Chapter 4

TRIGONOMETRY

Learning Objectives

After studying this chapter, the students will be able to understand

- trigonometric ratio of angles
- addition formulae, multiple and sub-multiple angles
- transformation of sums into products and vice versa
- basic concepts of inverse trigonometric functions
- properties of inverse trigonometric functions

Brahmagupta

Introduction

The word trigonometry is derived from the Greek word 'tri' (meaning three), 'gon' (meaning sides) and 'metron' (meaning measure). In fact, trigonometry is the study of relationships between the sides and angles of a triangle. Around second century A.D. George Rheticus was the first to define the trigonometric functions in terms of right angles. The study of trigonometry was first started in India. The ancient Indian Mathematician, Aryabhatta, Brahmagupta, Bhaskara I and Bhaskara II obtained important results. Bhaskara I gave formulae to find the values of sine functions for angles more than 90 degrees. The earliest applications of trigonometry were in the fields of navigation, surveying and astronomy.

IANBQB

DO YOU KNOW?

Currently, trigonometry is used in many areas such as electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analyzing a musical tone.

Recall

1. $\sin\theta = \dfrac{\text{side opposite to angle } \theta}{\text{Hypotenuse}}$

2. $\cos\theta = \dfrac{\text{Adjacent side to angle } \theta}{\text{Hypotenuse}}$

3. $\tan\theta = \dfrac{\text{side opposite to angle } \theta}{\text{Adjacent side to angle } \theta}$

4. $\cot\theta = \dfrac{\text{Adjacent side to angle } \theta}{\text{side opposite to angel } \theta}$

5. $\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Adjacent side to angle } \theta}$

6. $\operatorname{cosec}\theta = \dfrac{\text{Hypotenuse}}{\text{side opposite to angle } \theta}$

Relations between trigonometric ratios

1. $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ or $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

2. $\cos\theta = \dfrac{1}{\sec\theta}$ or $\sec\theta = \dfrac{1}{\cos\theta}$

3. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ or $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

4. $\tan\theta = \dfrac{1}{\cot\theta}$ or $\cot\theta = \dfrac{1}{\tan\theta}$

Trigonometric Identities

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

Angle

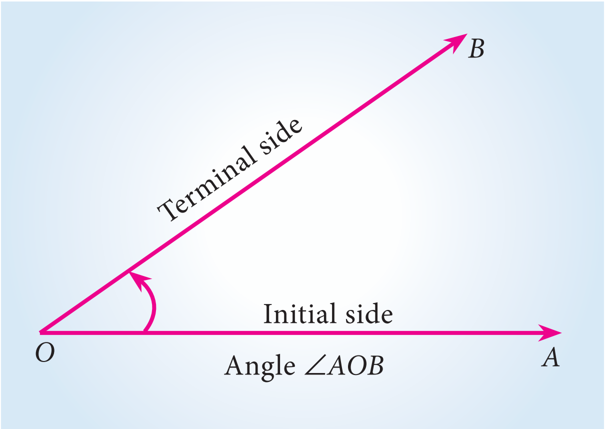


Fig. 4.1

Angle is a measure of rotation of a given ray about its initial point. The ray 'OA' is called the initial side and the final position of the ray 'OB' after rotation is called the terminal side of the angle. The point 'O' of rotation is called the vertex.

If the direction of rotation is anticlockwise, then angle is said to be positive and if the direction of rotation is clockwise, then angle is said to be negative.

Measurement of an angle

Two types of units of measurement of an angle which are most commonly used namely degree measure and radian measure.

Degree measure

If a rotation from the initial position to the terminal position $\left(\frac{1}{360}\right)^{th}$ of the revolution, the angle is said to have a measure of one degree and written as 1^o. A degree is divided into minutes and minute is divided into seconds.

One degree = 60 minutes $(60')$
One minute = 60 seconds $(60'')$

Radian measure

The angle subtended at the centre of the circle by an arc equal to the length of the radius of the circle is called a radian, and it is denoted by 1^c

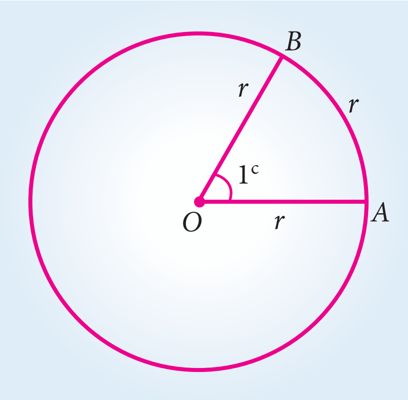


Fig. 4.2

NOTE

The number of radians in an angle subtended by an arc of a circle at the centre of a circle is

$$= \frac{\textit{length of the arc}}{\textit{radius}}$$

i.e., $\theta = \frac{s}{r}$, where θ is the angle subtended at the centre of a circle of radius r by an arc of length s.

Relation between degrees and radians

We know that the circumference of a circle of radius 1unit is 2π. One complete revolution of the radius of unit circle subtends 2π radians. A circle subtends at the centre an angle whose degree measure is 360°.

$$2\pi \text{ radians} = 360^\circ \qquad \pi \text{ radians} = 180^\circ$$

$$1 \text{radian} = \frac{180^\circ}{\pi}$$

Example 4.1

Convert: (i) 160° into radians

(ii) $\frac{4\pi}{5}$ radians into degree

(iii) $\frac{1}{4}$ radians into degree

Solution

i) $160^\circ = 160 \times \frac{\pi}{180} \textit{radians} = \frac{8}{9}\pi$

ii) $\frac{4\pi}{5} \textit{radians} = \frac{4}{5}\pi \times \frac{180^\circ}{\pi} = 144^\circ$

iii) $\frac{1}{4} \textit{radian} = \frac{1}{4}\frac{180}{\pi}$

$= \frac{1}{4} \times 180 \times \frac{7}{22} = 14^\circ 19' 5''$

4.1 Trigonometric Ratios

4.1.1 Quadrants:

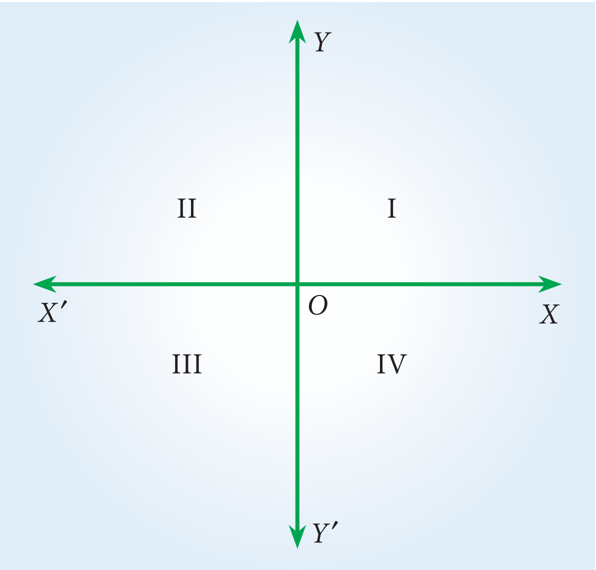


Fig. 4.3

Let $X'OX$ and $Y'OY$ be two lines at right angles to each other as in the figure. We call $X'OX$ and $Y'OY$ as X axis and Y axis respectively. Clearly these axes divided the entire plane into four equal parts called "Quadrants".

The parts XOY, YOX', $X'OY'$ and $Y'OX$ are known as the first, second, third and the fourth quadrant respectively.

Example 4.2

Find the quadrants in which the terminal sides of the following angles lie.

(i) -70° (ii) -320° (iii) 1325°

Solution (i)

(i) The terminal side of -70° lies in IV quadrant.

