Question Set 8

DIFFERENTIATION

(Marks with option : 09)

Remember :

1. **Definition :** If x and $x + h$ belong to the domain of f and $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists, then this limit is called the **derivative** of f at x and is denoted by $f'(x)$. Thus, $\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$.

2. **Derivatives of Standard Functions :**

$\frac{d}{dx}(k) = 0$, (where k is a constant); $\frac{d}{dx}(x^n) = nx^{n-1}$;

$\frac{d}{dx}(e^x) = e^x$; $\frac{d}{dx}(a^x) = a^x \log a$; $\frac{d}{dx}(\log x) = \frac{1}{x}$;

$\frac{d}{dx}(\sin x) = \cos x$; $\frac{d}{dx}(\cos x) = -\sin x$;

$\frac{d}{dx}(\tan x) = \sec^2 x$; $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$;

$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$; $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$;

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$; $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$;

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$; $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}$;

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$; $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2 - 1}}$.

8.1 DERIVATIVES OF COMPOSITE FUNCTIONS

Theory Question	3 or 4 marks each

Q. 1. If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, such that the composite function $y = f[g(x)]$ is a differentiable function of x, then prove that $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Hence, find $\frac{dy}{dx}$ if $y = \sqrt{x^2 + 5}$. ***(Sept. '21)***

Proof : Given that $y = f(u)$ and $u = g(x)$.

We assume that u is not a constant function.

Let δu and δy be the increments in u and y respectively, corresponding to the increment δx in x.

Now, y is a differentiable function of u and u is a differentiable function of x.

$$\therefore \frac{dy}{du} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \text{ and } \frac{du}{dx} = \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \quad \text{... (1)}$$

Also, $\lim_{\delta x \to 0} \delta u = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} . \delta x \right)$

$$= \left(\lim_{\delta x \to 0} \frac{\delta u}{\delta x} \right) \left(\lim_{\delta x \to 0} \delta x \right) = \frac{du}{dx} \times 0 = 0$$

This means that as $\delta x \to 0$, $\delta u \to 0$ ... (2)

Now, $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$ ... $(\delta u \neq 0)$

Taking limits as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right)$$

$$= \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

$$= \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \quad \text{... [By (2)]}$$

Now, both the limits on RHS exist ... [By (1)]

$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ exists and is equal to $\frac{dy}{dx}$.

$\therefore$ y is differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}. \quad \text{... [By (1)]}$$

To find $\frac{dy}{dx}$, if $y = \sqrt{x^2 + 5}$:

Let $u = x^2 + 5$. Then $y = \sqrt{u}$

$$\therefore \frac{dy}{du} = \frac{d}{du}(\sqrt{u}) = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{x^2+5}}$$

and $\frac{du}{dx} = \frac{d}{dx}(x^2 + 5) = 2x + 0 = 2x$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{x^2+5}} \times 2x = \frac{x}{\sqrt{x^2+5}}.$$

Solved Examples	2 marks each

Ex. 1. Differentiate the following w.r.t. x :

(1) $\sqrt{x^2 + 4x - 7}$ **(2)** $\sqrt{\tan\sqrt{x}}$

(3) $\cot^2(x^3)$ **(4)** $\log\left[\tan\left(\frac{x}{2}\right)\right]$.

Solution :

(1) Let $y = \sqrt{x^2 + 4x - 7}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sqrt{x^2 + 4x - 7})$$

$$= \frac{1}{2\sqrt{x^2+4x-7}} \cdot \frac{d}{dx}(x^2 + 4x - 7)$$

$$= \frac{1}{2\sqrt{x^2+4x-7}} \times (2x + 4 \times 1 - 0)$$

$$= \frac{2x+4}{2\sqrt{x^2+4x-7}} = \frac{x+2}{\sqrt{x^2+4x-7}}.$$

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(2) Let $y = \sqrt{\tan\sqrt{x}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{\tan\sqrt{x}})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \frac{d}{dx}(\tan\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \times \sec^2\sqrt{x} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \times \sec^2\sqrt{x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{\sec^2\sqrt{x}}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

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(3) Let $y = \cot^2(x^3)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[\cot(x^3)]^2$$

$$= 2\cot(x^3)\cdot\frac{d}{dx}[\cot(x^3)]$$

$$= 2\cot(x^3)\cdot[-\operatorname{cosec}^2(x^3)]\cdot\frac{d}{dx}(x^3)$$

$$= -2\cot(x^3)\operatorname{cosec}^2(x^3)\times 3x^2$$

$$= -6x^2\cot(x^3)\operatorname{cosec}^2(x^3).$$

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(4) Let $y = \log\left[\tan\left(\frac{x}{2}\right)\right]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\log\left[\tan\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{\tan\left(\frac{x}{2}\right)}\cdot\frac{d}{dx}\left[\tan\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{\tan\left(\frac{x}{2}\right)}\cdot\sec^2\left(\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\cdot\frac{1}{\cos^2\left(\frac{x}{2}\right)}\cdot\frac{1}{2}\times 1$$

$$= \frac{1}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$= \frac{1}{\sin x} = \operatorname{cosec} x.$$

Ex. 2. Differentiate the following w.r.t. x :

(1) $x^{\tan^{-1}x}$ **(2)** $x^x + x^a + a^x + a^a$ **(3)** $\log_x a$.

Solution :

(1) Let $y = x^{\tan^{-1}x}$

$\therefore\ \log y = \log(x^{\tan^{-1}x}) = (\tan^{-1}x)(\log x)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}[(\tan^{-1}x)(\log x)]$$

$$= (\tan^{-1} x) \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(\tan^{-1} x)$$

$$= (\tan^{-1} x) \times \frac{1}{x} + (\log x) \times \frac{1}{1+x^2}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2}\right]$$

$$= x^{\tan^{-1} x}\left[\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2}\right].$$

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(2) Let $y = x^x + x^a + a^x + a^a$

Let $u = x^x$.

