2t + 3)\ ms^{-1}$. If the angle of elevation of the top of the rocket from a point on the ground at time $t = 1$ sec is 30° and at $t = 3$ sec is 60°, then the value of h is

(a) $\frac{1}{2}$ (b) $\frac{5}{2}$ (c) 3 (d) 1

32. The slope of the tangent to the curve $y = \int_x^{x^2} \cos^{-1} t^2\, dt$ at $x = \frac{1}{\sqrt[4]{2}}$ is

(a) 0 (b) $\frac{\pi}{2}$

(c) $\left(\frac{\sqrt[4]{8}}{3} - \frac{1}{4}\right)\pi$ (d) $\left(\frac{\sqrt[6]{8}}{6} - \frac{1}{4}\right)$

33. Equation of the tangent to the curve $y = e^{-|x|}$ at the point where it cuts the line $x = 1$

(a) is $ey + x = 2$ (b) is $x + y = e$

(c) is $ex + y = 1$ (d) does not axist

34. A point is in a motion on the curve $12y = x^3$, the ordinate is changing at a slower rate than the abscissa in the inerval

(a) $(-2, 2)$ (b) $(-\infty, -2) \cup (2, \infty)$

(c) $(2, \infty)$ (d) $(0, 2)$

35. If m is slope of common tangents $y = x^2 - x + 1$, $y = x^2 - 3x + 1$ then m is

(a) 2 (b) -1 (c) $\frac{1}{2}$ (d) -2

MARK YOUR RESPONSE					
	23. ⓐⓑⓒⓓ	24. ⓐⓑⓒⓓ	25. ⓐⓑⓒⓓ	26. ⓐⓑⓒⓓ	27. ⓐⓑⓒⓓ
	28. ⓐⓑⓒⓓ	29. ⓐⓑⓒⓓ	30. ⓐⓑⓒⓓ	31. ⓐⓑⓒⓓ	32. ⓐⓑⓒⓓ
	33. ⓐⓑⓒⓓ	34. ⓐⓑⓒⓓ	35. ⓐⓑⓒⓓ		

36. Slope of the normal to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$, at the point P (2, –1) is given by

(a) $\frac{7}{6}$ (b) $-\frac{7}{6}$ (c) $\frac{6}{7}$ (d) $-\frac{6}{7}$

37. If m be slope of a tangent to the curve $e^y = 1 + x^2$ then

(a) $|m| > 1$ (b) $m \le 1$

(c) $|m| < 1$ (d) $|m| \le 1$

38. Angle formed by the positive Y-axis and the tangent to $y = x^2 + 4x - 17$ at $\left(\frac{5}{2}, \frac{-3}{4}\right)$ is

(a) $\tan^{-1} 9$ (b) $\frac{\pi}{2} - \tan^{-1} 9$

(c) $\frac{\pi}{2} + \tan^{-1} 9$ (d) $\frac{\pi}{2}$

39. If the line joining the points (0, 3) and (5 ,–2) is a tangent to the curve $y = \frac{c}{x+1}$, then value of c is

(a) 1 (b) -2 (c) 4 (d) -4

40. Coordinates of the point on $y^2 = 8x$ which is nearest to the circle $x^2 + (y+6)^2 = 1$ is;

(a) $(2, 4)$ (b) $(18, -12)$

(c) $(2, -4)$ (d) $(18, 12)$

41. The curve $x + y - \ell n\,(x + y) = 2x + 5$ has a vertical tangent at the point (α, β). Then $\alpha + \beta$ is equal to

(a) -1 (b) 1 (c) 2 (d) -2

42. If the fuction f: [0, 8] $\to R$ is differentiable then for $0 < \alpha, \beta < 2, \int_0^8 f(t)\; dt$ is equal to

(a) $3[\alpha^3 f(\alpha^2) + \beta^2 f(\beta^2)]$

(b) $3[\alpha^3 f(\alpha^2) + \beta^3 f(\beta)]$

(c) $3[\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3)]$

(d) $3[\alpha^2 f(\alpha^2) + \beta^2 f(\beta^2)]$

43. If $f(x) = 0$ is a cubic equation with positive and distinct roots α, β, γ such that β is the $H.M$ of the roots of $f'(x) = 0$. Then α, β, γ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

44. Let the equation of a curve be $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. If θ changes at a constant take k then rate of change of the slope of the tangent to the curve at $\theta = \frac{\pi}{3}$ is

(a) $\frac{2k}{\sqrt{3}}$ (b) $\frac{k}{\sqrt{3}}$ (c) k (d) $\frac{2k}{3}$

MARK YOUR RESPONSE					
	36. ⓐⓑⓒⓓ	37. ⓐⓑⓒⓓ	38. ⓐⓑⓒⓓ	39. ⓐⓑⓒⓓ	40. ⓐⓑⓒⓓ
	41. ⓐⓑⓒⓓ	42. ⓐⓑⓒⓓ	43. ⓐⓑⓒⓓ	44. ⓐⓑⓒⓓ	

B COMPREHENSION TYPE

This section contains groups of questions. Each group is followed by some multiple choice questions based on a paragraph. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

PASSAGE-1

Cauchy Theorem : If two functions f and g are

(i) continuous in $[a, b]$

(ii) derivable in (a, b)

(iii) $g'(x) \ne 0$ for any $x \in (a, b)$

then there exists at least one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

1. Which of the following inequalities is true

(a) $|\tan^{-1}x - \tan^{-1}y| \le |x-y|\ \forall\ x, y \in R$

(b) $|\tan^{-1}x - \tan^{-1}y| \ge |x-y|\ \forall\ x, y \in R$

(c) $|\sin x - \sin y| \ge |x-y|\ \forall\ x, y \in R$

(d) $|\cos x - \cos y| \ge |x-y|\ \forall\ x, y \in R$

2. Suppose α, β and θ are angles satisfying

$0 < \alpha < \theta < \beta < \dfrac{\pi}{2}$, then $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} =$

(a) $\tan\theta$ (b) $-\tan\theta$

(c) $\cot\theta$ (d) $-\cot\theta$

3. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) then there exists at least one $c \in (a,b)$ such that

$\dfrac{f(b)-f(a)}{b^3-a^3}$ equals

(a) $3c^2 f'(c)$ (b) $\dfrac{f'(c)}{3c^2}$

(c) $f(c)f'(c)$ (d) $3c^2 f(c)f'(c)$

PASSAGE-2

The relation between the distance of any point on a curve from the origin and the length of the perpendicular from the origin to the tangent at the point is called the Pedal equation of the curve. Let the cartesian equation of a curve be $y = f(x)$

Equation of the tangent at a point (h, k) is

$y - k = f'(h)(x-h) \Rightarrow xf'(h) - y + k - hf'(h) = 0$

If p, be the length of the perpendicular from (0, 0) to this tangent, then we have

$$p = \frac{|k - hf'(h)|}{\sqrt{1+\{f'(h)\}^2}} \qquad .....(1)$$

Further $k = f(h)$ .....(2)

and $r^2 = h^2 + k^2$ .....(3)

where r is the distance of the point (h, k) from origin.

Eliminating h, k between (1), (2) and (3), we obtain a relation between p and r, i.e. $p = f(r)$, which is the required pedal equation of the curve.

For example if the equation of a curve is $y^2 = 4a(x+a)$

Tangent at (h, k) is

$$y - k = \frac{2a}{k}(x-h) \Rightarrow 2ax - ky + k^2 - 2ah = 0 \qquad ...(i)$$

The length p of the perpendicular from the origin to the tangent (i) is given by

$$p = \frac{k^2 - 2ah}{\sqrt{4a^2+k^2}} = \frac{4a(h+a)-2ah}{\sqrt{4a^2+4a(h+a)}} = \sqrt{a(x+2a)} \qquad ....(ii)$$

Also, $r^2 = h^2 + k^2 = h^2 + 4a(h+a) = (x+2a)^2$ ....(iii)

From (ii) and (iii) we obtain $p^4 = a^2r^2$, which is the required pedal equation.

4. The pedal equation of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

(a) $p^2 = a^2 + b^2 - r^2$

(b) $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2+b^2}$

(c) $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2b^2}$

(d) $\dfrac{1}{p^2} + \dfrac{1}{r^2} = 1$

5. The equation of a curve is given in parametric form $x = a\cos^3\theta, y = a\sin^3\theta$

The pedal equation of this curve is

(a) $r^2 = a^2 + 3p^2$ (b) $3r^2 = a^2 + p^2$

(c) $r^2 = a^2 - 3p^2$ (d) $r^3 = a^2 - 3p^2$

6. The cartesian equation of a curve is given by $c^2(x^2+y^2) = x^2y^2$. The pedal equation of this curve is

(a) $\dfrac{1}{p^3} + \dfrac{3}{r^2} = \dfrac{1}{c^2}$ (b) $\dfrac{1}{p^3} + \dfrac{1}{r^2} = \dfrac{3}{c^2}$

(c) $\dfrac{1}{p^3} + \dfrac{1}{r^3} = \dfrac{1}{c^3}$ (d) $\dfrac{1}{p^2} + \dfrac{3}{r^2} = \dfrac{1}{c^2}$

PASSAGE-3

Read the following write up carefully:

If $f(x) = (x-\alpha)^n g(x)$,

then $f(\alpha) = f'(\alpha) = f''(\alpha) = ... = f^{n-1}(\alpha) = 0$ where $f(x)$ and $g(x)$ are polynomials.

For a polynomial $f(x)$ with rational coefficients, answer the following question.

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ
	6. ⓐⓑⓒⓓ				

7. If $f(x)$ touches x-axis at only one point, then the point of tangency is
(a) always a rational number
(b) may or may not be a rational number
(c) never a rational number
(d) nothing can be said

8. If $f(-3) = f(2) = 0$ and $f'(-3) < 0$, then the largest integral value of c can be (where $f(x) = x^3 + ax^2 + bx + c$)
(a) -18 (b) -19
(c) -12 (d) -6

9. If $f(\alpha) = f'(\alpha) = f''(\alpha) = 0$, $f(\beta) = f'(\beta) = f''(\beta) = 0$ where $\alpha < \beta$ and $f(x)$ is a polynomial of degree 6, then
(a) all the roots of $f''(x) = 0$ are real and distinct
(b) atleast two roots of $f''(x) = 0$ are always non-real
(c) exactly two roots of $f''(x) = 0$ are real
(d) $f''(x) = 0$ has exactly two coincident roots

PASSAGE-4

In second degree curves, a line which once touches the curve cannot meet the curve again but in cubic and other non-algebraic curves, the tangent can meet the curve again. If we solve the equation of tangent and a cubic curve, we will in general get three roots two of which will be equal since they will correspond to the point where the tangent was initially drawn.

10. The equation of tangent to the curve $y = x^3 + 1$ at (1, 2) is
(a) $y = 3x$ (b) $y = 3x + 1$
(c) $y = 3x - 1$ (d) none of these

11. The tangent at (1, 2) of $y = x^3 + 1$, will meet the curve $y = x^3 + 1$ at
(a) $(-2, -7)$ (b) $(2, 9)$
(c) $(3, 28)$ (d) $(-3, -27)$

12. The tangent at 't' of the curve $y = 8t^3 - 1, x = 4t^2 + 3$ meets the curve at t' and is normal to the curve at that point, then value of 't' must be
(a) $\pm\frac{1}{\sqrt{3}}$ (b) $\pm\frac{1}{\sqrt{2}}$
(c) $\pm\frac{\sqrt{2}}{3}$ (d) none of these

PASSAGE-5

$f: R \to R, f(x)$ is a differentiable function such that all its successive derivatives exist. $f'(x)$ can be zero at discrete points only and $f(x)f''(x) \le 0 \ \forall \ x \in R$.