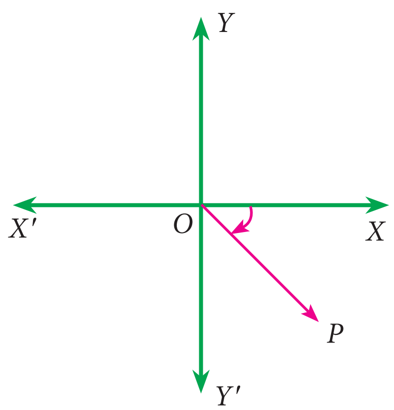


Fig. 4.4

(ii) The terminal side -320° of lies in I quadrant.

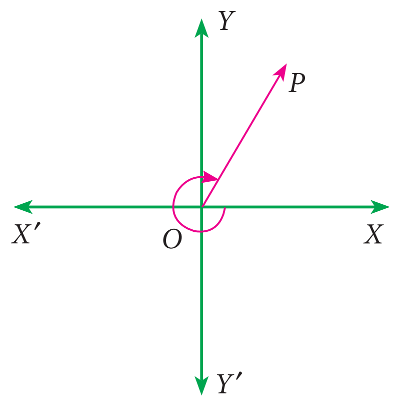


Fig. 4.5

(iii) $1325^\circ = (3 \times 360) + 180^\circ + 65^\circ$ the terminal side lies in III quadrant.

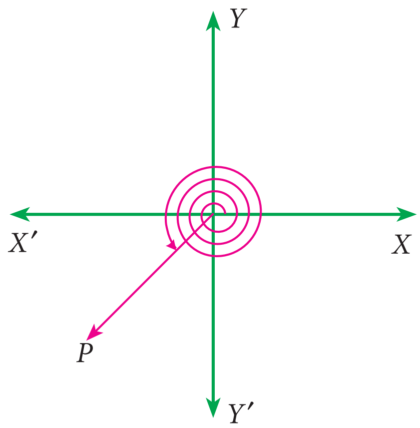
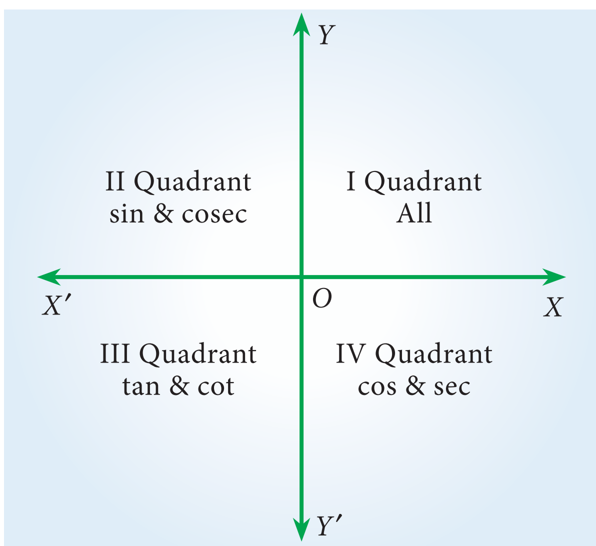


Fig. 4.6

4.1.2 Signs of the trigonometric ratios of an angle θ as it varies from 0° to 360°



ASTC : All Sin Tan Cos

Fig. 4.7

In the first quadrant both x and y are positive. So all trigonometric ratios are positive. In the second quadrant $(90^\circ < \theta < 180^\circ)$ x is negative and y is positive. So trigonometric ratios sin θ and cosec θ are positive. In the third quadrant $(180^\circ < \theta < 270^\circ)$ both x and y are negative. So trigonometric ratios tan θ and cot θ are positive. In the fourth quadrant $(270^\circ < \theta < 360^\circ)$ x is positive and y is negative. So trigonometric ratios cos θ and sec θ are positive.

A function $f(x)$ is said to be odd function if $f(-x) = -f(x)$. $\sin\theta$, $\tan\theta$, $\cot\theta$ and $\operatorname{cosec}\theta$ are all odd function. A function $f(x)$ is said to be even function if $f(-x) = f(x)$. $\cos\theta$ and $\sec\theta$ are even function.

4.1.3 Trigonometric ratios of allied angles:

Two angles are said to be allied angles when their sum or difference is either zero or a multiple of 90°. The angles $-\theta$, $90^\circ \pm \theta$, $180^\circ \pm \theta$, $360^\circ \pm \theta$ etc .,are angles allied to the angle θ. Using trigonometric ratios of the allied angles we can find the trigonometric ratios of any angle.

Angle/ Function	$-\theta$	$90^\circ-\theta$ or $\frac{\pi}{2}-\theta$	$90^\circ+\theta$ or $\frac{\pi}{2}+\theta$	$180^\circ-\theta$ or $\pi-\theta$	$180^\circ+\theta$ or $\pi+\theta$	$270^\circ-\theta$ or $\frac{3\pi}{2}-\theta$	$270^\circ+\theta$ or $\frac{3\pi}{2}+\theta$	$360^\circ-\theta$ or $2\pi-\theta$	$360^\circ+\theta$ or $2\pi+\theta$
Sine	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
Cosine	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tangent	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
cotangent	$-\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$
secant	$\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\sec\theta$
cosecant	$-\operatorname{cosec}\theta$	$\sec\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$	$\operatorname{cosec}\theta$

Table : 4.1

Example 4.3

Find the values of each of the following trigonometric ratios.

(i) $\sin 150^\circ$ (ii) $\cos(-210^\circ)$

(iii) $\text{cosec}\,390^\circ$ (iv) $\tan(-1215^\circ)$

(v) $\sec 1485^\circ$

Solution

(i) $\sin 150^\circ = \sin(1\times 90^\circ + 60^\circ)$

Since 150° lies in the second quadrant, we have

$$\sin 150^\circ = \sin(1\times 90^\circ + 60^\circ)$$

$$= \cos 60^\circ = \frac{1}{2}$$

(ii) we have $\cos(-210^\circ) = \cos 210^\circ$

Since 210° lies in the third quadrant, we have

$$\cos 210^\circ = \cos(180^\circ + 30^\circ)$$

$$= -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

(iii) $\text{cosec}\,390^\circ = \text{cosec}(360^\circ + 30^\circ)$

$= \text{cosec}\,30^\circ = 2$

(iv) $\tan(-1215^\circ) = -\tan(1215^\circ)$

$= -\tan(3\times 360^\circ + 135^\circ)$

$= -\tan 135^\circ$

$= -\tan(90^\circ + 45^\circ)$

$= -(-\cot 45^\circ) = 1$

(v) $\sec 1485^\circ = \sec(4\times 360^\circ + 45^\circ)$

$= \sec 45^\circ = \sqrt{2}$

Example 4.4

Prove that

$\sin 600^\circ \cos 390^\circ + \cos 480^\circ \sin 150^\circ = -1$

Solution

$\sin 600^\circ = \sin(360^\circ + 240^\circ) = \sin 240^\circ$

$= \sin(180^\circ + 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$

$\cos 390^\circ = \cos(360^\circ + 30^\circ)$

$= \cos 30^\circ = \frac{\sqrt{3}}{2}$

$\cos 480^\circ = \cos(360^\circ + 120)$

$= \cos 120^\circ$

$= \cos(180^\circ - 60^\circ) = -\cos 60^\circ$

$= -\frac{1}{2}$

$\sin 150^\circ = \sin(180^\circ - 30^\circ)$

$= \sin 30^\circ = \frac{1}{2}$

Now, $\sin 600^\circ \cos 390^\circ + \cos 480^\circ \sin 150^\circ$

$$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= -\frac{3}{4} - \frac{1}{4} = \frac{-3-1}{4} = -1$$

