CHAPTER 12

Transition Curve

12.1 Need for Transition Curve

- A transition curve is a horizontal curve of varying radius.
- It is provided between a straight path and the circular path so that jerks due to sudden introduction of centrifugal force at the junction of the straight and the circular paths are minimized while a vehicle is going from straight to the circular path.
- Transition curve provides a gradual change in the path from an initially straight one to the circular one.
- It can also be provided in between the two reverse curves and the compound curves.
- These are provided in the highways and railway to reduce the discomfort to passengers at the junction of straight and the circular path.

12.2 Requirements of a Transition Curve

1. It must originate and meet the straights at zero curvature i.e. at contact points T_1 and T_2, the radius of curvature must be infinite.
2. It must meet the circular curve of radius R tangentially i.e. radius of curvature at contact points C and C' is equal to the radius of curve i.e. R.
3. The length of transition curve must be such that full super-elevation is achieved at the contact points C and C'.
4. The rate of increase of curvature along the transition curve should be equal to the rate of increase of super-elevation.

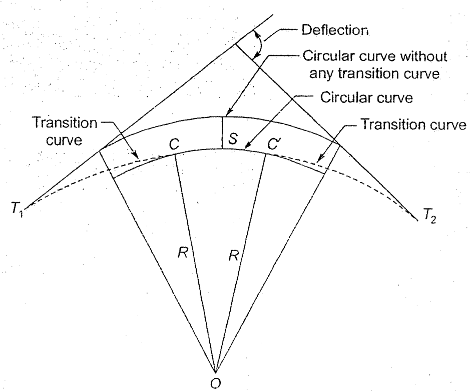


Fig. 12.1 *A typical transition curve*

12.3 Super-elevation

- Super-elevation also called as **cant** or **banking** is the raising of outer edge of road with respect to the inner edge so as to avoid skidding off of the vehicle due to centrifugal force while traversing a circular path.

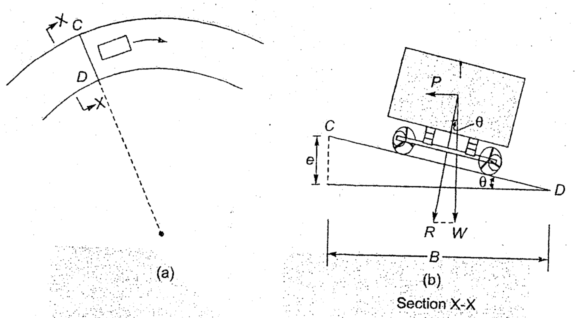


Fig. 12.2 *Super-elevation*

- As shown in **Fig. 12.2**, point *C* is on the outer edge while point *D* is on the inner edge. The difference of elevations of point *C* and *D* is called as **super-elevation** or **cant** or **banking**.

  Let a vehicle is traversing a curved path of radius *R*. The forces acting on this vehicle are:

  1. The weight *W* of the vehicle acting in the direction of gravity i.e. downwards.
  2. The centrifugal force *P* acting outwards.

     The above two forces *W* and *P* will meet at the center of gravity of the vehicle. In order to have a perfect equilibrium of the vehicle on the curved path, the resultant force *R* should be normal to the road (or rail) surface.

Now,
$$P = \frac{mv^2}{R} = \frac{Wv^2}{gR} \qquad (\because W = mg)$$

$$\Rightarrow \quad \frac{P}{W} = \frac{v^2}{gR} \qquad ...(12.1)$$

From Fig. 12.2,

$$\tan\theta = \frac{P}{W} = \frac{e}{B}$$

Thus,
$$\frac{P}{W} = \frac{e}{B} = \frac{v^2}{gR}$$

$$\Rightarrow \quad \frac{e}{B} = \frac{v^2}{gR}$$

$$\Rightarrow \quad e = \frac{Bv^2}{gR} \qquad ...(12.2)$$

Here the term $\left(\frac{P}{W}\right)$ is called as **centrifugal ratio**.

Thus
$$\text{Centrifugal ratio} = \frac{P}{W} = \frac{v^2}{gR}$$

NOTE: For roads, the maximum permissible centrifugal ratio is 1/4 and that for railways is 1/8.

Thus for roads,
$$\frac{P}{W} = \frac{1}{4} = \frac{v_1^2}{gR}$$

where v_1 = Design speed for maximum centrifugal ratio for roads i.e. $\frac{1}{4}$

For railways,
$$\frac{P}{W} = \frac{1}{8} = \frac{v_2^2}{gR}$$

where v_2 = Design speed for maximum centrifugal ratio for railways i.e. $\frac{1}{8}$

Thus,
$$\frac{v_1}{v_2} = \frac{\sqrt{gR/4}}{\sqrt{gR/8}} = \sqrt{2}$$

$$\Rightarrow \quad v_1 = \sqrt{2}v_2 \qquad ...(12.3)$$

Thus, for roads, maximum design speed can be $\sqrt{2}$ times (or 41.4%) higher than maximum design speed for railways.

12.3.1 Equilibrium Cant

When resultant R is normal to the road (or rail) surface, in that case, equal pressures act on the wheels on either side. This is achieved when super-elevation (e) is provided as per Eq. (12.2). When actual cant provided is same as that required as per Eq. (12.2), then this is called as **equilibrium cant**.

12.3.2 Cant Deficiency

When the actual cant provided is less than that given by Eq. (12.2) then this is called as **cant deficiency**.

12.4 Length of Transition Curve

(a) Method of Arbitrary Gradient

Here the length of the transition curve is determined by the arbitrary rate at which super-elevation is provided.

Let e = Total super-elevation provided at the junction of transition curve with the circular curve

Thus $L = ne$

where rate of super-elevation is 1 in n.

Also
$$e = \frac{Bv^2}{gR}$$

Thus
$$L = ne = \frac{nBv^2}{gR} \qquad ...(12.4)$$

(b) Method of Time Rate

In this method, the super-elevation is provided at an arbitrary rate as fixed usually by experience.

Let r_1 = Time rate of super-elevation i.e. a super-elevation of 'r_1' meters is provided in a distance travelled by a vehicle in one second.

Thus
$$t = \frac{L}{v}$$

Now
$$e = r_1 t = \frac{r_1 L}{v}$$

i.e
$$L = \frac{ev}{r_1} = \left(\frac{Bv^2}{gR}\right)\frac{v}{r_1} = \frac{Bv^3}{gRr_1} \qquad \text{...(12.5)}$$

(c) Method of Rate of Change of Radial Acceleration

This is the most commonly adopted method. In this method, the length of the transition curve is fixed based on the rate of change of radial acceleration.