13. If $f(a) = 0$, then which of the following is correct
(a) $f(a+h)f''(a-h) < 0$
(b) $f(a+h)f''(a-h) > 0$
(c) $f'(a+h)f''(a-h) < 0$
(d) $f'(a+h)f''(a-h) > 0$

14. If α and β are two consecutive roots of $f(x) = 0$, then
(a) $f''(\gamma) = 0$ $\gamma \in (\alpha, \beta)$
(b) $f'(\gamma) = 0$ $\gamma \in (\alpha, \beta)$
(c) $f'''(\gamma) = 0$ $\gamma \in (\alpha, \beta)$
(d) $f'''''(\gamma) = 0$ $\gamma \in (\alpha, \beta)$

15. If $f'(x) \ne 0$, then maximum number of real roots of $f''(x) = 0$ is/are
(a) no real root (b) one
(c) two (d) three

MARK YOUR RESPONSE					
	7. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ	10. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ
	12. ⓐⓑⓒⓓ	13. ⓐⓑⓒⓓ	14. ⓐⓑⓒⓓ	15. ⓐⓑⓒⓓ	

C REASONING TYPE

In the following questions two Statements (1 and 2) are provided. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct. Mark your responses from the following options:
(a) Both Statement-1 and Statement-2 are true and Statement-2 is the correct explanation of Statement-1.
(b) Both Statement-1 and Statement-2 are true and Statement-2 is not the correct explanation of Statement-1.
(c) Statement-1 is true but Statement-2 is false.
(d) Statement-1 is false but Statement-2 is true.

1. **Statement-1** : The tangent to the curve $y = 4+\sin^2 x$ at $x = 0$ is parallel to x-axis.

Statement-2 : The range of function $y(x)$ is [4, 5].

2. **Statement-1** : The shortest distance of the line $y = x$ from the curve $(x-2)^2 + y^2 = 1$ is $\sqrt{2}-1$.

Statement-2 : Line joining (1, 1) and (2, 0) is perpendicular to the line $y = x$.

3. **Statement-1** : Wherever exists the slope of tangents to curve $\tan(x+y) = e^{x+y}$ is constant.

Statement-2 : Wherever defined the equation represents family of parallel lines.

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ		

D MULTIPLE CORRECT CHOICE TYPE

Each of these questions has 4 choices (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

1. For the curve $y = be^{x/a}$
(a) the subtangent is of constant length
(b) the subnormal is of constant length
(c) the subnormal varies as the square of ordinate
(d) the subtangent varies as the radius vector

2. If a, b, c, be non-zero real numbers such that

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\, dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)dx = 0,$$

then the equation $ax^2+bx+c=0$ has
(a) roots of opposite sign
(b) at least one fractional root
(c) one root in [0, 1] and other root in [1, 2]
(d) one root in $[-\infty, 1]$ and other root in $[2, \infty]$

3. The real values of a for which the equation $x^3 - 3x + a = 0$ has three real and distinct roots is
(a) $-2 < a < 2$ (b) $a \le -2$
(c) $a \ge 2$ (d) $-2 < a < \frac{1}{\sqrt{2}}$

4. Let α and β are two distinct positive numbers such that $\log_a \alpha = \alpha$ and $\log_a \beta = \beta$ then a can attain the values from the set(s)
(a) $\left(0, \frac{1}{e}\right)$ (b) $\left(1, \sqrt[3]{e}\right)$
(c) $(1, e^{1/e})$ (d) $(\sqrt{e}, \infty)$

5. The tangent lines for the curve $y = \int_0^x 2|t|\, dt$, which are parallel to the bisector of the first quadrant angle are
(a) $y = x+\frac{1}{4}$ (b) $y = x+\frac{3}{4}$
(c) $y = x-\frac{3}{4}$ (d) $y = x-\frac{1}{4}$

6. If $\frac{x}{a}+\frac{y}{b}=1$ is a tangent to the curve $x = 4t, y = \frac{4}{t}, t \in R$ then
(a) $a > 0, b > 0$ (b) $a > 0, b < 0$
(c) $a < 0, b > 0$ (d) $a < 0, b < 0$

7. The values of the parameter 'a' so that line $(3-a)x + ay + a^2 - 1 = 0$ is a normal to the curve $xy = 1$, is / are;
(a) $(3, \infty)$ (b) $(-\infty, 0)$
(c) $(0, 3)$ (d) $(-\infty, -3)$

MARK YOUR RESPONSE	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ
	6. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ			

E MATRIX-MATCH TYPE

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labeled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column -I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example:
If the correct matches are A–p, s and t; B–q and r; C–p and q; and D–s and t; then the correct darkening of bubbles will look like the given.

	p	q	r	s	t
A	(p)	(q)	(r)	(s)	(t)
B	(p)	(q)	(r)	(s)	(t)
C	(p)	(q)	(r)	(s)	(t)
D	(p)	(q)	(r)	(s)	(t)

1. Observe the following columns :

Column-I	Column-II
(A) The normal line to $y = be^{-x/a}$ where it crosses y-axis, has slope equal to	p. $\frac{a^2}{b^2}$
(B) Subnormal length to $xy = a^2 b^2$ at any point (x, y) is p then $\frac{1}{p}\left\|y^3\right\|$ is equal to	q. $\frac{a}{b}$
(C) The length of subtangent at any point (x, y) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is p then $\frac{p\,\|x\|}{y^2}$ is equal to	r. $a^2 b^2$
(D) If m be slope of tangent at any point (x, y) on the curve $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then $\frac{my}{x}$ is equal to	s. $\frac{b^2}{a}$

2. Observe the following columns :

Column-I	Column-II
(A) If the curves $y = 1 - \cos x, x \in (-\pi, \pi)$ and $y = \frac{\sqrt{3}}{2}\|x\| + a$ touch each other then the number of the possible values of a is equal to	p. 0
(B) If two curves $y^2 = 4a(x - b_1)$ and $x^2 = 4a(y - b_2)$,where a is a positive constant number and b_1 and b_2 are variables, touch each other then their point of contact lies on $xy = ka^2$, where k is equal to	q. 2
(C) The point on the parbola $y^2 = 4x$, which is nearest to the circle $x^2 + (y - 12)^2 = 1$ has the ordinate equal to	r. 3
(D) The ordinate of the point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is parallel to y-axis is/are	s. 4
	t. –2

MARK YOUR RESPONSE

1.

	p	q	r	s
A	(p)	(q)	(r)	(s)
B	(p)	(q)	(r)	(s)
C	(p)	(q)	(r)	(s)
D	(p)	(q)	(r)	(s)

2.

	p	q	r	s	t
A	(p)	(q)	(r)	(s)	(t)
B	(p)	(q)	(r)	(s)	(t)
C	(p)	(q)	(r)	(s)	(t)
D	(p)	(q)	(r)	(s)	(t)

3. Observe the following columns:

Column-I		**Column-II**
(A)	The intercept of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ on the axis of y is equal to	p. -2
(B)	Let f be a real function whose derivatives upto third order exist and for some pair $a, b \in R, a < b$ $\log \dfrac{f(a)+f'(a)+f''(a)}{f(b)+f'(b)+f''(b)} = a - b$, then $\exists c \in (a, b)$ for which $\dfrac{f'''(c)}{f(c)}$ is equal to	q. 0
(C)	Let $f(x) = (x^2 - 1)(x^2 - 4)$, and α, β, γ be the roots of the equation $f'(x) = 0$ then $[\alpha] + [\beta] + [\gamma]$ is equal to ($[t]$ represents the integral part of t)	r. 1
(D)	If three normals can be drawn to the curve $y^2 = x$ from the point $(c, 0)$ then c can be equal to	s. $\dfrac{5}{4}$
		t. 2

4.

Column-I		**Column-II**
(A)	Number of points on the curve $xy = 2$, where tangent makes an acute angle with x-axis is equal to	p. 4
(B)	If $y = \sin^{-1}(\ell n(\sin x))$ then $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{2}$ is equal to	q. 0
(C)	The slope of the tangent to the curve $x = e^t \sin t, y = e^t \cos t$ at the point $t = \dfrac{\pi}{4}$ is equal to	r. 1
(D)	If θ be the angle of intersection of the curves $y = x^2$ and $6y = 7 - x^3$ at $(1, 1)$ then $\sin\theta$ is equal to	s. $\dfrac{1}{2}$

MARK YOUR RESPONSE

3.

	p	q	r	s	t
A	p	q	r	s	t
B	p	q	r	s	t
C	p	q	r	s	t
D	p	q	r	s	t

4.

	p	q	r	s
A	p	q	r	s
B	p	q	r	s
C	p	q	r	s
D	p	q	r	s

F NUMERIC/INTEGER ANSWER TYPE

The answer to each of the questions is either numeric (eg. 304, 40, 3010 etc.) or a single-digit integer, ranging from 0 to 9.

The appropriate bubbles below the respective question numbers in the response grid have to be darkened.

For example, if the correct answers to a question is 6092, then the correct darkening of bubbles will look like the given.

For single digit integer answer darken the extreme right bubble only.

1. If the tangent at (a, b) to the curve $x^3 + y^3 = c^3$ meets the curve again at (a_1, b_1), then $\frac{a_1}{a} + \frac{b_1}{b} + 1$ is equal to

2. If the curves $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ may cut each other orthogonally such that $\frac{1}{a} - \frac{1}{a_1} = \lambda\left(\frac{1}{b} - \frac{1}{b_1}\right)$ then λ is equal to

3. If the acute angles between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection be θ such that $\tan\theta = \frac{m}{7}$ then m^2 is equal to

4. If the tangent at any point $P(4m^2, 8m^3)$ of $x^3 - y^2 = 0$ is a normal also to the curve $x^3 - y^2 = 0$, then $9m^2$ is equal to

5. If θ be the angle of intersection of curves $y = [|\sin x| + |\cos x|]$ and $x^2 + y^2 = 5$, where [.] denotes the greatest integer function, then $\tan^2\theta$ is equal to

6. If the point of intersection of the tangents drawn to the curve $x^2y = 1 - y$ at the points where it is met by the curve $xy = 1 - y$ be (α, β) then $\alpha + \beta$ is equal to

7. The segment of the tangent to the curve $x^{2/3} + y^{2/3} = 16$, contained between x and y axes, has length equal to

8. Tangent at $P_1(2,3)$ on the curve $3y = x^3 + 1$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 and so on. If the sum of the ordinates for $P_1, P_2, P_3, ...P_{60}$ be S then $S + \left(\frac{2^{183} - 8}{27}\right)$ is equal to

MARK YOUR RESPONSE

1. 2. 3. 4. 5. 6. 7. 8.