Example 4.5

Prove that

$$\frac{\sin(-\theta)\tan(90^\circ-\theta)\sec(180^\circ-\theta)}{\sin(180+\theta)\cot(360-\theta)\,\text{cosec}(90^\circ-\theta)} = 1$$

Solution

L.H.S

$$= \frac{\sin(-\theta)\tan(90^\circ-\theta)\sec(180^\circ-\theta)}{\sin(180+\theta)\cot(360-\theta)\,\text{cosec}(90^\circ-\theta)}$$

$$= \frac{(-\sin\theta)\cot\theta(-\sec\theta)}{(-\sin\theta)(-\cot\theta)\sec\theta} = 1$$

Exercise 4.1

1. Convert the following degree measure into radian measure

   (i) 60° (ii) 150°

   (iii) 240° (iv) -320°

2. Find the degree measure corresponding to the following radian measure.

   (i) $\frac{\pi}{8}$ (ii) $\frac{9\pi}{5}$

   (iii) -3 (iv) $\frac{11\pi}{18}$

04_11th_BM-STAT_Ch-4-EM.indd 80 21-04-2020 12:10:06 PM

3. Determine the quadrants in which the following degree lie.

(i) 380° (ii) –140° (iii) 1195°

4. Find the values of each of the following trigonometric ratios.

(i) sin300° (ii) cos(–210°)

(iii) sec390° (iv) tan(–855°)

(v) cosec1125°

5. Prove that:

(i) tan(–225°) cot(–405°) – tan(–765°) cot(675°) = 0

(ii) $2\sin^2\frac{\pi}{6}+\operatorname{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=\frac{3}{2}$

(iii) $\sec\left(\frac{3\pi}{2}-\theta\right)\sec\left(\theta-\frac{5\pi}{2}\right)+\tan\left(\frac{5\pi}{2}+\theta\right)\tan\left(\theta-\frac{5\pi}{2}\right)=-1$

6. If A, B, C, D are angles of a cyclic quadrilateral, prove that :

$\cos A + \cos B + \cos C + \cos D = 0$

7. Prove that:

(i) $\frac{\sin(180^\circ-\theta)\cos(90^\circ+\theta)\tan(270-\theta)\cot(360-\theta)}{\sin(360-\theta)\cos(360+\theta)\sin(270-\theta)\operatorname{cosec}(-\theta)}$ $= -1$

(ii) $\sin\theta\cdot\cos\theta\left\{\sin\left(\frac{\pi}{2}-\theta\right)\cdot\operatorname{cosec}\theta+\cos\left(\frac{\pi}{2}-\theta\right)\cdot\sec\theta\right\}=1$

8. Prove that : cos510° cos330° + sin390° cos120° = –1

9. Prove that:

(i) $\tan(\pi+x)\cot(x-\pi)-\cos(2\pi-x)\cos(2\pi+x)=\sin^2 x$

(ii) $\frac{\sin(180^\circ+A)\cos(90^\circ-A)\tan(270^\circ-A)}{\sec(540^\circ-A)\cos(360^\circ+A)\operatorname{cosec}(270^\circ+A)}=-\sin A\cos^2 A$

10. If $\sin\theta=\frac{3}{5}$, $\tan\varphi=\frac{1}{2}$ and $\frac{\pi}{2}<\theta<\pi<\varphi<\frac{3\pi}{2}$, then find the value of $8\tan\theta-\sqrt{5}\sec\varphi$

4.2 Trigonometric Ratios of Compound Angles

4.2.1 Compound angles

When we add or subtract angles, the result is called a compound angle. i.e., the algebraic sum of two or more angles are called compound angles

For example, If A, B, C are three angles then $A \pm B$, $A + B + C$, $A - B + C$ etc., are compound angles.

4.2.2 Sum and difference formulae of sine, cosine and tangent

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$

(vi) $\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$

Example 4.6

If $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, $\frac{3\pi}{2} < (A, B) < 2\pi$, find the value of

(i) $\sin(A-B)$

(ii) $\cos(A+B)$.

Solution

Since $\frac{3\pi}{2} < (A, B) < 2\pi$, both A and B lie in the fourth quadrant,

$\therefore \sin A$ and $\sin B$ are negative.

Given $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$,

Therefore,

$\sin A = -\sqrt{1-\cos^2 A}$

$= -\sqrt{1-\frac{16}{25}}$

$$= -\sqrt{\frac{25-16}{25}}$$

$$= -\frac{3}{5}$$

$$\text{Sin B} = -\sqrt{1-\cos^2 B}$$

$$= -\sqrt{1-\frac{144}{169}}$$

$$= -\sqrt{\frac{169-144}{169}}$$

$$= -\frac{5}{13}$$

(i) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \frac{4}{5}\times\frac{12}{13} - \left(\frac{-3}{5}\right)\times\left(\frac{-5}{13}\right)$$

$$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

(ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

$$= \left(\frac{-3}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{-5}{13}\right)$$

$$= \frac{-36}{65} + \frac{20}{65} = \frac{-16}{65}$$

Example 4.7

Find the values of each of the following trigonometric ratios.

(i) $\sin 15^o$ (ii) $\cos(-105^o)$

(iii) $\tan 75^o$ (iv) $-\sec 165^o$

Solution

(i) $\sin 15^o = \sin(45^o - 30^o)$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) $\cos(-105^o)$

$$= \cos 105^o$$

$$= \cos(60^o + 45^o)$$

$$= \cos 60^o \cos 45^o - \sin 60^o \sin 45^o$$

$$= \frac{1}{2}\times\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

(iii) $\tan 75^o$

$$= \tan(45^o + 30^o)$$

$$= \frac{\tan 45^\circ + \tan 30^\circ}{1-\tan 45^\circ \tan 30^\circ}$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$$

(iv) $\cos 165^o = \cos(180^o - 15^o)$

$$= -\cos 15^o$$

$$= -\cos(60^o - 45^o)$$

$$= -(\cos 60^o \cos 45^o + \sin 60^o \sin 45^o)$$

$$= -\left(\frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\right)$$

$$= -\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)$$

$$\therefore -\sec 165^o = \frac{2\sqrt{2}}{1+\sqrt{3}}$$

$$= \frac{2\sqrt{2}}{1+\sqrt{3}}\times\frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= \sqrt{2}(\sqrt{3}-1)$$

Example 4.8

If $\tan A = m \tan B$, prove that $\frac{\sin(A+B)}{\sin(A-B)} = \frac{m+1}{m-1}$

DO YOU KNOW?

Componendo and dividendo rule:

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Solution

Given $\tan A = m \tan B$

$$\frac{\sin A}{\cos A} = m\frac{\sin B}{\cos B}$$

$$\frac{\sin A \cos B}{\cos A \sin B} = m$$

Applying componendo and dividendo rule, we get

$$\frac{\sin A\cos B+\cos A\sin B}{\sin A\cos B-\cos A\sin B}=\frac{m+1}{m-1}$$

$$\frac{\sin(A+B)}{\sin(A-B)}=\frac{m+1}{m-1},$$

which completes the proof.

4.2.3 Trigonometric ratios of multiple angles

Identities involving sin2A, cos2A, tan2A, sin3A etc., are called multiple angle identities.