Then $\log u = \log x^x = x \log x$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x \log x)$$

$$= x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

$$\therefore \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$$

Now, $y = u + x^a + a^x + a^a$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{d}{dx}(x^a) + \frac{d}{dx}(a^x) + \frac{d}{dx}(a^a)$$

$$= x^x(1 + \log x) + ax^{a-1} + a^x \log a + 0$$

$$= x^x(1 + \log x) + ax^{a-1} + a^x \log a.$$

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(3) Let $y = \log_x a = \dfrac{\log a}{\log x}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log a}{\log x}\right)$$

$$= (\log a) \cdot \frac{d}{dx}(\log x)^{-1}$$

$$= (\log a)(-1)(\log x)^{-2} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{-\log a}{(\log x)^2} \times \frac{1}{x} = \frac{-\log a}{x(\log x)^2}.$$

Examples for Practice	2 marks each

Differentiate the following w.r.t. x :

1. $(x^3-2x-1)^5$ **2.** $\cos(x^2+a^2)$ **3.** $\sqrt{\sin x^3}$

4. $\sqrt{\cos x}+\sqrt{\cos\sqrt{x}}$ **5.** $\sin(\log x)$

6. $\log(x^5+4)$ **7.** $\tan(xe^x)$ **8.** $(4)^{\log_2(\sin x)}+(9)^{\log_3(\cos x)}$

9. $x^{1/x}$ **10.** x^{4x} **11.** $(\sin x)^x$

12. $x^e+x^x+e^x+e^e$.

ANSWERS

1. $5(3x^2-2)(x^3-2x-1)^4$ **2.** $-2x\sin(x^2+a^2)$

3. $\dfrac{3x^2\cos x^3}{2\sqrt{\sin x^3}}$ **4.** $\dfrac{-\sin x}{2\sqrt{\cos x}}-\dfrac{\sin\sqrt{x}}{4\sqrt{x}\sqrt{\cos\sqrt{x}}}$

5. $\dfrac{\cos(\log x)}{x}$ **6.** $\dfrac{5x^4}{x^5+4}$

7. $e^x(x+1)\sec^2(xe^x)$ **8.** 0

9. $x^{1/x}\left(\dfrac{1-\log x}{x^2}\right)$ **10.** $x^{4x}\cdot 4^x\left[\dfrac{1}{x}+(\log x)(\log 4)\right]$

11. $(\sin x)^x[x\cot x+\log\sin x]$ **12.** $ex^{e-1}+x^x(1+\log x)+e^x$.

Solved Examples	3 or 4 marks each

Ex. 3. Find $\dfrac{dy}{dx}$, if

(1) $y=\log\left[\dfrac{e^{x^2}(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right]$ **(2)** $y=\log\left[\dfrac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}-x}\right)$

(3) $y=\log[\tan^3x\cdot\sin^4x\cdot(x^2+7)^7]$.

Solution : (1) $y=\log\left[\dfrac{e^{x^2}(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right]$

$$=\log e^{x^2}+\log(5-4x)^{\frac{3}{2}}-\log(7-6x)^{\frac{1}{3}}$$

$$=x^2\log e+\frac{3}{2}\log(5-4x)-\frac{1}{3}\log(7-6x)$$

$$=x^2+\frac{3}{2}\log(5-4x)-\frac{1}{3}\log(7-6x) \qquad \ldots[\because \log e=1]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[x^2+\frac{3}{2}\log(5-4x)-\frac{1}{3}\log(7-6x)\right]$$

$$= \frac{d}{dx}(x^2) + \frac{3}{2}\frac{d}{dx}[\log(5-4x)] - \frac{1}{3}\frac{d}{dx}[\log(7-6x)]$$

$$= 2x + \frac{3}{2} \times \frac{1}{5-4x} \cdot \frac{d}{dx}(5-4x) - \frac{1}{3} \times \frac{1}{7-6x} \cdot \frac{d}{dx}(7-6x)$$

$$= 2x + \frac{3}{2(5-4x)} \times (0 - 4 \times 1) - \frac{1}{3(7-6x)} \times (0 - 6 \times 1)$$

$$= 2x - \frac{6}{5-4x} + \frac{2}{7-6x}.$$

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(2) $y = \log\left[\dfrac{x + \sqrt{x^2+a^2}}{\sqrt{x^2+a^2} - x}\right]$

Rationalizing the denominator, we get

$$y = \log\left[\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x} \times \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}+x}\right]$$

$$= \log\left[\frac{(\sqrt{x^2+a^2}+x)^2}{(x^2+a^2)-x^2}\right]$$

$$= \log\left[\frac{(\sqrt{x^2+a^2}+x)^2}{a^2}\right]$$

$$= 2\log(\sqrt{x^2+a^2}+x) - \log a^2$$

$$\therefore \frac{dy}{dx} = 2\frac{d}{dx}[\log(\sqrt{x^2+a^2}+x)] - \frac{d}{dx}(\log a^2)$$

$$= 2 \cdot \frac{1}{\sqrt{x^2+a^2}+x} \cdot \frac{d}{dx}(\sqrt{x^2+a^2}+x) - 0$$

$$= \frac{2}{\sqrt{x^2+a^2}+x} \times \left[\frac{1}{2\sqrt{x^2+a^2}} \cdot \frac{d}{dx}(x^2+a^2) + 1\right]$$

$$= \frac{2}{\sqrt{x^2+a^2}+x}\left(\frac{1}{2\sqrt{x^2+a^2}} \cdot 2x + 1\right)$$

$$= \frac{2}{\sqrt{x^2+a^2}+x}\left(\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)$$

$$= \frac{2}{\sqrt{x^2+a^2}}.$$

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(3) $y = \log[\tan^3 x \cdot \sin^4 x \cdot (x^2+7)^7]$

$$= \log \tan^3 x + \log \sin^4 x + \log (x^2+7)^7$$

$$= 3 \log \tan x + 4 \log \sin x + 7 \log (x^2+7)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[3 \log \tan x + 4 \log \sin x + 7 \log (x^2 + 7)]$$

$$= 3\frac{d}{dx}(\log \tan x) + 4\frac{d}{dx}(\log \sin x) + 7\frac{d}{dx}[\log (x^2 + 7)]$$

$$= 3 \times \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x) + 4 \times \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + 7 \times \frac{1}{x^2+7} \cdot \frac{d}{dx}(x^2 + 7)$$

$$= 3 \times \frac{1}{\tan x} \cdot \sec^2 x + 4 \times \frac{1}{\sin x} \cdot \cos x + 7 \times \frac{1}{x^2+7} \cdot (2x + 0)$$

$$= 3 \times \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x} + 4 \cot x + \frac{14x}{x^2+7}$$

$$= \frac{6}{2 \sin x \cos x} + 4 \cot x + \frac{14x}{x^2+7}$$

$$= \frac{6}{\sin 2x} + 4 \cot x + \frac{14x}{x^2+7}$$

$$= 6 \operatorname{cosec} 2x + 4 \cot x + \frac{14x}{x^2+7}.$$

Ex. 4. Differentiate the following w.r.t. x :

(1) $\dfrac{(x^2+3)^2 \cdot \sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}$ **(2)** $\dfrac{x^5 \cdot \tan^3 4x}{\sin^2 3x}$.