Answerkey

A SINGLE CORRECT CHOICE TYPE

1	(b)	**9**	(c)	**17**	(c)	**25**	(c)	**33**	(a)	**41**	(b)
2	(c)	**10**	(a)	**18**	(d)	**26**	(a)	**34**	(a)	**42**	(c)
3	(b)	**11**	(a)	**19**	(a)	**27**	(b)	**35**	(d)	**43**	(b)
4	(b)	**12**	(b)	**20**	(b)	**28**	(b)	**36**	(b)	**44**	(d)
5	(a)	**13**	(b)	**21**	(b)	**29**	(c)	**37**	(d)		
6	(a)	**14**	(a)	**22**	(d)	**30**	(b)	**38**	(b)		
7	(d)	**15**	(a)	**23**	(c)	**31**	(c)	**39**	(c)		
8	(d)	**16**	(c)	**24**	(b)	**32**	(c)	**40**	(c)		

B COMPREHENSION TYPE

1	(a)	**4**	(c)	**7**	(a)	**10**	(c)	**13**	(b)
2	(c)	**5**	(c)	**8**	(b)	**11**	(a)	**14**	(b)
3	(b)	**6**	(d)	**9**	(a)	**12**	(c)	**15**	(b)

C REASONING TYPE

1	(a)	**2**	(a)	**3**	(a)

D MULTIPLE CORRECT CHOICE TYPE

1	(a, c)	**3**	(a,d)	**5**	(b, d)	**7**	(a,b,d)
2	(b, c)	**4**	(b, c)	**6**	(a, d)		

E MATRIX-MATCH TYPE

1. **A - q; B - r; C - p; D - s**

2. **A - q; B - s; C - s; D - q**

3. **A - t; B - r; C - p; D - r, s, t**

4. **A - q; B - q; C - q; D - r**

F NUMERIC/INTEGER ANSWER TYPE

1	1	**3**	32	**5**	4	**7**	64
2	1	**4**	2	**6**	1	**8**	20

Solutions

A SINGLE CORRECT CHOICE TYPE

1. **(b)** The equation of tangent to $x^3+y^3=c^3$ at (a, b) is $a^2x+b^2y=a^3+b^3$, which passes through (p,q)

$\therefore \quad a^2p+b^2q=a^3+b^3$ .....(1)

Also $p^3+q^3=a^3+b^3$ .....(2)

Let $p=a\alpha$ and $q=b\beta$ On eliminating a^3 and b^3 from (1) and (2), we get

$\alpha+\beta+1=0 \Rightarrow aq+bp+ab=0$

2. **(c)** Given curve is $x+y=x^y$ .....(i)

at x-axis $y=0 \Rightarrow x+0=x^0 \Rightarrow x=1$

$\therefore$ Point is $A(1,0)$

Now to differentiate $x+y=x^y$ take log on both sides

$\Rightarrow \log(x+y)=y\log x$

$$\therefore \quad \frac{1}{x+y}\left\{1+\frac{dy}{dx}\right\}=y.\frac{1}{x}+(\log x)\frac{dy}{dx}$$

Putting $x=1, y=0$, we get

$$\left\{1+\frac{dy}{dx}\right\}=0 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)}=-1$$

$\therefore$ Slope of normal = 1

Equation of normal is, $\dfrac{y-0}{x-1}=1 \Rightarrow y=x-1$

3. **(b)** Let $y=f(x)=ax^2+bx+c$

$\therefore \quad f'(x)=2ax+b$

$f(0)=c$ and $f'(0)=b$

$f'(x)\{at(1,1)\}=2a+b=1 \qquad f(1)=a+b+c=1$

Solving, we have $a-c=0$ or $a=c$.

Now, $2f(0)+f'(0)=2c+b=2a+b=1$

4. **(b)** For the curve $\sqrt{x}+\sqrt{y}=2$, the parametric coordinates are given by

$\sqrt{x}=2\cos^2\theta$ and $\sqrt{y}=2\sin^2\theta$ *i.e.*, $x=4\cos^4\theta$

and $x=4\sin^4\theta$

$$\frac{dy}{dx}=\frac{4\times 4\sin^3\theta.\cos\theta}{4\times 4\cos^3\theta(-\sin\theta)}=-\tan^2\theta$$

$\Rightarrow$ Eq. of tangent

$y-4\sin^4\theta=-\tan^2\theta(x-4\cos^4\theta)$

$$\therefore \quad x\text{-intercept} = \left|4\cos^4\theta+\frac{4\sin^4\theta}{\tan^2\theta}\right|=4\cos^2\theta \text{ and}$$

$$\text{y-intercept} = \left|4\sin^4\theta+\frac{4\sin^4\theta}{\tan^2\theta}\right|=4\sin^2\theta$$

Hence, the sum of intercept made on the axes of coordinates is $4\cos^2\theta+4\sin^2\theta=4$

5. **(a)** Let $P(x, y)$ be a point on the curve $\ln(x^2+y^2)=c\tan^{-1}\dfrac{y}{x}$

Differentiating both sides with respect to x, we get

$$\frac{2x+2yy'}{(x^2+y^2)}=\frac{c(xy'-y)}{(x^2+y^2)}$$

$$\Rightarrow \quad y'=\frac{2x+cy}{cx-2y}=m_1 \quad \text{(say)}$$

and Slope of $OP=\dfrac{y}{x}=m_2$ (say)

Let the angle between the tangent at P and OP be θ

$$\therefore \quad \tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\dfrac{2x+cy}{cx-2y}-\dfrac{y}{x}}{1+\dfrac{2xy+cy^2}{cx^2-2xy}}\right|=\frac{2}{c}$$

$\theta=\tan^{-1}\left(\dfrac{2}{c}\right)$ which is independent of x and y.

6. **(a)** Here $y=\dfrac{c^2}{x+a} \Rightarrow \dfrac{dy}{dx}=-\dfrac{c^2}{(x+a)^2}$

Slope of normal is $\Rightarrow \dfrac{(x+a)^2}{c^2}>0$ (for all x)

$\therefore x\cos\alpha+y\sin\alpha=p$ is normal if, $-\dfrac{\cos\alpha}{\sin\alpha}>0$

or $\cot\alpha<0$

i.e., α lies in II or IV quadrant.

So, $\alpha\in\left(2n\pi+\dfrac{\pi}{2}, (2n+1)\pi\right)\cup\left(2n\pi+\dfrac{3\pi}{2}, (2n+2)\pi\right)$

7. **(d)** We have, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta(-\sin\theta)} = -\tan\theta$

Equation of tangent at 'θ' is

$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$

$\Rightarrow \frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1$

$\therefore$ P is ($a\cos\theta$,0) and Q (0, $a\sin\theta$). If mid point of PQ is (h, k), then

$h = \frac{a\cos\theta}{2}, k = \frac{a\sin\theta}{2}$ Eliminating θ, we get

$h^2 + k^2 = \frac{a^2}{4}$

$\therefore$ focus is $x^2 + y^2 = \frac{a^2}{4}$

8. **(d)** Here $by^2 = (x+a)^3$, differentiating both sides, we get

$2by\frac{dy}{dx} = 3(x+a)^2.1 \Rightarrow \frac{dy}{dx} = \frac{3}{2}\frac{(x+a)^2}{by}$

$\therefore$ Length of subnormal $\Rightarrow SN = y\frac{dy}{dx} = \frac{3}{2}\frac{(x+a)^2}{b}$ .....(1)

and length of subtangent $\Rightarrow ST = y\frac{dx}{dy} = \frac{2by^2}{3(x+a)^2}$ ......(2)

$\frac{p}{q} = \frac{(ST)^2}{(SN)}$ (given)

$\Rightarrow \frac{p}{q} = \frac{(2by^2).2b}{\{3(x+a)^2\}^2.3(x+a)^2}$ {using (i) and (ii)}

$= \frac{8b}{27}\frac{\{(x+a)^3\}^2}{(x+a)^6}$ {using, $by^2 = (x+a)^3$ } $= \frac{8b}{27}$

$\therefore \frac{p}{q} = \frac{8b}{27}$

9. **(c)** Given curve is $y = a^{1-n}.x^n$

Taking logarithm of both sides, we get

$\ln y = (1-n)\ln a + n\ln x$

Differentiating both sides, with resect to x, we get

$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = 0 + \frac{n}{x}$ or $\frac{dy}{dx} = \frac{ny}{x}$ ....(i)

$\therefore$ Length of sub-normal $= y\frac{dy}{dx} = y.\frac{ny}{x}$ {from (1)}

$= \frac{ny^2}{x} = n.\frac{(a^{1-n}x^n)^2}{x}$ $(\because y = a^{1-n}.x^n)$

$= n.a^{2-2n}.x^{2n-1}$

Since length of sub-normal is to be constant, so x should not appear in its value i.e., $2n - 1 = 0$

$\therefore n = \frac{1}{2}$

10. **(a)** Differentiating the given equation, we get

$\frac{dy}{dx} = \left[\frac{1}{\sqrt{1+t^2}}\right]_{t=x^3}\frac{d}{dx}(x^3) - \left[\frac{1}{\sqrt{1+t^2}}\right]_{t=x^2}$

$\frac{d}{dx}(x^2) = \frac{3x^2}{\sqrt{1+x^6}} - \frac{2x}{\sqrt{1+x^4}}$

Therefore the slope of the required tangent is

$\left(\frac{dy}{dx}\right)_{x=1} = \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

This tangent passes through the point $x = 1$ and

$y(1) = \int_1^1 \frac{dt}{\sqrt{1+t^2}} = 0$

So, its equation is

$y - 0 = \frac{1}{\sqrt{2}}(x-1) \Rightarrow \sqrt{2}\,y + 1 = x.$

11. **(a)** Since $f(x)$ is differentiable in [0,4], using Lagrange's Mean Value Theorem.

$f'(b) = \frac{f(4) - f(0)}{4}, b \in (0,4)$ ....(1)

Now, $\{f(4)^2 - \{f(0)\}^2 = \frac{4\{f(4) - f(0)\}}{4}\{f(4) + f(0)\}$

$= 4f'(b)\{f(4) + f(0)\}$ ....(2)

Also, from Intermediate Mean Value Theorem,

$\frac{f(4) + f(0)}{2} = f(a)$ for $a \in (0,4)$.

Hence, from (2) $[f(4)]^2 - [f(0)]^2 = 8f'(b)f(a)$,

12. **(b)** Since $f(x)$ is differentiable in $0 \le x \le 5$, so is $g(x)$ and hence applying Lagrange's Mean Value Theorem

$g'(c) = \frac{g(5) - g(0)}{5} = \frac{-1/6 - 4}{5} = -\frac{5}{6}$

13. **(b)** Given $s^2 = (at^2 + 2bt + c)$ ...(i)

or $s = \sqrt{at^2 + bt + c}$

Differentiating both sides with resepct to 't' we get

$\frac{ds}{dt} = \frac{(at+b)}{\sqrt{(at^2 + 2bt + c)}} = \frac{(at+b)}{s} = v$

(say) {From (i)} ...(2)

Again differentiating both sides with respect to 't' then

$$\frac{d^2s}{dt^2} = \frac{s(a) - (at+b).\frac{ds}{dt}}{s^2} = \frac{as - (at+b).\frac{(at+b)}{s}}{s^2}$$

{From (2)}

$$= \frac{as^2 - (at+b)^2}{s^3}$$

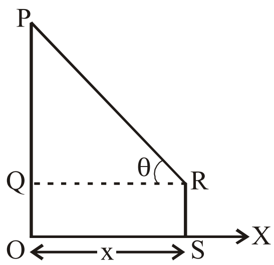
$$= \frac{a(at^2+2bt+c) - (a^2t^2+2abt+b^2)}{s^3} = \frac{ac-b^2}{s^3}$$

$\therefore$ acceleration $\propto \frac{1}{s^3}$

14. **(a)** Let RS be the position of man at any time t. Let $OS = x$, and $\angle QRP = \theta$

Given, $OP = 41.6\ m$, $RS = 1.6\ m$ and $\frac{dx}{dt} = 2m/\text{sec}$.

