7 Quadrilaterals

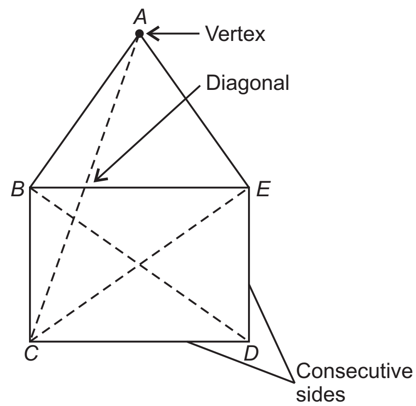
KEY FACTS

1. Polygon: A closed plane figure bounded by line segments is called a polygon. Poly means 'many', so a polygon is a closed many sided figure.  are all polygons.

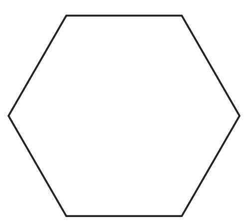
Polygons are classified as follows with reference to the number of sides they have:

5 sides	6 sides	7 sides	8 sides	9 sides	10 sides	12 sides
pentagon	**hexagon**	**heptagon**	**octagon**	**nonagon**	**decagon**	**dodecagon**

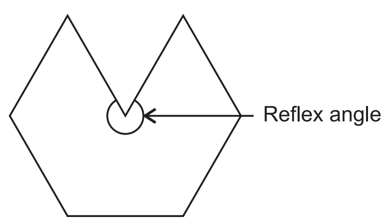
2. Some important terms used with polygons



1. **Vertex:** It is the point where two adjacent sides of a polygon meet.
   For example, *A*, *B*, *C*, *D*, *E*.
2. **Consecutive sides:** Are those sides which have a vertex in common, *viz*., *AB*, *BC*; *BC*, *CD*; *CD*, *DE*; *DE EA*.
3. **Diagonal:** The line joining any two non-adjacent or non-consecutive vertices of a polygon, *viz*., *AC*, *BD*, *CE*, *BE*.
4. **Perimeter:** The sum of the lengths of all sides of a polygon, *i.e.*, $AB + BC + CD + DE + EA$.
5. **Convex polygon:** In a convex polygon each interior angle is less than 180°.

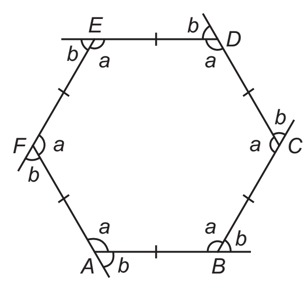


Convex Polygon



Concave Polygon

6. **Concave polygon:** In this type of a polygon at least one of the interior angles is a reflex angle, *i.e.*, greater than 180°. It is also called a **re-entrant polygon.**
7. **Regular polygon:** A polygon is regular if:
   (***i***) it is convex
   (***ii***) all of its sides are equal
   (***iii***) all of its interior angles are equal
   (***iv***) all of its exterior angles are equal.

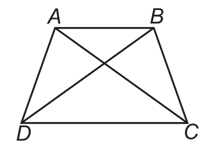


The figure shows a regular hexagon *ABCDEF*, with all equal interior angles marked **'*a*'** and all exterior angles marked **'*b*'**. The sides are equal, *i.e.*, $AB = BC = CD = DE = EF = FA$.

Some common examples of a regular polygon are: *an equilateral triangle, a square.*

8. **Sum of the angles of a polygon:**
   a. **Sum of the interior angles** of a convex polygon of ***n* sides = (2*n* – 4)rt. $\angle$*s* = (2n – 4) × 90°**
   b. If the sides of a convex polygon are produced in order, **the sum of the exterior angles so formed is 4 rt. $\angle$*s*, *i.e.*, 360°.**
9. **In a regular polygon:** ***a.*** Each exterior angle = $\dfrac{360°}{\text{Number of sides}}$
   b. Each interior angle = 180° – Exterior angle
   $= \dfrac{\mathbf{(2n-4)\times 90°}}{\boldsymbol{n}}$, where n is the number of sides.

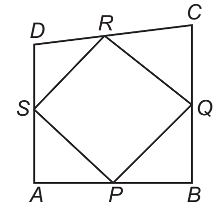
3. Quadrilateral: A quadrilateral is a plane figure bounded by four straight line segments.



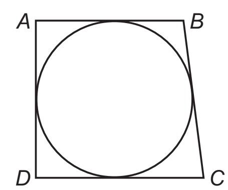
Ex. *ABCD* is a quadrilateral. The four line segments bounding it, *i.e.*, ***AB*, *BC*, *CD* and *DA*** are called its **sides**. ***A*, *B*, *C*, *D*** are the **vertices** and ***AC*** and ***BD*,** the line segments joining the opposite vertices are the **diagonals** of the given quadrilateral.

Properties:

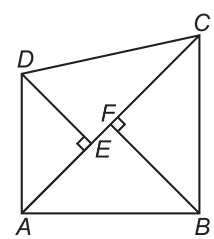
1. Sum of the four interior angles = 360°.
2. The figure formed by joining the mid-points of the adjacent sides of a quadrilateral is a parallelogram.
   Ex. *PQRS* is a parallelogram.



3. The sum of the opposite sides of a quadrilateral circumscribed about a circle is always equal.
   $$AB + DC = AD + BC$$



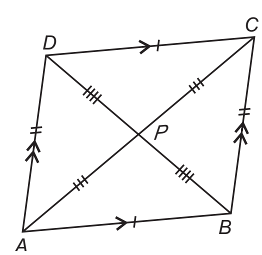
4. Area of quadrilateral = $\frac{1}{2}$ × one of the diagonals × sum of the perpendiculars drawn on the diagonal from opposite vertices
   Ex. Area of quad. $ABCD = \frac{1}{2} \times AC \times (BF + DE)$



4. Special Types of Qudrilaterals and their Properties

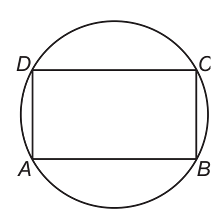
PARALLELOGRAM:

A quadrilateral in which opposite sides are equal and parallel is called a parallelogram.



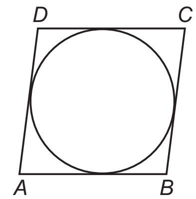
Properties:

1. *Both pairs of opposite sides are equal, i.e., AB = DC, AD = BC.*
2. *Opposite angles are equal, i.e.,* $\angle A = \angle C$, $\angle B = \angle D$.
3. *Sum of any two adjacent angles is 180°, i.e.,*
   $\angle A + \angle B = \angle B + \angle C = \angle C + \angle D = \angle D + \angle A = 180°$.
4. *Diagonals bisect each other, i.e., AP = PC, BP = PD.*
5. *Each diagonal divides a parallelogram into two congruent triangles, i.e.,*
   $\Delta ABC \cong \Delta ADC$, $\Delta DAB \cong \Delta BCD$.
6. *In case of a parallelogram, the diagonals need not be of equal length, need not bisect at right angles and need not bisect the angles at the vertices.*
7. *Lines joining the mid-points of the adjacent sides of a parallelogram is a parallelogram.*
8. *The parallelogram that is inscribed in a circle is a rectangle. ABCD is a rectangle.*



Property 8

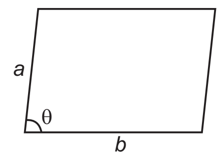
9. *The parallelogram that is circumscribed about a circle is a rhombus.*
   ABCD is a rhombus.



Property 9

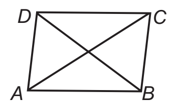
10. **1. Area of a parallelogram = *base* × *height***

    2. Area of a parallelogram = *product of any two adjacent sides* × *sine of included angle*
    = *ab sin* θ



11. Perimeter of a parallelogram = 2 (Sum of any two adjacent sides)

12. $AC^2 + BD^2 = 2\,(AB^2 + BC^2)$



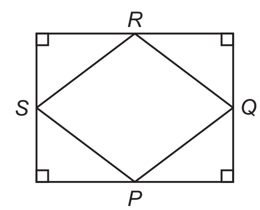
13. Parallelograms that lie on the same base and between the same parallel lines are equal in area.

RECTANGLE:

A rectangle is a parallelogram in which all the four angles at the vertices are right angles, *i.e.*, = 90°.

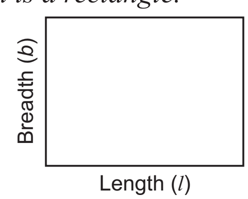
Properties:

1. *Opposite sides are equal and parallel.*
2. *All angles are each equal to 90°.*
3. *Diagonals are* **equal** *and bisect each other but are not necessarily at right angles.*



4. *The figure formed by joining the mid-points of the adjacent sides of a rectangle is a* **rhombus.**

   Thus, *PQRS* is a rhombus.
5. *The quadrilateral formed by the intersection of the angle bisectors of a parallelogram is a rectangle.*
6. **Area of a rectangle = length × breadth = $l \times b$**
7. **Diagonals of a rectangle = $\sqrt{\text{length}^2 + \text{breadth}^2} = \sqrt{l^2 + b^2}$**
8. **Perimeter of a rectangle = 2(length + breadth) = ($l + b$)**

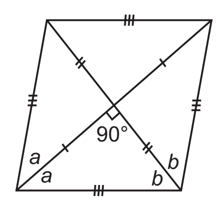


RHOMBUS:

A parallelogram with all sides equal (adjacent sides equal) is called a rhombus.

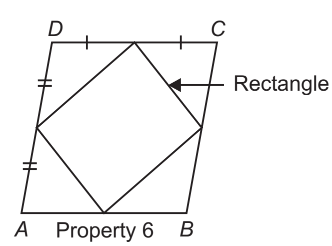
Properties:

1. *Opposite sides are parallel and all sides are equal.*
2. *Opposite angles are equal.*
3. *Diagonals bisect each other at right angles but they are not necessarily equal.*
4. *Diagonals bisect the vertex angles.*
5. *Sum of any two adjacent angles is 180°.*



Properties 1-5

6. *Figure formed by joining the mid-points of the adjacent sides of a rhombus is a rectangle.*
7. • **Area of rhombus = base × height**
   • **Area of rhombus = $\frac{1}{2}$ × product of diagonals**
   • **Area of rhombus = Product of adjacent sides × sine of included angle.**

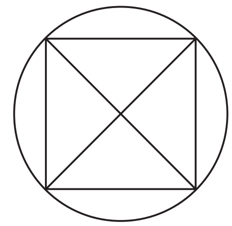


Property 6

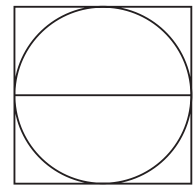
SQUARE:

A parallelogram whose all sides are equal and whose all angles are each equal to right angle is a square. Thus each square is a parallelogram, a rectangle and a rhombus.

Properties:



Property 4



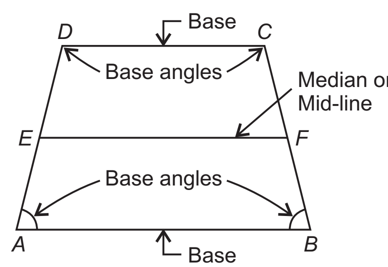
Property 5

1. *All sides are equal and parallel.*
2. *All angles are each equal to 90°.*
3. *Diagonals are equal and bisect each other at right angles.*
4. *Diagonal of an inscribed square is equal to the diameter of the inscribing circle.*
5. *Side of the circumscribed square is equal to the diameter of the inscribed circle.*
6. The figure formed by joining the mid-points of the adjacent sides of a square is a square.
7. **Area of a square = (side)^2 = $\frac{(\textbf{diagonal})^2}{2}$**
8. **Diagonal = side $\sqrt{2}$**
9. **Perimeter = 4 × side**

TRAPEZIUM:

A quadrilateral which has only one pair of opposite sides parallel and other pair of opposite sides not parallel is called a trapezium or a trapezoid.

- The parallel sides ***AB*** and ***DC*** of the trapezium *ABCD* shown in the given figure are called the **bases of the trapezium.**
- The pair of angles that contain the same base, *i.e.*, **∠*A* and ∠*B*; ∠*C* and ∠*D*** are called **base angles.**
- The line joining the mid-points of the non-parallel sides is called the mid-line or median, *i.e.*, here is *EF*.
- *If the non parallel sides of a trapezium are equal, it is an* **isosceles trapezium.**

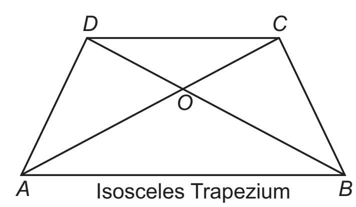


Trapezium

Properties:

1. The median is half the sum of the parallel sides, *i.e.*, $EF = \frac{1}{2}(AB + DC)$.
2. In case of an isosceles trapezium, the diagonals are also equal to each other, *i.e.*, $AC = BD$
3. Diagonals intersect each other proportionally in the ratio of lengths of parallel sides.

   $$\frac{AO}{OC} = \frac{BO}{OD} = \frac{AB}{CD}$$



Isosceles Trapezium (AC = BD)

4. By joining the mid-points of the adjacent sides of a trapezium four similar triangles are obtained.
5. If a trapezium is inscribed in a circle (cyclic trapezium), then it is an isosceles trapezium.
6. Area of a trapezium = $\frac{1}{2}$ × height × (sum of parallel sides)
7. Also, $AC^2 + BD^2 = BC^2 + AD^2 + 2AB \cdot CD$

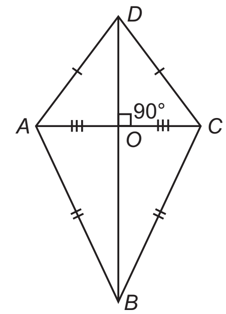
KITE:

A quadrilateral in which two pairs of adjacent sides are equal is called a kite.

Properties:

1. *Adjacent sides are equal, i.e.,* $AD = DC$; $AB = BC$.

2. *Shorter diagonal is bisected by the longer diagonal, i.e., $OA = OC$.*
3. *The diagonals are perpendicular to each other, i.e., $AC \perp BD$.*
4. *The angles at the vertices of the shorter diagonal are equal, i.e., $\angle A = \angle C$.*
5. *The longer diagonal, i.e., BD here, divides the kite into two congruent triangles, i.e., $\Delta ABD \cong \Delta DBC$.*
6. **Area of a kite = $\frac{1}{2}$ × product of diagonals.**



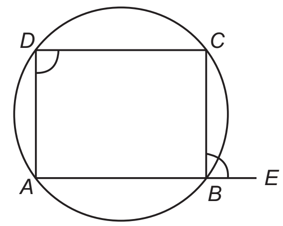
Kite

CYCLIC QUADRILATERAL:

A quadrilateral whose all four vertices lie on a circle is called cyclic quadrilateral.

Properties:

1. *The sum of pair of opposite angles of a cyclic quadrilateral is 180°, i.e.,* $\angle DAB + \angle BCD = 180°$; $\angle ADC + \angle ABC = 180°$
2. If a side of a cyclic quadrilateral is produced, then the exterior angle so formed is equal to the interior opposite angle, *i.e.*, $\angle CBE = \angle ADC$.
3. • **Area of a cyclic quadrilateral = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$,** where s is the semi-perimeter, i.e., $s = \frac{a+b+c+d}{2}$, and a, b, c, d denote the lengths of the sides of the quadrilateral.
   • **Area of a cyclic quadrilateral in which a circle can be inscribed = $\sqrt{a \times b \times c \times d}$,** where a, b, c and d are the lengths of the sides of the quadrilateral.



Cyclic Quadrilateral

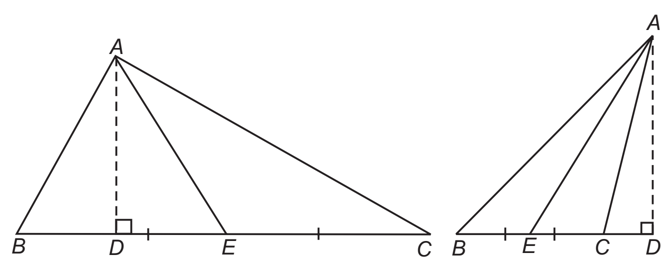
5. Apollonius' Theorem:

In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side, together with twice the square on the median which bisects the third side.