Let α = Rate of change of radial acceleration

Thus, radial acceleration (a) attained after time 't' is given by,

$$a = \alpha t = \frac{\alpha L}{v}$$

But
$$a = \frac{v^2}{R}$$

Thus
$$\frac{v^2}{R} = \frac{\alpha L}{v}$$

or
$$L = \frac{v^3}{\alpha R} \qquad \text{...(12.6)}$$

12.5 Ideal Transition Curve

- A transition curve is a curve of variable radius inserted between the straight and the circular path so that centrifugal force comes into play gradually which reduces the discomfort to the passengers.
- This centrifugal force is required to vary with time at a constant rate.

Let L = Length of transition curve traversed by a vehicle in time 't' as measured from the point of commencement of transition curve i.e. junction of straight and transition curve.

Thus
$$L = vt$$

or
$$t = \frac{L}{v}$$

Let r = Radius of transition curve at any point

Now centrifugal force is given by,

$$P = \frac{Wv^2}{gr}$$

In order to vary the centrifugal force (P) with time, P should vary with t i.e. L/v

Thus
$$P \propto \frac{L}{v}$$

or
$$\frac{Wv^2}{gr} \propto \frac{L}{v}$$

Now since, W, g and v are constant,

$$L \propto \frac{1}{r} \quad \text{i.e.} \quad Lr = \text{constant} \qquad \text{...(12.7)}$$

- Eq. (12.7) represents the equation of an ideal transition curve usually called as **clothoid, the Glover spiral** or **the Euler spiral.**
- At the end of the transition curve i.e., when the transition curve meets the circular curve, $r = R$

And thus,

$$LR = \text{constant}$$

$$\therefore \quad lr = LR = \text{Constant} \qquad \text{...(12.8)}$$

where, l = length of transition curve at radius of curvature 'r'

12.5.1 Intrinsic Equation of an Ideal Transition Curve

Fig. 12.3 shows the transition curve AC between the straight line AD and the curve of radius R.

Thus line AD is tangent to the transition curve at A. Let point B is on the transition curve at a curved distance 'l' from A.

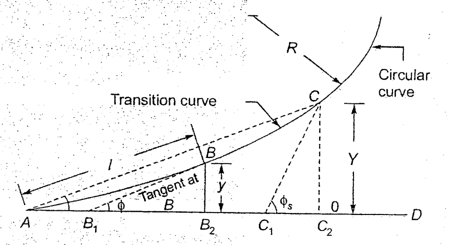


Fig. 12.3 *Intrinsic equation of an ideal transition curve*

Thus $lr = LR$ = Constant

where r = Radius of the transition curve at any point

Now, since $lr = LR$ = Constant

$$\Rightarrow \quad r = \frac{RL}{l}$$

Let ϕ = Inclination of the tangent at B to the initial tangent AD

Thus curvature is,

$$\frac{1}{r} = \frac{d\phi}{dl}$$

$$\Rightarrow \quad d\phi = \frac{dl}{r}$$

But
$$\frac{1}{r} = \frac{l}{RL}$$

Thus,
$$d\phi = \frac{l\,dl}{RL} \qquad \text{...(12.9)}$$

Integrating Eq. (12.9) we get,

$$\phi = \frac{l^2}{2RL} + C$$

Now at point A, $l = 0$, $\phi = 0$ and thus $C = 0$

Therefore
$$\boxed{\phi = \frac{l^2}{2RL}} \qquad \text{...(12.10)}$$

The Eq. (12.10) represents the equation of CLOTHOID which is an ideal transition curve.

Eq. (12.10) can also be written as

$$l = \sqrt{2RL\phi} = K\sqrt{\phi} \quad \text{...(12.11)}$$

where $K = \sqrt{2RL}$

At point '*C*', $l = L$ and $\phi = \phi_s$

Thus,

$$\phi_s = \frac{L^2}{2RL} = \frac{L}{2R} \quad \text{...(12.12)}$$

Here ϕ_s is the spiral angle i.e. the angle between the initial tangent and the final tangent points on the curve.

12.5.2 Equation of an Ideal Transition Curve in Cartesian Co-ordinates

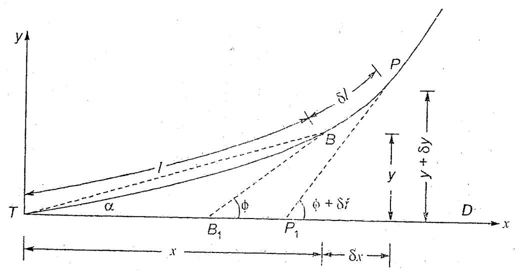


Fig. 12.4 *Ideal transition curve in Cartesian co-ordinates*

- The intrinsic equation of an ideal transition curve as arrived in preceding section is not very convenient to use. Thus the equation for an ideal transition curve in Cartesian co-ordinates is derived below.

  Let there is a point *B* on the transition curve distant *l* from the starting point *T*. Assuming point *T* as origin, the co-ordinates of point *B* is (x, y). Let BB_1 is the tangent at *B* which makes an angle of ϕ with the initial tangent *TD*.

  Let there is a point *P* distant δl from B. Thus the co-ordinates of *P* are $(x + \delta x)$ and $(y + \delta y)$. The tangent PP_1 at *P* makes an angle of $(\phi + \delta\phi)$ with the initial tangent *TD*.

From Fig. 12.4,

$$\frac{\delta x}{\delta l} = \cos\phi$$

$\Rightarrow$

$$dx = dl.\cos\phi = dl\left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - ....\right) \quad \text{...(12.13)}$$

Now from Eq. (12.11), $l = K\sqrt{\phi}$

Differentiating Eq. (12.11), $dl = \dfrac{K}{2\sqrt{\phi}}d\phi$

Substituting this value of dl in the expression for dx, in Eq. (12.13)

$$dx = dl.\cos\phi = \frac{K}{2\sqrt{\phi}}\left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - ....\right)d\phi$$

$\Rightarrow$

$$dx = \frac{K}{2}\left(\phi^{-1/2} - \frac{\phi^{3/2}}{2!} + \frac{\phi^{7/2}}{4!} - ....\right)d\phi$$

Integrating,

$$x = \frac{K}{2}\left(2\phi^{1/2} - \frac{1}{5}\phi^{5/2} + \frac{\phi^{9/2}}{108} - ....\right)$$