Solution :

(1) Let $y = \dfrac{(x^2+3)^2 \cdot \sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}$

Then $\log y = \log \left[\dfrac{(x^2+3)^2 \cdot (x^3+5)^{\frac{3}{2}}}{(2x^2+1)^{\frac{3}{2}}} \right]$

$$= \log (x^2+3)^2 + \log (x^3+5)^{\frac{2}{3}} - \log (2x^2+1)^{\frac{3}{2}}$$

$$= 2 \log (x^2+3) + \frac{2}{3} \log (x^3+5) - \frac{3}{2} \log (2x^2+1)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}\left[2 \log (x^2+3) + \frac{2}{3} \log (x^3+5) - \frac{3}{2} \log (2x^2+1)\right]$$

$$= 2\frac{d}{dx}[\log(x^2+3)] + \frac{2}{3}\frac{d}{dx}[\log(x^3+5)] - \frac{3}{2}\frac{d}{dx}[\log(2x^2+1)]$$

$$= 2 \times \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) + \frac{2}{3}\times\frac{1}{x^3+5}\cdot\frac{d}{dx}(x^3+5) - \frac{3}{2}\times\frac{1}{2x^2+1}\cdot\frac{d}{dx}(2x^2+1)$$

$$= \frac{2}{x^2+3}\times(2x+0) + \frac{2}{3(x^3+5)}\times(3x^2+0) - \frac{3}{2(2x^2+1)}\times(2\times 2x+0)$$

$$\therefore \frac{dy}{dx} = y\left[\frac{4x}{x^2+3} + \frac{2x^2}{(x^3+5)} - \frac{6x}{2x^2+1}\right]$$

$$= \frac{(x^2+3)^2\sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\left[\frac{4x}{x^2+3} + \frac{2x^2}{(x^3+5)} - \frac{6x}{2x^2+1}\right].$$

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(2) Let $y = \dfrac{x^5\cdot\tan^3 4x}{\sin^2 3x}$

Then $\log y = \log\left[\dfrac{x^5\cdot\tan^3 4x}{\sin^2 3x}\right]$

$$= \log x^5 + \log\tan^3 4x - \log\sin^2 3x$$

$$= 5\log x + 3\log(\tan 4x) - 2\log(\sin 3x)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = 5\frac{d}{dx}(\log x) + 3\frac{d}{dx}[\log(\tan 4x)] - 2\frac{d}{dx}[\log(\sin 3x)]$$

$$= 5\times\frac{1}{x} + 3\times\frac{1}{\tan 4x}\cdot\frac{d}{dx}(\tan 4x) - 2\times\frac{1}{\sin 3x}\cdot\frac{d}{dx}(\sin 3x)$$

$$= \frac{5}{x} + 3\times\frac{1}{\tan 4x}\times\sec^2 4x\cdot\frac{d}{dx}(4x) - 2\times\frac{1}{\sin 3x}\times\cos 3x\cdot\frac{d}{dx}(3x)$$

$$= \frac{5}{x} + 3\cdot\frac{\cos 4x}{\sin 4x}\times\frac{1}{\cos^2 4x}\times 4 - 2\cot 3x\times 3$$

$$= \frac{5}{x} + \frac{24}{2\sin 4x\cdot\cos 4x} - 6\cot 3x$$

$$\therefore \frac{dy}{dx} = y\left[\frac{5}{x} + \frac{24}{\sin 8x} - 6\cot 3x\right]$$

$$= \frac{x^5\cdot\tan^3 4x}{\sin^2 3x}\left[\frac{5}{x} + 24\operatorname{cosec} 8x - 6\cot 3x\right].$$

Ex. 5. Find $\dfrac{dy}{dx}$, if :

(1) $y = \sin x^x$ **(2)** $y = \left(\dfrac{x^2}{x+1}\right)^x$.

Solution :

(1) $y = (\sin x^x)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[(\sin x^x)]$$

$$\therefore \frac{dy}{dx} = \cos(x^x) \cdot \frac{d}{dx}(x^x) \qquad \text{... (1)}$$

Let $u = x^x$

Then $\log u = \log x^x = x \cdot \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \cdot \log x)$$

$$= x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

$$\therefore \frac{du}{dx} = u(1 + \log x)$$

$$\therefore \frac{d}{dx}(x^x) = x^x(1 + \log x) \qquad \text{... (2)}$$

From (1) and (2), we get

$$\frac{dy}{dx} = \cos(x^x) \cdot x^x(1 + \log x).$$

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(2) $y = \left(\frac{x^2}{x+1}\right)^x$

$$\therefore \log y = \log\left(\frac{x^2}{x+1}\right)^x = x \log\left(\frac{x^2}{x+1}\right)$$

$$= x[\log x^2 - \log(x+1)] = x \log x^2 - x \log(x+1)$$

$$= 2x \log x - x \log(x+1)$$

Differentiating both sides w.r.t x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2\frac{d}{dx}(x \log x) - \frac{d}{dx}[x \log(x+1)]$$

$$= 2\left[x \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)\right] - \left\{x \frac{d}{dx}[\log(x+1)] + \log(x+1) \cdot \frac{d}{dx}(x)\right\}$$

$$= 2\left[x \times \frac{1}{x} + (\log x) \times 1\right] - \left\{x \times \frac{1}{x+1} \cdot \frac{d}{dx}(x+1) + \log(x+1) \cdot 1\right\}$$

$$= 2(1 + \log x) - \frac{x}{x+1} \cdot (1+0) - \log(x+1)$$

$$= 2 + 2\log x - \frac{x}{x+1} - \log(x+1)$$

$$= \left(2 - \frac{x}{x+1}\right) + \log x^2 - \log(x+1)$$

$$= \left(\frac{2x+2-x}{x+1}\right) + \log\left(\frac{x^2}{x+1}\right)$$

$$\therefore \frac{dy}{dx} = y\left[\left(\frac{x+2}{x+1}\right) + \log\left(\frac{x^2}{x+1}\right)\right]$$

$$= \left(\frac{x^2}{x+1}\right)^x \left[\left(\frac{x+2}{x+1}\right) + \log\left(\frac{x^2}{x+1}\right)\right].$$

Ex. 6. Differentiate the following w.r.t. x :

(1) $x^{e^x} + (\log x)^{\sin x}$ **(2) $(\sin x)^{\tan x} - x^{\log x}$.**

Solution :