$PQ = 40\ m$



From ΔPQR, $\tan\theta = \frac{PQ}{QR} = \frac{40}{x}$ .....(1)

Differentiating w.r. to t, we get $\sec^2\theta\frac{d\theta}{dt} = \frac{40}{x^2}\frac{dx}{dt}$

or $\sec^2\theta\frac{d\theta}{dt} = -\frac{40}{x^2}.2$

$$\therefore \quad \frac{d\theta}{dt} = \frac{-80}{x^2}\cos^2\theta = -\frac{80}{x^2}.\frac{x^2}{x^2+40^2}$$

$$\left[\because \cos\theta = \frac{x}{\sqrt{x^2+40^2}}\right]$$

or $\frac{d\theta}{dt} = -\frac{80}{x^2+40^2}$ when $x = 30\ m$

$$\frac{d\theta}{dt} = -\frac{80}{30^2+40^2} = -\frac{4}{125}\ \text{radian/sec}$$

15. **(a)** Given curve is $2x^2y^2 - x^4 = c$ .....(1)

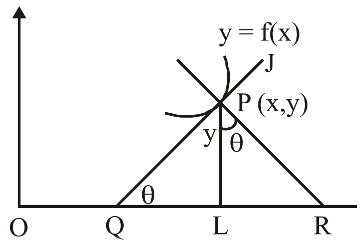
Subnormal at $P(x,y) = yy_1 = y\frac{dy}{dx}$ ......(2)

From (1), $2\left(x^2.2y\frac{dy}{dx} + 2xy^2\right) - 4x^3 = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x(x^2-y^2)}{x^2y} \quad ....(3)$$

Now, $x(x - yy_1) = x^2 - xy\frac{dy}{dx} = x^2 - (x^2 - y^2)$

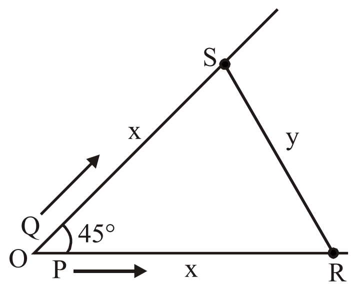
[from (3)]



$= y^2 \Rightarrow$ mean proportion $= \sqrt{x(x - yy_1)} = y$

16. **(c)** Let R and S be the positions of men P and Q at any time t. Since velocities are same

$\therefore \quad OR = OS = x$ (say) and given $\frac{dx}{dt} = v$ and let $SR = y$



Now in triangle ORS, applying cosine rule, we get

$$y^2 = x^2 + x^2 - 2x\,.\,x\cos 45^\circ = 2x^2 - x^2\sqrt{2}$$

$\therefore \quad y = x\sqrt{(2-\sqrt{2})}$

$\therefore \quad \frac{dy}{dt} = \{\sqrt{(2-\sqrt{2})}\}\frac{dx}{dt} = v\sqrt{(2-\sqrt{2})}$

Hence the required rate at which they are being separted is $v\sqrt{2-\sqrt{2}}$.

17. **(c)** Given x and y are sides of two squares thus the area of two squares are x^2 and y^2

We have to obtain $\frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x}.\frac{dy}{dx}$ ..(i)

where the given curve is, $y = x - x^2$

$$\Rightarrow \frac{dy}{dx} = 1 - 2x \quad ....(ii)$$

Thus $\frac{d(y^2)}{d(x^2)} = \frac{y}{x}(1-2x)$ [from (i) and (ii)]

or $\frac{d(y^2)}{d(x^2)} = \frac{(x-x^2)(1-2x)}{x} = 2x^2 - 3x + 1$

So, the rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$.

18. (d) Since $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \{s(s-a)(s-b)(s-c)\}^{1/2}$

Taking logarithm of both sides, we get

$\ln \Delta = \frac{1}{2}\{\ln s + \ln(s-a) + \ln(s-b) + \ln(s-c)\}$

$\therefore \frac{1}{\Delta}\frac{d\Delta}{dc} = \frac{1}{2}\left\{\frac{1}{s}\frac{ds}{dc} + \frac{1}{(s-a)}.\frac{d(s-a)}{dc} + \frac{1}{(s-b)}.\frac{d(s-b)}{dc} + \frac{1}{s-c}.\frac{d(s-c)}{dc}\right\}$ ....(1)

But $s = \frac{1}{2}(a+b+c)$

Now $\frac{ds}{dc} = \frac{1}{2}, \frac{d(s-a)}{dc} = \frac{ds}{dc} - \frac{da}{dc} = \frac{1}{2} - 0 = \frac{1}{2}$,

$\frac{d(s-b)}{dc} = \frac{ds}{dc} - \frac{db}{dc} = \frac{1}{2} - 0 = \frac{1}{2}$

and $\frac{d(s-c)}{dc} = \frac{ds}{dc} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$

Now from (1),

$\frac{1}{\Delta}.\frac{\delta\Delta}{\delta c} = \frac{1}{2}\left\{\frac{1}{s}.\frac{1}{2} + \frac{1}{(s-a)}.\frac{1}{2} + \frac{1}{(s-b)}.\frac{1}{2} - \frac{1}{(s-c)}.\frac{1}{2}\right\}$

$= \frac{1}{4}\left\{\frac{1}{s} + \frac{1}{(s-a)} + \frac{1}{(s-b)} - \frac{1}{(s-c)}\right\}$

Hence $\delta\Delta = \frac{\Delta}{4}\left\{\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right\}\delta c$

19. (a) Since in a triangle $A + B + C = \pi$

$\therefore \quad dA + dB + dC = 0$

If R is circumradius then $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\therefore \quad a = 2R \sin A$

On differentiating, we get $da = 2R\cos A dA$

or $\frac{da}{\cos A} = 2R dA$

Similarly, $\frac{db}{\cos B} = 2R dB$ and $\frac{dc}{\cos C} = 2R dC$

$\therefore \quad \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos c} = 2R(dA + dB + dC) = 0$

Hence, $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$

20. (b) Let $F(x) = Ag(x) + f(x)$ .....(1)

Since $f(x)$ and $g(x)$ are differentiable in [0, 1]

Therefore $f(x)$ and $g(x)$ are continuous in [0, 1].

Thus $F(x)$ is a continuous function in the closed interval [0, 1] and differentiable in the open interval (0, 1).

From (1) $F(x) = Ag'(x) + f'(x)$ ......(2)

$\therefore \quad F'(c) = Ag'(c) + f'(c)$

We choose A such that $F(0) = F(1)$

$\therefore \quad Af(0) + g(0) = Af(1) + g(l)$

$\Rightarrow \quad A = -\frac{g(1) - g(0)}{f(1) - f(0)}$ ......(3)

$F(x)$ satisfies all condition of mean value theorem in [0, 1], therefore there exists at least one c, $0 < c < 1$ such that $F'(c) = 0$

$\therefore Af'(c) + g'(c) = 0$

$\Rightarrow \frac{g'(c)}{f'(c)} = -A = \frac{g(1) - g(0)}{f(1) - f(0)} = \frac{2-0}{6-2} = \frac{1}{2}$

$\Rightarrow f'(c) = 2g'(c)$

21. (b) Consider the function $f(x) = x^3 + 2ax^2 + bx$

Obviously $f(x)$ being a polynomial function is continuous in [0, 1] and differentiable in (0, 1)

Also $f(0) = 0$

If $f(1) = 0$, then all the three condition of Rolle's theorem will be satisfied.

$\therefore \quad f'(c) = 0$, for at least one c in (0, 1)

Hence, $f'(x) = 3x^2 + 4ax + b = 0$ at least once in (0, 1)

i.e. the equation $3x^2 + 4ax + b = 0$ has at least one root in (0, 1) if $f(1) = 0$ i.e. $1 + 2a + b = 0$.

22. (d) Let α and β be two distinct roots of $f(x) = x^3 - 3x + k = 0$

then $0 < \alpha < \beta < 1$ and $f(\alpha) = f(\beta) = 0$

The function satisfies conditions of Rolle's MVT, so, for some $y \in (\alpha, \beta)$

$f'(y) = 0 \Rightarrow 3y^2 - 3y = 0 \Rightarrow y = 0$ or 1 but none of these lies in (α, β) so, $k \in \phi$.

23. (c) Given $e^{\alpha}\cos\alpha = 1$ ....(1)

and $e^{\beta}\cos\beta = 1$ .....(2)

Let $f(x) = e^{-x} - \cos x$, then $f(x)$ is continuous and differentiable.

Also, $f(\alpha) = f(\beta) = 0$ (from (1) and (2))
Therefore by Rolle's MVT, $f'(x) = 0$ has at least one root in (α, β).

$\Rightarrow -e^{-x} + \sin x = 0$ for at least one $x \in (\alpha, \beta)$

$\Rightarrow e^{x} \sin x = 1$ has at least one root in (α, β).

24. **(b)** Slope of normal at $x = 0$ is 3, so $f'(0) = -\frac{1}{3}$

Now, the required limit

$$= \lim_{x \to 0} \frac{x^2}{f(x^2) - 5f(4x^2) + 4f(7x^2)} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2x}{2xf'(x^2) - 40f'(4x^2)x + 56f'(7x^2)x}$$

(L' Hospital Rule)

$$= \lim_{x \to 0} \frac{1}{f'(x^2) - 20f'(4x^2) + 28f'(7x^2)}$$

$$= \frac{1}{f'(0) - 20f'(0) + 28f'(0)} = \frac{1}{9 \times (-1/3)} = -\frac{1}{3}$$

25. **(c)** Any point on $x^2 - y^2 = a^2$ is $(a \sec\theta, a \tan\theta)$
This point is neareast to $y = 2x$ if the tangent at this point is parallel to $y = 2x$

Now, $x^2 - y^2 = a^2 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(a\sec\theta, a\tan\theta)} = \operatorname{cosec}\theta$$

$$\operatorname{cosec}\theta = 2 \Rightarrow \theta = \frac{\pi}{6}$$

Hence the point is $\left(a \sec\frac{\pi}{6}, a \tan\frac{\pi}{6}\right)$ i.e. $\left(\frac{2a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$,

Clearly they lie on the line $2y = x$

26. **(a)** $ax^2 + by^2 = c$. Differentiating, $2ax + 2byy' = 0$
$\Rightarrow ax + by\, y' = 0$ ...... (1)

Differentiating again $a + by\,y'' + b(y')^2 = 0$

$$\Rightarrow a + by\,y'' + b\left(\frac{-ax}{by}\right)^2 = 0 \quad \text{[from (1)]}$$

$$\Rightarrow a + by\,y'' + b\frac{a^2x^2}{b^2y^2} = 0$$

$$\Rightarrow ay^2 + \frac{a^2x^2}{b} = -by^3y''$$

$$\Rightarrow \frac{a(ax^2 + by^2)}{b} = -by^3y'' \Rightarrow \frac{a(c)}{b} = -b(-4)$$

$$\Rightarrow ac = 4b^2 \Rightarrow b^2 - 4ac$$

$$= b^2 - 4(4b^2) = -15b^2 < 0$$

$\therefore$ $ax^2 + bx + c = 0$ has imaginary roots.

27. **(b)** Consider the function $\phi(x) = f(x) - 2g(x)$ defined on [0, 1].
As $f(x)$ and $g(x)$ are differentiable on (0,1) and hence continuous on [0, 1], hence $\phi(x)$ is also differentiable on (0, 1) and continuous on [0, 1].