(i) $\sin 2A = \sin(A+A)=\sin A\cos A+\cos A\sin A = 2\sin A\cos A.$

(ii) $\cos 2A = \cos(A+A)=\cos A\cos A-\sin A\sin A=\cos^2 A-\sin^2 A$

(iii) $\sin 3A = \sin(2A+A)$

$= \sin 2A\cos A+\cos 2A\sin A$

$= (2\sin A\cos A)\cos A+(1-2\sin^2 A)\sin A$

$= 2\sin A\cos^2 A+\sin A-2\sin^3 A$

$= 2\sin A(1-\sin^2 A)+\sin A-2\sin^3 A$

$= 3\sin A-4\sin^3 A$

Thus we have the following multiple angles formulae

1. (i) $\sin 2A = 2\sin A\cos A$

   (ii) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$

2. (i) $\cos 2A = \cos^2 A-\sin^2 A$

   (ii) $\cos 2A = 2\cos^2 A-1$

   (iii) $\cos 2A = 1-2\sin^2 A$

   (iv) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

3. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

4. $\sin 3A = 3\sin A-4\sin^3 A$

5. $\cos 3A = 4\cos^3 A-3\cos A$

6. $\tan 3A = \dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

NOTE

(i) $\cos^2 A=\frac{1}{2}(1+\cos 2A)$ (or)

$\cos A=\pm\sqrt{\dfrac{1+\cos 2A}{2}}$

(ii) $\sin^2 A=\frac{1}{2}(1-\cos 2A)$ (or)

$\sin A=\pm\sqrt{\dfrac{1-\cos 2A}{2}}$

Example 4.9

Show that $\dfrac{\sin 2\theta}{1+\cos 2\theta}=\tan\theta$

Solution

$$\frac{\sin 2\theta}{1+\cos 2\theta}=\frac{2\sin\theta\cos\theta}{2\cos^2\theta}=\frac{\sin\theta}{\cos\theta}=\tan\theta$$

Example 4.10

Using multiple angle identity, find tan60º

Solution

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

Put A = 30º in the above identity, we get

$$\tan 60^\circ=\frac{2\tan 30^\circ}{1-\tan^2 30^\circ}$$

$$=\frac{2\frac{1}{\sqrt{3}}}{1-\frac{1}{3}}=\frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}}=\frac{2}{\sqrt{3}}\times\frac{3}{2}=\sqrt{3}$$

Example 4.11

If $\tan A=\dfrac{1}{7}$ and $\tan B=\dfrac{1}{3}$, show that $\cos 2A=\sin 4B$

Solution

$$\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}=\frac{1-\frac{1}{49}}{1+\frac{1}{49}}=\frac{48}{49}\times\frac{49}{50}$$

$$=\frac{24}{25} \quad \ldots (1)$$

Now, $\sin 4B = 2\sin 2B \cos 2B$

$$= 2\frac{2\tan B}{1+\tan^2 B} \times \frac{1-\tan^2 B}{1+\tan^2 B}$$

$$= \frac{4\times\frac{1}{3}}{1+\frac{1}{9}} \times \frac{1-\frac{1}{9}}{1+\frac{1}{9}} = \frac{24}{25} \quad \ldots (2)$$

From(1) and (2) we get, $\cos 2A = \sin 4B$.

Example 4.12

If $\tan A = \frac{1-\cos B}{\sin B}$ then prove that $\tan 2A = \tan B$

Solution

Consider $\frac{1-\cos B}{\sin B} = \frac{2\sin^2\frac{B}{2}}{2\sin\frac{B}{2}\cos\frac{B}{2}}$

$$= \frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} = \tan\frac{B}{2}$$

$$\tan A = \frac{1-\cos B}{\sin B}$$

$$\tan A = \tan\frac{B}{2}$$

$$A = \frac{B}{2}$$

$$2A = B$$

$$\therefore \tan 2A = \tan B$$

Example 4.13

If $\tan\alpha = \frac{1}{3}$ and $\tan\beta = \frac{1}{7}$ then prove that $(2\alpha+\beta) = \frac{\pi}{4}$.

Solution

$$\tan(2\alpha+\beta) = \frac{\tan 2\alpha + \tan\beta}{1-\tan 2\alpha \cdot \tan\beta}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\times\frac{1}{3}}{1-\frac{1}{9}} = \frac{3}{4}$$

$$\tan(2\alpha+\beta) = \frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}} = \frac{\frac{25}{28}}{\frac{25}{28}}$$

$$= 1 = \tan\frac{\pi}{4}$$

$$2\alpha+\beta = \frac{\pi}{4}$$

Exercise 4.2

1. Find the values of the following:
   (i) cosec15° (ii) sin(–105°) (iii) cot75°
2. Find the values of the following
   (i) sin76° cos16° + cos76° sin16°
   (ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12}$
   (iii) cos70° cos10° – sin70° sin10°
   (iv) $\cos^2 15° - \sin^2 15°$
3. If $\sin A = \frac{3}{5}, 0 < A < \frac{\pi}{2}$ and $\cos B = \frac{-12}{13}$, $\pi < B < \frac{3\pi}{2}$ find the values of the following:
   (i) $\cos(A+B)$ (ii) $\sin(A-B)$
   (iii) $\tan(A-B)$
4. If $\cos A = \frac{13}{14}$ and $\cos B = \frac{1}{7}$ where A, B are acute angles prove that $A-B = \frac{\pi}{3}$
5. Prove that 2tan80° = tan85° –tan 5°
6. If $\cot\alpha = \frac{1}{2}$, $\sec\beta = \frac{-5}{3}$, where $\pi < \alpha < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$, find the value of $\tan(\alpha+\beta)$. State the quadrant in which $\alpha+\beta$ terminates.
7. If $A+B = 45°$, prove that $(1+\tan A)(1+\tan B) = 2$ and hence deduce the value of $\tan 22\frac{1}{2}^\circ$
8. Prove that
   (i) $\sin(A+60^\circ)+\sin(A-60^\circ) = \sin A$
   (ii) $\tan 4A\tan 3A\tan A + \tan 3A + \tan A - \tan 4A = 0$
9. (i) If $\tan\theta = 3$ find $\tan 3\theta$
   (ii) If $\sin A = \frac{12}{13}$, find $\sin 3A$
10. If $\sin A = \frac{3}{5}$, find the values of $\cos 3A$ and $\tan 3A$
11. Prove that $\frac{\sin(B-C)}{\cos B\cos C} + \frac{\sin(C-A)}{\cos C\cos A} + \frac{\sin(A-B)}{\cos A\cos B} = 0.$

12. If $\tan A - \tan B = x$ and $\cos B - \cos A = y$ prove that $\cot(A-B) = \frac{1}{x} + \frac{1}{y}$.

13. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, then prove that $\cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$.

14. Find the value of $\tan\frac{\pi}{8}$.

15. If $\tan\alpha = \frac{1}{7}$, $\sin\beta = \frac{1}{\sqrt{10}}$ Prove that $\alpha + 2\beta = \frac{\pi}{4}$ where $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$.

4.3 Transformation formulae

The two sets of transformation formulae are described below.

