CHAPTER 10

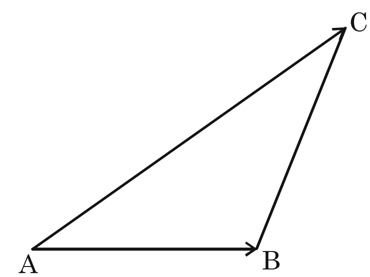
Vector Algebra

Algebra of Vectors

- A vector quantity has both magnitude and direction where the magnitude is a distance between the initial and terminal point of the vector. Let's assume a vector starts at a point A and ends at a point B. Therefore the magnitude of the vector is denoted by $|\overrightarrow{AB}|$.
- $\overrightarrow{OA} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of any point $A(x, y, z)$ having a magnitude equal to $\sqrt{x^2 + y^2 + z^2}$. Where O is the origin (0, 0, 0) and P is any point in the space.
- The angles α, β, γ are known as the direction angles which are made by the position vector and the positive x, y, z – axes respectively and their cosine values ($\cos\alpha$, $\cos\beta$, $\cos\gamma$) are known as direction cosines, denoted by l, m, n respectively.
- The projections of a vector along the respective axes are represented by the direction ratios which are the scalar components of the vector. They are denoted by a, b, c respectively.
- The direction cosines, direction ratios, and magnitude of a vector are related as:

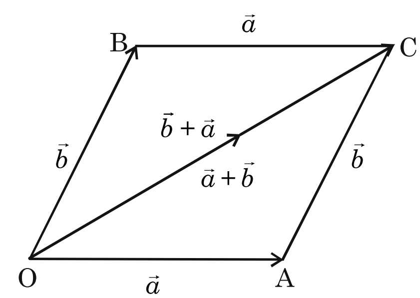
$$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$$

- In general, $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$.
- Zero vector (also known as a null vector) is symbolized by $\vec{0}$. Its initial and terminal points coincide.
- A unit vector has a magnitude equal to 1 and is denoted by $\hat{a}$.
- If two or more than two vectors have the same initial points, they are called as co-initial vectors.
- The vectors which are parallel to the same line are known as collinear vectors.
- The vectors having equal magnitude and same direction are called equal vectors.
- A vector having the same magnitude as the given vector but opposite direction is known as the negative of the given vector.
- **Triangle law of vector addition:** Let's say that A, B, and C are the vertices of a triangle then



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

- **Parallelogram law of vector addition:** If two vectors are represented by the two adjacent sides of a parallelogram, then their sum is represented by the diagonal of that parallelogram through their common point. For example, the



$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$

$\Rightarrow \overrightarrow{OC} = \vec{a} + \vec{b}$

- Vector addition is commutative as well as associative in nature and also has zero vector as an additive identity.
- The multiplication of any vector $\vec{a}$ by a scalar λ is denoted by $\lambda\vec{a}$ and has the same direction as the original vector if λ is positive and opposite direction if λ is negative. Its magnitude is $|\lambda\vec{a}| = |\lambda||\vec{a}|$.

- The unit vector of any vector $\vec{a}$ in its direction is written as $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$
- The unit vectors along the positive x, y, z axes are denoted by $\hat{i}, \hat{j}, \hat{k}$ respectively.
- **Component form of a vector:** The component form of any vector is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ where x, y, z are called as the scalar components and $x\hat{i}, y\hat{j}, z\hat{k}$ as the vector components of $\vec{r}$. x, y, z is also called the rectangular components.
- If two vectors are in their component form as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then
  - The sum of the vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$.
  - The difference of the vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$
  - The vectors $\vec{a}$ and $\vec{b}$ are equal if $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$.
  - The multiplication of a vector $\vec{a}$ by scalar λ is given by $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$.
- **Vector joining two points:** The magnitude of a vector $\overrightarrow{A_1A_2}$ joining two points $\overrightarrow{A_1}(x_1, y_1, z_1)$ and $\overrightarrow{A_2}(x_2, y_2, z_2)$ is
  $\overrightarrow{A_1A_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- **Section Formula:** The position vector of a point C dividing the line segment joining two points A and B (having position vectors $\vec{a}, \vec{b}$ respectively) in the ratio of $m : n$
  - Internally: $\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$
  - Externally: $\vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$
  - If C is the midpoint of A and B, then $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$