(1) Let $y = x^{e^x} + (\log x)^{\sin x}$

Put $u = x^{e^x}$ and $v = (\log x)^{\sin x}$

Then $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{... (1)}$$

Take $u = x^{e^x}$ $\quad \therefore \log u = \log x^{e^x} = e^x \cdot \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(e^x \log x)$$

$$= e^x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(e^x)$$

$$= e^x \cdot \frac{1}{x} + (\log x)(e^x)$$

$$\therefore \frac{du}{dx} = y\left[\frac{e^x}{x} + e^x \cdot \log x\right]$$

$$= e^x \cdot x^{e^x}\left[\frac{1}{x} + \log x\right]. \quad \text{... (2)}$$

Also, $v = (\log x)^{\sin x}$

$\therefore\ \log v = \log (\log x)^{\sin x} = (\sin x) \cdot (\log \log x)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx}[(\sin x) \cdot (\log \log x)]$$

$$= (\sin x) \cdot \frac{d}{dx}[(\log \log x)] + (\log \log x) \cdot \frac{d}{dx}(\sin x)$$

$$= \sin x \times \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) + (\log \log x) \cdot (\cos x)$$

$$\therefore\ \frac{dv}{dx} = v\left[\frac{\sin x}{\log x} \times \frac{1}{x} + (\cos x)(\log \log x)\right]$$

$$= (\log x)^{\sin x}\left[\frac{\sin x}{x \log x} + (\cos x)(\log \log x)\right] \quad \text{... (3)}$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = e^x \cdot x^{e^x}\left[\frac{1}{x} + \log x\right] + (\log x)^{\sin x}\left[\frac{\sin x}{x \log x} + (\cos x)(\log \log x)\right].$$

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(2) Let $y = (\sin x)^{\tan x} - x^{\log x}$

Put $u = (\sin x)^{\tan x}$ and $v = x^{\log x}$

Then $y = u - v$

$$\therefore\ \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \quad \text{... (1)}$$

Take $u = (\sin x)^{\tan x}$

$\therefore\ \log u = \log (\sin x)^{\tan x} = \tan x \cdot \log (\sin x)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}[\tan x \cdot \log (\sin x)]$$

$$= \tan x \cdot \frac{d}{dx}[\log (\sin x)] + \log (\sin x) \cdot \frac{d}{dx}(\tan x)$$

$$= \tan x \times \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \log (\sin x) \times \sec^2 x$$

$$\therefore\ \frac{du}{dx} = u\left[\tan x \times \frac{1}{\sin x} \cdot \cos x + \log (\sin x) \cdot \sec^2 x\right]$$

$$= (\sin x)^{\tan x}[1 + \sec^2 x \cdot \log (\sin x)] \quad \text{... (2)}$$

Also, $v = x^{\log x}$

$\therefore\ \log v = \log x^{\log x} = \log x \cdot \log x = (\log x)^2$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx} (\log x)^2$$

$$= 2 \log x \cdot \frac{d}{dx} (\log x) = 2 \log x \times \frac{1}{x}$$

$$\therefore \frac{dv}{dx} = v \left[\frac{2 \log x}{x}\right] = x^{\log x} \left[\frac{2 \log x}{x}\right]$$

$$= \frac{2x^{\log x} \cdot \log x}{x} \qquad \text{... (3)}$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = (\sin x)^{\tan x} [1 + \sec^2 x \cdot \log (\sin x)] - \frac{2x^{\log x} \cdot \log x}{x}.$$

Examples for Practice	3 or 4 marks each

1. Find $\frac{dy}{dx}$, if :

(1) $y = \log\left[e^{3x} \cdot \frac{(3x-4)^{\frac{2}{3}}}{\sqrt[3]{2x+5}}\right]$ (2) $y = \log\left[\frac{a^{\cos x}}{(x^2-3)^3 \log x}\right]$

(3) $y = \log\left[\frac{x + \sqrt{x^2+25}}{\sqrt{x^2+25} - x}\right]$.

2. Find $\frac{dy}{dx}$, if :

(1) $y = \sqrt[3]{\frac{4x+1}{(2x+3)(5-2x)^2}}$ (2) $y = \frac{(3x^2+1)\sqrt{1+x^2}}{x^3}$

(3) $y = \frac{e^{x^2} \cdot (\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}} \cdot \cos^3 x}$.

3. Find $\frac{dy}{dx}$, if :

(1) $y = \frac{x^{\cos x}}{x^2 + 4x + 5}$ (2) $y = \frac{(\cos x)^x}{1 + x - x^2}$.

4. Differentiate the following w.r.t. x :

(1) $x^{\sin^{-1} x} + (\sin^{-1} x)^x$ (2) $x^{\sin x} + (\sin x)^x$

(3) $x^{x^x} + e^{x^x}$ (4) $(\log x)^x - (\cos x)^{\cot x}$.

ANSWERS

1. (1) $3+\dfrac{2}{3x-4}-\dfrac{2}{3(2x+5)}$ (2) $-(\sin x)(\log a)-\dfrac{6x}{x^2-3}-\dfrac{1}{x\log x}$

(3) $\dfrac{2}{\sqrt{x^2+25}}$.

2. (1) $\sqrt[3]{\dfrac{4x-1}{(2x+3)(5-2x)^2}}\left[\dfrac{4}{3\,(4x-1)}-\dfrac{2}{3\,(2x+3)}+\dfrac{4}{3\,(5-2x)}\right]$

(2) $\dfrac{(3x^2+1)\sqrt{1+x^2}}{x^3}\left[\dfrac{6x}{3x^2+1}+\dfrac{x}{1+x^2}-\dfrac{3}{x}\right]$

(3) $\dfrac{e^{x^2}\cdot(\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}}\cdot\cos^3 x}\left[2x+x\operatorname{cosec}2x+\dfrac{1}{2}\log\tan x-\dfrac{3x}{1+x^2}+3\tan x\right]$

3. (1) $\dfrac{x^{\cos x}}{x^2+4x+5}\left[\dfrac{\cos x}{x}-(\sin x)(\log x)-\dfrac{2x+4}{x^2+4x+5}\right]$

(2) $\dfrac{(\cos x)^x}{1+x-x^2}\left[\dfrac{2x-1}{1+x-x^2}-x\tan x+\log\cos x\right]$

4. (1) $x^{\sin^{-1}x}\left[\dfrac{\sin^{-1}x}{x}+\dfrac{\log x}{\sqrt{1-x^2}}\right]$

$+(\sin^{-1}x)^x\left[\dfrac{x}{\sqrt{1-x^2}\cdot\sin^{-1}x}+\log(\sin^{-1}x)\right]$

(2) $x^{\sin x}\left[\dfrac{\sin x}{x}+(\log x)(\cos x)\right]+(\sin x)^x\,[x\cot x+\log\sin x]$

(3) $x^{x^x}\cdot x^x\cdot\log x\left[1+\log x+\dfrac{1}{x\log x}\right]+e^{x^x}\cdot x^x(1+\log x)$

(4) $(\log x)^x\left[\dfrac{1}{\log x}+\log(\log x)\right]+(\cos x)^{\cot x}\,[1+(\operatorname{cosec}^2 x)(\log\cos x)]$.