4.3.1 Transformation of the products into sum or difference

$\sin A\cos B + \cos A\sin B = \sin(A+B)$ (1)

$\sin A\cos B - \cos A\sin B = \sin(A-B)$ (2)

$\cos A\cos B - \sin A\sin B = \cos(A+B)$ (3)

$\cos A\cos B + \sin A\sin B = \cos(A-B)$ (4)

Adding (1) and (2), we get

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

Subtracting (1) from (2), we get

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

Adding (3) and (4), we get

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

Subtracting (3) from (4), we get

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Thus, we have the following formulae;

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

Also

$$\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$$

$$\cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)$$

4.3.2 Transformation of sum or difference into product

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

Example 4.14

Express the following as sum or difference

(i) $2\sin 2\theta\cos\theta$ (ii) $2\cos 3\theta\cos\theta$

(iii) $2\sin 4\theta\sin 2\theta$ (iv) $\cos 7\theta\cos 5\theta$

(v) $\cos\frac{3A}{2}\cos\frac{5A}{2}$ (vi) $\cos 9\theta\sin 6\theta$

(vii) $2\cos 13A\sin 15A$

Solution

(i) $2\sin 2\theta\cos\theta = \sin(2\theta+\theta) + \sin(2\theta-\theta)$

$= \sin 3\theta + \sin\theta$

(ii) $2\cos 3\theta\cos\theta = \cos(3\theta+\theta) + \cos(3\theta-\theta)$

$= \cos 4\theta + \cos 2\theta$

(iii) $2\sin 4\theta\sin 2\theta = \cos(4\theta-2\theta) - \cos(4\theta+2\theta)$

$= \cos 2\theta - \cos 6\theta$

(iv) $\cos 7\theta\cos 5\theta = \frac{1}{2}[\cos(7\theta+5\theta) + \cos(7\theta-5\theta)]$

$= \frac{1}{2}[\cos 12\theta + \cos 2\theta]$

(v) $\cos\frac{3A}{2}\cos\frac{5A}{2} = \frac{1}{2}\cos\left(\frac{3A}{2}+\frac{5A}{2}\right) + \cos\left(\frac{3A}{2}-\frac{5A}{2}\right)$

$= \frac{1}{2}\left[\cos\frac{8A}{2} + \cos\left(\frac{-2A}{2}\right)\right]$

$= \frac{1}{2}[\cos 4A + \cos(-A)]$

$= \frac{1}{2}[\cos 4A + \cos A]$

(vi) $\cos 9\theta \sin 6\theta = \frac{1}{2}[\sin(9\theta+6\theta) - \sin(9\theta-6\theta)]$

$= \frac{1}{2}[\sin(15\theta) - \sin 3\theta]$

(vii) $2\cos 13A \sin 15A = \sin(13A+15A) - \sin(13A-15A)$

$= \sin 28A + \sin 2A$

Example 4.15

Show that $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{\sqrt{3}}{8}$

Solution

LHS $= \sin 20^\circ \sin 40^\circ \sin 80^\circ$

$= \sin 20^\circ \sin(60^\circ - 20^\circ)\sin(60^\circ + 20^\circ)$

$= \sin 20^\circ[\sin^2 60^\circ - \sin^2 20^\circ]$

$= \sin 20^\circ\left[\frac{3}{4} - \sin^2 20^\circ\right]$

$= \sin 20^\circ\left[\frac{3-4\sin^2 20^\circ}{4}\right]$

$= \frac{3\sin 20^\circ - 4\sin^3 20^\circ}{4} = \frac{\sin 3(20^\circ)}{4}$

$= \frac{\sin 60^\circ}{4} = \frac{\sqrt{3}/2}{4} = \frac{\sqrt{3}}{8}$.

Example 4.16

Show that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$

Solution

L.H.S $= \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$

$= \sin 60^\circ \sin 20^\circ \sin(60^\circ - 20^\circ)\sin(60^\circ + 20^\circ)$

$= \frac{\sqrt{3}}{2}\sin 20^\circ(\sin^2 60^\circ - \sin^2 20^\circ)$

$= \frac{\sqrt{3}}{2}\sin 20^\circ\left(\frac{3}{4} - \sin^2 20^\circ\right)$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{4}\right)(3\sin 20^\circ - 4\sin^3 20^\circ)$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{4}\right)\sin 60^\circ = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16}$

= R.H.S.

Example 4.17

Prove that $\cos^2 A + \cos^2(A+120^\circ) + \cos^2(A-120^\circ) = \frac{3}{2}$

Solution

$\cos^2 A + \cos^2(A+120^\circ) + \cos^2(A-120^\circ)$

$= \cos^2 A + \cos^2(A+120^\circ) + (1 - \sin^2(A-120^\circ))$

(Since $\cos^2 A = 1 - \sin^2 A$)

$= \cos^2 A + 1 + \cos^2(A+120^\circ) - \sin^2(A-120^\circ)$

$= \cos^2 A + 1 + \cos[(A+120^\circ) + (A-120^\circ)]\cos[(A+120^\circ) - (A-120^\circ)]$

(Since $\cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)$)

$= \cos^2 A + 1 + \cos 2A \cos 240^\circ$

$= \cos^2 A + 1 + (2\cos^2 A - 1)\cos(180^\circ + 60^\circ)$

(since $2\cos^2 A = 1 + \cos 2A$)

$= \cos^2 A + 1 + (2\cos^2 A - 1)(-\cos 60^\circ)$

$= \cos^2 A + 1 + (2\cos^2 A - 1)\left(-\frac{1}{2}\right)$

$= \cos^2 A + 1 - \cos^2 A + \frac{1}{2} = \frac{3}{2}$

Example 4.18

Convert the following into the product of trigonometric functions.

(i) $\sin 9A + \sin 7A$ (ii) $\sin 7\theta - \sin 4\theta$

(iii) $\cos 8A + \cos 12A$ (iv) $\cos 4\alpha - \cos 8\alpha$

(v) $\cos 20^\circ - \cos 30^\circ$ (vi) $\cos 75^\circ + \cos 45^\circ$

(vii) $\cos 55^\circ + \sin 55^\circ$

Solution

(i) $\sin 9A + \sin 7A = 2\sin\left(\frac{9A+7A}{2}\right)\cos\left(\frac{9A-7A}{2}\right)$

$= 2\sin\left(\frac{16A}{2}\right)\cos\left(\frac{2A}{2}\right)$

$= 2\sin 8A \cos A$

(ii) $\sin 7\theta - \sin 4\theta = 2\cos\left(\frac{7\theta+4\theta}{2}\right)\sin\left(\frac{7\theta-4\theta}{2}\right)$

$= 2\cos\left(\frac{11\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)$

(iii) $\cos 8A + \cos 12A$

$= 2\cos\left(\frac{8A+12A}{2}\right)\cos\left(\frac{8A-12A}{2}\right)$

$= 2\cos\left(\frac{20A}{2}\right)\cos\left(\frac{-4A}{2}\right)$

$= 2\cos 10A\cos(-2A)$

$= 2\cos 10A\cos 2A$

(iv) $\cos 4\alpha - \cos 8\alpha$

$= -2\sin\left(\frac{4\alpha+8\alpha}{2}\right)\sin\left(\frac{4\alpha-8\alpha}{2}\right)$

$= 2\sin\left(\frac{12\alpha}{2}\right)\sin\left(\frac{4\alpha}{2}\right)$

$= 2\sin 6\alpha \sin 2\alpha$

(v) $\cos 20^\circ - \cos 30^\circ$

$= -2\sin\left(\frac{20^\circ+30^\circ}{2}\right)\sin\left(\frac{20^\circ-30^\circ}{2}\right)$

$= 2\sin\left(\frac{50^\circ}{2}\right)\sin\left(\frac{10^\circ}{2}\right)$

$= 2\sin 25^\circ \sin 5^\circ$

(vi) $\cos 75^\circ + \cos 45^\circ$

$= 2\cos\left(\frac{75^\circ+45^\circ}{2}\right)\cos\left(\frac{75^\circ-45^\circ}{2}\right)$

$= 2\cos\left(\frac{120^\circ}{2}\right)\cos\left(\frac{30^\circ}{2}\right)$

$= 2\cos 60^\circ \cos 15^\circ$

$= 2\times\frac{1}{2}\cos 15^\circ = \cos 15^\circ$

(vii) $\cos 55^\circ + \sin 55^\circ$

$= \cos 55^\circ + \cos(90^\circ - 55^\circ)$

$= \cos 55^\circ + \cos 35^\circ$

$= 2\cos\left(\frac{55^\circ+35^\circ}{2}\right)\cos\left(\frac{55^\circ-35^\circ}{2}\right)$