Also, $\phi(0) = f(0) - 2g(0) = 2 - 0 = 2$

$$\phi(1) = f(1) - 2g(1) = 6 - 2g(1)$$

Now, $\phi'(x) = f'(x) - 2g'(x)$

$\Rightarrow \phi'(c) = f'(c) - 2g'(c) = 0$

Hence, $\phi(x)$ satisfies all conditions of Roll's Mean Volue Theorem, on [0, 1], therefore,

$$\phi(0) = \phi(1) \Rightarrow 2 = 6 - 2g(1) \Rightarrow g(1) = 2$$

28. **(b)** Let $h(x) = \{f(x) - f(a)\}\,\{g(b) - g(x)\}$
Then $h(x)$ is continuous and differentiable and $h(a) = h(b) = 0$
Therefore $h(x)$ satisfies the conditions of Rolle's MVT.

Therefore for some $c \in (a, b)$, $f'(c) = 0$

$\Rightarrow f'(c)\{g(b) - g(c)\} - g'(c)\{f(c) - f(a)\} = 0$

$$\Rightarrow \frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$$

29. **(c)** Consider $f(x) = e^{x/3}(ax^2 + bx + c)$. Clearly $f(x) = 0$ has two distinct positive roots α and β. So, $f'(x) = 0$ has at least one root in (α, β) [Rolle's Theorem]

or $\frac{1}{3}e^{3x}\{ax^2 + (b + 6a)x + c + 3b\} = 0$ has at least one root in (α, β)

$\Rightarrow ax^2 + (b + 6a)x + c + 3b = 0$ has at least one positive root.

30. **(b)** Let $\phi(x) = x^2 \ln(4) - 16 \ln x$, which is continuous on [4, 5] and differentiable on (4, 5), so by LMVT,

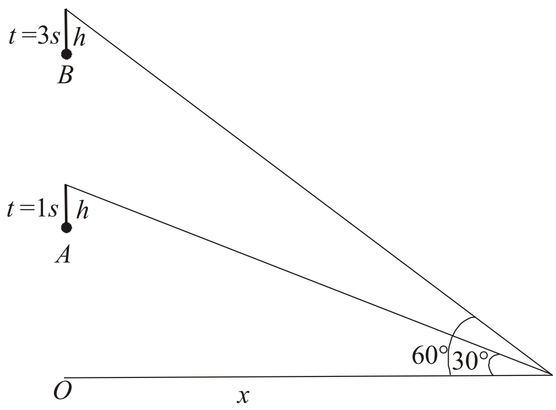
$$\frac{\phi(5) - \phi(4)}{5 - 4} = \phi'(c),\ c \in (4, 5)$$

Now, $\phi(5) - \phi(4) = \ln\left(\frac{4^{25}}{5^{16}}\right)$

and $\phi'(c) = \frac{2}{c}(c^2 \ln 4 - 8) \Rightarrow c \ln\left(\frac{4^{25}}{5^{16}}\right) = 2(c^2 \ln 4 - 8)$

31. **(c)** $OA = \int_0^1 v\,dt = (t^2 + 3t)\Big|_0^1 = 4$

$OB = \int_0^3 v\,dt = (t^2 + 3t)\Big|_0^3 = 18$



$\tan 30° = \dfrac{4+h}{x} \Rightarrow \dfrac{1}{\sqrt{3}}x = 4 + h$

$\tan 60° = \dfrac{18+h}{x} \Rightarrow \sqrt{3}x = 18 + h$

$\therefore \quad 4\sqrt{3} + \sqrt{3}h = \dfrac{18}{\sqrt{3}} + \dfrac{h}{\sqrt{3}}$

$\Rightarrow \quad h\left(\sqrt{3} - \dfrac{1}{\sqrt{3}}\right) = \dfrac{18}{\sqrt{3}} - 4\sqrt{3}$

$\Rightarrow \quad h = \dfrac{6}{2} = 3.$

32. **(c)** $\dfrac{dy}{dx} = \cos^{-1}(x^4)2x - \cos^{-1}(x^2)$

$\left.\dfrac{dy}{dx}\right|_{x=\frac{1}{\sqrt[4]{2}}} = 2\dfrac{1}{\sqrt[4]{2}} \times \dfrac{\pi}{3} - \dfrac{\pi}{4} = \left(\dfrac{\sqrt[4]{8}}{3} - \dfrac{1}{4}\right)\pi$

33. **(a)** The point of intersection is $(1, e^{-1})$

$\because$ $x = 1$, so equation of the curve is $y = e^{-x}$

$\Rightarrow \dfrac{dy}{dx} = -e^{-x}$

$\left[\dfrac{dy}{dx}\right]_{x=1} = -e^{-1}$. Hence equation of tangent is

$y - e^{-1} = -e^{-1}(x-1)$ or, $ey + x = 2$

34. **(a)** $\dfrac{dy}{dx} = \dfrac{x^2}{4} \quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{x^2}{4}\dfrac{dx}{dt}$

when $\dfrac{x^2}{4} < 1$ then $\dfrac{dy}{dt} < \dfrac{dx}{dt}$

$\Rightarrow \quad x^2 - 4 < 0 \quad \Rightarrow \quad x \in (-2, 2)$

35. **(d)** A point on first curve is $P(t_1, t_1^2 - t + 1)$, slope of tangent at $P = 2t_1 - 1$. So, the equation of tangent at P is $y - (t_1^2 - t + 1) = (2t_1 - 1)(x - t_1)$ ...(1)

A point on second curve is $Q(t_2, t_2^2 - 3t_2 + 1)$, slope of tangent at $Q = 2t_2 - 3$.
So, the equation of tangent at Q is

$y - (t_2^2 - 3t + 1) = (2t_2 - 3)(x - t_2)$ ...(2)

(1) and (2) are same lines, so,

$2t_1 - 1 = 2t_2 - 3 \quad \Rightarrow \quad t_2 - t_1 = 1$ ...(4)

and

$-t_1(2t_1 - 1) + t_1^2 - t + 1 = -t_2(2t_2 - 3) + t_2^2 - 3t_2 + 1$

$\Rightarrow \quad -t_1^2 + 1 = -t_2^2 + 1$

$\Rightarrow \quad t_1 = \pm t_2 \quad \Rightarrow \quad t_2 = -t_1 \quad (\because t_1 \neq t_2)$

$\therefore$ From (4), $t_1 = -\dfrac{1}{2}$

Desired slope $= 2t_1 - 1 = -2$

36. **(b)** $x = t^2 + 3t - 8,\ y = 2t^2 - 2t - 5$ at $(2, -1)$

$t^2 + 3t - 8 = 2$ and $2t^2 - 2t - 5 = -1$

$\Rightarrow t^2 + 3t - 10 = 0$ and $2t^2 - 2t - 4 = 0$

$\Rightarrow (t-2)(t+5) = 0$ and $(t+1)(t-2) = 0$

$\Rightarrow \quad t = -5$ or $2 \quad \Rightarrow \quad t = -1; t = 2$

$\Rightarrow \quad t = 2$

$$\left.\begin{aligned}\frac{dy}{dt} &= 4t - 2\\ \frac{dx}{dt} &= 2t + 3\end{aligned}\right\} \frac{dy/dt}{dx/dt} = \frac{4t-2}{2t+3} = \frac{6}{7} = \text{slope of tangent}$$

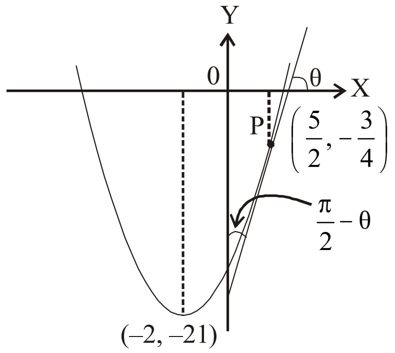
$\therefore$ slope of normal $= -\dfrac{7}{6}$

37. **(d)** $e^y = 1 + x^2 \Rightarrow y = \ell n(1 + x^2) \Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{1+x^2}$

$\therefore$ Slope of tangent at any point, $m = \dfrac{2x}{1+x^2}$

Now, $1 + x^2 \geq 2|x| \Rightarrow \dfrac{2|x|}{1+x^2} \leq 1 \Rightarrow |m| \leq 1$

38. (b)



$y = x^2 + 4x + 4 - 4 - 17$

$y = (x+2)^2 - 21 \Rightarrow$ Vertex is $(-2,-21)$

Also $y = x^2 + 4x - 17$

$\Rightarrow \frac{dy}{dx} = 2x + 4 \Rightarrow$ Slope of tangent at $\left(\frac{5}{2}, -\frac{3}{4}\right)$

$m = \frac{dy}{dx} = 2\times\frac{5}{2} + 4 = 9$

$\theta = \tan^{-1} 9$

$\therefore$ angle by y-axis $= \frac{\pi}{2} - \tan^{-1} 9 = \cot^{-1} 9$

39. (c) Equation of the line joining given points is

$(y+2) = \frac{-2-3}{5-0}(x-5)$

$\Rightarrow \quad y + 2 = -1(x-5)$

$\Rightarrow \quad y + x = 3$

Solving with the curve $y = \frac{c}{x+1}$, we get

$(3-x)(x+1) = c$

$\Rightarrow x^2 - 2x + c - 3 = 0$

For equal roots $4 - 4(c-3) = 0 \Rightarrow c = 4$

40. (c) General point on $y^2 = 8x$

$(2t, \pm 4t)\ t > 0$

$\frac{dy}{dx} = \frac{4}{y}$ at $(2t^2, -4t)$

$= -\frac{1}{t}$

Equation of normal at t

$(y+4t) = t(x - 2t^2)$

$y + 4t = tx - 2t^3$ it passes through (0, – 6)

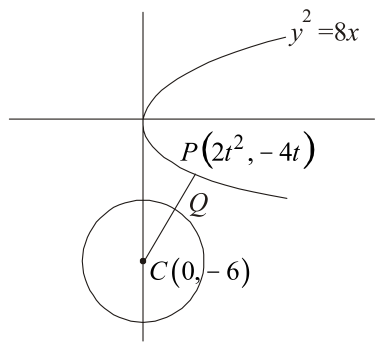
$\Rightarrow \quad -6 + 4t = -2t^3$

$\Rightarrow \quad 2t^3 + 4t - 6 = 0,\ t^3 + 2t - 3 = 0$

$\Rightarrow \quad (t-1)(t^2 + t + 3) = 0$

$\Rightarrow \quad t = 1$

point (2, – 4)



41. (b) Given $x + y - \ell n(x+y) = 2x + 5$

$\Rightarrow \quad 1 + \frac{dy}{dx} - \frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) = 2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y+1}{x+y-1}$

$\left.\frac{dt}{dx}\right|_{(\alpha,\beta)} = \frac{\alpha+\beta}{\alpha+\beta-1} = \infty$ when $\alpha + \beta = 1$.

42. (c) Let $g(x) = \int_0^{x^3} f(t)dt$

Now $\int_0^8 f(t)dt = g(2) = \frac{g(2)-g(1)}{2-1} + \frac{g(1)-g(0)}{1-0}$

$= g'(\alpha) + g'(\beta)$ where $1 < \alpha < 2, 0 < \beta < 1$

$= 3[\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3)]$.