$\Rightarrow$

$$x = K\sqrt{\phi}\left(1 - \frac{1}{10}\phi^2 + \frac{\phi^4}{216} - ....\right)$$

$\Rightarrow$

$$x = l\left(1 - \frac{1}{10}\phi^2 + \frac{\phi^4}{216} - ....\right) \qquad (\because l = K\sqrt{\phi})$$

Now $l = K\sqrt{\phi}$

Substituting this value,

$$x = l\left(1 - \frac{1}{10K^4}l^4 + \frac{l^8}{216K} - ....\right)$$

Also, $K = \sqrt{2RL}$

$\therefore$

$$x = l\left(1 - \frac{l^4}{40R^2L^2} + \frac{l^8}{3456R^4L^4} - ....\right) \quad \text{...(12.14)}$$

Similarly the expression for *y* can be obtained as follows.

$$\frac{\delta y}{\delta l} = \sin\phi$$

$\Rightarrow$

$$dy = dl.\sin\phi = dl\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - ....\right) \quad \text{...(12.15)}$$

Now, $dl = \dfrac{K}{2\sqrt{\phi}}d\phi$

$\therefore$

$$dy = dl.\sin\phi = dl\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - ....\right)$$

$\Rightarrow$

$$dy = \frac{K}{2}\left(\phi^{1/2} - \frac{\phi^{5/2}}{3!} + \frac{\phi^{9/2}}{5!} - ....\right)d\phi$$

Integrating,

$$y = K\left(\frac{\phi^{3/2}}{3} - \frac{\phi^{7/2}}{42} + \frac{\phi^{11/2}}{1320} - ....\right)d\phi$$

Now, $l = K\sqrt{\phi}$

Thus,

$$y = \frac{l^3}{3K^2}\left(1 - \frac{l^4}{14K^4} + \frac{l^8}{440K^8} - ....\right)d\phi$$

$$y = \frac{l^3}{6RL}\left(1 - \frac{l^4}{56R^2L^2} + \frac{l^8}{7040R^4L^4} - ....\right)d\phi \quad \text{...(12.16)}$$

- **The Cubic Spiral**

From Eq. (12.16)

$$y = \frac{l^3}{6RL}\left(1 - \frac{l^4}{56R^3L^2} + \frac{l^8}{7040R^4L^4} - ...\right)$$

Neglecting all the terms beyond the first term, we have

$$\boxed{y = \frac{l^3}{6RL}} \quad ...(12.17)$$

Eq. (12.17) represents the equation of a cubic spiral and the same is shown in **Fig. 12.5**

Alternatively, the equation of cubic spiral (Eq. 12.17) can also be derived as per the following procedure.

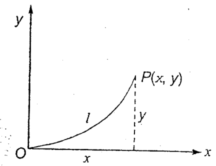


Fig. 12.5 *Cubic Spiral*

From Fig. 12.4,

$$\frac{\delta y}{\delta l} = \sin\phi \simeq \frac{dy}{dl}$$

For very small angles,

$$\sin\phi \simeq \phi$$

But, $\phi = \frac{l^2}{2RL}$ from eq. (12.10)

$$\therefore \quad \sin\phi \simeq \phi = \frac{dy}{dl} = \frac{l^2}{2RL}$$

$$\Rightarrow \quad \frac{dy}{dl} = \frac{l^2}{2RL}$$

$$\Rightarrow \quad dy = \frac{l^2}{2RL}dl$$

Integrating, $y = \frac{l^3}{6RL} + C$

At point *T*, of **Fig. 12.4** $l = 0$, $y = 0$ and thus $C = 0$

$$\therefore \quad y = \frac{l^3}{6RL}$$ which is same as Eq. (12.17).

12.6 The Cubic Parabola

From Eq. (12.14)

$$x = l\left(1 - \frac{l^4}{40R^2L^2} + \frac{l^8}{3456R^4L^4} - ...\right)$$

From Eq. (12.16)

$$y = \frac{l^3}{6RL}\left(1 - \frac{l^4}{56R^2L^2} + \frac{l^8}{7040R^4L^4} - ...\right)$$

In the above two equations, if all the terms after the first term are neglected then,

$$x = l$$

$$y = \frac{l^3}{6RL}$$

$$\Rightarrow \quad \boxed{y = \frac{x^3}{6RL}} \quad ...(12.18)$$

Eq. (12.18) is the equation of cubic parabola.

Here it has been assumed that

$$\sin\phi \simeq \phi$$
$$\cos\phi \simeq 1$$

But in case of cubic spiral, the only assumption made was,

$$\sin\phi \simeq \phi$$

- Thus cubic spiral is better than cubic parabola and consequently it represents the true transition curve.
- However, since cubic parabola is expressed entirely in terms of Cartesian coordinates (x, y) and thus setting out of cubic parabola is much easier than setting out the cubic spiral.

NOTE There is not any significant difference between cubic parabola and cubic spiral for deviation angles upto 12°. But for deviation angles larger than 12°, cubic parabola does not give accurate results and thus cubic spiral is resorted to.

12.6.1 Minimum Radius of Curvature of Cubic Parabola

- The cubic parabola has a very peculiar characteristic that its radius of curvature decreases from infinity at the point of intersection of straight path with circular path ($\phi = 0$) to a minimum value at $\phi = 24° 05' 41''$. After this point, the radius of curvature starts increasing again.
- Due to this characteristic of cubic parabola, it does not serve the purpose of transition curve beyond $\phi = 24° 05' 41''$.

The expression for minimum radius of curvature (r_{min}) is derived as below:

From Eq. (12.18)

$$y = \frac{x^3}{6RL} = Cx^3$$

where, $C = \frac{1}{6RL}$

Differentiating, $\frac{dy}{dx} = 3Cx^2$

But, $\frac{dy}{dx} = \tan\phi$

$$\therefore \quad 3Cx^2 = \tan\phi$$

$$\Rightarrow \quad x = \sqrt{\frac{\tan\phi}{3C}}$$

Also, $$\frac{d^2y}{dx^2} = 6Cx = 6C\sqrt{\frac{\tan\phi}{3C}} = \sqrt{\frac{36C^2\tan\phi}{3C}} = \sqrt{12C\tan\phi}$$

The radius of curvature (r) is given by

$$r = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\Rightarrow \quad r = \frac{\left(1+\tan^2\phi\right)^{3/2}}{\sqrt{12C\tan\phi}}$$

$$\Rightarrow \quad r = \frac{\sec^3\phi}{\sqrt{12C\tan\phi}}$$

$$\Rightarrow \quad r = \frac{1}{\sqrt{12C\sin\phi\cos^5\phi}} \qquad ...(12.19)$$