8.2 DERIVATIVES OF INVERSE FUNCTIONS

Theory Question	3 or 4 marks each

Q. 2. If $y=f(x)$ is a derivable function of x such that the inverse function $x=f^{-1}(y)$ is defined, then show that $\dfrac{dx}{dy}=\dfrac{1}{(dy/dx)}$, where $\dfrac{dy}{dx}\neq 0$.

OR

If $y = f(x)$ and $x = g(y)$, where g is the inverse of f, i.e. $g = f^{-1}$ and if $\frac{dy}{dx}$ and $\frac{dx}{dy}$ both exist and $\frac{dx}{dy} \neq 0$, show that $\frac{dy}{dx} = \frac{1}{(dx/dy)}$.

Hence, (1) find $\frac{d}{dx}(\tan^{-1}x)$.

(2) if $y = \sin^{-1}x$, $-1 \leqslant x \leqslant 1$, $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, then show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \text{ where } |x| < 1.$$

(3) if $y = \sec^{-1}x$, $|x| > 1$, $0 < y < \pi$, $y \neq \frac{\pi}{2}$, then show that

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}.$$

Proof : Let δy be the increment in y corresponding to an increment δx in x.

$\therefore$ as $\delta x \to 0$, $\delta y \to 0$.

Now, y is a differentiable function of x.

$$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Now, $\frac{\delta y}{\delta x} \times \frac{\delta x}{\delta y} = 1 \qquad \therefore \frac{\delta x}{\delta y} = \frac{1}{\left(\frac{\delta y}{\delta x}\right)}$

Taking limits on both sides as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta x}{\delta y} = \lim_{\delta x \to 0} \left[\frac{1}{\left(\frac{\delta y}{\delta x}\right)}\right] = \frac{1}{\lim_{\delta x \to 0} \frac{\delta y}{\delta x}}$$

$$\therefore \lim_{\delta y \to 0} \frac{\delta x}{\delta y} = \frac{1}{\lim_{\delta x \to 0} \frac{\delta y}{\delta x}} \qquad \text{... [as } \delta x \to 0, \delta y \to 0]$$

Since limit in R.H.S. exists, limit in L.H.S. also exists and we have

$$\lim_{\delta y \to 0} \frac{\delta x}{\delta y} = \frac{dx}{dy}$$

$$\therefore \frac{dx}{dy} = \frac{1}{(dy/dx)}, \text{ where } \frac{dy}{dx} \neq 0.$$

OR

Let δx and δy be the corresponding increments in x and y respectively.

$\therefore$ as $\delta x \to 0$, $\delta y \to 0$ and as $\delta y \to 0$, $\delta x \to 0$.

Now, $\dfrac{\delta y}{\delta x} \times \dfrac{\delta x}{\delta y} = 1.$ ... $[\because \ \delta x \neq 0, \delta y \neq 0]$

Taking limits as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \times \frac{\delta x}{\delta y}\right) = 1 \qquad \therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \times \lim_{\delta x \to 0} \frac{\delta x}{\delta y} = 1$$

$$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \times \lim_{\delta y \to 0} \frac{\delta x}{\delta y} = 1 \qquad \text{... [as } \delta x \to 0, \delta y \to 0]$$

$$\therefore \frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \qquad \text{...} \left[\because \frac{dy}{dx} \text{ and } \frac{dx}{dy} \text{ both exist}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{(dx/dy)}, \text{ as } \frac{dx}{dy} \neq 0.$$

(1) To find $\dfrac{d}{dx}(\tan^{-1}x)$:

Let $y = \tan^{-1} x$. Then $x = \tan y$, where $x \in R$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\therefore \frac{dy}{dx} = \frac{1}{(dx/dy)}, \quad \text{if } \frac{dx}{dy} \neq 0$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2} \qquad \therefore \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

---

(2) Let $y = \sin^{-1} x$. Then $x = \sin y$, where $-1 < x < 1$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

$\therefore \cos y > 0$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(dx/dy)}, \quad \text{if } \frac{dx}{dy} \neq 0$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad \text{if } |x| < 1.$$

---

(3) Let $y = \sec^{-1} x$. Then $x = \sec y$

Here $|x| > 1$ and $0 < y < \pi$, $y \neq \pi/2$

Also, $\sec y$ and $\tan y$ are of the same sign in the first and the second quadrants.

$\therefore$ $\sec y \cdot \tan y$ is positive.

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \sec y \cdot \tan y = \sec y \cdot \sqrt{\sec^2 y - 1} = x\sqrt{x^2 - 1}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(dx/dy)}, \text{ if } \frac{dx}{dy} \neq 0 \qquad \therefore \frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

Solved Examples	2 marks each

Ex. 7. Differentiate the following w.r.t. x :

(1) $\cos^{-1}(1 - x^2)$ **(2)** $\cot^{-1}\left(\frac{1}{x^2}\right)$

Solution :

(1) Let $y = \cos^{-1}(1 - x^2)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\cos^{-1}(1 - x^2)]$$

$$= \frac{-1}{\sqrt{1 - (1 - x^2)^2}} \cdot \frac{d}{dx}(1 - x^2)$$

$$= \frac{-1}{\sqrt{1 - (1 - 2x^2 + x^4)}} \cdot (0 - 2x)$$

$$= \frac{2x}{\sqrt{2x^2 - x^4}}$$

$$= \frac{2x}{x\sqrt{2 - x^2}} = \frac{2}{\sqrt{2 - x^2}}.$$

---

(2) Let $y = \cot^{-1}\left(\frac{1}{x^2}\right) = \tan^{-1} x^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1} x^2]$$

$$= \frac{1}{1 + (x^2)^2} \cdot \frac{dy}{dx}(x^2)$$

$$= \frac{1}{1 + x^4} \times 2x = \frac{2x}{1 + x^4}.$$

Ex. 8. Differentiate the following w.r.t. x :

(1) $\cos^{-1}\left(\sqrt{\frac{1+\cos x}{2}}\right)$ **(2)** $\operatorname{cosec}^{-1}(\sec 5^x)$ ***(Sept. '21)***

(3) $\tan^{-1}\left(\frac{1-\cos 3x}{\sin 2x}\right)$.