Product of Two Vectors, Scalar Triple Product

- **Scalar (or dot) product of two vectors $\vec{a}$ and $\vec{b}$:** The scalar product of two vectors having an angle θ between them is denoted by $\vec{a} \cdot \vec{b}$ and is defined by
  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$
  $\Rightarrow \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$
  or $\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)$
- If $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$
- Properties of scalar product:
  - Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors then $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
  - Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then $(\lambda\vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\lambda\vec{b})$
- Projection of a vector $\vec{a}$ on another vector $\vec{b}$ is given as $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right)$.
- Projection of a vector $\vec{b}$ on another vector $\vec{a}$ is given as $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right)$.
- **Vector (or cross) product of two vectors $\vec{a}$ and $\vec{b}$:** The vector product of two vectors having an angle θ between them is denoted by $\vec{a} \times \vec{b}$ and is defined by:
  $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$
  $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$
- If $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$
- If the vectors are in their component form as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then their cross product is given by

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

The dot product is given by

$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$

- Properties of vector product:
  - Let $\vec{a},\vec{b}$ and $\vec{c}$ be any three vectors then $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.
  - Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then $\left(\lambda\vec{a}\right)\times\vec{b}=\lambda\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\left(\lambda\vec{b}\right)$
- Scalar triple product of three vectors [*A*, *B*, *C*] is given by $\vec{A}.\left(\vec{B}\times\vec{C}\right)$ which is the volume of a parallelepiped whose sides are given by vectors $\vec{A},\vec{B}$ and $\vec{C}$.

Exercise

1. The value of m for which the vectors $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=-2\hat{i}+m\hat{j}$ are collinear, is

(*a*) $\frac{1}{2}$ (*b*) 2

(*c*) 3 (*d*) –3

2. If the position vectors of A and B are $\left(3\hat{i}-8\hat{j}\right)$ and $\left(-6\hat{i}+4\hat{j}\right)$, then a unit vector in the direction of $\overline{AB}$ is

(*a*) $-\frac{1}{5}\left(3\hat{i}+4\hat{j}\right)$ (*b*) $\frac{1}{5}\left(-3\hat{i}+4\hat{j}\right)$

(*c*) $\frac{1}{15}\left(3\hat{i}-4\hat{j}\right)$ (*d*) None of these

3. The unit vector parallel to the resultant of the vectors $\left(2\hat{i}+4\hat{j}-5\hat{k}\right)$ and $\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ is.

(*a*) $\frac{\left(\hat{i}+2\hat{j}-8\hat{k}\right)}{49}$ (*b*) $\frac{\left(3\hat{i}+6\hat{j}-2\hat{k}\right)}{49}$

(*c*) $\frac{\left(\hat{i}+2\hat{j}-8\hat{k}\right)}{7}$ (*d*) $\frac{\left(3\hat{i}+6\hat{j}-2\hat{k}\right)}{7}$

4. If θ is the angle between the vectors $\vec{a}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ and $\vec{b}=\left(-3\hat{i}+2\hat{j}+\hat{k}\right)$, then cos θ = ?

(*a*) $\frac{2}{5}$ (*b*) $\frac{3}{7}$

(*c*) $\frac{2}{7}$ (*d*) $\frac{3}{5}$

5. If $\vec{a}=\left(\hat{i}+\hat{j}\right),\vec{b}=\left(\hat{j}+\hat{k}\right)$ and $\vec{c}=\left(\hat{i}+\hat{k}\right)$, then a unit vector in the direction of $\left(\vec{a}-2\vec{b}+3\vec{c}\right)$ is

(*a*) $\frac{1}{3\sqrt{2}}\left(4\hat{i}+\hat{j}-\hat{k}\right)$ (*b*) $\frac{1}{3\sqrt{2}}\left(4\hat{i}-\hat{j}+\hat{k}\right)$

(*c*) $\frac{1}{\sqrt{2}}\left(4\hat{i}-\hat{j}+\hat{k}\right)$ (*d*) None of these

6. If $\left(2\hat{i}-3\hat{j}+\hat{k}\right)$ and $\left(\hat{i}-\hat{j}-\hat{k}\right)$ are the position vectors of the points A and B respectively, then the unit vector along $\overline{AB}$ is

(*a*) $\frac{\left(\hat{i}-2\hat{j}+2\hat{k}\right)}{3}$ (*b*) $\frac{\left(-\hat{i}+2\hat{j}-2\hat{k}\right)}{3}$

(*c*) $\frac{\left(-\hat{i}+2\hat{j}-2\hat{k}\right)}{9}$ (*d*) None of these

7. If $\vec{a}=4\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=4\hat{i}-2\hat{j}+\hat{k}$, then find a unit vector parallel to the vector $\vec{a}+\vec{b}$.