For 'r' to be minimum,

$$\frac{dr}{d\phi} = 0$$

From Eq. (12.19), it is evident that 'r' will be minimum when denominator is maximum i.e., $12\,C\sin\phi\cos^5\phi$ is maximum i.e.,

$$\frac{d}{d\phi}\left(12C\sin\phi\cos^5\phi\right) = 0$$

$$\Rightarrow \quad \cos^6\phi - 5\cos^4\phi\sin^2\phi = 0$$

$$\Rightarrow \quad \cos^4\phi\,(\cos^2\phi - 5\sin^2\phi) = 0$$

Thus either $\cos^4\phi = 0$

or $(\cos^2\phi - 5\sin^2\phi) = 0$

$$\Rightarrow \quad \phi = 90^\circ$$

or $$\tan^2\phi = \frac{1}{5}$$

But $\phi = 90°$ is not possible.

$$\therefore \quad \tan^2\phi = \frac{1}{5}$$

$$\Rightarrow \quad \tan\phi = \frac{1}{\sqrt{5}} \qquad \text{(Neglecting –ve sign)}$$

$$\Rightarrow \quad \phi = 24.0948^\circ = 24^\circ\,05'\,41.4'' \qquad ...(12.20)$$

Thus, r_{min} is

$$r_{min} = \frac{1}{\sqrt{12C\sin\phi\cos^5\phi}} = \frac{1}{\sqrt{12C(\sin 24.0948^\circ)(\cos^5 24.0948^\circ)}}$$

$$= \frac{1}{1.762285\sqrt{C}}$$

But, $$C = \frac{1}{6RL}$$

$$\therefore \quad r_{min} = \frac{1}{1.762285\sqrt{\frac{1}{6RL}}}$$

$$\Rightarrow \quad r_{min} = 1.38995\sqrt{RL} \simeq 1.39\sqrt{RL} \qquad ...(12.21)$$

Therefore for deflection (or deviation) angles greater than 24° 05′ 41″, cubic parabola will not serve the purpose of transition curve.

12.6.2 Deflection Angle

- As shown in **Fig. 12.6**, let there be a transition curve starting from point T_1. P and Q are two points on transition curve with Cartesian coordinates (x, y) and $(x + \delta x, y + \delta y)$ respectively.
- Chord T_1P makes an angle of α with initial tangent at T_1. The tangent at $P(x, y)$ makes an angle of ϕ with initial tangent at T_1.

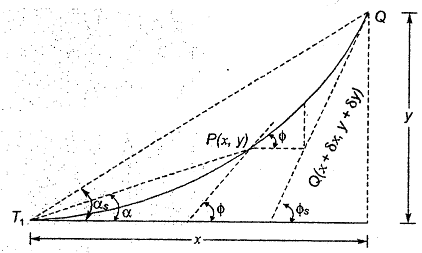


Fig. 12.6 *Deflection angle*

$$\therefore \quad \tan\alpha = \frac{y}{x}$$

But $$y = \frac{x^3}{6RL}$$

$$\therefore \quad \tan\alpha = \frac{x^3/6RL}{x} = \frac{x^2}{6RL} \qquad ...(12.22)$$

$$\tan\phi = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3}{6RL}\right) = \frac{x^2}{2RL} \qquad ...(12.23)$$

From Eqs. (12.22) and (12.23)

$$\tan\phi = 3\tan\alpha \qquad ...(12.24)$$

For small values of ϕ and α,

$$\phi \simeq 3\alpha$$

$$\Rightarrow \quad \alpha \simeq \frac{\phi}{3} \qquad ...(12.25)$$

$$\Rightarrow \quad \alpha = \frac{\phi}{3}\left(\frac{l^2}{2RL}\right) = \frac{l^2}{6RL} \qquad \text{(Where } \alpha \text{ is in radians) ...(12.26)}$$

$$\Rightarrow \quad \alpha = \frac{l^2}{6RL}\text{ rad} = \frac{l^2}{6RL}\times\frac{180}{\pi}\text{deg}$$

$$= \frac{l^2}{6RL}\times\frac{180}{\pi}\times 60\text{ min.} = 572.96\left(\frac{l^2}{RL}\right)\text{ min.} = 577\left(\frac{l^2}{RL}\right)\text{ min.}$$

At point $D(x, y)$ where $l = L$

$$\alpha_s = \frac{R^2}{6RL} = \frac{L^2}{6RL} = \frac{L}{6R} \text{ rad} \qquad 12.27)$$

$$\phi_s = \frac{l^2}{2RL} = \frac{L^2}{6RL} = \frac{L}{6R} = 3\alpha_s \text{ rad} \qquad ...(12.28)$$

12.7 Insertion of the Transition Curve

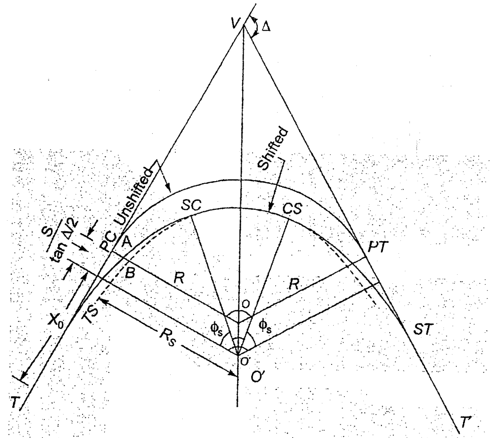


Fig. 12.7 *Insertion of transition curve*

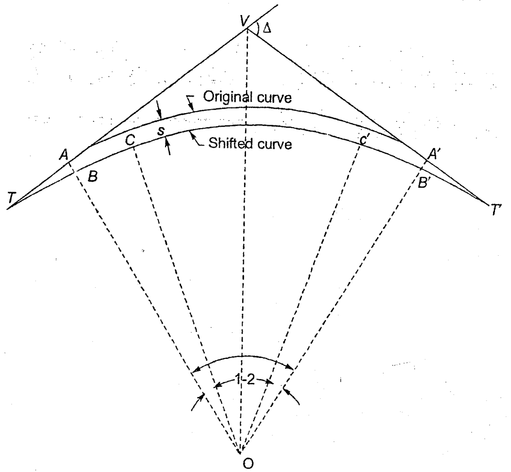


Fig. 12.8 *Shift in transition curve for insertion*

- The transition curves are inserted between the straight lines *TV* and *T'V* by shifting the circular curve slightly inwards.
- As shown in **Fig. 12.6**, the original circular curve is from the point of curvature (*PC*) to the point of tangency (*PT*) with its center at *O*. Now the shifted position of the circular curve is from *SC* to *CS* with the new center at *O'*. The transition curves are the two from *TS* to *SC* and *CS* to *ST*.