Solution :

(1) Let $y = \cos^{-1}\left(\sqrt{\frac{1+\cos x}{2}}\right)$

$$= \cos^{-1}\left(\sqrt{\frac{2\cos^2\left(\frac{x}{2}\right)}{2}}\right)$$

$$= \cos^{-1}\left[\cos\left(\frac{x}{2}\right)\right] = \frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\frac{d}{dx}(x) = \frac{1}{2} \times 1 = \frac{1}{2}.$$

---

(2) Let $y = \operatorname{cosec}^{-1}(\sec 5^x)$

$$= \operatorname{cosec}^{-1}\left[\operatorname{cosec}\left(\frac{\pi}{2} - 5^x\right)\right]$$

$$= \frac{\pi}{2} - 5^x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 5^x\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(5^x)$$

$$= 0 - 5^x \cdot \log 5 = -5^x \cdot \log 5.$$

---

(3) Let $y = \tan^{-1}\left(\frac{1-\cos 3x}{\sin 3x}\right)$

$$= \tan^{-1}\left[\frac{2\sin^2\left(\frac{3x}{2}\right)}{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{3x}{2}\right)\right] = \frac{3x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x}{2}\right) = \frac{3}{2}\frac{d}{dx}(x)$$

$$= \frac{3}{2} \times 1 = \frac{3}{2}.$$

Ex. 9. Differentiate the following w.r.t. x :

(1) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ (2) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

Solution :

(1) Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$. Then $\theta = \tan^{-1}x$

$$\therefore\ y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1}x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

$$= 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

---

(2) Let $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Put $x = \tan\theta$. Then $\theta = \tan^{-1}x$

$$\therefore\ y = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3\frac{d}{dx}(\tan^{-1}x)$$

$$= 3 \times \frac{1}{1+x^2} = \frac{3}{1+x^2}.$$

Examples for Practice	2 marks each

Differentiate the following w.r.t. x :

1. (1) $\sin^{-1}(x^3)$ (2) $\cot^{-1}\left(\frac{1}{\sqrt{x}}\right)$

(3) $\sin^{-1}(2^x)$ (4) $\sin^4[\sin^{-1}(\sqrt{x})]$.

2. (1) $\sec\left[\cos^{-1}\left(\frac{7}{x}\right)\right]$ (2) $\cos^{-1}[\sin(4^x)]$ (3) $\tan^{-1}(\cot 4x)$.

3. (1) $x^3\tan^{-1}x$ (2) $5^x\cdot\sec^{-1}2x$ (3) $\frac{\cos^{-1}x}{x^2+1}$.

4. (1) $\sin^{-1}(2\cos^2x-1)$ (2) $\tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

(3) $\cos^{-1}(4\cos^3x-3\cos x)$ (4) $\tan^{-1}\left(\frac{a+b\tan x}{b-a\tan x}\right)$.

5. (1) $\sin^{-1}\left(\frac{1-25x^2}{1+25x^2}\right)$ (2) $\cos^{-1}\left(\frac{2\sqrt{x}}{1+x}\right)$

(3) $\tan^{-1}\left(\frac{2e^x}{1-e^{2x}}\right)$ (4) $\cot^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

(5) $\operatorname{cosec}^{-1}\left(\frac{1}{3x-4x^3}\right)$.

ANSWERS

1. (1) $\frac{3x^2}{\sqrt{1-x^6}}$ (2) $\frac{1}{2\sqrt{x}(1+x)}$ (3) $\frac{2^x\cdot\log 2}{\sqrt{1-4^x}}$ (4) $2x$.

2. (1) $\frac{1}{7}$ (2) $-4^x\cdot\log 4$ (3) -4

3. (1) $\frac{x^3}{1+x^2}+3x^2\tan^{-1}x$

(2) $5^x\left[\frac{1}{x\sqrt{4x^2-1}}+(\sec^{-1}2x)(\log 5)\right]$

(3) $-\left[\frac{(x^2+1)+2x\sqrt{1-x^2}\cos^{-1}x}{\sqrt{1-x^2}\cdot(x^2+1)^2}\right]$

4. (1) -2 (2) $-\frac{1}{2}$ (3) 3 (4) 1

5. (1) $-\frac{10}{1+25x^2}$. Put $5x=\tan\theta$ (2) $\frac{-1}{\sqrt{x}(1+x)}$. Put $\sqrt{x}=\tan\theta$

(3) $\dfrac{2e^x}{1-e^{2x}}\cdot$ Put $e^x = \tan\theta$ (4) $\dfrac{1}{2\sqrt{x}(1+x)}$

(5) $\dfrac{3}{\sqrt{1-x^2}}\cdot$ Put $x = \sin\theta$.

Solved Examples	3 marks each

Ex. 10. Differentiate the following w.r.t. x :

(1) $\tan^{-1}(\sec x + \tan x)$ **(2) $\sin^{-1}\left(\dfrac{4\sin x + 5\cos x}{\sqrt{41}}\right)$**

(3) $\sin^{-1}\left(\dfrac{5x + 12\sqrt{1-x^2}}{13}\right)$.

Solution :

(1) Let $y = \tan^{-1}(\sec x + \tan x)$

$$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)$$

$$= \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$$

$$= \tan^{-1}\left[\frac{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right]$$

... $\left[\text{Dividing numerator and denominator by } \cos\frac{x}{2}\right]$

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{2}}{1 - \tan\frac{\pi}{4} \cdot \tan\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] = \frac{\pi}{4} + \frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{1}{2}\frac{d}{dx}(x)$$

$$= 0 + \frac{1}{2} \times 1 = \frac{1}{2}.$$

---

(2) Let $y = \sin^{-1}\left(\frac{4\sin x + 5\cos x}{\sqrt{41}}\right)$

$$= \sin^{-1}\left[(\sin x)\left(\frac{4}{\sqrt{41}}\right) + (\cos x)\left(\frac{5}{\sqrt{41}}\right)\right]$$

Since $\left(\frac{4}{\sqrt{41}}\right)^2 + \left(\frac{5}{\sqrt{41}}\right)^2 = \frac{16}{41} + \frac{25}{41} = 1$, we can write

$\frac{4}{\sqrt{41}} = \cos\alpha$ and $\frac{5}{\sqrt{41}} = \sin\alpha$.