ΔABC is the given triangle where the median AE is drawn from A to E.

According to the given theorem,

$$AB^2 + AC^2 = 2BE^2 + 2AE^2$$



SOLVED EXAMPLES

Ex. 1. *ABCD* is a parallelogram. *AB* is produced to *E* such that *BE = AB*. Prove that *ED* bisects *BC*.

Sol. *ABCD* is a parallelogram.

$\Rightarrow AB \parallel CD \Rightarrow AE \parallel CD$, CB is the transversal $\Rightarrow \angle 3 = \angle 4$ (alt. $\angle s$ are equal)

Now in Δs OCD and OBE,

$CD = AB = BE$

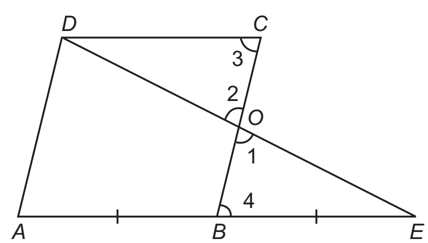
$\angle 2 = \angle 1$ (Vertically opposite $\angle s$)

$\angle 3 = \angle 4$ (Proved above)

$\therefore \Delta OCD \cong \Delta OBE$ (*AAS*)

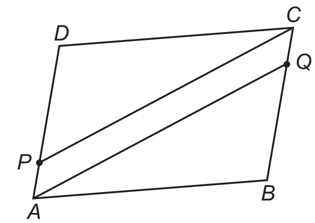
$\Rightarrow BO = OC \Rightarrow O$ is the mid-point of BC

$\Rightarrow ED$ bisects BC.



Ex. 2. *ABCD* is a parallelogram. *P* is a point on *AD* such that $AP = \frac{1}{3}$ *AD* and *Q* is a point on *BC* such that $CQ = \frac{1}{3}$ *BC*. Prove that *AQCP* is a parallelogram.

Sol. *ABCD* is a parallelogram



$\Rightarrow AD = BC$ and $AD \parallel BC$

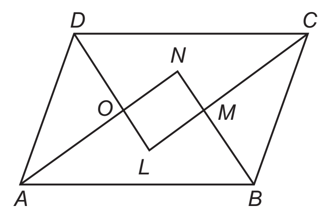
$\Rightarrow \frac{1}{3}AD = \frac{1}{3}BC$ and $AD \parallel BC$

$\Rightarrow AP = CQ$ and $AP \parallel CQ$

$\Rightarrow$ *APCQ* is a parallelogram.

Ex. 3. Prove that the angle bisectors of a parallelogram form a rectangle.

Sol. Let *ABCD* be the given parallelogram, whose angle bisectors form the quadrilateral *LMNO* which needs to be proved a rectangle.



$AB \parallel DC$ and *AD* is the transversal

$\Rightarrow \angle A + \angle D = 180°$ (Co-interior angles)

$\Rightarrow \frac{1}{2}\angle A + \frac{1}{2}\angle D = 90°$ (*AO* and *DO* are angle bisectors of $\angle A$ and $\angle D$ respectively)

$\Rightarrow \angle DAO + \angle ADO = 90°$

$\therefore$ In ΔADO,

$\angle DAO + \angle ADO + \angle AOD = 180°$

$\Rightarrow \angle AOD = 180° - 90° = 90°$

$\Rightarrow \angle NOL = \angle AOD = 90°$ (vert. opp. $\angle s$)

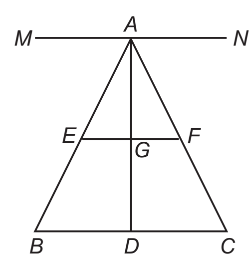
Similarly, we can show,

$\angle ONM = \angle NML = \angle MLO = 90°$

Hence, $\angle MNO$ is a rectangle.

Ex. 4. Prove that any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.

Sol. Given, a triangle *ABC* with *E* and *F* respectively as mid-points of *AB* and *AC*. The line *AD* from vertex *A* meets *EF* in *G* and we are to prove that $AG = GD$.



Draw a line *MAN* through *A* parallel to *BC*.

Since, *E* and *F* are mid-points of *AB* and *AC* respectively, $EF \parallel BC$. (Mid pt. Theorem)

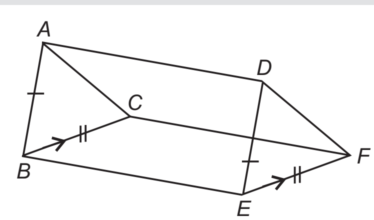
Given, $MAN \parallel BC$

$\Rightarrow MAN \parallel EF \parallel BC$

Thus, by the intercept theorem, the intercepts *AE* and *EB* made by the transversal *AB* on the parallel lines *MAN*, *EF* and *BC* are equal, so the intercepts made by the transversal *AD* on these three parallel lines will also be equal, *i.e.*, $AG = GD$.

Ex. 5. In ΔABC and ΔDEF, $AB = DE$, $AB \parallel DE$, $BC = EF$ and $BC \parallel EF$. Vertices *A*, *B* and *C* are joined to vertices *D*, *E* and *F* respectively. Show that $\Delta ABC \cong \Delta DEF$.

Sol. **Given:** $AB = DE$, $AB \parallel DE$



As one pair of opposite sides is equal and parallel

$\Rightarrow$ *ABED* is a parallelogram.

Given: $BC = EF$ and $BC \parallel EF$

$\Rightarrow$ *BCFE* is a parallelogram

Now, $ABED$ is a parallelogram $\Rightarrow$ $AD \parallel BE$ and $AD = BE$ ...(i)

$BCFE$ is a parallelogram $\Rightarrow$ $CF \parallel BE$ and $CF = BE$ ...(ii)

$\therefore$ From (i) and (ii),

$AD = CF$ and $AD \parallel CF$

$\Rightarrow$ $ADFC$ is a parallelogram.

$\Rightarrow$ $AC = DF$.

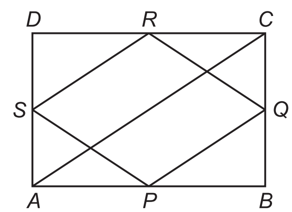
Now in Δs ABC and DEF

$AB = DE$ (Given), $BC = EF$ (Given) and $AC = DF$ (Proved above)

$\therefore$ $\Delta ABC \cong \Delta DEF$ (SSS)

Ex. 6. If *ABCD* is a rectangle and *P*, *Q*, *R* and *S* are the mid-points of the sides *AB*, *BC*, *CD* and *DA* respectively, then quadrilateral *PQRS* is a rhombus.

Sol. $ABCD$ is the given rectangle. Join AC. In ΔABC, P and Q are the mid-points of sides AB and BC respectively.



$\therefore$ By mid-point theorem,

$PQ \parallel AC$ and $PQ = \frac{1}{2}AC$ ...(i)

In ΔADC, R and S are the mid-points of CD and AD respectively

$\therefore$ $SR \parallel AC$ and $SR = \frac{1}{2}AC$ ...(ii)

From (i) and (ii), we get

$PQ \parallel SR$ and $PQ = SR \Rightarrow PQRS$ is a parallelogram

$ABCD$ is a rectangle, $AD = BC \Rightarrow \frac{1}{2}AD = \frac{1}{2}BC \Rightarrow AS = BQ$

$\therefore$ In Δs APS and BPQ

$AS = BQ$ (Proved above)

$AP = PB$ (P is mid-point of AB)

$\angle SAP = \angle QBP = 90°$

$\therefore$ $\Delta APS \cong \Delta BPQ$ (SAS)

$\Rightarrow$ $PS = PQ$

$\therefore$ $PQRS$ is a parallelogram with adjacent sides equal

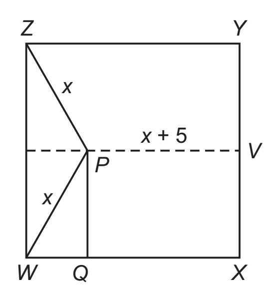
$\Rightarrow$ $PQRS$ is a rhombus.

Ex. 7. *WXYZ* is a square of side length 30. *V* is a point on *XY* and *P* is a point inside the square with *PV* perpendicular to *XY*. *PW* = *PZ* = *PV* – 5. Find *PV*.

Sol. $PZ = PW$

$\Rightarrow$ P lies half-way between ZY and WX. Let $PZ = PW = x \Rightarrow PV = x + 5$

$\because$ $PV \perp XY$ (Given)



$\therefore$ V is also the mid-point of XY

$\Rightarrow$ $VX = VY = 15$

$WQ = WX - QX = WX - PV = 30 - (5 + x) = 25 - x$.

$PQ = 15$

$\therefore$ $PW = \sqrt{WQ^2 + PQ^2}$

$\Rightarrow$ $x = \sqrt{(25 - x)^2 + 15^2}$

$\Rightarrow \quad x^2 = 625 - 50x + x^2 + 225$

$\Rightarrow \quad 50x = 850 \quad \Rightarrow \quad x = 17$

$\therefore \quad PV = x + 5 = \mathbf{22}.$

Ex. 8. *ABCD* is a trapezium in which side *AB* is parallel to side *DC* and *E* is the mid-point of side *AD*. If *F* is a point on side *BC* such that segment *EF* is parallel to side *DC*. Prove that $EF = \frac{1}{2}(AB + DC)$.

Sol. Let *ABCD* be the given trapezium in which $AB \parallel DC$. In ΔADC, *E* is the mid-point of *AD* and $EG \parallel DC$

$\Rightarrow$ *G* is the mid-point of *AC*

$\Rightarrow EG = \frac{1}{2} DC$ ...(*i*)

($\therefore$ Line segment joining the mid-points of to sides of a triangle is parallel to the third side and half of it).

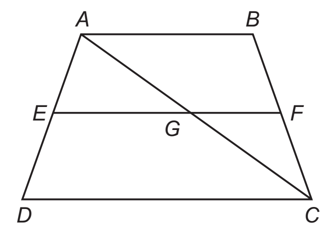
Also, $AB \parallel DC$, $EF \parallel DC \quad \Rightarrow \quad EF \parallel AB \Rightarrow GF \parallel AB$

$\therefore$ In ΔABC, $GF \parallel AB$ and *G* is the mid-point of *AC*

$\Rightarrow$ *F* is the mid-point of *BC*

$\Rightarrow GF = \frac{1}{2} AB$ ...(*ii*)

(By mid-point theoram)



$\therefore$ From (*i*) and (*ii*), $GE + GF = \frac{1}{2} DC + \frac{1}{2} AB$

$\Rightarrow \quad \boldsymbol{EF = \frac{1}{2}(AB + DC)}$.

Ex. 9. A point *O* in the interior of a rectangle *ABCD* is joined with each of the vertices *A*, *B*, *C* and *D*. Then, show that $OA^2 + OC^2 = OB^2 + OD^2$.

Sol. As the diagonals of a rectangle are equal and bisect each other, $AM = DM$

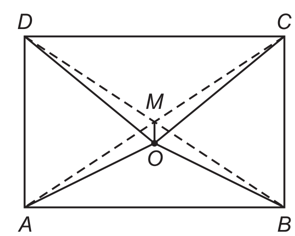
Applying Applonius theorem,

In ΔAOC, $OA^2 + OC^2 = 2(OM^2 + AM^2)$

In ΔBOD, $OB^2 + OD^2 = 2(OM^2 + DM^2)$

$\because \quad AM = DM$

$\therefore \quad \boldsymbol{OA^2 + OC^2 = OB^2 + OD^2}$.



Ex. 10. The median of a trapezoid cuts the trapezoid into two regions whose areas are in the ratio 1 : 2. Compute the ratio of the smaller base of the trapeziod to the larger base.

(Median of a trapezoid means the line joining the mid-points of the 2 non-parallel sides of the trapezoid)

Sol. *ABCD* is the given trapezoid, whose median or midline is *EF*.

Let *AH* be drawn perpendicular to *DC*

$EG \parallel DH \qquad (\because EF \parallel DC)$

$\therefore$ By Basic Proportionality Theorem,

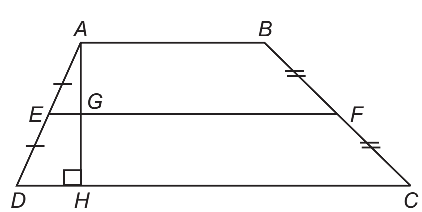
$$\frac{AE}{ED} = \frac{AG}{GH} \Rightarrow \frac{AG}{GH} = 1 \qquad (\because E \text{ is the mid-point of } AD)$$

$\Rightarrow AG = GH$

Let $AG = GH = h$ and $AB = x$, $EF = y$ and $CD = z$.

Then, Area (*ABFE*) $= h/2\ (x + y)$

Area (*EFCD*) $= h/2\ (y + z)$



Area $(ABCD) = h\,(x+z)$

Now given, $\dfrac{\text{Area }(ABFE)}{\text{Area }(EFCD)} = \dfrac{1}{2} \Rightarrow \dfrac{h/2\ (x+y)}{h/2\ (y+z)} = \dfrac{1}{2}$

$\Rightarrow \quad 2x + 2y = y + z \Rightarrow 2x + y = z$ ...(*i*)

Also, $\dfrac{\text{Area }(ABFE)}{\text{Area }(ABCD)} = \dfrac{1}{3} \Rightarrow \dfrac{h/2\ (x+y)}{h(x+z)} = \dfrac{1}{3}$

$\Rightarrow \quad 3x + 3y = 2x + 2z \Rightarrow x + 3y = 2z$ ...(*ii*)

$\Rightarrow$ From (*i*) and (*ii*)

$x + 3y = 2(2x + y) \Rightarrow x + 3y = 4x + 2y$

$\Rightarrow \quad 3x = y$

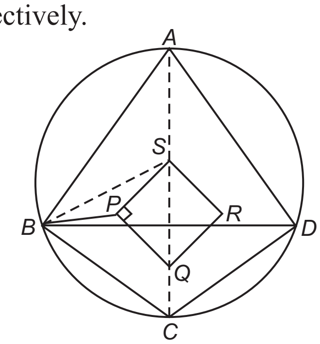
$\Rightarrow$ Putting in (*i*), we get

$2x + 3x = z \Rightarrow 5x = z \Rightarrow \boldsymbol{x : z = 1 : 5.}$

Ex. 11. Let *ABCD* be a cyclic quadrilateral. Show that the incentres of the triangles *ABC*, *BCD*, *CDA* and *DAB* form a rectangle.

Sol. We know that incentre is the point of concurrence of the internal bisectors of the angles of a triangle.

Let P, Q, R and S be the incentres of the triangles ABC, BCD, DAC and DAB respectively.



$\Rightarrow$ P is point of intersection of $\angle BAC$ and $\angle ABC$

In ΔAPB, $\angle APB = 180° - [\angle PAB + \angle PBA]$

$\angle APB = 180° - \left[\dfrac{\angle BAC}{2} + \dfrac{\angle ABC}{2}\right]$ ...(*i*)

But in ΔABC, $\angle ABC + \angle BAC + \angle BCA = 180°$

$\Rightarrow \dfrac{1}{2}(\angle ABC + \angle BAC) = 90° - \dfrac{\angle BCA}{2}$ ...(*ii*)

From (*i*) and (*ii*), $\angle APB = 180° - \left[90° - \dfrac{\angle BCA}{2}\right] = 90° + \dfrac{\angle BCA}{2}$ ...(*iii*)

Similarly, in ΔASB, $\angle ASB = 180° - [\angle SAB + \angle SBA]$

$\Rightarrow \angle ASB = 180° - \left[\dfrac{\angle DAB}{2} + \dfrac{\angle ABD}{2}\right]$ ...(*iv*)

Also in ΔADB,

$\angle ADB + \angle DAB + \angle ABD = 180°$

$\Rightarrow \dfrac{\angle DAB}{2} + \dfrac{\angle ABD}{2} = 90° - \dfrac{\angle ADB}{2}$ ...(*v*)

$\therefore$ From (*iv*) and (*v*)

$\angle ASB = 180^{\circ} - \left(90° - \dfrac{\angle ADB}{2}\right) = 90° + \dfrac{\angle ADB}{2}$ ...(*vi*)

But $\angle BCA = \angle ADB$ (Angles in the same segment are equal)

$\therefore$ From (*iii*) and (*vi*)

$\angle APB = \angle ASB$

$\Rightarrow$ $ASPB$ is a cyclic quadrilateral.