43. (b) $f(x) = (x-\alpha)(x-\beta)(x-\gamma)$

$\Rightarrow \quad f'(x) = 3x^2 - 2x(\alpha+\beta+\gamma) + \alpha\beta + \beta\gamma + \gamma\alpha$

$\Rightarrow \quad \beta = \frac{2\alpha_1\beta_1}{\alpha_1 + \beta_1}$

(where α_1, β_1 are the roots of $f'(x) = 0$

$\Rightarrow \quad \beta = \frac{2(\alpha\beta + \beta\gamma + \gamma\alpha)}{2(\alpha+\beta+\gamma)}$

$\Rightarrow \quad \beta^2 = \gamma\alpha.$

44. (d) $\frac{dx}{d\theta} = a(1+\cos\theta)$ and $\frac{dy}{d\theta} = a\sin\theta$

$\therefore$ Slope of tangent is given by

$f(\theta) = \frac{dy}{dx} = \tan\frac{\theta}{2}$

$$\Rightarrow \frac{df(\theta)}{dt} = \frac{d\tan\frac{\theta}{2}}{d\theta} \times \frac{d\theta}{dt} = \frac{k}{2}\sec^2\frac{\theta}{2}$$

$$\Rightarrow \left(\frac{df(\theta)}{dt}\right)_{\text{at } \theta=\frac{\pi}{3}} = \frac{2k}{3}$$

B COMPREHENSION TYPE

1. (a) Let $f(x) = \tan^{-1}x$, then using Lagrange's MVT, for some

$$\alpha \in (x, y);\ f'(\alpha) = \frac{f(x)-f(y)}{x-y}$$

$$\Rightarrow \left|\frac{1}{1+\alpha^2}\right| = \left|\frac{\tan^{-1}x - \tan^{-1}y}{x-y}\right|$$

$$\because 1+\alpha^2 \geq 1 \Rightarrow \frac{1}{1+\alpha^2} \leq 1$$

$$\Rightarrow \text{So, } \left|\frac{\tan^{-1}x - \tan^{-1}y}{x-y}\right| \leq 1$$

Again if $f(x) = \sin x$, then as above

$$\left|\frac{\sin x - \sin y}{x-y}\right| = |\cos\alpha| \leq 1$$

Similarly, for $f(x) = \cos x$

2. (c) Let $f(x) = \sin x$ and $g(x) = \cos x$, then f and g are continuous and derivable. Also, $\sin x \neq 0$ for any $x \in \left(0, \frac{\pi}{2}\right)$ so by Cauchy's theorem,

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(\theta)}{g'(\theta)} \Rightarrow \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}$$

3. (b) $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b)

Let $g(x) = x^3$ then $g(x)$ is everywhere continuous and differentiable.

Now using Cauchy's theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \text{ for some } c \in (a, b)$$

$$\text{Thus } \frac{f(b)-f(a)}{b^3-a^3} = \frac{f'(c)}{3c^2} \quad (\because g'(x) = 3x^2)$$

ALTERNATE SOLUTION:

Let us define a function

$h(x) = f(x) - f(a) + k(x^3 - a^3)$, where k is selected in such a way that $h(b) = 0$. Also $h(a) = 0$. So, the function $h(x)$ satisfies all conditions of Rolle's theorem. There must, thus exists $c \in (a, b)$ such that $h'(c) = 0$

$$\Rightarrow f'(c) + k(3c^2) = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b^3-a^3}\ (3c^2)$$

$$\left[\text{where } k = -\frac{f(b)-f(a)}{b^3-a^3}\right]$$

4. (c) The equation of tangent at (h, k) is

$$y - k = -\frac{b^2h}{a^2k}(x-h).$$

$$\text{So, } p = \frac{a^2k^2+b^2h^2}{\sqrt{b^4h^2+a^4k^2}} = \frac{a^2b^2}{\sqrt{b^4h^2+a^4k^2}}$$

$$\left[\because \frac{h^2}{a^2} + \frac{k^2}{b^2} = 1 \qquad .....(1)\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{h^2}{a^4} + \frac{k^2}{b^4} = \frac{b^2(a^2b^2 - a^2k^2) + a^2(a^2b^2 - b^2h^2)}{a^4b^4}$$

$$= \frac{a^2b^2(a^2+b^2-h^2-k^2)}{a^4b^4}$$

$$= \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2} \quad (\because r^2 = h^2+k^2)$$

5. (c) $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

and $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta \Rightarrow \frac{dy}{d\theta} = -\tan\theta$

So, the equation of tangent at θ is

$$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$$

$$\therefore p = \frac{|a\sin^3\theta + a\sin\theta\cos^2\theta|}{\sqrt{1+\tan^2\theta}} = \left|\frac{a\sin\theta}{\sec\theta}\right| = |a\sin\theta\cos\theta|$$

....(1)

Also,

$r^2 = a^2(\cos^6\theta + \sin^6\theta) = a^2[1 - 3\cos^2\theta\sin^2\theta]$ ...(2)

From (1) and (2), $r^2 = a^2 - 3p^2$.

6. **(d)** The equation is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{c^2}$.

Let $x = c\sec\theta$, $y = c\,\mathrm{cosec}\,\theta$.

[Note that it is always easier to solve such questions with the help of parametric forms, as only one unknown remains for elimination. You can solve question (1) using parametric forms. Try!]

So, $\frac{dy}{dx} = \frac{c\cot\theta\cos\theta}{c\sec\theta\tan\theta} = -\cot^3\theta$

Equation of tangeut at θ is

$y - c\,\mathrm{cosec}\,\theta = -\cot^3\theta\,(x - c\sec\theta)$

$$\therefore p = \frac{c\,\mathrm{cos}ec\,\theta + c\sec\theta\cot^3\theta}{\sqrt{1+\cot^6\theta}} = \frac{c}{\sqrt{\cos^6\theta + \sin^6\theta}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{c^2}\left[\cos^6\theta + \sin^6\theta\right] = \frac{1}{c}\left[1 - 3\cos^2\theta\sin^2\theta\right]$$

$$= \frac{1}{c^2} - \frac{3}{r^2}$$

$$\therefore \frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$$

$$\left[\because \quad r^2 = c^2\sec^2\theta + c^2\,\mathrm{cos}ec^2\theta = \frac{c^2}{\sin^2\theta\cos^2\theta}\right]$$

7. **(a)** If $f(x)$ touches x-axis at only one irrational point, then

$f(x) = (x-\alpha)^2 g(x)$, where α is irrational.

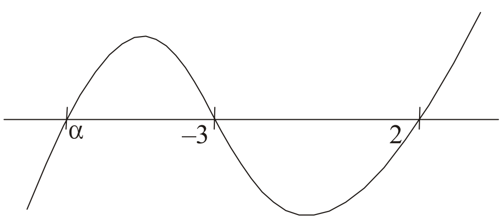
$\Rightarrow$ coefficients of $f(x)$ can't be rational

$\Rightarrow$ for $f(x)$ with rational coefficients, the point of tangency is rational.

8. **(b)** A possible graph of $y = f(x)$ is shown.

The third root α is lesser than – 3.

$\Rightarrow$ $c < -18$.



9. **(a)** $f(x) = (x-\alpha)^3(x-\beta)^3$

$\Rightarrow$ $f'(x) = (x-\alpha)^2(x-\beta)^2(2x - (\alpha+\beta))$

$\Rightarrow$ $f''(x)$ has roots α, β and a root in each of

$\left(\alpha, \frac{\alpha+\beta}{2}\right)$ and $\left(\frac{\alpha+\beta}{2}, \beta\right)$.

10. **(c)** $\left.\frac{dy}{dx}\right|_{(1,2)} = 3x^2 = 3$

$\therefore$ Equation of tangent is $y - 2 = 3\,(x - 1)$

$\Rightarrow$ $y = 3x - 1$.

11. **(a)** Solving $y = 3x - 1$ and $y = x^3 + 1$

$\Rightarrow$ $3x - 1 = x^3 + 1$ $\Rightarrow$ $x = 1, 1, -2$

The equal roots (1, 1) corresponding to (1, 2).

So the tangent meet the curve again at

$(-2, (-2)^3 + 1)$ or $(-2, -7)$.

12. **(c)** Equation of tangent at t

$y - (8t^3 - 1) = 3t[x - (4t^2 + 3)]$

If this meet the curve again at t' then

$(8t'^3 - 1) - (8t^3 - 1) = 3t[4t'^2 + 3 - (4t^2 + 3)]$

we will get $t' = -\frac{t}{2}$ ($t' = t$ will be rejected)

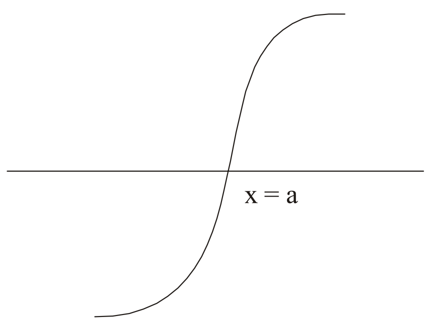
Normal at t' is $-\frac{1}{3t'} = \frac{2}{3t}$

But slope of tangent at $t = 3t$

since they are the same line $\Rightarrow$ $3t = \frac{2}{3t}$

$\Rightarrow$ $t = \pm\frac{\sqrt{2}}{3}$.

13. (b) According to graph $f(a+h)\,f''(a-h) > 0$.



14. (b) According to above graph if $f(\alpha) = 0$ and $f(\beta) = 0$

$\Rightarrow f''(a) = 0$ and $f''(b) = 0$ (these are points of inflection)

$\Rightarrow f''(\gamma) = 0$

$\alpha \le \gamma < \beta$ (using Rolle's theorem).

15. (b) $f'(x) \ne 0 \Rightarrow f(x) = 0$ has atmost one real.

$\Rightarrow f'(x) = 0$ has atmost one solution.

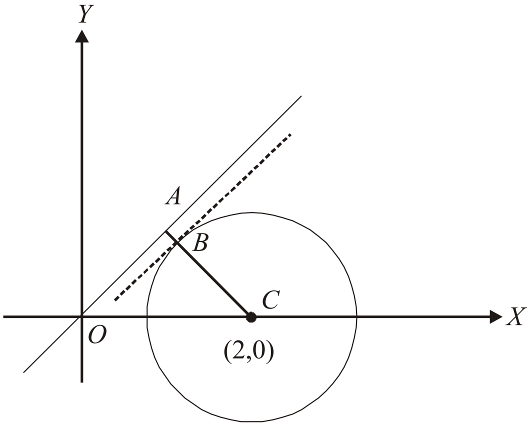
C REASONING TYPE

1. **(a)** $y = 4 + \sin^2 x \geq 4$ and absolute minima occurs at $x = 0$

So, the tangent to the curve at $x = 0$ must be parallel to x-axis.

(***NOTE*** that the function is differentiable everywhere)

2. **(a)**



The centre of the circle is $C(2, 0)$. If $A(a, a)$ be the point on $y = x$, which is closest to the circle then AC must be perpendicular to $y = x$

$$\Rightarrow \frac{a}{a-2} \times 1 = -1 \Rightarrow a = 1$$

So, the point A is $(1, 1)$ and the minimum distance $= AC - BC = \sqrt{2} - 1$

3 **(a)** $\sec^2(x+y).\left[1+\frac{dy}{dx}\right] = e^{x+y}\left[1+\frac{dy}{dx}\right]$

$$\Rightarrow \left[1+\frac{dy}{dx}\right][\sec^2(x+y) - e^{x+y}] = 0$$

$$\Rightarrow \frac{dy}{dx} = -1$$

Further, $\tan(x+y) = e^{x+y}$

$$\Rightarrow (x+y) = \ln \tan(x+y)$$

$$\Rightarrow u = \ln \tan u, \text{ where } u = x+y$$

If the above equation has roots say $u = \alpha, \beta, \gamma$....... then the family of curves represented by the given implicit equation are

$x + y = \alpha,\ x + y = \beta,\ x + y = \gamma$,...............

D MULTIPLE CORRECT CHOICE TYPE

1. **(a, c)**

The given curve is $y = be^{x/a}$

Let us consider a point (x_1, y_1) on the curve.