$$\therefore\ y = \sin^{-1}(\sin x\cos\alpha + \cos x\sin\alpha)$$

$$= \sin^{-1}[\sin(x + \alpha)]$$

$$= x + \alpha, \quad \text{where } \alpha \text{ is a constant}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x + \alpha)$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}(\alpha)$$

$$= 1 + 0 = 1.$$

---

(3) Let $y = \sin^{-1}\left(\frac{5x + 12\sqrt{1 - x^2}}{13}\right)$

Put $x = \sin\theta$. Then $\theta = \sin^{-1}x$

$$\therefore\ y = \sin^{-1}\left(\frac{5\sin\theta + 12\sqrt{1 - \sin^2\theta}}{13}\right)$$

$$= \sin^{-1}\left(\frac{5\sin\theta + 12\cos\theta}{13}\right)$$

$$= \sin^{-1}\left[(\sin\theta)\left(\frac{5}{13}\right) + (\cos\theta)\left(\frac{12}{13}\right)\right]$$

Since $\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = \frac{25}{169} + \frac{144}{169} = 1$, we can write

$\frac{5}{13} = \cos\alpha$ and $\frac{12}{13} = \sin\alpha$

$$\therefore\ y = \sin^{-1}(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$

$$= \sin^{-1}[\sin(\theta + \alpha)] = \theta + \alpha$$

$= \sin^{-1} x + \alpha$, where α is a constant

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\alpha)$$

$$= \frac{1}{\sqrt{1-x^2}} + 0 = \frac{1}{\sqrt{1-x^2}}.$$

Ex. 11. Differentiate the following w.r.t. x :

(1) $\tan^{-1}\left(\sqrt{\frac{3-x}{3+x}}\right)$ **(2)** $\sin^{-1}(2x\sqrt{1-x^2})$.

Solution :

(1) Let $y = \tan^{-1}\left(\sqrt{\frac{3-x}{3+x}}\right)$

Put $x = 3\cos 2\theta$. $\therefore\ \frac{x}{3} = \cos 2\theta$

$\therefore\ 2\theta = \cos^{-1}\left(\frac{x}{3}\right)$ $\therefore\ \theta = \frac{1}{2}\cos^{-1}\left(\frac{x}{3}\right)$

$$\therefore\ y = \tan^{-1}\left(\sqrt{\frac{3 - 3\cos 2\theta}{3 + 3\cos 2\theta}}\right) = \tan^{-1}\left(\sqrt{\frac{3(1-\cos 2\theta)}{3(1+\cos 2\theta)}}\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right) = \tan^{-1}(\sqrt{\tan^2\theta})$$

$$= \tan^{-1}(\tan\theta) = \theta = \frac{1}{2}\cos^{-1}\left(\frac{x}{3}\right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left[\cos^{-1}\left(\frac{x}{3}\right)\right]$$

$$= \frac{1}{2} \times \frac{-1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{3}\right)$$

$$= \frac{-1}{2\sqrt{1-\frac{x^2}{9}}} \times \frac{1}{3} \times 1$$

$$= \frac{-1}{6\sqrt{9-x^2}} \times 3 = \frac{-1}{2\sqrt{9-x^2}}.$$

---

(2) Let $y = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin\theta$. Then $\theta = \sin^{-1}x$.

$\therefore\ y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$

$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta)$

$= 2\theta = 2\sin^{-1}x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\sin^{-1}x) = 2\frac{d}{dx}(\sin^{-1}x)$$

$$= 2 \times \frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}.$$

We can also put $x = \cos\theta$. Then $\theta = \cos^{-1}x$.

$\therefore\ y = \sin^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$

$= \sin^{-1}(2\cos\theta\sin\theta) = \sin^{-1}(\sin 2\theta)$

$= 2\theta = 2\cos^{-1}x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\cos^{-1}x) = 2\frac{d}{dx}(\cos^{-1}x)$$

$$= 2 \times \frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

Hence, $\frac{dy}{dx} = \pm\frac{2}{\sqrt{1-x^2}}$.

Ex. 12. Differentiate the following w.r.t. x :

(1) $\tan^{-1}\left(\dfrac{x}{1+6x^2}\right)+\cot^{-1}\left(\dfrac{1-10x^2}{7x}\right)$

(2) $\tan^{-1}\left(\dfrac{5-x}{6x^2-5x-3}\right)$.

Solution :

(1) Let $y=\tan^{-1}\left(\dfrac{x}{1+6x^2}\right)+\cot^{-1}\left(\dfrac{1-10x^2}{7x}\right)$

$$=\tan^{-1}\left(\frac{x}{1+6x^2}\right)+\tan^{-1}\left(\frac{7x}{1-10x^2}\right) \quad \ldots \left[\because \cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$=\tan^{-1}\left[\frac{3x-2x}{1+(3x)(2x)}\right]+\tan^{-1}\left[\frac{5x+2x}{1-(5x)(2x)}\right]$$

$$=\tan^{-1}3x-\tan^{-1}2x+\tan^{-1}5x+\tan^{-1}2x$$

$$=\tan^{-1}3x+\tan^{-1}5x$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}[\tan^{-1}3x+\tan^{-1}5x]$$

$$=\frac{d}{dx}(\tan^{-1}3x)+\frac{d}{dx}(\tan^{-1}5x)$$

$$=\frac{1}{1+(3x)^2}\cdot\frac{d}{dx}(3x)+\frac{1}{1+(5x)^2}\cdot\frac{d}{dx}(5x)$$

$$=\frac{1}{1+9x^2}\times 3\times 1+\frac{1}{1+25x^2}\times 5\times 1$$

$$=\frac{3}{1+9x^2}+\frac{5}{1+25x^2}.$$

---

(2) Let $y=\tan^{-1}\left(\dfrac{5-x}{6x^2-5x-3}\right)$

$$=\tan^{-1}\left[\frac{5-x}{1+(6x^2-5x-4)}\right]$$

$$=\tan^{-1}\left[\frac{(2x+1)-(3x-4)}{1+(2x+1)(3x-4)}\right]$$

$$=\tan^{-1}(2x+1)-\tan^{-1}(3x-4)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\tan^{-1}(2x+1)]-\frac{d}{dx}[\tan^{-1}(3x-4)]$$

$$= \frac{1}{1+(2x+1)^2} \cdot \frac{d}{dx}(2x+1) - \frac{1}{1+(3x-4)^2} \cdot \frac{d}{dx}(3x-4)$$

$$= \frac{1}{1+(2x+1)^2} \cdot (2 \times 1 + 0) - \frac{1}{1+(3x-4)^2} \cdot (3 \times 1 - 0)$$

$$= \frac{2}{1+(2x+1)^2} - \frac{3}{1+(3x-4)^2}.$$

Examples for Practice	3 marks each

Differentiate the following w.r.t. x :

1. (1) $\sin^{-1}\left(\frac{2\cos x + 3\sin x}{\sqrt{13}}\right)$ (2) $\cos^{-1}\left(\frac{\sqrt{3}\cos x - \sin x}{2}\right)$

(3) $\sin^{-1}\left(\frac{5\sqrt{1-x^2} - 12x}{13}\right)$ (4) $\operatorname{cosec}^{-1}\left[\frac{10}{6\sin(2^x) - 8\cos(2^x)}\right]$.