$\Rightarrow \quad \angle SPB + \angle SAB = 180°$ (Opposite angles of a cyclic quadrilateral are supp.)

$\Rightarrow \quad \angle SPB = 180° - \angle SAB$

$\angle SPB = 180° - \angle A/2$ ($\because$ SA bisects $\angle A$)

Similarly, $\angle BPQ = 180° - C/2$

$$\Rightarrow \quad \angle SPQ = 360° - (\angle SPB + \angle BPQ)$$

$$= 360° - \left(180° - \frac{\angle A}{2} + 180° - \frac{\angle C}{2}\right)$$

$$= \frac{\angle A + \angle C}{2} = \frac{180°}{2} = 90° \qquad (\because ABCD \text{ is cylic quadrilateral})$$

Thus, in quadrilateral $SPQR$, $\angle P = 90°$

Similarly it can be shown that $\angle S = \angle Q = \angle R = 90°$

$\Rightarrow$ Quadrilateral is a rectangle.

Ex. 12. Let *ABCD* be a quadrilateral. Let *X* and *Y* be the mid-points of *AC* and *BD* respectively and the lines through *X* and *Y* respectively parallel to *BD* and *AC* meet in *O*. Let *P*, *Q*, *R*, *S* be the mid-points of *AB*, *BC*, *CD* and *DA* respectively. Prove that:

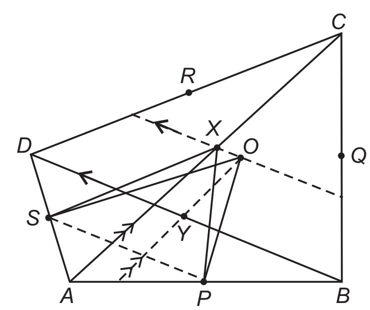
(*a*) Quadrilateral *APOS* and *APXS* have the same area.

(*b*) The areas of quadrilaterals *APOS*, *BQOP*, *CROQ* and *DSOR* are all equal.

Sol. *ABCD* is the given quadrilateral.

P, Q, R and S are the mid-points of sides AB, BC, CD and DA respectively.

$OX \parallel BD$ and $OY \parallel AC$



(*a*) In ΔABD, the line joining the mid-points P and S of sides AB and AD respectively is parallel to the third side.

$\therefore$ $PS \parallel BD$.

Now $PS \parallel BD$ and $OX \parallel BD \Rightarrow OX \parallel PS$.

$\because$ Area of triangles on the same base and between the same parallel lines are equal,

So, Area (ΔPXS) = Area (ΔPOS)

$\Rightarrow$ Area (ΔPAS) + Area (ΔPXS) = Area (ΔPAS) + Area (ΔPOS)

$\Rightarrow$ **Area (*APXS*) = Area (*APOS*)**

(*b*) Area ($APXS$) = Area (ΔAPX) + Area (ΔASX)

$$= \frac{1}{2}\text{ Area }(\Delta ABX) + \frac{1}{2}\text{ Area }(\Delta AXD)$$

($\because$ XP is the median of ΔAXB and XS is the median of ΔADX and a median divides a triangle into two triangles of equal area)

$\because$ X is the mid-point of AC, XB is the median of ΔABC and XD is the median of ΔADC.

$$\text{So, Area }(APXS) = \frac{1}{2}\times\frac{1}{2}\text{ Area }(\Delta ABC) + \frac{1}{2}\times\frac{1}{2}\text{ Area }(\Delta ADC)$$

$$= \frac{1}{4}\,[\text{Area }(\Delta ABC) + \text{Area }(\Delta ADC)]$$

$$= \frac{1}{4}\text{ Area (quad. }ABCD)$$

Now in part (*a*), we have proved Area ($APXS$) = Area ($APOS$)

$$\Rightarrow \text{Area }(APOS) = \frac{1}{4}\text{ Area (quad. }ABCD)$$

Similarly, it can be shown by symmetry that

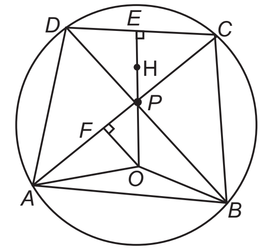
$$\text{Area }(BPOQ) = \text{Area }(CROQ) = \text{Area }(DSOR) = \frac{1}{4}\text{ Area }(ABCD)$$

$\Rightarrow$ Area ($APOS$) = Area ($BPOQ$) = Area ($CROQ$) = Area ($DSOR$).

Ex. 13. The diagonals *AC* and *BD* of a cyclic quadrilateral *ABCD* intersect at *P*. Let *O* be the circumcentre of ΔAPB and *H* be the orthocentre of ΔCPD. Show that the points *H*, *P*, *O* are collinear.

Sol. Let O be the circumcentre (point of intersection of the perpendicular bisectors of the sides of a Δ) of ΔAPB

Join A to O and draw $OF \perp AP$



O being the circumcentre of ΔAPB, it is equidistant from the three vertices A, B and P

$\therefore$ $AO = OP \Rightarrow \Delta AOP$ is isosceles $\Rightarrow OF$ bisects $\angle AOP$

$\Rightarrow$ $\angle FOP = \frac{1}{2}\angle AOP$ ...(*i*)

Also, O being equidistant form the three vertices, we can consider a circle passing through the vertices A, P and B with O as its centre.

Then, $\angle ABP = \frac{1}{2}\angle AOP$ …(*ii*) (Angle subtended by chord AP at the centre is twice the angle subtended at any other part of the circle)

Also, $\angle ABD = \angle ACD$ (Angles in the same segment)

$\Rightarrow$ $\angle ABP = \angle PCD$ ...(*iii*)

$\therefore$ From (*i*), (*ii*) and (*iii*)

$\angle FOP = \angle PCD$

Now in ΔFOP and ΔEPC

$\angle FOP = \angle PCE$ (Proved above)

$\angle OFP = \angle PEC = 90°$

$\Rightarrow$ $\angle FOP \sim \angle EPC$ (*AA* similarity)

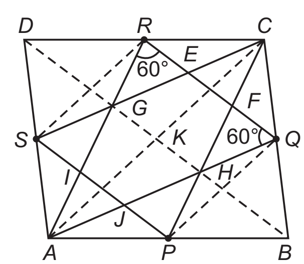
$\Rightarrow$ $\angle FPO = \angle EPC$, which can only be equal if they are vertically opposite angles.

$\Rightarrow$ EO is a straight line.

$\Rightarrow$ H, P and O are collinear.

Ex. 14. Let *ABCD* be a convex quadrilateral. *P*, *Q*, *R* and *S* are the mid-points of *AB*, *BC*, *CD* and *DA* respectively such that the triangles *AQR* and *CSP* are equilateral. Prove that *ABCD* is a rhombus and determine its angles. (*RMO 2005*)

Sol. In ΔDCB, by mid-point theoram,



$QR = \frac{1}{2}BD$

Also, in ΔABD, by mid-point theoram,

$PS = \frac{1}{2}BD$

$\Rightarrow$ $QR = PS$.

Also, given ΔAQR and ΔCSP are equilateral Δs and $QR = PS$

$\Rightarrow$ ΔAQR is congruent to ΔCSP

$\Rightarrow$ $AR = RQ = AQ = CS = CP = PS$ ...(*i*)

Also, $\Delta CEF \sim \Delta CSP \Rightarrow \angle CEF = 60°$ (ΔCSP is equilateral)

ΔAQR being equilateral $\Rightarrow \angle AQR = 60°$

$\therefore$ $\angle CEF = \angle AQR \Rightarrow$ alternate angles are equal $\Rightarrow$ $CS \parallel QA$

So, $CS = QA$ (from (*i*)) and $CS \parallel QA \Rightarrow CSAQ$ is a parallelogram

$\Rightarrow$ $SA \parallel CQ$ and $SA = CQ \Rightarrow AD \parallel BC$ and $AD = BC \Rightarrow ABCD$ is a parallelogram.

Let diagonals AC and BD bisect each other at K. Then,

$$DK = \frac{BD}{2} = QR = CS = AR.$$

$\Rightarrow$ In ΔADC, medians, AR, DK and CS are all equal

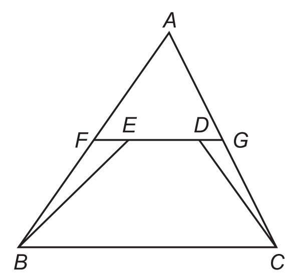
$\Rightarrow$ ΔADC is equilateral $\Rightarrow AD = AB$.

$\therefore$ $ABCD$ is a parallelogram with equal adjacent sides

$\Rightarrow$ $ABCD$ is a rhombus.

Ex. 15. If a triangle and a convex quadrilateral are drawn on the same base and no part of the quadrilateral is outside the triangle, show that the perimeter of the triangle is greater than the perimeter of the quadrilateral. **(*SAT 2000*)**

Sol. Let ABC be the given triangle and $BCDE$, the given quadrilateral.



Produce ED to meet AB in F and AC in G.

Then, in ΔBEF, $BE < FE + BF$ (Triangle inequality)

In ΔDGC, $DC < DG + GC$ (Triangle inequality)

Now perimeter of quadrilateral $BEDC$

$= BE + ED + DC + BC$

$< BF + \underbrace{FE + ED + DG} + GC + BC$

$= BF + FG + GC + BC < \underbrace{BF + AF} + \underbrace{AG + GC} + BC$ ($\because$ In ΔAFG, $FG < AF + AG$)

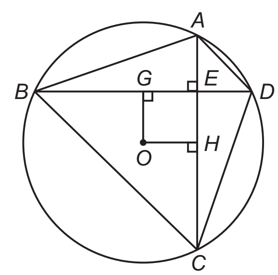
$= AB + AC + BC$

$=$ Perimeter of ΔABC.

$\therefore$ Perimeter of quad. $BEDC <$ Perimeter of ΔABC.

Ex. 16. *ABCD* is a cyclic quadrilateral such that $AC \perp BD$. *AC* meet *BD* at *E*. Prove that $EA^2 + EB^2 + EC^2 + ED^2 = 4R^2$, where *R* is the radius of the circle.

Sol. Let O be the centre of the circle, circumscribing the cyclic quadrilateral $ABCD$.



Let OG and OH be the perpendiculars from O on BD and AC respectively.

Then OG bisects BD, *i.e.*, $BG = GD$

and OH bisects AC, *i.e.*, $AH = HC$

($\because$ The perpendicular from the centre of the chord, bisects the chord)

Now, $BE^2 + ED^2 = (BG + GE)^2 + (GD - GE)^2$

$= BG^2 + GE^2 + 2BG \cdot GE + GD^2 + GE^2 - 2GD \cdot GE$

$= 2BG^2 + 2GE^2$ ...(*i*) ($\because BG = GD$)

Also, $AE^2 + EC^2 = (AH - EH)^2 + (CH + EH)^2$

$= AH^2 + EH^2 - 2AH \cdot EH + CH^2 + EH^2 - 2CH \cdot EH$

$= 2AH^2 + 2EH^2$ ...(*ii*) ($\because AH = CH$)

$\therefore$ $EA^2 + EB^2 + EC^2 + ED^2 = 2AH^2 + 2EH^2 + 2BG^2 + 2GE^2$

$= 2(AH^2 + GO^2 + BG^2 + OH^2)$ ($\therefore EH = GO, GE = OH$)

$= 2(AO^2 + BO^2) = 2(R^2 + R^2) = \mathbf{4R^2}$.

PRACTICE SHEET

LEVEL–1

1. Which one of the following statements is correct?

If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is:

(*a*) A rhombus but not a square
(*b*) A square but not a rhombus
(*c*) Either a rhombus or a square
(*d*) A rectangle but not a square **(*CDS 2005*)**

2. If the diagonals of a quadrilateral are equal and bisect each other at right angles, then the quadrilateral is a

(*a*) rectangle (*b*) square
(*c*) rhombus (*d*) trapezium **(*CDS 2012-II*)**

3. If the bisectors of the angles A and B of a quadrilateral $ABCD$ meet at O, then $\angle AOB$ is equal to :

(*a*) $\angle C + \angle D$ (*b*) $\frac{1}{2}(\angle C + \angle D)$

(*c*) $\frac{1}{2}\angle C + \frac{1}{3}\angle D$ (*d*) $\frac{1}{3}\angle C + \frac{1}{2}\angle D$

(*CDS 2004*)

4. If two parallel lines are intersected by a transversal, then the bisectors of the interior angles form which one of the following?

(*a*) Rectangle (*b*) Square
(*c*) Rhombus (*d*) Parallelogram

(*CDS 2006*)

5. $ABCD$ is a cyclic quadrilateral such that $\angle A + \angle B = 2(\angle C + \angle D)$. If $\angle C > 30°$, then which one of the following is correct?

(*a*) $\angle D \geq 90°$ (*b*) $\angle D < 90°$ (*c*) $\angle D \leq 90°$ (*d*) $\angle D > 90°$

(*CDS 2006*)

6. In a parallelogram $ABCD$, M is the mid-point of BD and BM bisects $\angle B$. Then what is $\angle AMB$ equal to?

(*a*) 45° (*b*) 60° (*c*) 90° (*d*) 120°

(*CDS 2006*)

7. If $ABCD$ is a rhombus, then

(*a*) $AC^2 + BD^2 = 6AB^2$ (*b*) $AC^2 + BD^2 = 5AB^2$
(*c*) $AC^2 + BD^2 = 4AB^2$ (*d*) $AC^2 + BD^2 = 3AB^2$

8. In a trapezium $ABCD$, AB is parallel to CD and the diagonals intersect each other at O. In this case, the ratio OA/OC is equal to:

(*a*) $\frac{OB}{OD}$ (*b*) $\frac{BC}{CD}$ (*c*) $\frac{AD}{AB}$ (*d*) $\frac{AC}{BD}$

9. $ABCD$ is a trapezium where AB and CD are non-parallel sides. If the vertices A, B, C and D are concyclic, then

(*a*) AB is also parallel to CD

(*b*) $AB = \frac{1}{2}CD$

(*c*) $\frac{1}{2}AB = CD$

(*d*) $AB = CD$

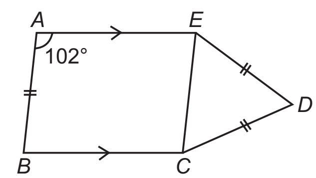
10. X, Y are the mid-points of opposite sides AB and DC of a parallelogram $ABCD$. AY and DX are joined intersecting in P. CX and BY are joined intersecting in Q. Then $PXQY$ is

(*a*) rectangle (*b*) rhombus
(*c*) square (*d*) parallelogram

11. If the sum of the diagonals of a rhombus is 12 cm and its perimeter is $8\sqrt{5}$ cm, then the lengths of the diagonals are:

(*a*) 6 cm and 6 cm (*b*) 7 cm and 5 cm
(*c*) 8 cm and 4 cm (*d*) 9 cm and 3 cm

12. In the given figure $AE = BC$ and $AE \parallel BC$ and the three sides AB, CD and ED are equal in length. If $\angle A = 102°$, then find the measure of $\angle BCD$.



(*a*) 138° (*b*) 162°
(*c*) 88° (*d*) 150°

13. If a rectangle and a parallelogram are equal in area and have the same base and are situated on the same side, and the ratio of the perimeter of the rectangle and that of the parallelogram is k, then

(*a*) $k > 1$ (*b*) $k < 1$
(*c*) $k = 1$ (*d*) Cannot be determined

14. $ABCD$ is a parallelogram. The diagonals AC and BD intersect at the point O. If E, F, G and H are the mid-points of AO, DO, CO and BO respectively, then the ratio of $(EF + FG + GH + HE)$ to $(AD + DC + CB + BA)$ is

(*a*) 1 : 1 (*b*) 1 : 2 (*c*) 1 : 3 (*d*) 1 : 4

15. A cyclic parallelogram having unequal adjacent sides is necessarily a:

(*a*) Square (*b*) Rectangle (*c*) Rhombus (*d*) Trapezium

(*MAT*)

LEVEL–2

16. In a quadrilateral $ABCD$, $\angle B = 90°$ and $AD^2 = AB^2 + BC^2 + CD^2$. Then $\angle ACD$ is equal to:

(*a*) 90° (*b*) 60° (*c*) 30° (*d*) None of these

(*MAT 2003*)

17. In a cyclic quadrilateral $PQRS$, $\angle P$ is double its opposite angle and difference between the other two angles is one-third of $\angle P$. The minimum difference between any two angles of this quadrilateral is

(*a*) 30° (*b*) 10° (*c*) 20° (*d*) 40°

(*MAT 2010*)

18. $ABCD$ is a square. P, Q, R and S are points on the sides AB, BC, CD and DA respectively such that $AP = BQ = CR = DS$. What is $\angle SPQ$ equal to?