  From **Fig. 12.8** let, *TS* = Tangent to spiral point (*T*)

  SC = Spiral to circular point (*C*)

  CS = Circular curve to spiral point (*C'*)

  ST = Spiral to tangent point (*T'*)

  Now the distance *AB* through which the main circular curve is shifted inwards in order to accommodate the transition curve is called as **shift (*s*)** and also sometimes called as **throw**.

  The distance between the original center *O* and the new center *O'* is usually very small and thus it is a usual practice to represent both the points by one point as *O*.

12.8 Characteristics of Transition Curve

- The angle between the initial tangent *TV* and the tangent CC_1 (which is common to both the transition curve and the circular curve at point *C*) is known as **spiral angle (ϕ_s)**.

Starting with the initial tangent *TV*, draw a line *OA* perpendicular to *TV* at *A*. Also draw *CE* perpendicular to *OA* at *E*.

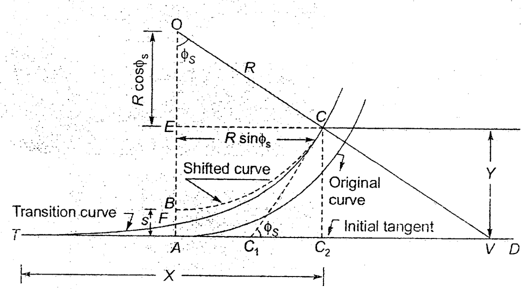


Fig. 12.9 *Characteristics of transition curve*

In □OAC_1C,

$$\angle OAC_1 = \angle C_1CO = 90°$$

$$\therefore \quad \angle AOC + \angle AC_1C = 180°$$

Also, $\angle AC_1C + \angle CC_1C_2 = 180°$

$$\Rightarrow \quad \angle AC_1C = 180° - \phi_s$$

$$\angle AOC = 180° - AC_1C$$

$$= 180° - (180° - \phi_s) = \phi_s$$

Thus angle subtended by half of the transition curve at the center is equal to the spiral angle. Therefore angle subtended by the shifted circular curve at the center will be $(\Delta - 2\phi_s)$.

From the **Fig. 12.9**

$$BC = R\phi_s$$

But from Eq. (12.12), $\phi_s = \frac{L}{2R}$

Thus, $BC = \frac{L}{2}$

Assuming arc *CF* to be approximately equal to the curve *BC*, $CF \approx \frac{L}{2}$. Thus the shift bisects the transition curve *TC*.

$$\text{Shift } (s) = AB = EA - EB$$
$$= Y - (R - R\cos\phi_s)$$
$$= Y - 2R\sin^2\left(\frac{\phi_s}{2}\right)$$
$$\simeq Y - 2R\left(\frac{\phi_s^2}{4}\right)$$
$$= Y - R\left(\frac{\phi_s^2}{2}\right)$$

Now from Eq. 12.17 for $l = L$, $y = Y$

$$\therefore \qquad Y = \frac{L^3}{6RL} = \frac{L^2}{6R}$$

Therefore,

$$\text{Shift } (s) = Y - R\left(\frac{\phi_s^2}{2}\right) = \frac{L^2}{6R} - R\left(\frac{\phi_s^2}{2}\right)$$
$$= \frac{L^2}{6R} - \left(\frac{R}{2}\right)\left(\frac{L}{2R}\right)^2 = \frac{L^2}{24R} \qquad \text{...(12.29)}$$

Also, y co-ordinate of $F = FA$

$$= \frac{(L/2)^3}{6RL} = \frac{s}{2}$$

Thus the transition curve TC bisects the shift AB at F.

12.8.1 Total Tangent Length

The distance TV is known as the tangent length.

$$TV = TA + AV$$
$$= (X - R\sin\phi_s) + (R + s)\tan\left(\frac{\Delta}{2}\right)$$

where $X = x$ co-ordinate or abscissa of point C

Now, $\qquad \phi_s = \dfrac{L}{2R}$ and $X \approx L$

Thus, $\qquad \text{Tangent length} = (R+s)\tan\left(\dfrac{\Delta}{2}\right) + \left(L - R \times \dfrac{L}{2R}\right)$

$$= (R+s)\tan\left(\frac{\Delta}{2}\right) + \frac{L}{2} \qquad \text{...(12.30)}$$

12.8.2 Total Length of Curve

The total length of the curve is given by,

$$\text{Total length of curve} = TB + BB' + B'T$$
$$= L + \left(\frac{\pi r}{180}\right)(\Delta - 2\phi_s) + L$$
$$= 2L + \left(\frac{\pi r}{180}\right)(\Delta - 2\phi_s) \qquad \text{...(12.31)}$$

Alternatively, the total length can be found by considering the arc BB'.

Thus, Total length of the curve $= TB + BB' + B'T$

$$= \frac{L}{2} + \frac{\pi RI}{180} + \frac{L}{2}$$
$$= L + \frac{\pi RI}{180} \qquad \text{...(12.32)}$$

12.8.3 Length of Long Chord

In transition curve, long chord is the line joining the points T and C as shown in **Fig. 12.8**. Its length is given by,

$$\text{Length of long chord} = \sqrt{(X^2 + Y^2)}$$
$$= \sqrt{L^2 + \left(\frac{L^2}{6R}\right)^2} = \sqrt{L^2 + 4s^2} \qquad \text{...(12.33)}$$

12.9 Lemniscate as a Transition Curve (at the end of Circular Curve)

For a lemniscate to be transitional throughout, the polar deflection angle (α_s) should be $\phi/6$. However if $\alpha_s < \phi/6$ then it becomes essential to introduce a circular curve between the two lemniscate transition curves.