$= 2\cos 45^\circ \cos 10^\circ = 2\left(\frac{1}{\sqrt{2}}\right)\cdot\cos 10^\circ$

$= \sqrt{2}\cos 10^\circ$

Example 4.19

If three angles A, B and C are in arithmetic progression, Prove that

$$\cot B = \frac{\sin A - \sin C}{\cos C - \cos A}$$

Solution

$$\frac{\sin A - \sin C}{\cos C - \cos A} = \frac{2\sin\left(\frac{A-C}{2}\right)\cos\left(\frac{A+C}{2}\right)}{2\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{A-C}{2}\right)}$$

$$= \frac{\cos\left(\frac{A+C}{2}\right)}{\sin\left(\frac{A+C}{2}\right)} = \cot\left(\frac{A+C}{2}\right) = \cot B$$

(since A, B, C are in $A.P.$, $B=\frac{A+C}{2}$)

Example 4.20

Prove that $\frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$

Solution

$$\frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \frac{\sin 5x + \sin x - 2\sin 3x}{\cos 5x - \cos x}$$

$$= \frac{2\sin 3x\cos 2x - 2\sin 3x}{\cos 5x - \cos x}$$

$$= \frac{2\sin 3x(\cos 2x - 1)}{-2\sin 3x\sin 2x} = \frac{1-\cos 2x}{\sin 2x}$$

$$= \frac{2\sin^2 x}{2\sin x\cos x} = \tan x$$

Example 4.21

Find $\sin 105^\circ + \cos 105^\circ$

Solution

$\sin 105^\circ + \cos 105^\circ$

$= \sin(90^\circ + 15^\circ) + \cos 105^\circ$

$= \cos 15^\circ + \cos 105^\circ$

$= \cos 105^\circ + \cos 15^\circ$

$$= 2\cos\left(\frac{105^\circ + 15^\circ}{2}\right)\cos\left(\frac{105^\circ - 15^\circ}{2}\right)$$

$$= 2\cos\left(\frac{120^\circ}{2}\right)\cos\left(\frac{90^\circ}{2}\right)$$

$$= 2\cos 60^\circ \cos 45^\circ = \frac{1}{\sqrt{2}}$$

Example 4.22

Prove that

$$(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

Solution

$\cos\alpha + \cos\beta =$

$$2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \ldots.(1)$$

$\sin\alpha + \sin\beta =$

$$2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \ldots.(2)$$

Squaring and adding (1) and (2)

$(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$$= 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right) +$$

$$4\sin^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left[\cos^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right)\right]$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

Exercise 4.3

1. Express each of the following as the sum or difference of sine or cosine:

   (i) $\sin\frac{A}{8}\sin\frac{3A}{8}$

   (ii) $\cos(60^\circ + A)\sin(120^\circ + A)$

   (iii) $\cos\frac{7A}{3}\sin\frac{5A}{3}$

   (iv) $\cos 7\theta \sin 3\theta$

2. Express each of the following as the product of sine and cosine

   (i) $\sin A + \sin 2A$

   (ii) $\cos 2A + \cos 4A$

   (iii) $\sin 6\theta - \sin 2\theta$

   (iv) $\cos 2\theta - \cos\theta$

3. Prove that

   (i) $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$

   (ii) $\tan 20^\circ \tan 40^\circ \tan 80^\circ = \sqrt{3}$

4. Prove that

   (i) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$

   (ii) $\sin A \sin(60^\circ + A)\sin(60^\circ - A) = \frac{1}{4}\sin 3A$

5. Prove that

   (i) $\sin(A-B)\sin C + \sin(B-C)\sin A + \sin(C-A)\sin B = 0$

   (ii) $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$

6. Prove that

(i) $\frac{\cos 2A-\cos 3A}{\sin 2A+\sin 3A}=\tan\frac{A}{2}$

(ii) $\frac{\cos 7A+\cos 5A}{\sin 7A-\sin 5A}=\cot A$

7. Prove that

$\cos 20^{\circ}\cos 40^{\circ}\cos 60^{\circ}\cos 80^{\circ}=\frac{1}{16}$

8. Evaluate

(i) $\cos 20^{\circ}+\cos 100^{\circ}+\cos 140^{\circ}$

(ii) $\sin 50^{\circ}-\sin 70^{\circ}+\sin 10^{\circ}$

9. If $\cos A+\cos B=\frac{1}{2}$ and $\sin A+\sin B=\frac{1}{4}$, prove that $\tan\left(\frac{A+B}{2}\right)=\frac{1}{2}$

10. If $\sin(y+z-x)$, $\sin(z+x-y)$, $\sin(x+y-z)$, are in A.P, then prove that $\tan x$, $\tan y$ and $\tan z$ are in A.P

11. If $\operatorname{cosec}A+\sec A=\operatorname{cosec}B+\sec B$ prove that $\cot\left(\frac{A+B}{2}\right)=\tan A\tan B$

4.4 Inverse Trigonometric Functions

4.4.1 Inverse Trigonometric Functions

The quantities such as $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, etc., are known as inverse trigonometric functions.

i.e., if $\sin\theta=x$, then $\theta=\sin^{-1}x$. The domains and ranges (Principal value branches) of trigonometric functions are given below

Function	Domain x	Range y
$\sin^{-1}x$	$[-1, 1]$	$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$R=(-\infty,\infty)$	$\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1}x$	$R-(-1, 1)$	$\left[\frac{-\pi}{2},\frac{\pi}{2}\right], y\neq 0$
$\sec^{-1}x$	$R-(-1, 1)$	$[0, \pi], y\neq\frac{\pi}{2}$
$\cot^{-1}x$	R	$(0, \pi)$

4.4.2 Properties of Inverse Trigonometric Functions

Property (1)

(i) $\sin^{-1}(\sin x)=x$

(ii) $\cos^{-1}(\cos x)=x$

(iii) $\tan^{-1}(\tan x)=x$

(iv) $\cot^{-1}(\cot x)=x$

(v) $\sec^{-1}(\sec x)=x$

(vi) $\operatorname{cosec}^{-1}(\operatorname{cosec} x)=x$

Property (2)

(i) $\sin^{-1}\left(\frac{1}{x}\right)=\operatorname{cosec}^{-1}(x)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right)=\sec^{-1}(x)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right)=\cot^{-1}(x)$

(iv) $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)=\sin^{-1}(x)$

(v) $\sec^{-1}\left(\frac{1}{x}\right)=\cos^{-1}(x)$

(vi) $\cot^{-1}\left(\frac{1}{x}\right)=\tan^{-1}(x)$

Property (3)

(i) $\sin^{-1}(-x)=-\sin^{-1}(x)$

(ii) $\cos^{-1}(-x)=\pi-\cos^{-1}(x)$

(iii) $\tan^{-1}(-x)=-\tan^{-1}(x)$

(iv) $\cot^{-1}(-x)=-\cot^{-1}(x)$

(v) $\sec^{-1}(-x)=\pi-\sec^{-1}(x)$

(vi) $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}(x)$

Property (4)

(i) $\sin^{-1}(x)+\cos^{-1}(x)=\frac{\pi}{2}$

(ii) $\tan^{-1}(x)+\cot^{-1}(x)=\frac{\pi}{2}$

(iii) $\sec^{-1}(x)+\operatorname{cosec}^{-1}(x)=\frac{\pi}{2}$

Property (5)