(*a*) 30° (*b*) 45° (*c*) 60° (*d*) 90°

(*CDS 2010*)

19. The middle points of the parallel sides AB and CD of a parallelogram $ABCD$ are P and Q respectively. If AQ and CP divide the diagonal BD into three parts BX, XY and YD, then which one of the following is correct?

(*a*) $BX \neq XY \neq YD$ (*b*) $BX = YD \neq XY$

(*c*) $BX = XY = YD$ (*d*) $XY = 2BX$ **(*CDS 2010*)**

20. Let $ABCD$ be a parallellogram. Let m and n be positive integers such that $n < m < 2n$. Let $AC = 2\,mn$ and $BD = m^2 - n^2$ and $AB = \dfrac{(m^2 + n^2)}{2}$

Statement I. $AC > BD$.

Statement II. $ABCD$ is a rhombus.

Which one of the following is correct in respect of the above statements?

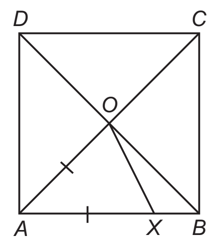
(*a*) Both the statement I and II are true and statement II is the correct explanation of statement I.

(*b*) Both the statements I and II are true but statement II is not the correct explanation of statement I.

(*c*) Statement I is true but statement II is false.

(*d*) Statement II is true but statement I is false.

21. In the given figure, $ABCD$ is a square in which $AO = AX$. What is $\angle XOB$?



(*a*) 22.5° (*b*) 25°

(*c*) 30° (*d*) 45°

(*CDS 2009*)

22. If $PQRS$ is trapezium such that $PQ > RS$ and L, M are the mid-points of the diagonals PR and QS respectively then what is LM equal to?

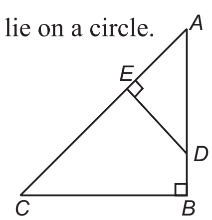
(*a*) $\dfrac{PQ}{2}$ (*b*) $\dfrac{RS}{2}$

(*c*) $\dfrac{PQ + RS}{2}$ (*d*) $\dfrac{PQ - RS}{2}$ **(*CDS 2006*)**

23. In the given figure, $\angle ABC = \angle AED = 90°$. Consider the following statements:

I. ABC and ADE are similar triangles.

II. The four points B, C, E and D may lie on a circle.



(*a*) Only I

(*b*) Only II

(*c*) Both I and II

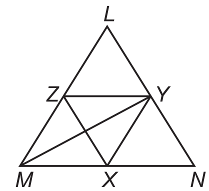
(*d*) Neither I nor II

24. Let X be any point within a square $ABCD$. On AX a square $AXYZ$ is described such that D is within it. Which one of the following is correct?

(*a*) $AX = DZ$ (*b*) $\angle ADZ = \angle BAX$

(*c*) $AD = DZ$ (*d*) $BX = DZ$ **(*CDS 2012*)**

25. In the given figure, YZ is parallel to MN, XY is parallel is LM and XZ is parallel to LN. Then MY is



(*a*) Median of ΔLMN

(*b*) The anguler bisector of $\angle LMN$

(*c*) Perpendicular to LN

(*d*) Perpendicular bisector of LN.

26. The locus of a point in rhombus $ABCD$ which is equidistant from A and C is

(*a*) a fixed point on diagonal BD

(*b*) diagonal BD

(*c*) diagonal AC

(*d*) None of the above

27. $ABCD$ is a square. The diagonals AC and BD meet at O. Let K, L be the points on AB such that $AO = AK$, $BO = BL$. If $\theta = \angle LOK$, then what is the value of tan θ?

(*a*) $\dfrac{1}{\sqrt{3}}$ (*b*) $\sqrt{3}$ (*c*) 1 (*d*) $\dfrac{1}{2}$

(*CDS 2008*)

28. $ABCD$ is a trapezium in which $AB \parallel DC$ and $AD = BC$. If P, Q, R and S be respectively the mid-points of BA, BD, CD and CA, then $PQRS$ is a

(*a*) Rhombus (*b*) Rectangle

(*c*) Parallelogram (*d*) Square

29. A rigid square plate $ABCD$ of unit side rotates in its own plane about the middle-point of CD until the new position of A coincides with the old position of B. How far is the new position of B from the old position of A?

(*a*) 4 units (*b*) $5\sqrt{5}$ units

(*c*) $\dfrac{4\sqrt{5}}{5}$ units (*d*) $4\sqrt{5}$ units **(*RMO*)**

30. A trapezium $ABCD$ in which $AB \parallel CD$ is inscribed in a circle with centre O. Suppose the diagonals AC and BD of the trapezium intersect at M and $OM = 2$. If $\angle AMB = 60°$, the difference between the lengths of the parallel sides is:

(*a*) 2 (*b*) $\sqrt{3}$ (*c*) $3\sqrt{3}$ (*d*) $2\sqrt{3}$

31. $ABCD$ is a trapezium with AB and CD as parallel sides. The diagonals intersect at O. The area of the triangle ABO is $\boldsymbol{p}$ and that of triangle CDO is $\boldsymbol{q}$. The area of the trapezium is:

(*a*) $\sqrt{p} + \sqrt{q}$ (*b*) $\dfrac{1}{3}\left(\sqrt{p} + \sqrt{q}\right)^3$

(*c*) $\dfrac{1}{2}\left(\sqrt{p} + \sqrt{q}\right)^2$ (*d*) $\left(\sqrt{p} + \sqrt{q}\right)^2$

32. $PQRS$ is a rectangle in which $PQ = 2PS$. T and U are the mid-points of PS and PQ respectively. QT and US intersect at V. Find the ratio of the area of quadrilateral $QRSV$ to the area of triangle PQT.

(*a*) 4 : 1 (*b*) 8 : 3 (*c*) 5 : 2 (*d*) 7 : 4

33. A rhombus has sides of length 1 and area $\dfrac{1}{2}$. Find the angle between the two adjacent sides of the rhombus.

(*a*) 60° (*b*) 75° (*c*) 45° (*d*) 30°

34. In the given figure $ABCD$ is a quadrilateal whose diagonals intersect at O. $\angle AOB = 30°$, $AC = 24$ and $BD = 22$. The area of quadrilateral $ABCD$ is:

(a) 132 (b) 264 (c) 528 (d) 66

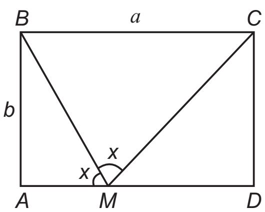
35. $ABCD$ is a square and AOB is an equilateral triangle. What is the value of $\angle DOC$?

(a) 120° (b) 150° (c) 125° (d) 80°

36. In a trapezium $ABCD$, AB is parallel to CD, BD is perpendicular to AD. AC is perpendicular to BC. If $AD = BC$ = 15 cm and AB = 25 cm, then the area of the trapezium is

(a) 192 cm² (b) 232 cm² (c) 162 cm² (d) 172 cm²

37. $ABCD$ is a rectangle with $BC = a$, $AB = b$ and $a > b$. If M is a point on AD such that $\angle BMA = \angle BMC$, then MD is equal to:



(a) $\sqrt{a^2 + b^2}$

(b) $\sqrt{a^2 - b^2}$

(c) $\sqrt{ab}$

(d) $a - b$

38. Let $ABCD$ be a square. M, N, R are the points on AB, BC and CD respectively such that $AM = BN = CR$. If $\angle MNR$ is a right angle, then $\angle MRN$ is equal to

(a) 30° (b) 45° (c) 60° (d) 75°

39. If for a regular pentagon $ABCDE$, the lines AD and BE intersect at points P, then $\angle BAD$ and $\angle APE$ respectively are

(a) 36° and 72° (b) 54° and 108°
(c) 72° and 108° (d) 36° and 108°

LEVEL–3

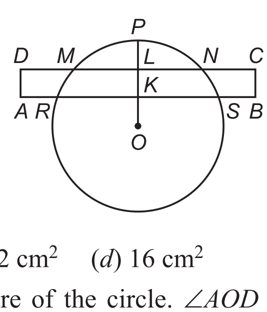
40. Let $ABCD$ be a parallelogram. P is any point on the side AB. If DP and CP are joined in such a way that they bisect the angles ADC and BCD respectively, then DC is equal to

(a) CB (b) $2CB$ (c) $3CB$ (d) $4CB$

41. The adjacent sides of a parallelogram are $2a$ and a. If the angle between them is 60°, then one of the diagonals of the parallelogram is

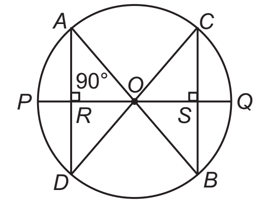
(a) $3a$ (b) $\sqrt{5}a$ (c) $2a$ (d) $\sqrt{3}a$

42. In the given figure, O is the centre of the circle. The radius OP bisects a rectangle $ABCD$ at right angles. $DM = NC = 2$ cm and $AR = SB = 1$ cm, $KS = 4$ cm and $OP = 5$ cm. What is the area of the rectangle?



(a) 8 cm² (b) 10 cm² (c) 12 cm² (d) 16 cm²

43. In the given figure O is the centre of the circle. $\angle AOD = 120°$. If the radius of the circle be 'r', then find the sum of the areas of quadrilateral $AODP$ and $OBQC$



(a) $\frac{\sqrt{3}}{2} r^2$ (b) $3\sqrt{3}\, r^2$

(c) $\sqrt{3}\, r^2$ (d) $2\sqrt{3}\, r^2$

44. Let $ABCD$ be a quadrilateral with $\angle CBD = 2\,\angle ADB$, $\angle ABD = 2\,\angle CDB$ and $AB = CB$. Then which of the following statement hold true.

I $AD = CD$

II $AD = BD$

III $BEDF$ is a parallelogram

IV $AB = CD$

(a) I and III only (b) All of above
(c) I, II and IV (d) I, III and IV

(*Canadian Mathematical Olympiad 2000*)

45. A square sheet of paper $ABCD$ is so folded that B falls on the mid-point M of CD. The crease will divide BC in the ratio

(a) 7 : 4 (b) 5 : 3 (c) 8 : 5 (d) 4 : 1

46. $ABCD$ is a square. P, Q, R, S are the mid-points of AB, BC, CD and DA respectively. By joining AR, BS, CP, DQ, we get a quadrilateral which is a

(a) trapezium (b) rectangle (c) square (d) rhombus

47. Perpendiculars are drawn from the vertex of the obtuse angles of a rhombus to its sides. The length of each perpendicular is equal to a units. The distance between their feet being equal to b units. The area of the rhombus is

(a) $\dfrac{\sqrt{a^2+b^2}}{2\sqrt{b^2-a^2}}$ (b) $\dfrac{2ab}{2\sqrt{b^2-a^2}}$

(c) $\dfrac{ab^2}{2\sqrt{b^2-a^2}}$ (d) $\dfrac{2a^2b^2}{2\sqrt{b^2-a^2}}$

48. In a trapezoid $ABCD$, side BC is parallel to side AD. Also, the lengths of the sides AB, BC, CD and AD are 8, 2, 8 and 10 units respectively. Find the radius of the circle that passes through all four of the points A, B, C and D?

(a) $2\sqrt{3}$ (b) $2\sqrt{5}$ (c) $2\sqrt{11}$ (d) $2\sqrt{7}$

49. The length of the midline of a trapezoid equals 4 cm and the base angles are 40° and 50°. The length of the bases if the distance of their mid-points equals 1 cm is equal to

(a) 5 cm, 3 cm (b) 4 cm, 3 cm
(c) 7 cm, 4 cm (d) 6 cm, 5 cm

50. In the figure, find the value of

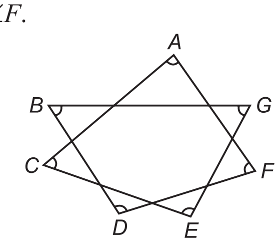
$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F$.

(a) 120°

(b) 720°

(c) 360°

(d) 540°



ANSWERS

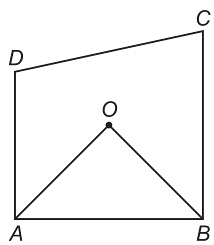
1. (*c*)	**2.** (*b*)	**3.** (*b*)	**4.** (*a*)	**5.** (*b*)	**6.** (*c*)	**7.** (*c*)	**8.** (*a*)	**9.** (*d*)	**10.** (*d*)
11. (*c*)	**12.** (*b*)	**13.** (*a*)	**14.** (*b*)	**15.** (*b*)	**16.** (*a*)	**17.** (*b*)	**18.** (*d*)	**19.** (*c*)	**20.** (*b*)
21. (*a*)	**22.** (*d*)	**23.** (*c*)	**24.** (*d*)	**25.** (*a*)	**26.** (*a*)	**27.** (*c*)	**28.** (*a*)	**29.** (*c*)	**30.** (*d*)
31. (*d*)	**32.** (*b*)	**33.** (*d*)	**34.** (*a*)	**35.** (*b*)	**36.** (*a*)	**37.** (*b*)	**38.** (*b*)	**39.** (*c*)	**40.** (*b*)
41. (*d*)	**42.** (*b*)	**43.** (*c*)	**44.** (*a*)	**45.** (*b*)	**46.** (*c*)	**47.** (*c*)	**48.** (*d*)	**49.** (*a*)	**50.** (*d*)

HINTS AND SOLUTIONS

3. In ΔAOB, $\angle AOB = 180° - \left(\frac{1}{2}\angle A + \frac{1}{2}\angle B\right)$ ...(*i*)

(Angle sum property of a Δ, OA bisects $\angle A$ and OB bisects $\angle B$)

In quadrilateral $ABCD$,



$\angle A + \angle B + \angle C + \angle D = 360°$

$\Rightarrow \angle C + \angle D = 360° - (\angle A + \angle B)$

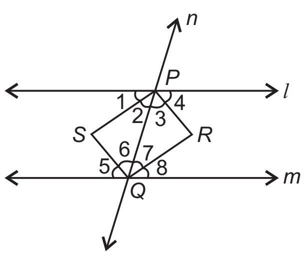
$\Rightarrow \frac{1}{2}(\angle C + \angle D)$

$= 180° - \frac{1}{2}(\angle A + \angle B)$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*)

$$\angle AOB = \frac{1}{2}(\angle C + \angle D).$$

4. Let l and m be the two given parallel lines and n the transversal intersecting them at points P and Q respectively. PS, PR and QS, QR are the bisectors of the interior angles at P and Q respectively.



Given, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, $\angle 5 = \angle 6$, $\angle 7 = \angle 8$

Also, $\angle 1 + \angle 2 = \angle 7 + \angle 8$, (alternate angles)

$\Rightarrow 2\angle 2 = 2\angle 7 \quad \Rightarrow \quad \angle 2 = \angle 7$ ...(*i*)

Also, $\angle 3 + \angle 4 = \angle 5 + \angle 6$, (alternate angles)

$\Rightarrow 2\angle 3 = 2\angle 6 \quad \Rightarrow \quad \angle 3 = \angle 6$ ...(*ii*)

Adding (*i*) and (*ii*), we get $\angle 2 + \angle 3 = \angle 6 + \angle 7$

$\Rightarrow \angle SPR = \angle SQR$

Also, $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 180°$ (St. $\angle$)

$\Rightarrow 2(\angle 2 + \angle 3) = 180° \quad \Rightarrow \quad \angle 2 + \angle 3 = 90°$

$\Rightarrow \angle SPR = \angle SQR = 90°$

Also, $\angle 1 + \angle 2 + \angle 5 + \angle 6 = 180°$

(co-int. $\angle s$ are supplementary)

$\Rightarrow 2(\angle 2 + \angle 6) = 180° \quad \Rightarrow \quad \angle 2 + \angle 6 = 90°$

$\therefore$ In ΔPSQ, $\angle PSQ = 180° - (\angle 2 + \angle 6)$

$= 180° - 90° = 90°$

Similarly, $\angle PRQ = 90°$.