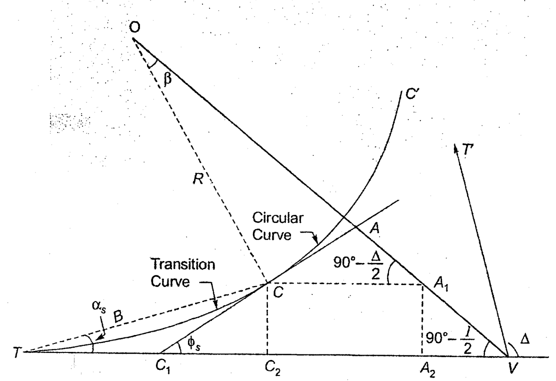


Fig. 12.10 *Lemniscate as a transition curve*

Fig.12.10 shows a lemniscate curve TC which is used as a transition curve between the circular curve CC' and the tangential point T.

Line OV bisects the $\Delta TVT'$ at vertex V (due to symmetry) where TV and $T'V$ are the tangents of the curve.

Draw CA_1 parallel to tangent TV intersecting OV at A_1. Draw CC_2 and A_1A_2 normal to tangent TV.

Let $\qquad \alpha_s$ = Polar deflection angle

$\qquad \phi_s$ = Total deviation (or deflection) angle at junction C

Thus, $\qquad \angle CTV = \alpha_s$

$$\angle CC_1V = \phi_s$$

Tangent length, $TV = TC_2 + C_2A_2 + A_2V$

Let length of chord $TC = B$ = Length of extreme polar ray at $\alpha_s = \dfrac{\phi_s}{3}$

Thus $TC_2 = B\cos\alpha_s$

$$\angle TVO = \left(\frac{180° - \Delta}{2}\right) = 90° - \frac{\Delta}{2}$$

In ΔAC_1V, ext. $\angle OAC = \angle AC_1V + \angle C_1VA$

$$= \phi_s + \left(90° - \frac{\Delta}{2}\right)$$

Thus, $\angle AOC = 90° - \angle AOC = 90° - \left[\phi_s + \left(90° - \frac{\Delta}{2}\right)\right]$

$\Rightarrow$ $$\beta = \frac{\Delta}{2} - \phi_s$$

From ΔOCA_1

$$\frac{CO}{CA_1} = \frac{\sin\left(90° - \frac{\Delta}{2}\right)}{\sin\left(\frac{\Delta}{2} - \phi_s\right)}$$

$$CA_1 = \frac{R\sin\left(\frac{\Delta}{2} - \phi_s\right)}{\cos\left(\frac{\Delta}{2}\right)}$$

$$= R\sec\left(\frac{\Delta}{2}\right)\left[\sin\left(\frac{\Delta}{2}\right)\cos\phi_s - \cos\left(\frac{\Delta}{2}\right)\sin\phi_s\right]$$

$$CA_1 = C_2A_2 = R\left[\tan\left(\frac{\Delta}{2}\right)\cos\phi_s - \sin\phi_s\right]$$

Now, $$A_2V = A_1A_2\cot\left(90° - \frac{\Delta}{2}\right)$$

$$= CC_2\cot\left(90° - \frac{\Delta}{2}\right)$$

$$= B\sin\alpha_s\tan\left(\frac{\Delta}{2}\right)$$

Adding the above equations,

$$TV = TC_2 + C_2A_2 + A_2V$$

$$= B\cos\alpha_s + R\left[\tan\left(\frac{\Delta}{2}\right)\cos\phi_s - \sin\phi_s\right] + B\sin\alpha_s\tan\left(\frac{\Delta}{2}\right) \quad ...(12.34)$$

Thus by measuring back the distance TV from the point of intersection V, the location of tangent point T can be established.

12.10 Comparison of Transition Curves

Clothoid $\qquad \phi = \dfrac{l^2}{2RL}$

It is an ideal transition curve but is difficult to set out in field.

Cubic Spiral $\qquad y = \dfrac{l^3}{6RL}$

This transition curve is arrived at by neglecting higher order terms of the series and is thus not an ideal transition curve. But it is relatively easy to set out as compared to clothoid.

Cubic Parabola $\qquad y = \dfrac{x^3}{6RL}$

Here more assumptions are made as compared to cubic spiral and thus it also does not represent an ideal transition curve. However it is the most easy curve to set out in field.

Illustrative Examples

Example 12.1 Two straight alignments intersect at a chainage of 4687.50 m with a deflection angle of 43°. A circular curve of radius 380 m with transition curves of 50 m each on the two ends. Compute the data for setting out the curve with page at 20 m for circular curve and 10 m for transition curve.

Solution:

$$\text{Spiral angle } (\phi_s) = \frac{L}{2R} = \frac{50}{2\times 380} = 0.0658\,\text{rad} = 3.77°$$

Thus central angle of circular curve $= \Delta - 2\phi_s$

$= 43° - 2(3.77°) = 35.46°$

$$\text{Thus shift of circular curve } (s) = \frac{L^2}{24R} = \frac{50^2}{24\times 380} = 0.2741\,\text{m}$$

$\therefore$ $$\text{Total tangent length } (T_t) = (R+s)\tan\left(\frac{\Delta}{2}\right) + \frac{50}{2}$$

$$= (380 + 0.2741)\tan\left(\frac{43}{2}\right) + \frac{50}{2} = 174.79\,\text{m}$$

$\therefore$ $$\text{Length of circular curve} = \frac{\pi R(\Delta - 2\phi_s)}{180°} = \frac{\pi\times 380°(43° - 2\times 3.77°)}{180°} = 235.18\,\text{m}$$

Computation of chainages

Chainage of start of transition curve (T_1)

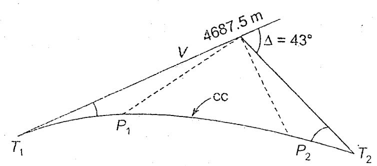
= 4687.50 − Total tangent length

= 4687.5 − 174.79 = 4512.71 m

Chainage of junction of transition curve and circular curve (P_1)

= Chainage of T_1 + Length of transition curve

= 4512.71 + 50 = 4562.71 m



Chainage of junction of circular curve and transition curve (P_2)

= Chainage of T_1 + Length of transition curve

= 4512.71 + 50 = 4562.71 m

Chainage of junction of circular curve and transition curve (P_2)

= Chainage of P_1 + Length of circular curve

= 4562.76 + 235.18 = 4797.89 m

Chainage of end of transition curve (T_2) = Chainage of P_2 + Length of transition curve

= 4797.89 + 50 = 4847.89 m

Example 12.2 A straight road alignment takes a bend and gets deflected by 78°. The road is to be designed for a design speed of 80 km/hr with a circular curve associated with two cubic spirals (as transition curves). The maximum rate of change of radial acceleration is 0.3 m/s²/s with a maximum permissible centrifugal ratio of 1/4.