If $xy<1$, then

(i) $\tan^{-1}(x)+\tan^{-1}(y)=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(ii) $\tan^{-1}(x)-\tan^{-1}(y)=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Property (6)

$\sin^{-1}(x)+\sin^{-1}(y)$
$=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

Example 4.23

Find the principal value of

(i) $\sin^{-1}(1/2)$ (ii) $\tan^{-1}(-\sqrt{3})$

Solution

(i) Let $\sin^{-1}(1/2)=y$ where $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$

$\therefore \sin y = (1/2) = \sin\left(\frac{\pi}{6}\right)$

$y=\frac{\pi}{6}$

The principal value of $\sin^{-1}(1/2)$ is $\frac{\pi}{6}$

(ii) Let $\tan^{-1}(-\sqrt{3})=y$ where $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$

$\therefore \tan y = -\sqrt{3} = \tan\left(-\frac{\pi}{3}\right)$

$y=-\frac{\pi}{3}$

The principal value of $\tan^{-1}(-\sqrt{3})$ is $-\frac{\pi}{3}$

Example 4.24

Evaluate the following

(i) $\cos\left(\sin^{-1}\frac{5}{13}\right)$ (ii) $\tan\left(\cos^{-1}\frac{8}{17}\right)$

Solution

(i) Let $\left(\sin^{-1}\frac{5}{13}\right)=\theta$ ... (1)

$\sin\theta=\frac{5}{13}$

$\cos\theta=\sqrt{1-\sin^2\theta}$

$=\sqrt{1-\frac{25}{169}}$

$=\frac{12}{13}$ ... (2)

From (1) and (2), we get

Now $\cos\left(\sin^{-1}\frac{5}{13}\right)=\cos\theta=\frac{12}{13}$

(ii) Let $\left(\cos^{-1}\frac{8}{17}\right)=\theta$

$\cos\theta=\frac{8}{17}$

$\sin\theta=\sqrt{1-\cos^2\theta}$

$=\sqrt{1-\frac{64}{289}}$

$=\frac{15}{17}$ ... (2)

From (1) and (2), we get

Now $\tan\left(\cos^{-1}\frac{8}{17}\right)=\tan\theta=\frac{\sin\theta}{\cos\theta}$

$=\frac{\frac{15}{17}}{\frac{8}{17}}=\frac{15}{8}$

Example 4.25

Prove that

(i) $\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)$

(ii) $\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{27}{11}\right)$

Solution

(i) $\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)$

$=\tan^{-1}\left[\frac{\frac{1}{7}+\frac{1}{13}}{1-\left(\frac{1}{7}\right)\left(\frac{1}{13}\right)}\right]$

$=\tan^{-1}\left(\frac{20}{90}\right)$

$=\tan^{-1}\left(\frac{2}{9}\right)$

(ii) Let $\cos^{-1}\left(\frac{4}{5}\right) = \theta$. Then $\cos\theta = \frac{4}{5}$

$$\Rightarrow \quad \sin\theta = \frac{3}{5}$$

$$\therefore \tan\theta = \frac{3}{4} \Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\therefore \cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\text{Now, } \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right)$$

$$= \tan^{-1}\left(\frac{27}{11}\right)$$

Example 4.26

Find the value of $\tan\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{8}\right)\right]$

Solution

$$\text{Let } \tan^{-1}\left(\frac{1}{8}\right) = \theta$$

$$\text{Then } \tan\theta = \frac{1}{8}$$

$$\therefore \tan\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{8}\right)\right] = \tan\left(\frac{\pi}{4} - \theta\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}$$

$$= \frac{1-\frac{1}{8}}{1+\frac{1}{8}} = \frac{7}{9}$$

Example 4.27

Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

Solution

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right)$$

$$= \tan^{-1}\left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right] = \tan^{-1}\left(\frac{2x^2-4}{-3}\right)$$

Given that

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\therefore \tan^{-1}\left(\frac{2x^2-4}{-3}\right) = \frac{\pi}{4}$$

$$\frac{2x^2-4}{-3} = \tan\frac{\pi}{4} = 1$$

$$2x^2 - 4 = -3$$

$$\Rightarrow 2x^2 - 1 = 0$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

Example 4.28

Simplify: $\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{3}\right)$

Solution

We know that

$$\sin^{-1}(x) + \sin^{-1}(y) = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

$$\therefore \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{3}\right) = \sin^{-1}\left[\frac{1}{3}\sqrt{1-\frac{4}{9}} + \frac{2}{3}\sqrt{1-\frac{1}{9}}\right]$$

$$= \sin^{-1}\left[\frac{1}{3}\sqrt{\frac{5}{9}} + \frac{2}{3}\sqrt{\frac{8}{9}}\right]$$

$$= \sin^{-1}\left(\frac{\sqrt{5}+4\sqrt{2}}{9}\right)$$

Example 4.29

Solve $\tan^{-1}(x+2) + \tan^{-1}(2-x) = \tan^{-1}\left(\frac{2}{3}\right)$

Solution

$$\tan^{-1}\left[\frac{(x+2)+(2-x)}{1-(x+2)(2-x)}\right] = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{4}{1-(4-x^2)}\right) = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow 2x^2 - 6 = 12$$

$$\Rightarrow x^2 = 9$$

$$\therefore x = \underline{+}3$$

Example 4.30

If $\tan(x + y) = 42$ and $x = \tan^{-1}(2)$, then find y

04_11th_BM-STAT_Ch-4-EM.indd 91 21-04-2020 12:11:08 PM

Solution

$$\tan(x+y) = 42$$
$$x + y = \tan^{-1}(42)$$
$$\tan^{-1}(2)+y = \tan^{-1}(42)$$
$$y = \tan^{-1}(42) - \tan^{-1}(2)$$
$$= \tan^{-1}\left[\frac{42-2}{1+(42\times 2)}\right]$$
$$= \tan^{-1}\left(\frac{40}{85}\right)$$
$$y = \tan^{-1}\left[\frac{8}{17}\right]$$

Exercise 4.4

1. Find the pricipal value of the following

   (i) $\sin^{-1}\left(-\frac{1}{2}\right)$ (ii) $\tan^{-1}(-1)$

   (iii) $\mathrm{cosec}^{-1}(2)$ (iv) $\sec^{-1}(-\sqrt{2})$

2. Prove that

   (i) $2\tan^{-1}(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

   (ii) $\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

3. Show that

   $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{11}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

4. Solve $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

5. Solve

   $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{4}{7}\right)$

6. Evaluate (i) $\cos\left[\tan^{-1}\left(\frac{3}{4}\right)\right]$

   (ii) $\sin\left[\frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)\right]$

7. Evaluate: $\cos\left(\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right)$

8. Prove that

   $\tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}$

9. Show that

   $\sin^{-1}\left(-\frac{3}{5}\right) - \sin^{-1}\left(-\frac{8}{17}\right) = \cos^{-1}\frac{84}{85}$

10. Express

    $\tan^{-1}\left[\frac{\cos x}{1-\sin x}\right], -\frac{\pi}{2} < x < \frac{3\pi}{2}$, in the simplest form.