As all four angles of $PRQS$ are each $= 90° \Rightarrow PRQS$ is a rectangle.

5. In a cyclic quadrilateral $ABCD$,

$\Rightarrow \angle A + \angle C = 180° \quad \Rightarrow \quad \angle A = 180° - \angle C$ ...(*i*)

$\Rightarrow \angle B + \angle D = 180° \quad \Rightarrow \quad \angle B = 180° - \angle D$ ...(*ii*)

Also, given that

$\angle A + \angle B = 2(\angle C + \angle D)$

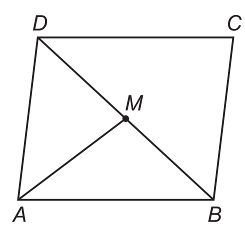
$\Rightarrow 180° - \angle C + 180° - \angle D = 2(\angle C + \angle D)$

$\Rightarrow 360° = 3(\angle C + \angle D)$

$\Rightarrow \angle C + \angle D = 120°$

Given $\angle C > 30° \quad \Rightarrow \quad \angle D = 120° - \angle C \quad \Rightarrow \quad \boldsymbol{\angle D < 90°}$.

6. If BM bisects $\angle B$, then AM bisects $\angle A$ as diagonals of a parallelogram bisect each other and here M is the point of intersection of the diagonals AC and BD. Also, diagonals of a prallelogram bisect the angles at the vertices they join.



$\Rightarrow \angle A + \angle B = 90°$

(Sum of adjacent angles of a parallelogram)

$\Rightarrow \frac{1}{2}(\angle A + \angle B) = 90°$

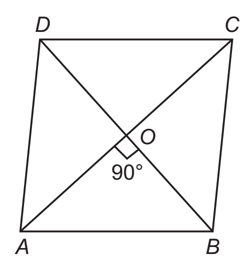
$\Rightarrow \angle MAB + \angle MBA = 90°$

In ΔAMB, $\angle MAB + \angle MBA + \angle AMB = 180°$

$\Rightarrow 90° + \angle AMB = 180° \quad \Rightarrow \quad \angle AMB = \mathbf{90°}$.

7. Diagonals of a rhombus bisect each other at right angles.

$\therefore$ In ΔAOB,



$OA^2 + OB^2 = AB^2$

($\because \angle AOB = 90°$)

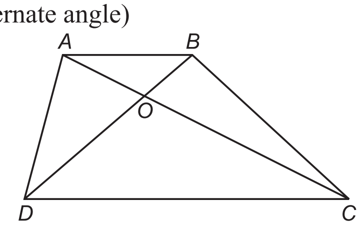
$\Rightarrow \left(\frac{1}{2}AC\right)^2 + \left(\frac{1}{2}BD\right)^2 = AB^2$

$\Rightarrow \frac{1}{4}AC^2 + \frac{1}{4}BD^2 = AB^2$

$\Rightarrow AC^2 + BD^2 = 4AB^2$.

8. In ΔAOB and ΔCOD,

$\left.\begin{array}{l}\angle BAO = \angle DCO \\ \angle ABO = \angle CDO\end{array}\right\}$ (alternate angle)



$\angle AOB = \angle COD$

(vert. opp. $\angle s$)

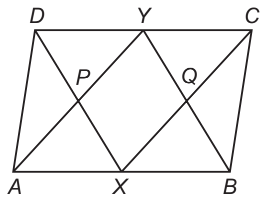
$\Rightarrow \Delta AOB \sim \Delta COD$

$\therefore \frac{OA}{OB} = \frac{OC}{OD}$

$\Rightarrow \dfrac{OA}{OC} = \dfrac{\mathbf{OB}}{\mathbf{OD}}$.

9. If a trapezium is inscribed in a circle, *i.e.*, it is a cyclic trapezium, then it is an isosceles trapezium, *i.e.*, non-parallel sides are equal $\Rightarrow AB = CD$.

10. Since X and Y are the mid-points of AB and DC respectively.



$AX = \frac{1}{2}AB$ and $CY = \frac{1}{2}DC$

But $AB = DC$

$\Rightarrow \frac{1}{2}AB = \frac{1}{2}DC$

$\Rightarrow AX = CY$

Also, $AB \parallel DC \Rightarrow AX \parallel CY$

Since a pair of opposite sides, AX and YC are equal and parallel, $AXCY$ is a parallelogram. Similarly, we can show that $BXDY$ is a parallelogram.

Now, $AXCY$ is a parallelogram.

$\Rightarrow AY \parallel CX$ (opp. sides are parallel)

$\Rightarrow PY \parallel QX$ ...(*i*)

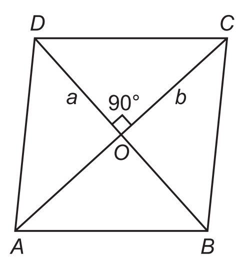
Also, $BXDY$ is a parallelogram $\Rightarrow BY \parallel XD$

$\Rightarrow QY \parallel PX$ ...(*ii*)

Thus, in quadrilateral $PYQX$, $PY \parallel QX$ and $QY \parallel PX$.

When both pair of opposite sides are parallel in a quadrilateral, it is a prallelogram.

11. Each side of the rhombus



$= \dfrac{8\sqrt{5}}{4} = 2\sqrt{5}$

Let $OD = a$ and $OC = b$ Then, in ΔCOD,

$a^2 + b^2 = \left(2\sqrt{5}\right)^2 = 20$ ...(*i*)

Also, $2a + 2b = 12$

$\Rightarrow a + b = 6$ ...(*ii*)

$\therefore$ From (*i*), $(a + b)^2 - 2ab = 20$

$\Rightarrow 36 - 2ab = 20 \Rightarrow 2ab = 16 \Rightarrow ab = 8$ ...(*iii*)

Now from (*ii*) and (*iii*), we get $a = 4$, $b = 2$

$\therefore$ Diagonals are 8 cm and 4 cm.

12. $AE \parallel BC$ and $AE = BC \Rightarrow AECB$ is a parallelogram.

(For a parallelogram one pair of opposite sides can be shown equal and parallel)

$\therefore \angle BCE = \angle BAE = 102°$

(Opposite angles of a parallelogram are equal)

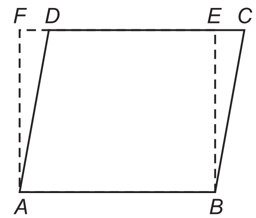
$AB = EC$ (Opposite sides of a parallelogram)

$ED = CD = EC$ ($\because AB = ED = CD$)

$\therefore \Delta EDC$ is equilateral $\Rightarrow \angle ECD = 60°$

$\therefore \angle BCD = \angle BAE + \angle ECD = 102° + 60° = 162°$.

13. $ABCD$ and $ABEF$ are parallelogram and rectangle respectively on the same base AB having equal perpendicular height. Therefore, the area of both the figures is equal.



Now, $AD > AF$ and $BC > BE$

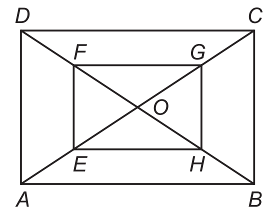
(hypotenuse > perpendicular)

$\therefore 2(AB + BC) > 2(AB + BE)$

$\Rightarrow$ Perimeter of parallelogram > Perimeter of rectangle

$\Rightarrow \boldsymbol{k > 1}$.

14. By mid-point theorem, in Δs ODA, OCD, OBC, OBA respectively



$\dfrac{EF}{AD} = \dfrac{FG}{DC} = \dfrac{GH}{CB} = \dfrac{HE}{BA} = \dfrac{1}{2}$

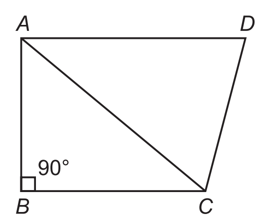
If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = ...,$

then each ratio is equal to $\dfrac{a + c + e + ...}{b + d + f + ...}$.

$\Rightarrow \dfrac{(EF + FG + GH + HE)}{(AD + DC + CB + BA)} = \dfrac{\mathbf{1}}{\mathbf{2}}$.

15. In a cyclic parallelogram each angle is equal to 90° as, sum of opposite angles is 180° and opposite angles are equal. So it is definitely either a square or a rectangle. As the given cyclic parallelogram has unequal adjacent sides, it is a rectangle.

16. Given,



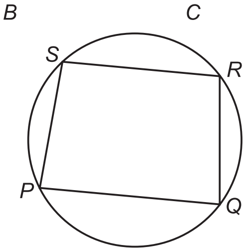
$AD^2 = AB^2 + BC^2 + CD^2$

$= AC^2 + CD^2$

(In ΔABC, $AB^2 + BC^2 = AC^2$)

$\Rightarrow \angle ACD = \mathbf{90°}$.

17. Given, $\angle P = 2\angle R$ ...(*i*)



Also, $\angle P + \angle R = 180°$ ...(*ii*)

$\angle Q + \angle S = 180°$ ...(*iii*)

$\therefore$ From (*i*) and (*ii*)

$\Rightarrow \angle R = 60°$

$\Rightarrow \angle P = 120°$

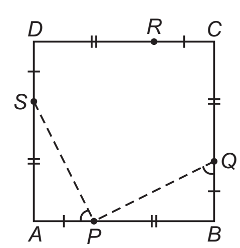
Also, given $\angle Q - \angle S = \frac{1}{3}\angle P = \frac{1}{3} \times 120° = 40°$ ...(*iv*)

Solving (*iii*) and (*iv*) simultaneously, we get

$\angle Q = 110°$, $\angle S = 70°$

$\therefore$ Minimum difference between any two angles of the quadrilateral is 10°.

18. In Δs SAP and PBQ,



$\left.\begin{array}{l} AS = BP \\ AP = BQ \end{array}\right\}$ Given

$\angle A = \angle B = 90°$

$\therefore \Delta SAP \cong \Delta PBQ$ (*SAS*)

$\Rightarrow SP = PQ$

$\angle SPA = \angle BQP = x$ (say)

and $\angle PSA = \angle QPB = y$ (say)

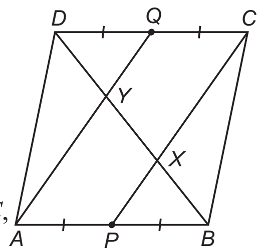
In ΔSAP or ΔPBQ

$x + y = 90°$

$\therefore\ \angle SPQ = 180° - (\angle SPA + \angle BPQ)$

$= 180° - (x + y) = 180° - 90° = \mathbf{90°}$.

19. $ABCD$ is a parallelogram.



$\Rightarrow\ AB = CD$ and $AB \parallel CD$

$\Rightarrow\ AP = QC$ and $AP \parallel QC$

$\Rightarrow\ APCQ$ is a parallelogram.

$\Rightarrow\ AQ \parallel PC$

$\therefore$ By intercept theorem in ΔDXC,

we have $\dfrac{DY}{YX} = \dfrac{DQ}{QC} = 1$

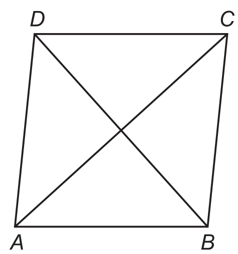
$\Rightarrow\ DY = YX$

Also using intercept theorem in ΔABY,

$\dfrac{BX}{XY} = \dfrac{BP}{PA} = \dfrac{1}{1} \ \Rightarrow\ BX = XY.$

$\therefore\ BX = XY = DY.$

20. Given, $AC = 2$ mn



$BD = m^2 - n^2$

$AB = \dfrac{m^2 + n^2}{2}$

By Apollonius theorem, we know that

$AC^2 + BD^2 = 2(AB^2 + BC^2)$

$\Rightarrow (2\,mn)^2 + (m^2 - n^2)^2 = 2\left\{\dfrac{1}{4}(m^2 + n^2)^2 + BC^2\right\}$

$\Rightarrow 4\,m^2n^2 + m^4 + n^4 - 2m^2n^2 = \dfrac{1}{2}(m^2 + n^2)^2 + 2BC^2$

$\Rightarrow (m^2 + n^2)^2 = \dfrac{1}{2}\ (m^2 + n^2)^2 + 2BC^2$

$\Rightarrow\ 2BC^2 = \dfrac{1}{2}(m^2 + n^2)^2 \ \Rightarrow\ BC^2 = \dfrac{(m^2 + n^2)^2}{4}$

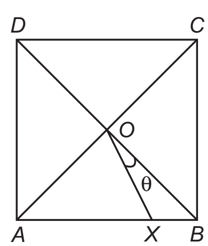
$\Rightarrow\ BC = \dfrac{m^2 + n^2}{2} = AB$

$\Rightarrow ABCD$ is a rhombus as adjacent sides are equal.

Let $AC > BD \ \Rightarrow\ 2mn > m^2 - n^2$

$\Rightarrow (m + n)^2 > 2m^2$, which is always true for every positive integer m, n where $n < m < 2n$.

21. Let $\angle XOB = \theta$.



In ΔOXB,

$\angle XOB + \angle XBO + \angle OXB = 180°$

$\Rightarrow \theta + 45° + \angle OXB = 180°$

$\Rightarrow \angle OXB = 180° - 45° - \theta = 135° - \theta$.

Also, $\angle OXB + \angle OXA = 180°$

$\Rightarrow \angle OXA = 180° - \angle OXB = 180° - (135° - \theta) = 45° + \theta$.

In ΔOAX, $AO = OX$

$\Rightarrow \angle OXA = \angle AOX = 45° + \theta$

Also, we know that diagonals of a square intersect each other at right angles, so

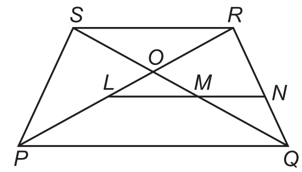
$\Rightarrow \angle AOX + \angle XOB = 90°$

$\Rightarrow 45° + \theta + \theta = 90°$

$\Rightarrow 2\theta = 45° \ \Rightarrow\ \theta = \mathbf{22.5°}$.

22. Produce LM to N which intersects side RQ at N.

By mid-point theorem,



In ΔPQR, $LN \parallel PQ$

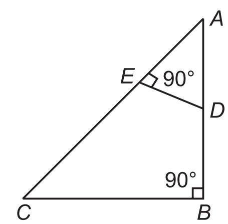
and $LN = \dfrac{1}{2} PQ$ and

in ΔQRS, $MN \parallel RS$

and $MN = \dfrac{1}{2} RS$

$\therefore\ LM = LN - MN = \mathbf{\dfrac{1}{2}\ (PQ - RS)}$.

23. In Δs ABC and ΔAED



$\angle ABC = \angle AED = 90°$

$\angle A = \angle A$

$\therefore\ \angle ABC \sim \angle AED$ (AA similarly)

Also, $\angle DEC = 180° - 90° = 90°$.

In quadrilateral $EDBC$

$\angle E + \angle D + \angle C + \angle B = 360° \ \Rightarrow \angle C + \angle D = 180°$

$\therefore$ Opposite angles are supplementary, $EDBC$ is cylic quadrilateral.

24. In ΔABX and ΔADZ,

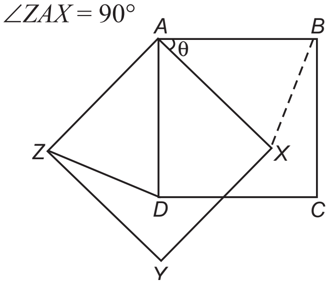
$AB = AD$ (Sides of square $ABCD$)

$AX = AZ$ (Sides of suqare $AXYZ$)

Let $\angle BAX = \theta$

$\therefore\ \angle XAD = 90º - \theta$

Also, $AXYZ$ being a square, $\angle ZAX = 90°$



$\Rightarrow \angle ZAD + \angle XAD = 90°$

$\Rightarrow \angle ZAD = 90° - \angle XAD$

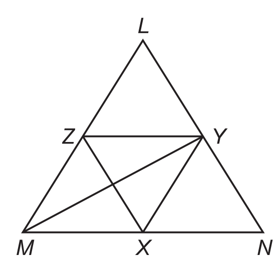
$= 90° - (90° - \theta) = \theta$

$\Rightarrow \angle BAX = \angle ZAD$

$\therefore\ \Delta ABX \cong \Delta ADZ$

$\Rightarrow BX = DZ.$

25. Since $ZY \parallel MN$ and $ZX \parallel YN$,



$XNYZ$ is a parallelogram

$\therefore\ ZX = YN$ ...(i)

Also, $ZX \parallel YN$ and $XY \parallel ZL$

($\because\ XY \parallel ZM$)

$\Rightarrow XYLZ$ is a parallelogram

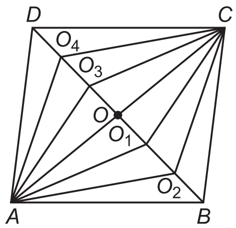
$\therefore\ ZX = LY$ ...(ii)

From (i) and (ii), $YN = LY$

$\Rightarrow\ MY$ is the median of ΔLMN.