Then, $y_1 = be^{x_1/a}$ ......(i)

Differentiating the curve $y = be^{x/a}$ with respect to x we get

$$\frac{dy}{dx} = be^{x/a}.\frac{1}{a}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{b}{a}e^{x_1/a}$$

Thus, the length of subtangent $= \left|y_1\left(\frac{dx}{dy}\right)_{(x_1,y_1)}\right|$

$$= \left|y_1.\frac{a}{be^{x_1/a}}\right| = \left|be^{x_1/a}.\frac{a}{be^{x_1/a}}\right| = |a| \text{ (constant)}$$

[using (i) and (ii)]

$\Rightarrow$ Subtangent is of constant length $|a|$

Again, length of subnormal $= \left|y_1\left(\frac{dy}{dx}\right)_{(x_1,y_1)}\right|$

$$= \left|be^{x_1/a}.\frac{be^{x_1/a}}{a}\right|$$

$$= \frac{1}{|a|}(be^{x_1/a})^2 = \frac{1}{|a|}y^2 \quad \text{[using (i)]}$$

Therefore, subnormal varies as the square of ordinate.

2. **(b, c)**

Let $f(t) = \int_0^t (1+\cos^8 x)(ax^2 + bx + c)dx$

then $f(1) = 0$ and $f(2) = 0$ (Given)

Also $f(0) = 0$

Thus by Rolle's Mean value theorem $f'(t) = 0$ has at least one root in (0, 1) and another in (1, 2)

Now $f'(t) = (1+\cos^8 t)(at^2 + bt + c)$

But $1 + \cos^8 t \neq 0$

$$\therefore f'(t) = 0 \Rightarrow at^2 + bt + c = 0$$

Thus $ax^2 + bx + c = 0$ has a root in (0, 1) and other in (1, 2)

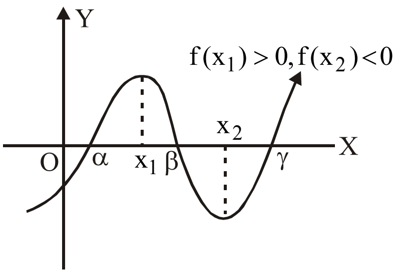
3. **(a,d)**

Let $f(x) = x^3 - 3x + a \Rightarrow f'(x) = 3x^2 - 3$

$= 3(x-1)(x+1)$

Now, $f(1) = a-2, f(-1) = a+2$

All the roots would be real and distinct if,



$f(1)f(-1) < 0 \Rightarrow (a-2)(a+2) < 0 \quad \Rightarrow -2 < a < 2$

Thus the given equation would have real and distinet roots if $a \in (-2, 2)$

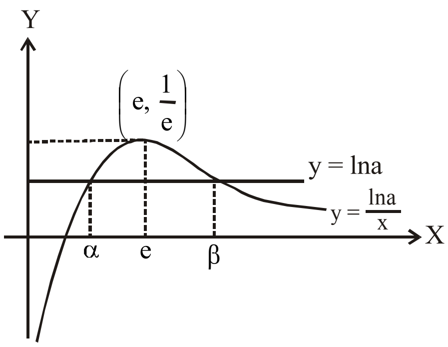
4. **(b, c)**

The given equations can be rewritten as

$\frac{\ln\alpha}{\alpha} = \frac{\ln\beta}{\beta} = \ln a$. So the graph of $y = \frac{\ln x}{x}$ and $y = \ln a$ must intersect at two distinct points. From the adjacent graph we get

$0 < \ln a < \frac{1}{e} \Rightarrow 1 < a < e^{1/e}$

$[\because \sqrt[3]{e} < e^{1/e}$, so, $a \in (1, \sqrt[3]{e})]$



5. **(b, d)**

We have the given curve, $y = \int_0^x 2|t|\,dt,$

Differentiating the curve with respect to x we get

$\frac{dy}{dx} = 2|x|$ [using Leibnitz-rule]

Since, the tangent lines are parallel to the bisector of the first coordinate angle we have,

$\frac{dy}{dx} = 1 \Rightarrow |x| = \frac{1}{2} \Rightarrow x = \pm\frac{1}{2}$

$\Rightarrow y = 2\int_0^{1/2} t\,dt = 2\left(\frac{t^2}{2}\right)_0^{1/2} = \frac{1}{4}$

Thus the points are $\left(\pm\frac{1}{2}, \frac{1}{4}\right) \Rightarrow$ Equation of tangents are $y - \frac{1}{4} = 1\left(x \pm \frac{1}{2}\right)$

There for, $y = x + \frac{3}{4}$ and $y = x - \frac{1}{4}$ are required equation of tangents.

6. **(a, d)**

Slope of tangent $= \frac{\frac{-1}{a}}{\frac{1}{b}} = -\frac{b}{a}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-4}{t^2}}{4} = -\frac{1}{t^2} < 0.$$

So, $\frac{-b}{a} < 0 \Rightarrow \frac{b}{a} > 0.$

7. **(a,b,d)**

Slope of the given line $= \frac{a-3}{a} > 0.$

$\Rightarrow (a-3)a > 0 \Rightarrow a < 0$ or $a > 3$

E MATRIX-MATCH TYPE

1. **A - q; B - r; C - p; D - s**

(A) Point of intersection $(0, b)$, $\frac{dy}{dx} = be^{-\frac{x}{a}}\left(-\frac{1}{a}\right)$;

$m = \left(\frac{dy}{dx}\right)_{(0,\, b)} = -\frac{b}{a}$

$\Rightarrow$ Slope of normal $= \frac{a}{b}$

(B) $\frac{dy}{dx} = -\frac{y}{x}$,

$$\text{Subnormal} = \left|y\frac{dy}{dx}\right| = \left|y.\left(\frac{-y}{x}\right)\right| = \left|\frac{y^2}{x}\right| = \left|\frac{y^2}{\frac{a^2b^2}{y}}\right| = \frac{|y^3|}{a^2b^2}$$

(C) $m = \frac{dy}{dx} = \frac{xb^2}{ya^2}$;

$$\text{Length of subtangent} = \left|\frac{y}{\frac{dy}{dx}}\right| = \left|\frac{y}{\frac{xb^2}{ya^2}}\right| = \frac{y^2\, a^2}{|x|\, b^2}$$

(D) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$

2. A - q; B - s; C - s; D - q

(A) $\frac{d}{dx}(1-\cos x) = \frac{d}{dx}(\frac{\sqrt{3}}{2}x + a)$ assuming $x \geq 0$

$\Rightarrow \sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3}$ and $\frac{2\pi}{3}$

Due to symmetry $x = -\frac{\pi}{3}$ and $-\frac{2\pi}{3}$ can also be the points and so the points

$(\pm\frac{\pi}{3}, \frac{1}{2})$ and $(\pm\frac{2\pi}{3}, \frac{3}{2})$ satisfy

$y = \frac{\sqrt{3}}{2}|x| + a$ giving $a = \frac{1}{2} - \frac{\pi}{2\sqrt{3}}$ or $\frac{3}{2} - \frac{\pi}{\sqrt{3}}$

(B) Slope of the curves at the point of contact are

$\frac{2a}{y}$ and $\frac{x}{2a}$ Thus $\frac{2a}{y} = \frac{x}{2a} \Rightarrow xy = 4a^2$

(C) The least distance will occur along the common nornal. Normal to parabola at $(t^2, 2t)$ is $y = -tx + 2t + t^3$ which is normal to circle if $12 = 2t + t^3 \Rightarrow t = 2$ Hence the point is (4,4)

(D) Differentiating we get

$3y^2\frac{dy}{dx} + 6x = 12\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{2x}{y^2 - 4}$

$\Rightarrow y^2 - 4 = 0$ or $y = \pm 2$

If $y = 2$, then $x = \pm\frac{4}{\sqrt{3}}$ if $y = -2$, then $x \notin R$. so the points are $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

3. A - t; B - r; C - p; D - r, s, t

(A) Equation of tangent to $xy = -1$ at $\left(t, -\frac{1}{t}\right)$ is

$y = \frac{1}{t^2}x - \frac{2}{t}$ and any tangent to $y^2 = 8x$ is

$y = mx + \frac{2}{m}$.

$\therefore m = \frac{1}{t^2}$ and $-\frac{2}{t} = \frac{2}{m} \Rightarrow t = -1$

So, the common tangent is $y = x + 2$.

(B) Let $\phi(x) = [f(x) + f'(x) + f''(x)]\, e^{-x}$. Clearly $\phi(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Also,

$$\frac{\phi(a)}{\phi(b)} = \left\{\frac{f(a) + f'(a) + f''(a)}{f(b) + f'(b) + f''(b)}\right\} e^{b-a} = e^{a-b}.e^{b-a} = 1$$

[From the given condition]

$\therefore\ \phi(a) = \phi(b)$

Hence by Rolle's theorem, these exists $c \in (a, b)$ such that

$\phi'(c) = 0 \Rightarrow -\{f(c) + f'(c) + f''(c)\}e^{-c} + e^{-c}\{f'(c) + f''(c) + f'''(c)\} = 0 \Rightarrow \{f'''(c) - f(c)\}\, e^{-c} = 0$

$\Rightarrow f'''(c) = f(c)$

(C) $f(-2) = f(-1) = f(1) = f(2) = 0$

Therefore by Rolle's mean value theorem the equation $f'(x) = 0$ has roots in $(-2, -1)$, $(-1, 1)$ and $(1, 2)$.

$\therefore [\alpha] = -2, [\beta] = -1, [\gamma] = 1$.

(D) $\frac{dy}{dx} = \frac{1}{2y}$, so the normal at the point (t^2, t) is

$y - t = -2t\,(x - t^2)$. It passes through $(c, 0)$ if $-t = -2t\,(c - t^2)$

$t^2 = c - \frac{1}{2}$ for three distinct values of

$t^2 > 0 \Rightarrow c > \frac{1}{2}$.

4. A - q; B - q; C - q; D - r

(A) $y = \frac{2}{x}$

$\Rightarrow \frac{dy}{dx} = -\frac{2}{x^2} < 0$, for all non zero real number x.

Thus angle of any tangent with x-axis is obtuse.