(*a*) $\frac{6\hat{i}-3\hat{j}+2\hat{k}}{7}$ (*b*) $\frac{8\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{77}}$

(*c*) $\frac{8\hat{i}-3\hat{j}+2\hat{k}}{9}$ (*d*) None of these

8. Find the sum of three vectors $\vec{a}=\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}-6\hat{j}+3\hat{k}$ is

(*a*) $2\hat{i}+5\hat{j}-8\hat{k}$ (*b*) $5\hat{i}-8\hat{j}-6\hat{k}$

(*c*) $6\hat{i}-8\hat{j}-6\hat{k}$ (*d*) None of these

9. If a line has direction ratios 3, –2, –4 then what are its direction?

(*a*) $\frac{1}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{-2}{\sqrt{29}}$ (*b*) $\frac{3}{\sqrt{29}}, \frac{-2}{\sqrt{29}}, \frac{-4}{\sqrt{29}}$

(*c*) $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$ (*d*) None of these

10. What is the positive vector of the mid-point of vector joining the points P(3, 8, 4) and Q(5, 2, –2) ?

(*a*) $4\hat{i}+5\hat{j}+\hat{k}$ (*b*) $3\hat{i}+5\hat{j}+\hat{k}$

(*c*) $2\hat{i}+\hat{j}+5\hat{k}$ (*d*) None of these

11. The angle between the vectors $\left(\hat{i}+3\hat{j}+2\hat{k}\right)$ and $\left(2\hat{i}-4\hat{j}-\hat{k}\right)$ is

(*a*) $\sin^{-1}\left(\frac{5}{7}\right)$ (*b*) $\sin^{-1}\left(\frac{6}{7}\right)$

(*c*) $\sin^{-1}\left(\frac{3}{5}\right)$ (*d*) $\sin^{-1}\left(\frac{4}{7}\right)$

12. If θ is acute and the vector $(\sin\theta)\hat{i}+(\cos\theta)\hat{j}$ is perpendicular to the vector $\left(\hat{i}-\sqrt{3}\hat{j}\right)$ then θ = ?

(*a*) $\frac{\pi}{6}$ (*d*) $\frac{\pi}{5}$

(*c*) $\frac{\pi}{4}$ (*d*) $\frac{\pi}{3}$

13. The projection of $\vec{a}=\left(2\hat{i}+3\hat{j}+3\hat{k}\right)$ on $\vec{b}=\left(\hat{i}-2\hat{j}+\hat{k}\right)$ is

(*a*) $\hat{i}-2\hat{j}+\hat{k}$ (*b*) $-\hat{i}+2\hat{j}-\hat{k}$

(*c*) $\frac{1}{\sqrt{6}}\left(-\hat{i}+2\hat{j}-\hat{k}\right)$ (*d*) $\frac{1}{6}\left(-\hat{i}+2\hat{j}-\hat{k}\right)$

14. If $\left|\vec{a}\times\vec{b}\right|^2+\left(\vec{a}.\vec{b}\right)^2=144$ and $|\vec{a}|=4$, then $|\vec{b}|$ = ?

(*a*) 16 (*b*) 8

(*c*) 12 (*d*) 3

15. If $\left|\vec{a}\times\vec{b}\right|=\vec{a}.\vec{b}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(*a*) $\frac{\pi}{6}$ (*b*) $\frac{\pi}{4}$

(*c*) $\frac{\pi}{3}$ (*d*) $\frac{\pi}{2}$

16. The length of projection of the vector $\vec{a}=\left(7\hat{i}+\hat{j}-4\hat{k}\right)$ on the vector $\vec{b}=\left(2\hat{i}+6\hat{j}+3\hat{k}\right)$ is