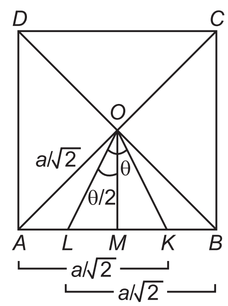
26. In a rhombus, diagonals bisect each other. Suppose in the given rhombus $ABCD$, diagonals bisect each other at point O. Then, the distances of O from the four vertices A, B, C

and D are equal. Thus, even if we take fixed point on diagonal BD as O_1, O_2, O_3, O_4, ..., they all are equidistant from the vertices A and C (by property of congruent triangles). Hence the locus of a point in rhombus $ABCD$ which is equidistant from A and C is a fixed point on diagonal BD.



Alternatively, since diagonals of a rhombus bisect each at rt. $\angle s$, therefore, BD is the single bisector of AC, and therefore, all points equidistant from A and C lie on it.

27. Let each side of the square be a.



Then, $AC = a\sqrt{2}$ and $AO = OC = \dfrac{a}{\sqrt{2}}$

$AM = \dfrac{a}{2}$, $AK = AO = \dfrac{a}{\sqrt{2}}$

$\text{LM} = \dfrac{a}{\sqrt{2}} - \dfrac{a}{2}$ and $OM = \dfrac{a}{2}$

In ΔOML, $\tan\dfrac{\theta}{2} = \dfrac{LM}{OL}$

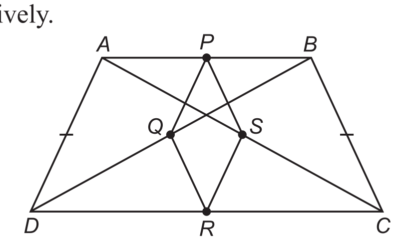
$$= \frac{\frac{a}{\sqrt{2}} - \frac{a}{2}}{\frac{a}{2}} = \frac{\frac{\sqrt{2}-1}{2}}{\frac{1}{2}} = \sqrt{2}-1$$

Now, $\tan\theta = \dfrac{2\tan\theta/2}{1-\tan^2\theta/2}$ $\left(\because\ \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}\right)$

$$= \frac{2(\sqrt{2}-1)}{1-(\sqrt{2}-1)^2} = \frac{2(\sqrt{2}-1)}{1-3+2\sqrt{2}} = \frac{2(\sqrt{2}-1)}{2\sqrt{2}-2}$$

$\Rightarrow\ \tan\theta = 1.$

28. In ΔBDC, Q and R are the mid-points of BD and CD respectively.



$\therefore\ QR \parallel BC$ and $QR = \dfrac{1}{2}BC$

Similarly, in ΔABC, $PS \parallel BC$ and $PS = \dfrac{1}{2}BC$

$\therefore\ PS = QR$

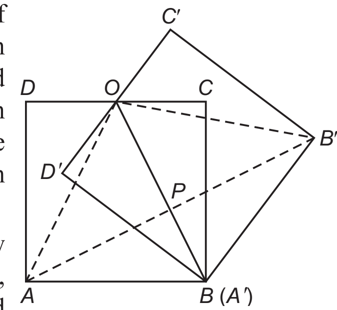
Also in ΔABD, $PQ \parallel AD$ and $PQ = \dfrac{1}{2}AD$

In ΔADC, $SR \parallel AD$ and $SR = \dfrac{1}{2}AD$

$\therefore\ PQ = SR$

($AD = BC \Rightarrow PS = QR = PQ = SR \Rightarrow PQRS$ is a rhombus.

29. Let O be the mid-point of CD. Since the new position of A coincides with the old position of B, the rotation is in the counter-clockwise direction about O, through the angle AOB.



Let B', C', D' be the new positions of B, C, D, respectively. Also, let OB and AB' intersect at point P. We can also see that $\angle AOB = \angle BOB'$

Also, $OA = OB = OB'$. This means that OB is the angle bisector of $\angle AOB'$ of isosceles triangle AOB'.

This implies that $OP \perp AB'$ and OP bisects AB'

$\Rightarrow\ AP = PB'$

Area of $\Delta AOB = \dfrac{1}{2}$ (Area of square $ABCD$)

$= \dfrac{1}{2}$ square units ... (ii)

But area of $\Delta AOB = \dfrac{1}{2} \times AP \times OB$ ...(ii) ($\because\ AP \perp OP$)

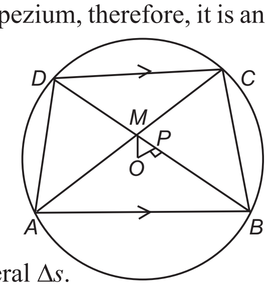
$$OB = \sqrt{OC^2 + CB^2} = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$\therefore$ From (i) and (ii),

$$\frac{1}{2} = \frac{1}{2} \times AP \times \frac{\sqrt{5}}{2}$$

$$\Rightarrow AP = \frac{2}{\sqrt{5}} \quad \Rightarrow \quad AB' = 2AP = 2 \times \frac{2}{\sqrt{5}} = \frac{4}{\sqrt{5}} = \mathbf{\frac{4\sqrt{5}}{5}}.$$

30. Since the trap. ABCD is cyclic trapezium, therefore, it is an isosceles trapezium.



It is given that,

$\angle AMB = 60°$

$\Rightarrow \angle CMD = 60°$ (vert. opp. $\angle s$)

Since $AM = BM$ and $DM = CM$,

(*Property of Isos. trap.*)

ΔAMB and so ΔCMD are equilateral Δs.

Draw $OP \perp BD$. Now OM bisects $\angle AMB$ and so $\angle OMP = 30°$

In ΔOMP,

$\sin 30° = \dfrac{OP}{OM} \ \Rightarrow\ OP = \dfrac{OM}{2} = \dfrac{2}{2} = 1$

$\therefore \angle OPM = 90°$, $\therefore\ PM = \sqrt{OM^2 - OP^2} = \sqrt{2^2 - 1^2} = \sqrt{3}$

$\therefore\ AB - CD = AM - CD = BM - MD$

($\because\ AB = AM$ and $CD = MD$)

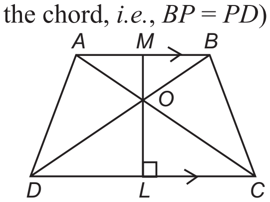
$= (BP + PM) - (PD - PM) = BP + PM - BP + PM$

($\because$ Perpendicular drawn from the centre bisects the chord, *i.e.*, $BP = PD$)

$= 2\,PM = 2 \times \sqrt{3} = \mathbf{2\sqrt{3}}.$

31. $\angle BDC = \angle DBA$
(alternate angles)

$\angle ACD = \angle CAB$
(alternate angles)



$\angle DOC = \angle AOB$ (vert. opp. $\angle s$)

$\Rightarrow \Delta DCO \sim \Delta BAO$

$\Rightarrow \dfrac{\text{Area of } \Delta ABO}{\text{Area of } \Delta DCO} = \dfrac{AB^2}{DC^2}$

(By property of similar triangles)

$\Rightarrow \dfrac{p}{q} = \dfrac{AB^2}{DC^2} \quad \Rightarrow \quad \dfrac{AB}{DC} = \dfrac{\sqrt{p}}{\sqrt{q}}$

$\Rightarrow \; AB = \sqrt{p} \cdot k$ and $DC = \sqrt{q} \cdot k$ for some constant k.

Area of $\Delta DOC = \dfrac{1}{2} \times DC \times OL$

$\Rightarrow q = \dfrac{1}{2} \times \sqrt{q} \cdot k \times OL \quad \Rightarrow OL = \dfrac{2q}{\sqrt{q} \cdot k} = \dfrac{2\sqrt{q}}{k}$

Area of $\Delta AOB = \dfrac{1}{2} \times AB \times OM$

$\Rightarrow p = \dfrac{1}{2} \times \sqrt{p} \cdot k \times OM \quad \Rightarrow \quad OM = \dfrac{2q}{\sqrt{p} \cdot k} = \dfrac{2\sqrt{p}}{k}$

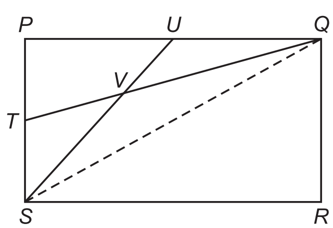
Area of a trapezium $= \dfrac{1}{2} \times$ height of trapezium $\times$ sum of parallel sides

$= \dfrac{1}{2} \times (OL + OM) \times (DC + AB)$

$= \left(\dfrac{\dfrac{2\sqrt{q}}{k} + \dfrac{2\sqrt{p}}{k}}{2} \right) \left(\sqrt{q}\, k + \sqrt{p}\, k \right)$

$= \boldsymbol{\left(\sqrt{p} + \sqrt{q}\right)^2}$**.**

32. U and T are mid-points of PQ and PS respectively.



$\Rightarrow$ SU and QT are medians of ΔPSQ.

$\Rightarrow$ V is the centroid of ΔPSQ.

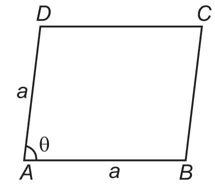
$\therefore \dfrac{\text{Area of quad. } QRSV}{\text{Area of } \Delta PQT} = \dfrac{\text{Area of } \Delta QRS + \text{Area of } \Delta QSV}{\dfrac{\text{Area of } \Delta PQS}{2}}$

$= \dfrac{\text{Area of } \Delta PQS + \dfrac{\text{Area of } \Delta PQS}{3}}{\dfrac{\text{Area of } \Delta PQS}{2}}$

$\left[\begin{array}{l} \because QS \text{ divides rectangle } PQRS \text{ into} \\ \quad \text{two equal } \Delta s\ PQS \text{ and } QRS. \\ \text{Also, Area of } \Delta QSV = \text{Area of } \dfrac{\Delta PQS}{3} \end{array} \right]$

$= \dfrac{\frac{4}{3}}{\frac{1}{2}} = \boldsymbol{\dfrac{8}{3}}$**.**

33. Let acute the angle between adjacent sides of the rhombus be θ and let each side of the rhombus $= \boldsymbol{a}$ units. Then,



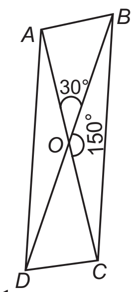
Area of rhombus $ABCD = a^2 \sin \theta$

Given, area $= \dfrac{1}{2}$ and $a = 1$

$\therefore \; \dfrac{1}{2} = 1 \times \sin \theta$

$\Rightarrow \quad \sin \theta = \dfrac{1}{2} \quad \Rightarrow \quad \theta = \mathbf{30°.}$

34. Area of quad. $ABCD$



$=$ Area (ΔAOB) + Area (ΔBOC) + Area (ΔCOD) + Area (ΔDOA)

$= \dfrac{1}{2} \cdot OA \cdot OB \sin 30° + \dfrac{1}{2}\, OB \cdot OC \sin 150°$

$+ \dfrac{1}{2}\, OC \cdot OD \sin 30° + \dfrac{1}{2}\, OD \cdot OA \sin 150°$

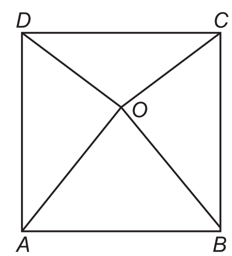
$= \dfrac{1}{4}\, OA \cdot OB + \dfrac{1}{4}\, OB \cdot OC + \dfrac{1}{4}\, OC \cdot OD + \dfrac{1}{4}\, OD \cdot OA$

$\left(\because \sin 30° = \sin 150° = \dfrac{1}{2} \right)$

$= \dfrac{1}{4}\, (OA + OC)\, (OB + OD) = \dfrac{1}{4} \times AC \times BD$

$= \dfrac{1}{4} \times 24 \times 22 = \mathbf{132.}$

35. ΔAOB is equilateral



$\Rightarrow AO = OB = AB$ and

$\angle OAB = \angle OBA = \angle AOB = 60°$

Also, $ABCD$ being a square,

$AB = BC = CD = AD$

$\Rightarrow \quad AO = AD \quad$ and $\quad BO = BC$

$\angle DAO = \angle CBO = 30° \quad (\because \angle OAB = \angle OBA = 60°)$

In ΔADO, $AO = AD \quad \Rightarrow \quad \angle ADO = \angle AOD$

$= \dfrac{1}{2}\, (180° - \angle DAO) = \dfrac{1}{2}\, (180° - 30°) = 75°$

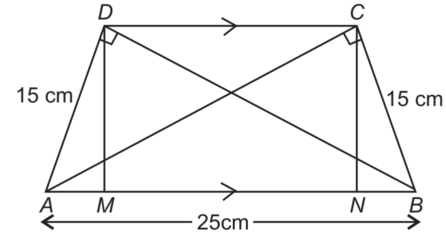
Similarly, $\angle BOC = 75°$.

$\therefore \; \angle DOC = 360° - (\angle AOB + \angle AOD + \angle BOC)$

$= 360° - (60° + 75° + 75°) = 360° - 210° = \mathbf{150°.}$

36. By Pythagoras' Theorem,

$AC = \sqrt{AB^2 - BC^2}$ and $BD = \sqrt{AB^2 - AD^2}$



$\Rightarrow \; AC = BD = \sqrt{625 - 225} = \sqrt{400} = 2$ cm

(By prop.of isos. trap. AC = BD)

Now, Area of $\Delta DAB = \frac{1}{2} \times AD \times BD$

Also, Area of $\Delta DAB = \frac{1}{2} \times DM \times AB$

$\therefore\ AD \times BD = DM \times AB$

$\Rightarrow\ DM = \frac{AD \times BD}{AB} = \frac{15 \times 20}{25} = 12$ cm

Also, $CN = DM = 12$ cm

$AM = \sqrt{AD^2 - DM^2} = \sqrt{15^2 - 12^2} = \sqrt{225 - 144}$

$= \sqrt{81} = 9$ cm

Also, $BN = AM = 9$ cm

$MN = AB - (AM + BN) = 25 - (9 + 9) = 25 - 18 = 7$ cm

$\Rightarrow CD = MN = 7$ cm

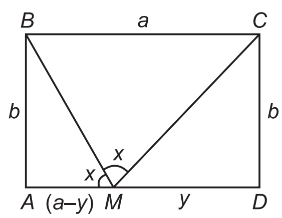
$\therefore$ Area of trapezium $ABCD = \frac{1}{2} \times DM \times (AB + CD)$

$= \frac{1}{2} \times 12 \times (25 + 7) = \frac{1}{2} \times 12 \times 32 =$ **192 cm^2.**

37. Let $\angle BMA = \angle BMC = x$

Then $\angle CMD = \pi - 2x$

Let $MD = y$. Then $AM = (a - y)$



In ΔCMD,

$\tan(\pi - 2x) = \frac{b}{y}$ ...(*i*)

In ΔBAM,

$\tan x = \frac{b}{a - y}$ ...(*ii*)

From (*i*), we have $\tan(\pi - 2x) = -\tan 2x = \frac{b}{y}$

$\Rightarrow -\frac{2\tan x}{1 - \tan^2 x} = \frac{b}{y}$

$\Rightarrow -\frac{2b/(a-y)}{1 - \left(b/(a-y)\right)^2} = \frac{b}{y}$

$\Rightarrow -\frac{2b(a-y)}{(a-y)^2 - b^2} = \frac{b}{y}$

$\Rightarrow -\frac{2ba - 2by}{a^2 + y^2 - 2ay - b^2} = \frac{b}{y}$

$\Rightarrow -2bya + 2by^2 = ba^2 + by^2 - 2aby - b^3$

$\Rightarrow by^2 = b(a^2 - b^2)$

$\Rightarrow\ y = \boldsymbol{\sqrt{a^2 - b^2}}$**.**

38. Let each side of the square $= x$ units

Let $AM = BN = CR = y$ units

$\Rightarrow MB = CN = DR$

$= (x - y)$ units

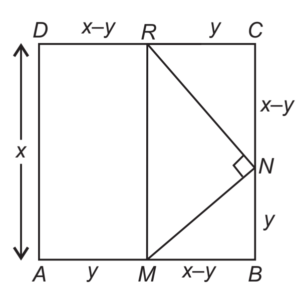
In Δs MBN and NCR

$MB = NC = (x - y)$

$BN = CR = y$

$\angle MBN = \angle NCR = 90°$

$\Rightarrow \Delta MBN \cong \Delta NCR$



$\Rightarrow MN = NR$

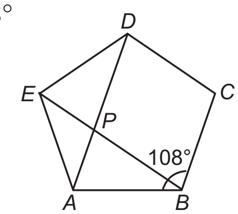
Given, $\angle MNR = 90°$

$\Rightarrow\ \Delta MNR$ is an isosceles right angled triangle.