(B) $\sin y = \ln x(\sin x)$, $x = \frac{\pi}{2}, y = 0$

$$\cos y \frac{dy}{dx} = \frac{1}{\sin x}.\cos x = \cot x$$

$$\frac{dy}{dx} = 0.$$

(C) $$\frac{dy/dt}{dx/dt} = \frac{e^t[-\sin t] + \cos t\, e^t}{e^t \cos t + e^t \sin t} = \frac{(\cos t - \sin t)}{(\cos t + \sin t)} = 0$$

(D) $$\frac{dy}{dx} = 2x, \frac{6dy}{dx} = -3x^2$$

At $x = 1$

$$m_1 = \frac{dy}{dx} = 2,\ m_2 = -\frac{1}{2}$$

$$m_1 m_2 = -1 \Rightarrow \theta = \frac{\pi}{2}$$

F NUMERIC/INTEGER ANSWER TYPE

1. **Ans : 1**

Given curve is $x^3 + y^3 = c^3$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2}{y^2} \Rightarrow \left.\frac{dy}{dx}\right|_{(a,b)} = -\frac{a^2}{b^2} \qquad \text{..... (1)}$$

and tangent at (a,b) cuts the curve again at (a_1, b_1)

$$\therefore \quad \text{Slope of tangent} = \frac{b_1 - b}{a_1 - a} = -\frac{a^2}{b^2} \ \{\text{From (1)}\} \text{......(2)}$$

Also $a^3 + b^3 = c^3$ .....(3)

and $a_1^3 + b_1^3 = c^3$ .....(4)

Subtracting (4) from (3), we get $(a^3 - a_1^3) + (b^3 - b_1^3) = 0$

$$\Rightarrow \quad (a - a_1)(a^2 + aa_1 + a_1^2) + (b - b_1)\ (\mathrm{b}^2 + \mathrm{bb}_1 + \mathrm{b}_1^2) = 0$$

$$\Rightarrow \quad \frac{b_1 - b}{a_1 - a} = -\frac{a^2 + aa_1 + a_1^2}{b^2 + bb_1 + b_1^2} \qquad \text{......(5)}$$

From (2) and (5), $$-\frac{a^2}{b^2} = -\frac{a^2 + aa_1 + a_1^2}{b^2 + bb_1 + b_1^2}$$

$$\Rightarrow \quad a^2b^2 + a^2bb_1 + a^2b_1^2 = a^2b^2 + ab^2a_1 + a_1^2b^2$$

$$\Rightarrow \quad ab(ab_1 - ba_1) + a^2b_1^2 - a_1^2b^2 = 0$$

$$\Rightarrow \quad ab(ab_1 - ba_1) + (ab_1 + a_1b)(ab_1 - a_1b) = 0$$

$$\Rightarrow \quad (ab_1 - a_1b)(ab + ab_1 + a_1b) = 0$$

If $ab_1 - a_1b = 0$ then $$\frac{a}{a_1} = \frac{b}{b_1} = \frac{\sqrt[3]{a^3 + b^3}}{\sqrt[3]{a_1^3 + b_1^3}} = \frac{c}{c} = 1$$

(Law of proportion)

$\therefore \quad a = a_1$ and $b = b_1$ which is impossible.

$\therefore \quad ab_1 - a_1b \neq 0$

Hence, $ab + ab_1 + a_1b = 0$ or $\frac{a_1}{a} + \frac{b_1}{b} = -1$

2. **Ans : 1**

Equation of given curves are

$$ax^2 + by^2 = 1 \qquad \text{.....(i)}$$

and $$a_1x^2 + b_1y^2 = 1 \qquad \text{.....(ii)}$$

From (i) $2ax + 2by\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{a}{b}\frac{x}{y} \qquad \text{.....(iii)}$$

From (ii) $2a_1x + 2b_1y\frac{dy}{dx} = 0,$

$$\therefore \frac{dy}{dx} = -\frac{a_1}{b_1}\frac{x}{y} \qquad \text{......(iv)}$$

Curve (i) and (ii) will cut each other at right angles if the product of the values of $\frac{dy}{dx}$ for the two curves is –1

i.e., $$\left(-\frac{a}{b}\frac{x}{y}\right)\left(-\frac{a_1x}{b_1y}\right) = -1 \text{ or } \frac{aa_1}{bb_1}\frac{x^2}{y^2} = -1 \qquad \text{......(v)}$$

from (i) and (ii), $ax^2 + by^2 = a_1x^2 + b_1y^2$

or $(a - a_1)x^2 = (b_1 - b)y^2$

$$\therefore \quad \frac{x^2}{y^2} = \frac{b_1 - b}{a - a_1} \qquad \text{......(vi)}$$

Putting the value of $\frac{x^2}{y^2}$ in (v), we get $\frac{aa_1}{bb_1}\left(\frac{b_1 - b}{a - a_1}\right) = -1$

or $$\frac{a - a_1}{aa_1} = -\frac{b_1 - b}{bb_1} = \frac{b - b_1}{bb_1} \text{ or } \frac{1}{a_1} - \frac{1}{a} = \frac{1}{b_1} - \frac{1}{b}$$

3. **Ans : 32**

Given curves are $y=|x^2-1|$ ....(1)

$y=|x^2-3|$ ....(2)

$$y=\begin{cases}x^2-1, & x\le -1 \text{ or } \ge 1\\ 1-x^2, & -1\le x\le 1\end{cases} \quad ....(3)$$

and $$y=\begin{cases}x^2-3, & x\le -\sqrt{3} \text{ or } x\ge \sqrt{3}\\ 3-x^2, & -\sqrt{3}\le x\le \sqrt{3}\end{cases} \quad .....(4)$$

Equating the two values of y from (1) and (2) we get

$|x^2-1|=|x^2-3|$

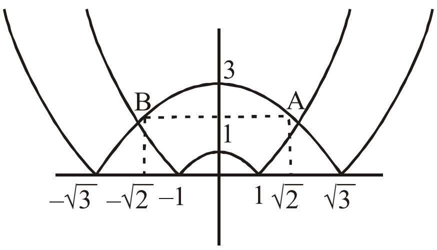
or $x^2-1=\pm(x^2-3)\Rightarrow x=\pm\sqrt{2}$

From (1), when $x=\pm\sqrt{2}, y=1$

Let $A\equiv(\sqrt{2},1)$ and $B\equiv(-\sqrt{2},1)$

Here A and B are the points of intersection of curves (1) and (2)

Angle of intersection between curves (1) and (2) at $A(\sqrt{2},1)$:



From (3), $\left(\frac{dy}{dx}\right)_{at(\sqrt{2},1)}=(2x-0)=2\sqrt{2}=m_1$ (say)

From(4), $\left(\frac{dy}{dx}\right)_{at(\sqrt{2},1)}=(-2x)=-2\sqrt{2}=m_2$ (say)

Let θ be the acute angle between curves (1) and (2) at A, then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{2\sqrt{2}+2\sqrt{2}}{1-8}\right|=\frac{4\sqrt{2}}{7}$$

$\Rightarrow m=4\sqrt{2}$

4. **Ans : 2**

$y^2=x^3$ ......(1)

$\Rightarrow 2y\times\frac{dy}{dx}=3x^2$

Slope of tangent at $P=\left.\frac{dy}{dx}\right|_P=\left.\frac{3x^2}{2y}\right|_{(4m^2,8m^3)}=3m$

$\therefore$ Equation of tangent at $P: y-8m^3=3m(x-4m^2)$

$y=3mx-4m^3$ ......(2)

It cuts curve again at point Q, solve (1) and (2) we get $x=4m^2, m^2$

Put $x=m^2$ in equation (2)

$\Rightarrow y=3m(m^2)-4m^3=-m^3$

$\therefore Q$ is $(m^2,-m^3)$

Slope of tangent at $Q=\left.\frac{dy}{dx}\right|_{(m^2,-m^3)}=\frac{3(m^4)}{2\times(-m^3)}=\frac{-3}{2}m$

Slope of normal at $Q=\frac{1}{(-3/2)m}=\frac{2}{3m}$. Since tangent at P is normal at Q

$\therefore \frac{2}{3m}=3m\Rightarrow 9m^2=2$

5. **Ans : 4**

(a) We know that, $1\le|\sin x|+|\cos x|\le\sqrt{2}$, for all real values of x

[Note that $(|\sin x|+|\cos x|)^2 1+|\sin 2x|\ge 1$

$\therefore\ y=[|\sin x|+|\cos x|]=1$

Let P and Q be the points of intersection of given curves

Clearly the given curves meet at points where $y=1$ so, we get

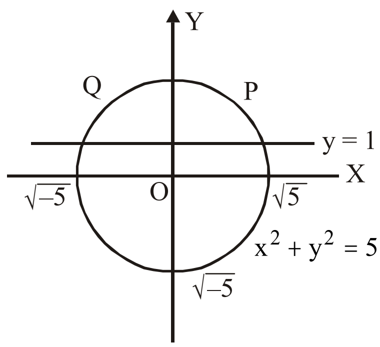
$x^2+1=5, x=\pm 2$

Now, $P(2,1)$ and $Q(-2,1)$

Now, $x^2+y^2=5$

Differentiating the above equation with respect to x we get

$2x+2y\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{x}{y}\ \therefore\left(\frac{dy}{dx}\right)_{(2,1)}=-2$



$\left(\frac{dy}{dx}\right)_{(-2,1)}=2$

Clearly the slope of line $y=1$ is zero and the slope of the tangents at P and Q are (-2) and (2) respectively. Thus, the angle of intersection is $\tan^{-1}(2)$.

6. Ans : 1

(c) Solving the two equations, we get

$x^2 y = xy \Rightarrow xy(x-1) = 0$

$\Rightarrow x = 0, y = 0, x = 1$

$\because\ y \neq 0$, So, points of intersection ot two curves are

$(0, 1)$ and $\left(1, \frac{1}{2}\right)$

Now, $x^2 y = 1 - y \Rightarrow x^2 \frac{dy}{dx} + 2xy = -\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = -\frac{2xy}{x^2+1}$

$\therefore \left(\frac{dy}{dx}\right)_{(0,1)} = 0$ and $\left(\frac{dy}{dx}\right)_{\left(1,\frac{1}{2}\right)} = -\frac{1}{2}$

The equations of the tangents are

$y - 1 = 0(x-0)$ and $y - \frac{1}{2} = -\frac{1}{2}(x-1)$

$\Rightarrow y = 1$ and $x + 2y - 2 = 0$

Clearly, these intersect at (0, 1)

7. Ans : 64

Let $x = 64\cos^3 t$

$y = 64\sin^3 t$

$\frac{dx}{dt} = -192\cos^2 t(-\sin t)$

$\frac{dy}{dt} = -192\sin^2 t\cos t$

$\frac{dy}{dx} = -\frac{\sin t}{\cos t}$

Equation of tangent at t is

$y - 64\sin^3 t = -\frac{\sin t}{\cos t}(x - 64\cos^3 t)$

$\Rightarrow \frac{y}{\sin t} - 64\sin^2 t = -\frac{x}{\cos t} + 64\cos^2 t$

$\Rightarrow \frac{x}{64\cos t} + \frac{y}{64\sin t} = 1$

$\Rightarrow$ segment of tangent between x and y axes

$= \sqrt{(64)^2\cos^2 t + 64^2\sin^2 t} = 64$

Length = 64.

8. Ans : 20

$\frac{dy}{dx} = x^2$

$\Rightarrow$ Equation of tangent at $P_1(x_1, y_1)$ is

$y - y_1$

$= x_1^2(x_2 - x_1)$

i.e. $x_2^3 - x_1^3 = 3x_1^2(x_2 - x_1^2)$

$\Rightarrow x_2^2 + x_1^2 + x_1x_2 = 3x_1^2$

$\Rightarrow 2x_1^2 - x_1x_2 - x_2^2 = 0$

$\Rightarrow x_2 = -2x_1$

Similarly $x_3 = -2x_2 = 4x_1$

$x_4 = -2x_3 = -8x_1$

....

$x_{2n} = -2^{2n-1}x_1$

$3(y_1 + y_2 + y_3 .... + y_{2n})$

$= (x_1^3 + x_2^3 + x_3^3 + ... + x_{2n}{}^3) + 2n$

$= x_1^3\frac{((-8)^{2n}-1)}{-8-1} + 2 = -\frac{x_1^3}{9}(8^{2n}-1) + 2n$

i.e. $y_1 + y_2 + y_3 + ... + y_{2n} = -\left(\frac{x_1}{3}\right)^3(2^{6n}-1) + \left(\frac{2n}{3}\right),$

Now put $n = 30$ and $x_1 = 2$ then

$y_1 + y_2 + y_3 + ... + y_{60} = -\left(\frac{2}{3}\right)^3\left(2^{180}-1\right) + 20$

$\Rightarrow S = -\frac{2^{183}-8}{27} + 20$ or $S + \left(\frac{2^{183}-8}{27}\right) = 20$

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