(*a*) $\frac{6}{7}$ (*b*) 1

(*c*) $\frac{8}{7}$ (*d*) None

17. The value of λ for which the vectors $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{c}=\hat{j}+\lambda\hat{k}$ are coplanar, is

(*a*) 1 (*b*) –1

(*c*) 2 (*d*) –3

18. If $\vec{a},\vec{b}$ and $\vec{c}$ be any three vectors, then $\vec{a}\times\left(\vec{b}+\vec{c}\right)$ = ?

(*a*) $\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (*b*) $\vec{a}.\vec{b}+\vec{a}.\vec{c}$

(*c*) $\vec{a}\times\vec{b}+\vec{a}.\vec{c}$ (*d*) None

19. If $\vec{a}=2\hat{i}+3\hat{j}+3\hat{k}, \vec{b}=-4\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+4\hat{k}$, then find $\vec{a}.\left(\vec{b}\times\vec{c}\right)$

(*a*) 33 (*b*) 39

(*c*) 41 (*d*) None of these

20. The vector of magnitude 3 and perpendicular to each one of the vectors $\left(4\hat{i}-\hat{j}+3\hat{k}\right)$ and $\left(-2\hat{i}+\hat{j}-2\hat{k}\right)$ is

(*a*) $\left(\hat{i}-2\hat{j}+2\hat{k}\right)$ (*b*) $\left(-\hat{i}+2\hat{j}+2\hat{k}\right)$

(*c*) $\frac{\left(-3\hat{i}+6\hat{j}+6\hat{k}\right)}{\sqrt{3}}$ (*d*) None of these

Answer Keys

1. (*b*)	**2.** (*b*)	**3.** (*d*)	**4.** (*c*)	**5.** (*b*)	**6.** (*b*)	**7.** (*b*)	**8.** (*d*)	**9.** (*b*)	**10.** (*a*)
11. (*a*)	**12.** (*d*)	**13.** (*d*)	**14.** (*d*)	**15.** (*b*)	**16.** (*c*)	**17.** (*b*)	**18.** (*a*)	**19.** (*c*)	**20.** (*b*)

Solutions

1. Since $\vec{a}$ and $\vec{b}$ are collinear, we have $\vec{a} = t\vec{b}$ for some scalar t.

$\Rightarrow \hat{i} - \hat{j} = \hat{t}\left(-2\hat{i} + m\hat{j}\right)$

$\Rightarrow (-2t)\hat{i} + (mt)\hat{j} = \left(\hat{i} - \hat{j}\right)$

$\Rightarrow -2t = 1, mt = -1$

$\therefore\ t = -\frac{1}{2}$ and $m = \frac{-1}{\left(\frac{-1}{2}\right)} = 2$

2. $\overline{AB}$ = position vector of B – positive vector of A

$= \left(-6\hat{i} + 4\hat{j}\right) - \left(3\hat{i} - 8\hat{j}\right)$

$= -6\hat{i} + 4\hat{j} - 3\hat{i} + 8\hat{j}$

$= -9\hat{i} + 12\hat{j}$

$\left|\overline{AB}\right| = \sqrt{(-9)^2 + (12)^2} = \sqrt{81 + 144} = \sqrt{225} = 15$

$\therefore$ Required vector $= \frac{-9\hat{i} + 12\hat{i}}{15} = \frac{-3\hat{i} + 4\hat{j}}{5}$

$= \frac{1}{5}\left(-3\hat{i} + 4\hat{j}\right)$

3. Let $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$

Resultant $= \left(\vec{a} + \vec{b}\right) = \left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)$

$\left|\vec{a} + \vec{b}\right| = \sqrt{3^2 + 6^2 + (-2)^2}$

$= \sqrt{9 + 36 + 4} = \sqrt{49} = 7$

$\therefore$ Required vector $= \frac{\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)}{7}$

4. $\left|\vec{a}\right| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

and $\left|\vec{b}\right| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$

$\vec{a}\,.\,\vec{b} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right).\left(-3\hat{i} + 2\hat{j} + \hat{k}\right)$