$\Rightarrow \angle MRN = \angle NMR =$ **45°.**

39. Sum of the interior angles of a regular pentagon = 540°

$\Rightarrow$ Each interior angle $= \frac{540°}{5} = 108°$



As $AD \parallel BC$

$\angle BAD + \angle ABC = 180°$ (co-int. $\angle s$)

$\Rightarrow \angle BAD = 180° - 108° =$ **72°**

$\Rightarrow \angle PAE = 108° - 72° = 36°$

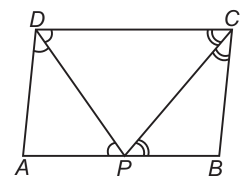
Also, applying $BE \parallel DC$, $\angle BED = 72° \Rightarrow \angle PEA = 36°$

$\therefore\ \angle APE = 180° - (\angle PAE + \angle PEA)$

(Angle sum property of a Δ)

$= 180° - (36° + 36°) = 180° - 72° =$ **108°.**

40. Given, DP bisects $\angle CDA$



$\Rightarrow \angle CDP = \angle ADP$

$\Rightarrow \angle APD = \angle CDP$

(alt. $\angle s$ $CD \parallel AB$)

$\Rightarrow AD = AP$ (Isosceles Δ property)

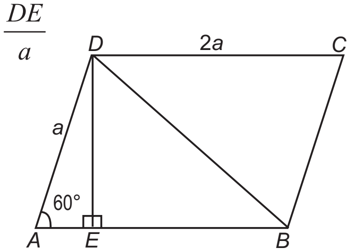
Similarly, $\angle CPB = \angle PCD = \angle PCB$

$\Rightarrow BP = BC$

$DC = AB = AP + BP = AD + BC =$ **2*BC*.**

41. In ΔDAE,

$\cos 60° = \frac{AE}{a}$, $\sin 60° = \frac{DE}{a}$



$\Rightarrow\ AE = \frac{1}{2}a,\ DE = \frac{\sqrt{3}}{2}a$

$\therefore\ BE = BA - AE$

$= 2a - \frac{1}{2}a = \frac{3a}{2}$

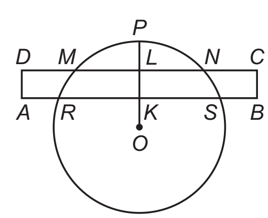
$\therefore$ In right angled ΔBDE,

$BD = \sqrt{BE^2 + DE^2} = \sqrt{\left(\frac{3a}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}a\right)^2}$

$= \sqrt{\frac{9a^2}{4} + \frac{3a^2}{4}} = \sqrt{\frac{12a^2}{4}} = \boldsymbol{\sqrt{3}a}$**.**

42. $OS = OP = ON =$ radii of the circle = 5 cm

$(OS)^2 = (OK)^2 + (KS)^2$



$\Rightarrow\ 25 = (OK)^2 + 16$

$\Rightarrow\ (OK)^2 = 25 - 16 = 9$

$\Rightarrow\ OK = 3$ cm

$LC = KB$

$\Rightarrow LN + NC = KS + SB$

$\Rightarrow\ LN + 2 = (4 + 1)$ cm $\Rightarrow\ LN = 3$ cm

Similarly, $(ON)^2 = (OL)^2 + (LN)^2$

$\Rightarrow\ 25 = (OL)^2 + 9$

$\Rightarrow \quad (OL)^2 = 16 \quad \Rightarrow \quad OL = 4$ cm

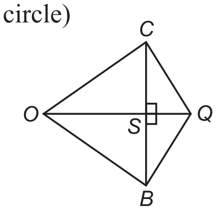
$\therefore \quad LK = OL - OK = (4 - 3)$ cm $= 1$ cm

Also, $\quad AB = DC = 10$ cm

$\therefore$ Area of rectangle $ABCD = AB \times BC = 10$ cm $\times$ 1 cm

$= \mathbf{10\ cm^2}$.

43. Let $OQ = OB = OC = r$ (radius of the circle)

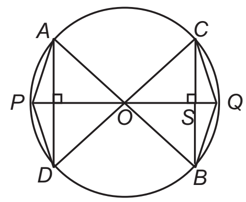


Given, $\angle AOD = \angle BOC = 120°$

$\therefore \angle BOQ = \angle COQ = 60°$

$\Rightarrow \dfrac{SB}{OB} = \sin 60° = \dfrac{\sqrt{3}}{2}$

$\Rightarrow SB = \dfrac{r\sqrt{3}}{2}$



$\therefore BC = 2\,SB = r\sqrt{3}$

$\therefore$ Area of quadrilateral $BQCO$

$= \dfrac{1}{2} \times$ product of diagonals

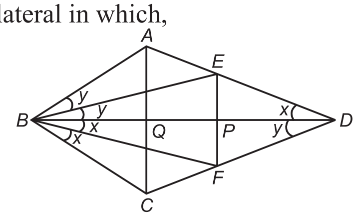
$= \dfrac{1}{2} \times BC \times OQ$

$= \dfrac{1}{2} \times r\sqrt{3} \times r = \dfrac{r^2\sqrt{3}}{2}$ cm^2

$\therefore$ Area of both the quadrilaterals

$= 2 \times \dfrac{r^2\sqrt{3}}{2} = \mathbf{r^2\sqrt{3}\ cm^2}$.

44. Given, $ABCD$ is a quadrilateral in which,



(1) $AB = BC$

(2) $\angle ABD = 2\angle BDC$

(3) $\angle DBC = 2\angle ADB$

Let BE and BF be the bisectors of $\angle ABD$ and $\angle CBD$ respectively.

Then, $\angle ABE = \angle EBD = \angle BDF = y$ and

$\angle CBF = \angle FBD = \angle BDE = x$

Now, $\angle EDB = \angle FBD$ and $\angle EBD = \angle BDF$

$\Rightarrow$ alternate angles are equal $\Rightarrow ED \parallel BF$ and $BE \parallel FD$

$\Rightarrow BEDF$ is a parallelogram.

Let EF and BD cut each other at P and AC and BD cut each other at Q

$\therefore$ diagonals of a parallelogram bisect each other,

$EP = PF$

Now by the bisector theoram

$\dfrac{AE}{ED} = \dfrac{AB}{BD} = \dfrac{BC}{BD} = \dfrac{CF}{FD}$ $\quad (\because AB = BC)$

$\Rightarrow \dfrac{AE}{ED} = \dfrac{CF}{FD}$

$\Rightarrow$ By the converse of Basic proportionality theoram,

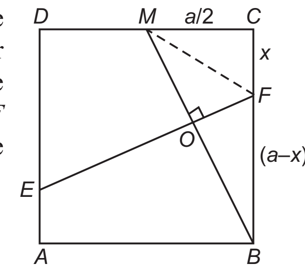
$EF \parallel AC$

$\therefore EP = PF \quad \Rightarrow \quad AQ = QC$

$\therefore \dfrac{AB}{AQ} = \dfrac{BC}{QC} \quad \Rightarrow \quad BQ$ bisects $\angle ABC \quad \Rightarrow \quad x = y$

$\Rightarrow \Delta ABD \cong \Delta BCD \quad \Rightarrow \quad \mathbf{AD = CD.}$

45. By physically folding the square sheet of paper as per the conditions stated in the question, we can see that EF is the crease along which the fold is made. Also,



$MF = FB$ and $MB \perp EF$.

We need to find $BF : FC$.

Let each side of the square be a units and $CF = x$ units

Then, $BF = (a - x)$ units and $MF = (a - x)$ units.

In ΔMFC, $MF^2 = MC^2 + FC^2$

$\Rightarrow FB^2 = MC^2 + FC^2$ $\quad$ (Since $MF = FB$)

$\Rightarrow (a - x)^2 = (a/2)^2 + x^2$

$\Rightarrow a^2 + x^2 - 2ax = a^2/4 + x^2$

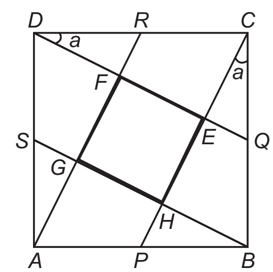
$\Rightarrow 2ax = a^2 - \dfrac{a^2}{4} = \dfrac{3a^2}{4}$

$\Rightarrow x = \dfrac{3a}{8} \quad \Rightarrow \quad a - x = a - \dfrac{3a}{8} = \dfrac{5a}{8}$

$\therefore BF : FC = (x - a) : x$

$= \dfrac{5a}{8} : \dfrac{3a}{8} = \mathbf{5 : 3.}$

46. The quadrilateral formed by joining AR, BS, CP and DQ is $EFGH$.



Now, $\Delta DCQ \cong \Delta PBC$ as,

$CD = BC$ $\quad$ (sides of a square)

$\angle DCQ = \angle PBC = 90°$

$CQ = PB$

(Half of sides of a square)

$\therefore \angle CDQ = \angle BCP = a$ (say)

Then, $\angle DQC = \angle EQC = 90° - a$

$\therefore$ In ΔEQC, $\angle CEQ = 180° - (\angle QCE + \angle CQE)$

$= 180° - (a + 90° - a) = 90°$

$\Rightarrow \angle FEH = 90°$

Similarly, $\angle EFG = \angle FGH = \angle GHE = 90°$

$\Rightarrow$ Vertex angles of quadrilateral $EFGH$ are 90° each

$\Rightarrow EFGH$ is a rectangle.

Now, $\Delta DRF \cong \Delta CQE \quad \Rightarrow \quad RF = EQ, \; DF = EC$

Following the same concept,

$BH = CE = AG = DF$ and

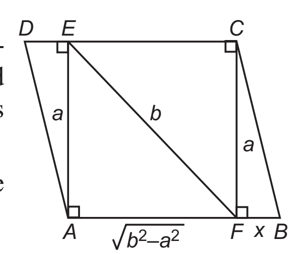
$RF = EQ = PH = GS$

$\therefore EF = FG$ $\quad$ (Adjacent sides of $EFGH$ are equal)

$\therefore$ A rectangle whose adjacent sides are equal is a square

$\Rightarrow EFGH$ is a square.

47. Let AE and CF be the perpendiculars drawn from vertex A and vertex C respectively on sides CD and AB.



$\Rightarrow E$ and F are the feet of the perpendiculars on the sides.

Given, $AE = CF = a$ and $EF = b$.

$$AF = EC = \sqrt{b^2 - a^2}$$

$\therefore$ Area of $AFCE$ = AE × AF = $a \times \sqrt{b^2 - a^2}$

Let $FB = x$. Then, $CB = \sqrt{a^2 + x^2}$

As $ABCD$ is a rhombus,

$$CB = AB$$

$$\Rightarrow \quad \sqrt{a^2 + x^2} = \sqrt{b^2 - a^2} + x$$

$$\Rightarrow \quad a^2 + x^2 = b^2 - a^2 + x^2 + 2x\sqrt{b^2 - a^2}$$

$$\Rightarrow \quad 2a^2 - b^2 = 2x\sqrt{b^2 - a^2}$$

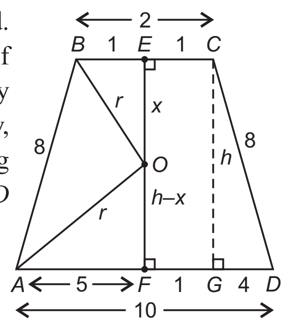
$$\Rightarrow x = \frac{2a^2 - b^2}{2\sqrt{b^2 - a^2}}$$

$\therefore$ Area of rhombus $ABCD$ = base × height = $AB \times AE$

$$= AB \times a = \left(\sqrt{b^2 - a^2} + \frac{2a^2 - b^2}{2\sqrt{b^2 - a^2}}\right) \times a$$

$$= \left(\frac{2(b^2 - a^2) + 2a^2 - b^2}{2\sqrt{b^2 - a^2}}\right) a = \frac{\boldsymbol{ab^2}}{\boldsymbol{2\sqrt{b^2 - a^2}}}.$$

48. Let $ABCD$ be the given trapezoid. Let E and F be the mid-points of the sides BC and AD respectively of trapezoid $ABCD$. By symmetry, the centre of the circle passing through the points A, B, C and D lies on EF. Let the centre be O.



Then, $OB = OA = r$ (say) (radii of the circle)

Let h be the height of the altitude EF of the trapezoid $ABCD$. Then, as we can see in the diagram, in ΔCGD,

$$h^2 + 4^2 = (8)^2 \quad \Rightarrow \quad h^2 = 64 - 16 = 48 \quad \Rightarrow \quad h = 4\sqrt{3}$$

Now let $EO = x$, so $OF = h - x = 4\sqrt{3} - x$

In ΔOBE, $OB = r = \sqrt{OE^2 + BE^2} = \sqrt{x^2 + 1}$ ...(i)

In ΔOAF, $OA = r = \sqrt{OF^2 + AF^2} = \sqrt{(4\sqrt{3} - x)^2 + 5^2}$ ...(ii)

$\therefore$ From (i) and (ii)

$$\sqrt{x^2 + 1} = \sqrt{(4\sqrt{3} - x)^2 + 5^2}$$

$$\Rightarrow x^2 + 1 = 48 - 8\sqrt{3}x + x^2 + 25$$

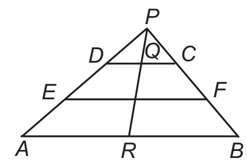
$$\Rightarrow 8\sqrt{3}x = 72 \quad \Rightarrow \quad x = \frac{72}{8\sqrt{3}} = 3\sqrt{3}$$

$$\therefore r = \sqrt{x^2 + 1} = \sqrt{(3\sqrt{3})^2 + 1} = \sqrt{28}$$

$$\Rightarrow r = \mathbf{2\sqrt{7}}.$$

49. Let $ABCD$ be the given trapezoid, whose base angles $\angle A$ and $\angle B$ equal 40° and 50°. Also mid-line EF = 4 cm.

Extend AD and BC to meet at P.



Now in ΔAPB,

$\angle APB = 180° - (40° + 50°) = 90°$

Also, $\Delta ABP \sim \Delta DCP$

$$\Rightarrow \frac{AB}{DC} = \frac{AP}{DP} = \frac{BP}{CP} \qquad ...(i)$$

Let Q and R be the mid-points of DC and AB respectively.

$\therefore DC = 2DQ$ and $AB = 2AR$

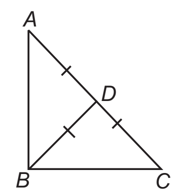
$\therefore$ From (i)

$$\frac{AB}{DC} = \frac{AP}{DP} = \frac{BP}{CP} \Rightarrow \frac{2\,AR}{2\,DQ} = \frac{AP}{DP} = \frac{BP}{CP}$$

$$\Rightarrow \frac{AR}{DQ} = \frac{AP}{DP} = \frac{BP}{CP}$$

$\Rightarrow \Delta PDQ$ and ΔPAR are similar $\Rightarrow$ P, Q, R are collinear.

Now, as ΔAPB and ΔDPC are right angled Δs



$AR = RB = PR$ and $DQ = QC = PQ$.