Exercise 4.5

Choose the correct answer

1. The degree measure of $\frac{\pi}{8}$ is

   (a) $20°60'$
   (b) $22°30'$
   (c) $22°60'$
   (d) $20°30'$

IAX7RY

2. The radian measure of $37°30'$ is

   (a) $\frac{5\pi}{24}$ (b) $\frac{3\pi}{24}$
   (c) $\frac{7\pi}{24}$ (d) $\frac{9\pi}{24}$

3. If $\tan\theta = \frac{1}{\sqrt{5}}$ and θ lies in the first quadrant then $\cos\theta$ is

   (a) $\frac{1}{\sqrt{6}}$ (b) $\frac{-1}{\sqrt{6}}$
   (c) $\frac{\sqrt{5}}{\sqrt{6}}$ (d) $\frac{-\sqrt{5}}{\sqrt{6}}$

4. The value of sin 15º is

   (a) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$
   (c) $\frac{\sqrt{3}}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2\sqrt{2}}$

5. The value of $\sin(-420°)$ is

   (a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{\sqrt{3}}{2}$
   (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

6. The value of $\cos(-480°)$ is

   (a) $\sqrt{3}$ (b) $-\frac{\sqrt{3}}{2}$
   (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

7. The value of sin28º cos17º + cos28º sin17º is

(a) $\frac{1}{\sqrt{2}}$ (b) 1

(c) $\frac{-1}{\sqrt{2}}$ (d) 0

8. The value of sin15º cos15º is

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{4}$

9. The value of sec$A$ sin(270º + A) is

(a) –1 (b) $\cos^2 A$

(c) $\sec^2 A$ (d) 1

10. If sin$A$ + cos$A$ =1 then sin2A is equal to

(a) 1 (b) 2

(c) 0 (d) $\frac{1}{2}$

11. The value of $\cos^2 45^\circ - \sin^2 45^\circ$ is

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$

(c) 0 (d) $\frac{1}{\sqrt{2}}$

12. The value of $1-2\sin^2 45^\circ$ is

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) 0

13. The value of $4\cos^3 40^\circ - 3\cos 40^\circ$ is

(a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{1}{2}$

(c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

14. The value of $\frac{2\tan 30^\circ}{1+\tan^2 30^\circ}$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$

15. If sin A= $\frac{1}{2}$ then $4\cos^3 A - 3\cos A$ is

(a)1 (b) 0

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$

16. The value of $\frac{3\tan 10^\circ - \tan^3 10^\circ}{1-3\tan^2 10^\circ}$ is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{2}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$

17. The value of $\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

18. $\sec^{-1}\frac{2}{3} + \operatorname{cosec}^{-1}\frac{2}{3} =$

(a) $\frac{-\pi}{2}$ (b) $\frac{\pi}{2}$

(c) π (d) $-\pi$

19. If α and β be between 0 and $\frac{\pi}{2}$ and if $\cos(\alpha+\beta) = \frac{12}{13}$ and $\sin(\alpha-\beta) = \frac{3}{5}$ then $\sin 2\alpha$ is

(a) $\frac{16}{15}$ (b) 0

(c) $\frac{56}{65}$ (d) $\frac{64}{65}$

20. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$ then $\tan(2A+B)$ is equal to

(a) 1 (b) 2

(c) 3 (d) 4

21. $\tan\left(\frac{\pi}{4}-x\right)$ is

(a) $\left(\frac{1+\tan x}{1-\tan x}\right)$ (b) $\left(\frac{1-\tan x}{1+\tan x}\right)$

(c) $1-\tan x$ (d) $1+\tan x$

22. $\sin\left(\cos^{-1}\frac{3}{5}\right)$ is

(a) $\frac{3}{5}$ (b) $\frac{5}{3}$

(c) $\frac{4}{5}$ (d) $\frac{5}{4}$

23. The value of $\frac{1}{\operatorname{cosec}(-45^\circ)}$ is

(a) $\frac{-1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\sqrt{2}$ (d) $-\sqrt{2}$

24. If $p\sec 50^\circ = \tan 50^\circ$ then p is

(a) $\cos 50^\circ$ (b) $\sin 50^\circ$

(c) $\tan 50^\circ$ (d) $\sec 50^\circ$

25. $\left(\dfrac{\cos x}{\operatorname{cosec} x}\right) - \sqrt{1-\sin^2 x}\sqrt{1-\cos^2 x}$ is

(a) $\cos^2 x - \sin^2 x$ (b) $\sin^2 x - \cos^2 x$

(c) 1 (d) 0

Miscellaneous Problems

1. Prove that
$$\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$$

2. Prove that $\sqrt{3}\operatorname{cosec} 20^\circ - \sin 20^\circ = 4$

3. Prove that $\cot 4x(\sin 5x + \sin 3x)$
$= \cot x(\sin 5x - \sin 3x)$.

4. If $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$ then find the value of $\sin\frac{x}{2}$ and $\cos\frac{x}{2}$

5. Prove that
$$2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 4\sec^2\frac{3\pi}{4} = 10$$

6. Find the value of (i) $\sin 75^\circ$
(ii) $\tan 15^\circ$

7. If $\sin A = \frac{1}{3}$, $\sin B = \frac{1}{4}$ then find the value of $\sin(A+B)$, where A and B are acute angles.

8. Show that
$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

9. If $\cos(\alpha+\beta) = \frac{4}{5}$ and $\sin(\alpha-\beta) = \frac{5}{13}$ where $\alpha+\beta$ and $\alpha-\beta$ are acute, then find $\tan 2\alpha$

10. Express $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$, $0 < x < \pi$ in the simplest form.

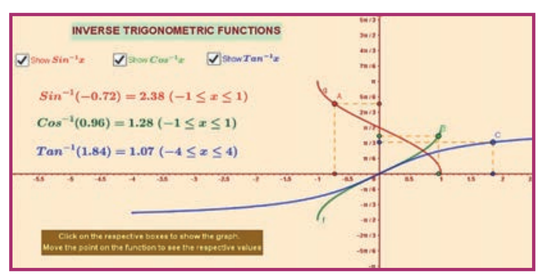
Summary

- If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\,\theta$
- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\cot^2 x + 1 = \operatorname{cosec}^2 x$
- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\cos(\alpha-\beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\sin\beta$
- $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$
- $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$
- $\sin 2\alpha = 2\sin\alpha\cos\alpha$
- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
  $= 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$
- $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$
- $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$
- $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$
- $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$
- $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$
- $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$
- $\cos x + \cos y = 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$
- $\cos x - \cos y = -2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$
- $\sin x\cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$
- $\cos x\sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$
- $\cos x\cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$
- $\sin x\sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

GLOSSARY (கலைச்சொற்கள்)	
Allied angles	துணைக் கோணங்கள்
Angle	கோணம்
Componendo and dividendo.	கூட்டல் மற்றும் கழித்தல் விகித சமம்
Compound angle	கூட்டுக் கோணம்
Degree measure	பாகை அளவை
Inverse function	நேர்மாறு சார்பு
Length of the arc.	வில்லின் நீளம்
Multiple angle	மடங்கு கோணம்
Quadrants	கால் பகுதிகள்
Radian measure	ரேடியன் அளவு / ஆரையன் அளவு
Transformation formulae	உருமாற்று சூத்திரங்கள்
Trignometric identities	திரிகோணமிதி முற்றொருமைகள்
Trignometry Ratios	திரிகோணமிதி விகிதங்கள்

ICT Corner

Expected final outcomes



Step – 1

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called "**11th BUSINESS MATHEMATICS and STATISTICS**" will appear.
In this several work sheets for Business Maths are given, Open the worksheet named "**Inverse Trigonometric Functions**"

Step - 2

Inverse Trigonometric Functions page will open. Click on the check boxes on the Left-hand side to see the respective Inverse trigonometric functions. You can move and observe the point on the curve to see the x-value and y-value along the axes. Sine and cosine inverse are up to the max limit. But Tangent inverse is having customised limit.

Browse in the link

11th Business Mathematics and Staistics: https://ggbm.at/qKj9gSTG
(or) scan the QR Code

B184_11_BM_EM