$= [1 . (-3) + 2 . 2 + 3 . 1] = [-3 + 4 + 3] = 4$

$\therefore\ \cos\theta = \frac{\vec{a}\,.\,\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|} = \frac{4}{\sqrt{14}\sqrt{14}} = \frac{2}{7}$

5. $\left(\vec{a} - 2\vec{b} + 3\vec{c}\right) = \left(\hat{i} + \hat{j}\right) - 2\left(\hat{j} + \hat{k}\right) + 3\left(\hat{i} + \hat{k}\right)$

$= 4\hat{i} - \hat{j} + \hat{k}$

and $\left|\vec{a} - 2\vec{b} + 3\vec{c}\right| = \sqrt{(4)^2 + (-1)^2 + (1)^2}$

$= \sqrt{16 + 1 + 1} = \sqrt{18} = 3\sqrt{2}$

$\therefore$ Required vector $= \frac{\left(4\hat{i} - \hat{j} + \hat{k}\right)}{3\sqrt{2}}$

6. $\overline{AB}$ = Position vector of B – position vector of A

$= \left(\hat{i} - \hat{j} - \hat{k}\right) - \left(2\hat{i} - 3\hat{j} + \hat{k}\right)$

$= \left(-\hat{i} + 2\hat{j} - 2\hat{k}\right)$

$\therefore\ \left|\overline{AB}\right| = \sqrt{(-1)^2 + 2^2 + (-2)^2}$

$= \sqrt{1 + 4 + 4} = \sqrt{9} = 3$

$\therefore$ Required vector $= \frac{\left(-\hat{i} + 2\hat{j} - 2\hat{k}\right)}{3}$

7. $\vec{a} + \vec{b} = 4\hat{i} - \hat{j} + \hat{k} + 4\hat{i} - 2\hat{j} + \hat{k}$

$= 8\hat{i} - 3\hat{j} + 2\hat{k}$

and $\left|\vec{a} + \vec{b}\right| = \sqrt{(8)^2 + (-3)^2 + (2)^2}$

$= \sqrt{64 + 9 + 4} = \sqrt{77}$

$\therefore$ Unit vector parallel to

$\left(\vec{a} + \vec{b}\right) = \frac{8\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{77}}$

8. Given, $\vec{a} = \hat{i} - 3\hat{j} + 4\hat{k}$

$\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - 6\hat{j} + 3\hat{k}$

$\therefore\ \left(\vec{a} + \vec{b} + \vec{c}\right) = \left(\hat{i} - 3\hat{j} + 4\hat{k}\right) + \left(2\hat{i} + j - \hat{k}\right) +$

$\left(3\hat{i} - 6\hat{j} + 3\hat{k}\right) = 6\hat{i} - 8\hat{j} + 6\hat{k}$

9. Given a = 3, b = –2 and c = –4

$\therefore\ \sqrt{a^2 + b^2 + c^2} = \sqrt{(3)^2 + (-2)^2 + \left(-4^2\right)}$

$= \sqrt{9 + 4 + 16} = \sqrt{29}$

$\therefore$ Direction cosines are

$l = \frac{3}{\sqrt{29}}, m = \frac{-2}{\sqrt{29}}, n = \frac{-4}{\sqrt{29}}$

10. Given P = (3, 8, 4) and Q = (5, 2, –2)

$\therefore\ \overline{OR} = \frac{\left(3\hat{i} + 8\hat{j} + 4\hat{k}\right) + \left(5\hat{i} + 2\hat{j} - 2\hat{k}\right)}{2}$

$= \frac{8\hat{i} + 10\hat{j} + 2\hat{k}}{2} = 4\hat{i} + 5\hat{j} + \hat{k}$

11. Let $\vec{a} = \hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 4\hat{j} - \hat{k}$

$$\therefore\ \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ 2 & -4 & -1 \end{vmatrix} = 5\hat{i} + 5\hat{j} - 10\hat{k}$$

$$\therefore\ |\vec{a} \times \vec{b}| = \sqrt{5^2 + 5^2 + (-10)^2} = \sqrt{25 + 25 + 100}$$

$$= \sqrt{150} = 5\sqrt{6}$$

Also, $|\vec{a}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{1 + 9 + 4} = \sqrt{14}$