[If D is the mid-pt. of hyp. AC, then $BD = AD = CD$]

$$\therefore \frac{AB}{2} - \frac{DC}{2} = AR - DQ = PR - PQ = QR = 1$$

$$\Rightarrow AB - CD = 2 \qquad ...(i)$$

$$\text{Also, } \frac{AB + CD}{2} = 4 \quad \Rightarrow \quad AB + CD = 8 \qquad ...(ii)$$

Solving (i) and (ii) simultaneously, we get

AB = **5** and CD = **3.**

50. Sum of the exterior angles of a polygon = 360°

Sum of all the angles marked **'*x*'**

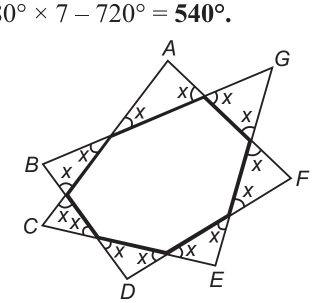
= 2 × Sum of ext. $\angle s$ of the polygon

= 2 × 360° = 720°

$\therefore \angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G$

= Sum of angles of the 7 triangles – 720°

= 180° × 7 – 720° = **540°.**



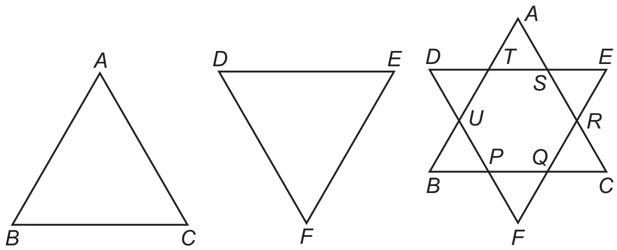
SELF ASSESSMENT SHEET

1. Let ABC be a right angled triangle with AC as its hypotenuse. Then,

(a) $AC^3 > AB^3 + AC^3$ (b) $AC^2 > AB^2 + BC^2$

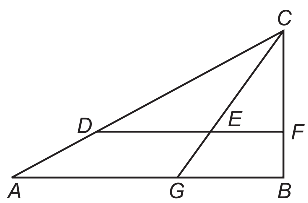
(c) $AC > AB + BC$ (d) None of these

2. There are two congruent triangles each with area 198 cm^2. Triangle DEF is placed over triangle ABC in such a way that the centroid of both the triangles coincide with each other and $AB \parallel DE$ as shown in the figure forming a star. What is the area of the common region $PQRSTU$?



(a) 49 cm^2 (b) 66 cm^2 (c) 132 cm^2 (d) 162 cm^2

3. In the given figure, $DF \parallel AB$, $BC \perp AB$, $BC = 5$, $BG = 4$, $BA = 12$, $DA = 3$. Then CE is:



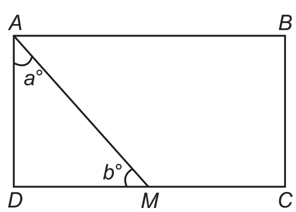
(a) 5.44

(b) 0.54

(c) 4.9

(d) 0.42

4. If $ABCD$ is a rectangle with $AB = 4$ and $BC = 2\sqrt{3}$ and M is the mid-point of CD, find $b - a$.



(a) 15° (b) 20°

(c) 45° (d) 30°

5. OD, OE and OF are the perpendicular bisectors to the three sides of the triangle ABC. What is the relationship between $m\angle BAC$ and $m\angle BOC$?

(a) $m\angle BAC = 180° - m\angle BOC$

(b) $m\angle BOC = 2m\angle BAC$

(c) $m\angle BOC = 90° + \frac{1}{2}\, m\angle BAC$

(d) $m\angle BOC = 90° + \frac{1}{2}\, m\angle BAC$

6. In a ΔABC, $CA = CB$. On CB square $BCDE$ is constructed away from the triangle. If x is the number of degrees in angle DAB, then

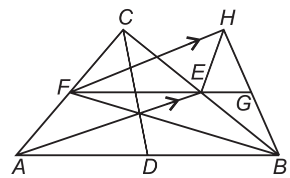
(a) x depends upon ΔABC

(b) x is independent of the triangle

(c) x may equal $\angle CAD$

(d) x is greater than 45° but less than 90°.

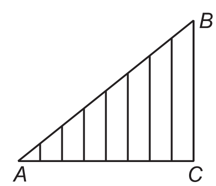
7. Given triangle ABC with medians AE, BF, CD; FH parallel and equal in length to AE; BH and HE are drawn; FE extended to meet BH in G. Which one of the following statements is not necessarily correct?



(a) $HE = HG$ (b) $BH = DC$

(c) $FG = \frac{3}{4} AB$ (d) FG is the median of ΔBFH.

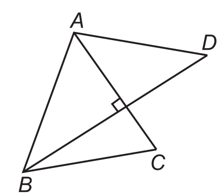
8. Side AC of a right triangle ABC is divided into 8 equal parts. Seven line segments parallel to BC are drawn to AB from the points of division. If $BC = 10$, then the sum of the lengths of the seven line segments is



(a) 33 (b) 34

(c) 35 (d) 45

9. Given $AB = AC = BD$, $BD \perp AC$ in the figure shown alongside, the sum of the measures of angles C and D is



(a) 120° (b) 140°

(c) 130° (d) 135°

10. Constructed externally on the sides AB, AC of ΔABC are equilateral triangle ABX and ACY. If P, Q, R are the mid-points of AX, AY and BC respectively, then ΔPQR is

(a) right angle (b) equilateral

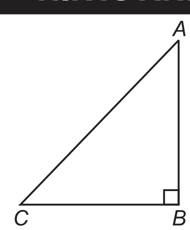
(c) isosceles (d) None of these

ANSWERS

1. (a) **2.** (c) **3.** (c) **4.** (d) **5.** (b) **6.** (b) **7.** (a) **8.** (c) **9.** (d) **10.** (b)

HINTS AND SOLUTIONS

1. In a right angled triangle, hypotenuse is the longest side, so in ΔABC, $AC > BC$ and $AC > AB$



$AC^3 = AC^2 \cdot AC$

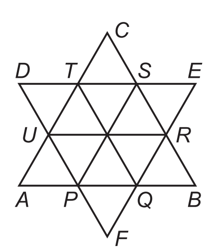
$= (AB^2 + BC^2) \cdot AC$

$= AB^2 \cdot AC + BC^2 \cdot AC$

$> AB^2 \cdot AB + BC^2 \cdot BC \quad \because \; AC > BC, AC > AB$

$\Rightarrow$ $\boldsymbol{AC^3 > AB^3 + BC^3}$**.**

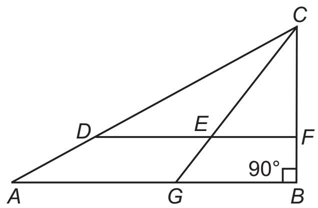
2. There are 12 similar triangles in the figure forming a star, each with equal area. But a larger triangle ABC (or DEF) has only 9 smaller triangles. Out of the 9 triangles only 6 triangles are common.



$\therefore$ Area of common region $PQRSTU$

$= \frac{6}{9} \times 198 \text{ cm}^2 = \mathbf{132 \text{ cm}^2.}$

3. By Pythagoras' Theorem,



$GC^2 = (BC)^2 + (GB)^2$

$= 25 + 16 = 41$

$\Rightarrow GC = \sqrt{41}$

$AC^2 = (BC)^2 + (AB)^2$

$= 24 + 144 = 169$

$\Rightarrow AC = 13$

$\therefore DC = AC - AD = 13 - 3 = 10$

$\Delta DEC \sim \Delta AGC$ (AA similarly $\because DF \parallel AB$)

$\therefore \frac{DC}{AC} = \frac{CE}{GC}$

$\Rightarrow \frac{10}{13} = \frac{CE}{\sqrt{41}} \Rightarrow CE = \frac{10\sqrt{41}}{13} = \mathbf{4.9 \text{ (approx)}}$

4. $CD = AB = 4 \Rightarrow DM = 2$

$AD = BC = 2\sqrt{3}$

We know that for ΔADM, $\angle ADM = 90°$, $AD = 2\sqrt{3}$, $DM = 2$

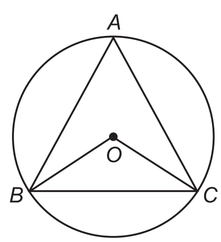
$\Rightarrow AM = \sqrt{AD^2 + DM^2} = \sqrt{12+4} = \sqrt{16} = 4$

Now By the $30° - 60° - 90°$ triangle theorem, the side half the hypotenuse is opposite the $30°$ angle and the side $\frac{\sqrt{3}}{2}$ times the hypotenuse is opposite the $60°$ angle.

$\therefore \Delta ADB$ is a $30° - 60° - 90°$ triangle.

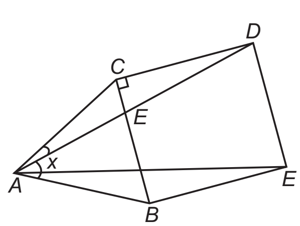
$\Rightarrow b = 60°, a = 30° \Rightarrow b - a = 60° - 30° = \mathbf{30°.}$

5. O being the point of intersection of the perpendicular bisectors of the sides of ΔABC, it is the circumcentre of ΔABC.



$\therefore \angle BOC = 2\angle BAC$ (Angle subtended by an arc at the centre of a circle is twice the angle subtended by it at any other point on the remaining part of the circumference)

6. In ΔABC, $CA = CB$



$\Rightarrow \angle CBA = \angle CAB = \angle A$

$\therefore \angle ACB = 180° - 2\angle A$

$\therefore \angle ACD = 180° - 2\angle A + 90°$

$= 270° - 2\angle A$

$\because AC = AB = CD$

$\Rightarrow AC = CD$

$\Rightarrow$ In ΔCAD, $\angle CDA = \angle CAD$

$= \frac{1}{2}(180° - (270° - 2\angle A)) = \angle A - 45°.$

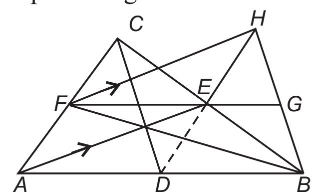
$= \angle A - x$ $\quad (\therefore \angle CAE = \angle CAB - \angle EAB = \angle A - x)$

$\Rightarrow x = 45°$

So X is independent of the triangle.

7. $FH = AE$ and $FH \parallel AE$

$\Rightarrow AEHF$ is a parallelogram.



$\Rightarrow EH \parallel AC$, when extended meets AB in D. In congruent Δs ACD and HDB, $DC = BH$ (corresponding sides)

$\therefore$ (b) is true.

Now $FG = FE + EG$

$= \frac{1}{2}AB + \frac{1}{2}DB$ (By mid-point theorem in Δs ABC and HDB respectively)

$= \frac{1}{2}AB + \frac{1}{2} \times \frac{1}{2}AB$ ($\because D$ is the mid-point of AB)

$= \frac{3}{4}AB$

(c) is true.

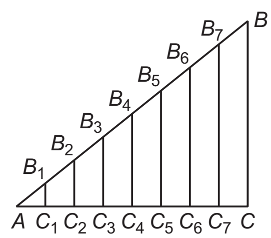
(d) $FE \parallel AB$, when extended to $G \Rightarrow EG \parallel AB$

$\Rightarrow G$ is the mid-point of HB (By converse of mid-point theorem)

$\therefore$ In ΔBFH, FG is the median, so (d) is also true.

(e) Cannot be proved true with the given information.

8. All the triangles formed with A as vertex of AB_1C_1, AB_2C_2, AB_3C_3, AB_4C_4, AB_5C_5, AB_6C_6, AB_7C_7, and ABC are similar.



Let h_k be the length of the side parallel to BC at a distance $k/8$. AC from A,

where $k = 1, 2, ..., 7$, *i.e.*,

$B_1C_1 = h_1 = \frac{AC}{8}$, $B_2C_2 = h_2 = \frac{2}{8}AC$, $B_3C_3 = h_3 = \frac{3}{8}AC$, ..., $B_7C_7 = h_7 = \frac{7}{8}AC$

$\therefore \Delta AB_1C_1 \sim \Delta ABC$, their sides are proportional

$\Rightarrow \frac{B_1C_1}{AC_1} = \frac{BC}{AC} \Rightarrow \frac{h_1}{1/8\,AC} = \frac{10}{AC}$

Similarly, $\Delta AB_2C_2 \sim \Delta ABC$, $\Delta AB_3C_3 \sim \Delta ABC$, ..., $\Delta AB_7C_7 \sim \Delta ABC$

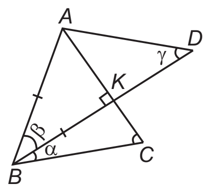
$\therefore$ In general, $\frac{h_k}{k/8\,AC} = \frac{10}{AC} \Rightarrow h_k = \frac{10\,k}{8}$

$\therefore$ Required sum $= h_1 + h_2 + h_3 + ... + h_7$

$= \frac{10}{8}(1 + 2 + 3 + ... + 7) = \frac{10}{8} \times \frac{7}{2} \times (1 + 7) = 35$

$\left(\because \text{Sum of } n \text{ terms} = \frac{n}{2}(a + l)\right)$

9. Let AC and BD intersect at point K and $\angle ABK = \beta$, $\angle KBC = \alpha$, $\angle ADK = \gamma$.



Then, In rt. ΔAKB, $\angle BAK = 90° - \beta$

Also, in rt. ΔAKD, $\angle DAK = 90° - \gamma$

$\therefore AB = BD$

$\therefore \angle BAD = \angle BDA$

$\Rightarrow (90° - \beta) + (90° - \gamma) = \gamma$

$\Rightarrow 180° - \beta - \gamma = \gamma$

$\Rightarrow 2\gamma = 180° - \beta \quad \Rightarrow \quad \gamma = 90° - \beta/2.$ ...(*i*)

In rt. ΔBKC, $\angle C = 90° - \alpha = \angle ABC = \alpha + \beta$ ($\because AB = AC$)

$\therefore 90° - \alpha = \alpha + \beta \quad \Rightarrow \quad 2\alpha = 90° - \beta$

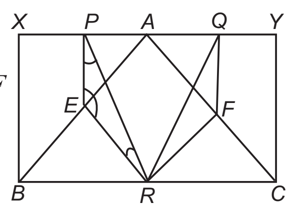
$\Rightarrow \alpha = 45° - \beta/2.$ ...(*ii*)

$\angle C + \angle D = (90° - \alpha) + \gamma = 90° - (45° - \beta/2) + 90° - \beta/2$

From (*i*) and (*ii*)

$= 45° + \beta/2 + 90° - \beta/2 =$ **135°.**

10. Let E and F be the mid-points of AB and AC respectively. Then, by mid-point theorem,



In ΔBAC,

$ER \parallel AC$ and $ER = \frac{1}{2}AC = AF$

In ΔCAB,

$FR \parallel AB$ and $FR = \frac{1}{2}AB = EA$

$\therefore AERF$ is a parallelogram.

$\therefore \Delta ABX$ being an equilateral Δ and $AERF$ a parallelogram,

$PE = AE = RF.$

Similarly for ΔACY and ||gm $AERF$

$QF = AF = ER$

Also, $\angle PER = \angle PEA + \angle AER$

$= 60° + \angle AER$ ...(*i*)

(opp. $\angle s$ of a prallelogram are equal)

$= 60° + \angle AFR = \angle QFR$ ...(*ii*)

$\therefore PE = RF$, $QF = ER$ and $\angle PER = \angle QFR$ ...(*iii*)

$\Rightarrow \Delta EPR \cong \Delta QRF$

$\Rightarrow PR = QR$ and $\angle EPR = \angle QRF$

Also, $\angle PRQ = \angle ERF - \angle ERP - \angle QRF$ ...(*iv*)

$= 180° - \angle AER - \angle ERP - \angle QRF$

$= 180° - \angle AER - \angle ERP - \angle EPR$ (From (*iv*)

$= 180° - \angle AER - (\angle ERP + \angle EPR)$

$= 180° - \angle AER - (180° - \angle PER)$

$= \angle PER - \angle AER =$ **60°.**

Also, $PR = QR \quad \Rightarrow \quad \angle RPQ = \angle RQP = \frac{1}{2}(180° - \angle PRQ)$

$= \frac{1}{2}(180° - 60°) = 60°.$

$\Rightarrow \Delta PQR$ is equilateral.