Determine:

(a) The radius of circular curve

(b) The length of transition curve

(c) The total length of the combined curve

(d) The chainages of prominent points if the chainage of point of intersection is 1245.65 m.

Solution:

$$\text{Design speed in m/s } (v) = \frac{80 \times 1000}{3600} \text{ m/s} = 22.22 \text{ m/s}$$

(a) $$\text{Max. centrifugal ratio} = \frac{p}{w} = \frac{v^2}{gh}$$

$$\Rightarrow \quad \frac{1}{4} = \frac{22.22^2}{9.81 \times R_{min}}$$

$$\Rightarrow \quad R_{min} = 201.32 \text{ m}$$

(b) Length of transition curve, $$L = \frac{v^2}{\alpha R} = \frac{22.22^3}{0.3 \times 201.32} = 181.65\text{m}$$

(c) Total length of combined curve

$$L_c = \text{Length of circular curve} + 2 \times \text{Length of transition curve}$$

$$= \frac{\pi R(\Delta - 2\phi_s)}{180°} + 2 \times 181.65$$

Now, $$\phi_s = \frac{L}{2R} = \frac{181.65}{2 \times 201.32} = 0.45115 \text{rad} = 25.85°$$

$$\therefore \quad L_c = \frac{\pi(201.32)(78° - 2 \times 25.85°)}{180°} + 2 \times 181.65$$

$$= 92.41 + 363.3 = 455.71 \text{ m}$$

(d) $$\text{Total tangent length } (T_t) = (R + s)\tan\left(\frac{\Delta}{2}\right) + \frac{L}{2}$$

Now, $$\text{Shift } (s) = \frac{L^2}{24R} = \frac{181.65^2}{24 \times 201.32} = 6.83\text{m}$$

$$\therefore \quad T_t = (201.32 + 6.83)\tan\left(\frac{78}{2}\right) + 181.65/2 = 259.38\text{m}$$

Chainage of start of curve (T_1)

= Chainage of point of intersection (v) – Total tangent length (T_t)

= 1245.65 – 259.38

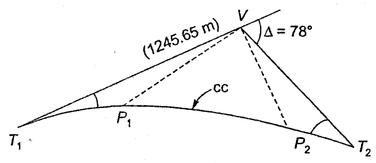
= 986.27 m

Chainage of junction of circular curve and transition curve (P_2)

= Chainage of point P_1 + Length of circular curve

= 1167.92 + 92.41

= 1260.33 m



Chainage of end of curve (T_2)

= Chainage of point P_2 + Length of transition curve

= 1260.33 + 181.65

= 1441.98 m

Example 12.3 A 7.5 m wide road deflects through an angle of 51°35′. The forward chainage of intersection is 8778.5 m. A circular curve of 185 m radius is to be inserted for a design speed of 75 km/hr at a rate of change of radial acceleration of 0.47 m/s²/s. Find the length of transition curve, the maximum super-elevation of outer curve. If pegs are at 10 m interval then determine the chainages of points.

Solution:

$$\text{Design speed } (v) = 75 \text{ km/hr} = 20.833 \text{ m/s}$$

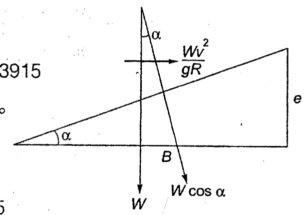
But $$\tan\alpha = \frac{v^2}{gR} = \frac{20.833^2}{9.81 \times 185} = 0.23915$$

$$\Rightarrow \quad \alpha = \tan^{-1}(0.23915) = 13.45°$$

Now, $$\tan\alpha = \frac{e}{B} = \frac{v^2}{gR}$$

$$\therefore \quad e = B\tan\alpha = 7.5 \times 0.23915$$

$$= 1.79 \text{ m} \simeq 1.8 \text{ m}$$



Rate of change of radial acceleration,

$$\alpha = \frac{v^2}{LR}$$

$$\Rightarrow \quad 0.47 = \frac{20.833^3}{L(185)}$$

$$\Rightarrow \quad L = 103.99 \text{ m} \simeq 104 \text{ m}$$

Shift of circular curve due to transition curve

$$s = \frac{L^2}{24R} = \frac{104^2}{24 \times 185} = 2.436\text{m}$$

Let equation of transition curve is, $$y = \frac{x^2}{6RL} = \frac{x^2}{6 \times 185 \times 104} = \frac{x^3}{115440}$$

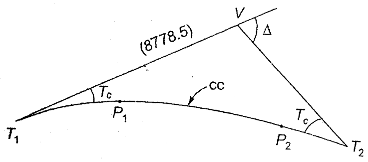
$$\text{Total length of tangent } (T_t) = (R+s)\tan\left(\frac{\Delta}{2}\right)+\frac{L}{2}$$

$$= (1.85+2.436)\tan\left(\frac{51°35'}{2}\right)+\frac{104}{2}$$

$$= 90.58 + 52 = 142.58 \text{ m}$$

Chainage of T_1 = Chainage of V – Total length of tangent (T_t)
= 8778.5 – 142.58 = 8635.92 m

Chainage of P_1 = Chainage of T_1 + Length of transition curve
= 8635.92 + 104 = 8739.92 m



Chainage of P_2 = Chainage of P_1 + Length of transition curve
= 8739.92 + 90.58 = 8830.5 m

Chainage of T_2 = Chainage of P_1 + Length of transition curve
= 8830.5 + 104 = 8934.5 m

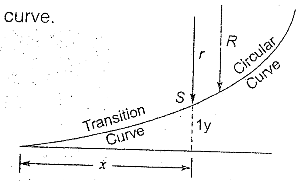
Example 12.4 A transition curve is required for a circular curve of radius 210 m and the gauge is 1.4 m. The permissible super-elevation is 70 mm. The transition curve is to be so designed that no lateral pressure is imposed on the wheels. The rate of change of radial acceleration is to be kept at 0.25 m/s²/s. Starting from the first principles find the length of transition curve and the design speed.

Solution:

Let r = Radius of curvature at any point $s(x,y)$ on the transition curve.

Thus, $$x \propto \frac{1}{r}$$

At $$x = L, \quad r = R$$



The radius of curvature 'r' is given by,

$$r = \frac{\left[1+\left(\frac{dy}{dx}\right)^3\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

For small values of $\frac{dy}{dx}, \left(\frac{dy}{dx}\right)^2$ can be ignored.