and $|\vec{b}| = \sqrt{2^2 + (-4^2) + (-1)^2} = \sqrt{4 + 16 + 1} = \sqrt{21}$

$$\therefore\ \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\hat{b}|} = \frac{5\sqrt{6}}{\sqrt{14}\sqrt{21}} = \frac{5}{7}$$

$$\therefore\ \theta = \sin^{-1}\left(\frac{5}{7}\right)$$

12. Let $\vec{a} = (\sin\theta)\hat{i} + (\cos\theta)\hat{j}$ and $\vec{b} = \hat{i} - \sqrt{3}\hat{j}$, then

$\because \vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0$

$$\Rightarrow \left[(\sin\theta)\hat{i} + (\cos\theta)\hat{j}\right].\left(\hat{i} - \sqrt{3}\hat{j}\right) = 0$$

$$\Rightarrow \sin\theta - \sqrt{3}\cos\theta = 0$$

$$\Rightarrow \sin\theta - \sqrt{3}\cos\theta \Rightarrow \tan\theta = \sqrt{3}$$

$$\Rightarrow \tan\theta = \tan\frac{\pi}{3}$$

$$\therefore\ \theta = \frac{\pi}{3}$$

13. $\vec{a}.\vec{b} = \left(2\hat{i} + 3\hat{j} + 3\hat{k}\right).\left(\hat{i} - 2\hat{j} + \hat{k}\right) = 2 - 6 + 3 = -1$

$$|\vec{b}|^2 = (1)^2 + (-2)^2 + (1)^2 = 1 + 4 + 1 = 6$$

$$\therefore\ \text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{(\vec{a}.\vec{b})\vec{b}}{|\vec{b}|^2}$$

$$= \frac{-1}{6}\left(\hat{i} - 2\hat{j} + \hat{k}\right) = \frac{1}{6}\left(-\hat{i} + 2\hat{j} - \hat{k}\right)$$

14. $|\vec{a} \times \vec{b}|^2 + (\vec{a}.\vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$ (We know)

$$\Rightarrow |\vec{a}|^2 |\vec{b}|^2 = 144$$

$$\Rightarrow (4)^2 |\vec{b}|^2 = 144$$

$$\Rightarrow |\vec{b}|^2 = \frac{144}{16} = 9$$

$$\therefore\ |\vec{b}| = \sqrt{9} = 3$$

15. $\because |\vec{a} \times \vec{b}| = \vec{a}.\vec{b} \Rightarrow$ ab sin θ = ab cos θ

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = 1$$

$$\Rightarrow \tan\theta = 1 \Rightarrow \tan\theta = \tan\frac{\pi}{4}$$

16. $\vec{a}.\vec{b} = \left(7\hat{i} + \hat{j} - 4\hat{k}\right).\left(2\hat{i} + 6\hat{j} + 3\hat{k}\right)$

$= 14 + 6 - 12 = 8$

and $|\vec{b}| = \sqrt{(2)^2 + (6)^2 + (3)^2} = \sqrt{4 + 36 + 9} = \sqrt{49} = 7$

$\therefore$ Length of projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}.\vec{b}}{|\vec{b}|} = \dfrac{8}{7}$

17. $\because \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

$$\therefore \begin{vmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 0 & 1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow 2(2\lambda + 3) + 3\lambda + 1 = 0$

$\Rightarrow 4\lambda + 6 + 3\lambda + 1 = 0$

$\Rightarrow 7\lambda + 7 = 0 \Rightarrow 7\lambda = -7$

$$\therefore \lambda = \frac{-7}{7} = -1$$

18. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

19. $$\vec{a}.(\vec{b} + \vec{c}) = \begin{vmatrix} 2 & 3 & 3 \\ -4 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix}$$

On expanding along R_1,

$= 2(8 - 1) - 3(-16 - 3) + 3(-4 - 6)$

$= 2 \times 7 - 3 \times (-19) + 3 \times (-10)$

$= 14 + 57 - 30 = 14 + 27 = 41$

20. Let $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$, then a vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $(\vec{a} \times \vec{b})$

$$\therefore (\vec{a} \times \vec{b}) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = \left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

and $|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$

$\therefore$ Unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$$= \frac{\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)}{3}$$

$\therefore$ Required vector

$$= \frac{3\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)}{3} = \left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)$$