Thus, $$r = \frac{1}{(d^2y/dx^2)}$$

$$\Rightarrow \quad \frac{1}{r} = \frac{d^2y}{dx^2}$$

$$\theta = \frac{\text{Arc}}{\text{Radius}}$$

$$\Rightarrow \quad \frac{1}{\text{Radius}} = \frac{\theta}{\text{Arc}}$$

$$\Rightarrow \quad \frac{1}{r} = \frac{(x/R)}{L} = \frac{x}{RL}$$

$$\therefore \quad \frac{1}{r} = \frac{d^2y}{dx^2} = \frac{x}{RL}$$

Integrating, $$\frac{dy}{dx} = \frac{x^2}{2RL} + C_1$$

and, $$y = \frac{x^2}{6RL} + C_1x + C_2$$

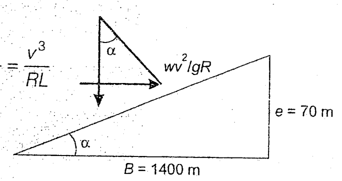
At $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$

$$\therefore \quad C_1 = 0, \quad C_2 = 0$$

$$\therefore \quad y = \frac{x^3}{6RL}$$

Now, rate of change of radial acceleration $(\alpha) = \frac{d}{dt}\left(\frac{v^2}{r}\right)$

$$= \frac{d}{dt}\left(\frac{v^2x}{RL}\right) = \frac{v^2}{RL}\frac{dx}{dt} = \frac{v^3}{RL}$$



For no lateral pressure, $$\frac{v^2}{gR} = \tan\alpha = \frac{e}{B} = \frac{70}{1400}$$

$$\Rightarrow \quad v^2 = \frac{70}{1400}gR = \frac{70}{1400}\times 9.81\times 210$$

$$= 103.005 \text{ m}^2/\text{s}^2$$

$$\therefore \quad v = \sqrt{103.005} \text{ m/s} = 10.149 \text{ m/s}$$

$$= 36.54 \text{ km/hr} \simeq 36 \text{ km/hr}$$

Now, $$\alpha = 0.25 \text{ m/s}^2/\text{s}$$

$$\Rightarrow \quad \frac{v^3}{RL} = 0.25$$

$$\Rightarrow \quad L = \frac{v^3}{0.25R} = \frac{(10.149)^3}{0.25\times 210} = 19.91 \text{ m} = \text{Length of transition curve}$$

Objective Brain Teasers

Q.1 Super-elevation (e) can be expressed as:

(a) $\frac{Bv^2}{gR^2}$ (b) $\frac{Bv^2}{gR}$

(c) $\frac{Bv^3}{gR}$ (d) $\frac{Bv}{gR}$

Q.2 The equation of cubic spiral is given by:

(a) $y = \frac{x^3}{6RL}$ (b) $y = \frac{l^3}{6RL}$

(c) $y^2 = \frac{l^3}{6RL}$ (d) None of these

Q.3 Spiral angle (ϕ_s) is equal to:

(a) $\frac{L}{4R}$ (b) $\frac{L}{2R}$

(c) $\frac{L}{R}$ (d) $\frac{L}{6R}$

Q.4 The shift(s) of a circular curve is given by:

(a) $s=\frac{L^2}{24R}$ (b) $s=\frac{L^2}{6R}$

(c) $s=\frac{L^2}{12R}$ (d) None of these

Q.5 For an ideal transition curve:

(a) Centrifugal force must be proportional to the curvature

(b) Length of curve must be inversely proportional to curvature

(c) Intrinsic equation is $\phi=\frac{L^2}{2RL}$

(d) All of these

Q.6 An ideal transition curve is:

(a) a clothoid

(b) a cubic parabola

(c) a parabola

(d) Bernoullis lemniscate

Q.7 Perpendicular offset from the junction of a transition curve and circular curve to the tangent is equal to:

(a) 4 times the shift

(b) 2 times the shift

(c) Half times the shift

(d) Shift

Q.8 Generally, the transition curve used in highways is:

(a) Cubic parabola

(b) Cubic spiral

(c) Clothoid

(d) Bernoullis lemniscate

Q.9 A transition curve is provided between a straight and a curve because:

(a) it bisects the shift

(b) a gradual change in super elevation can be accommodated in an easy way

(c) its radius of curvature increases or decreases gradually

(d) its eliminates the possibility of derailment

Q.10 The maximum value of centrifugal ratio on roads is usually taken as:

(a) 1/8 (b) 1/2

(c) 1/4 (d) 1/16

Q.11 For roads, the maximum design speed is ______ % ______ than railway.

(a) 41.4, lower (b) 41.4, higher

(c) 30, higher (d) 10, higher

Q.12 Cubic parabola cannot be used beyond deviation angle of

(a) 26° 25′ 11″ (b) 24° 05′ 41″

(c) 20° 05′ 16″ (d) 35° 43′ 11″

Q.13 Pick out the correct statement(s)

(i) Cubic parabola is easy to set out in field than cubic spiral.

(ii) Due to less number of assumptions, cubic spiral is better than cubic parabola.

(a) (i) only (b) (i) and (ii)

(c) (ii) only (d) Neither (i) nor (ii)

Q.14 The term "SHIFT" is also known as

(a) Displacement (b) Extension

(c) Adjustment (d) Throw

Q.15 The shift ______ the transition curve

(a) Trisects (b) Quadsects

(c) Pentsects (d) Bisects

Q.16 For a lemniscate to be transitional throughout, polar deflection angle (α) should be

(a) $\phi/6$ (b) $\phi/3$

(c) $\phi/2$ (d) ϕ

Answers

1. (b)	2. (b)	3. (b)	4. (a)	5. (d)
6. (a)	7. (c)	8. (d)	9. (b)	10. (c)
11. (b)	12. (b)	13. (b)	14. (d)	15. (d)
16. (a)				

Student's Assignments

Ex.1 Compute the required tangential offsets for setting out a transition curve of 125 m length with 20 m peg interval. The radius of circular curve is 235 m. Assume transition curve to be (a) cubic spiral and (b) cubic parabola.

Ex.2 Three straights *PQ*, *QR* and *RS* have *WCBs* of 45°, 90° and 30° respectively. The straight *PQ* is required to be connected to *RS* by a reverse curve made of two circular curves of equal radii. The straight *QR* is the common tangent for the two inner transition curves and is 905 m long. If maximum speed limit is 100 km/hr then calculate the radius of circular curve. The rate of change of radial acceleration is 0.3 m/sec^3.

■■■■