2. (1) $\tan^{-1}(\operatorname{cosec} x + \cot x)$ (2) $\tan^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$.

3. (1) $\tan^{-1}\left[\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}\right]$ (2) $\cos^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{2}\right)$

(3) $\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)$ (4) $\tan^{-1}\left(\frac{\sqrt{x}(3-x)}{1-3x}\right)$

(5) $\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]$.

4. (1) $\tan^{-1}\left(\frac{8x}{1-15x^2}\right)$ (2) $\cot^{-1}\left(\frac{1+35x^2}{2x}\right)$

(3) $\tan^{-1}\left(\frac{2\sqrt{x}}{1+3x}\right)$ (4) $\cot^{-1}\left(\frac{a^2-6x^2}{5ax}\right)$.

5. (1) $\tan^{-1}\left(\frac{5x+1}{3-x-6x^2}\right)$ (2) $\cot^{-1}\left(\frac{4-x-2x^2}{3x+2}\right)$.

ANSWERS

1. (1) 1 (2) 1 (3) $-\frac{1}{\sqrt{1-x^2}}$ (4) $2^x \cdot \log 2$.

2. (1) $-\frac{1}{2}$ (2) $-\frac{1}{2}$

3. (1) $\frac{1}{2(1+x^2)}$. Put $x = \tan\theta$ (2) $-\frac{1}{2\sqrt{1-x^2}}$. Put $x = \cos\ \theta$

(3) $-\dfrac{1}{2\sqrt{1-x^2}}$. Put $x = \cos\theta$ (4) $\dfrac{3}{2\sqrt{x}(1+x)}$. Put $\sqrt{x} = \tan\theta$

(5) $-\dfrac{x}{\sqrt{1-x^2}}$. Put $x = \cos\theta$.

4. (1) $\dfrac{5}{1+25x^2} + \dfrac{3}{1+9x^2}$ (2) $\dfrac{7}{1+49x^2} - \dfrac{5}{1+25x^2}$

(3) $\dfrac{1}{2\sqrt{x}}\left[\dfrac{3}{1+9x} - \dfrac{1}{1+x}\right]$ (4) $\dfrac{3a}{a^2+9x^2} + \dfrac{2a}{a^2+4x^2}$

5. (1) $\dfrac{3}{9x^2+12x+5} + \dfrac{1}{2x^2-2x+1}$ (2) $\dfrac{2}{1+(2x+3)^2} + \dfrac{1}{1+(x-1)^2}$.

8.3 DERIVATIVES OF IMPLICIT FUNCTIONS

Solved Examples | 2 marks each

Ex. 13. Find $\dfrac{dy}{dx}$ in the following cases :

(1) $x\sqrt{x} + y\sqrt{y} = a\sqrt{a}$ (2) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(3) $xy = \log(xy)$.

Solution :

(1) $x\sqrt{x} + y\sqrt{y} = a\sqrt{a}$

$\therefore\ x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Differentiating both sides w.r.t. x, we get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\cdot\frac{dy}{dx} = 0$$

$$\therefore\ \frac{2}{3}y^{-\frac{1}{3}}\cdot\frac{dy}{dx} = -\frac{2}{3}x^{-\frac{1}{3}}$$

$$\therefore\ \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = -\left(\sqrt[3]{\frac{y}{x}}\right).$$

---

(2) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Differentiating w.r.t. x, we get

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} + 0 = 0$$

Cancelling 2 throughout, we get

$$ax + hx\frac{dy}{dx} + hy + by\frac{dy}{dx} + g + f\frac{dy}{dx} = 0$$

$$(hx + by + f)\frac{dy}{dx} = -ax - hy - g$$

$$\therefore \frac{dy}{dx} = -\frac{(ax + hy + g)}{hx + by + f}.$$

---

(3) $xy = \log(xy)$

$\therefore\ xy = \log x + \log y$

Differentiating both sides w.r.t. x, we get

$$x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(x) = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\therefore\ x\frac{dy}{dx} + y \times 1 = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\therefore\ \left(x - \frac{1}{y}\right)\frac{dy}{dx} = \frac{1}{x} - y$$

$$\therefore\ \left(\frac{xy - 1}{y}\right)\frac{dy}{dx} = \frac{1 - xy}{x} = \frac{-(xy - 1)}{x}$$

$$\therefore\ \frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} \qquad \therefore\ \frac{dy}{dx} = -\frac{y}{x}.$$

Ex. 14. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \infty}}}$, then show that $\frac{dy}{dx} = \frac{\sin x}{1 - 2y}$.

Solution : $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \infty}}}$

$\therefore\ y^2 = \cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \infty}}$

$\therefore\ y^2 = \cos x + y$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$\therefore\ (1 - 2y)\frac{dy}{dx} = \sin x$$

$$\therefore\ \frac{dy}{dx} = \frac{\sin x}{1 - 2y}.$$

Ex. 15. If $\log_{10}\left(\frac{x^3-y^3}{x^3+y^3}\right)=2$, show that $\frac{dy}{dx}=-\frac{99x^2}{101y^2}$.

Solution : $\log_{10}\left(\frac{x^3-y^3}{x^3+y^3}\right)=2$

$\therefore \frac{x^3-y^3}{x^3+y^3}=10^2=100$

$\therefore x^3-y^3=100x^3+100y^3$

$\therefore 101y^3=-99x^3 \qquad \therefore y^3=-\frac{99}{101}x^3$

Differentiating both sides w.r.t. x, we get

$3y^2\frac{dy}{dx}=-\frac{99}{101}\times 3x^2$

$\therefore \frac{dy}{dx}=-\frac{99x^2}{101y^2}.$

Examples for Practice	2 marks each

1. Find $\frac{dy}{dx}$ in the following cases :

(1) $\sqrt{x}+\sqrt{y}=\sqrt{a}$

(2) $x+\sqrt{xy}+y=1$

(3) $x^3+x^2y+xy^2+y^3=81$

(4) $\sin y=\log(x+y)$

(5) $xe^y+ye^x=1$

(6) $y=\sin(x+y)$

(7) $\cos(xy)=x+y$

(8) $xy+y\sec^{-1}x=1.$

2. If $y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+\ldots\infty}}}$, then show that $\frac{dy}{dx}=\frac{\sec^2 x}{2y-1}$.

3. If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots\infty}}}$, then show that $\frac{dy}{dx}=\frac{1}{x(2y-1)}$.

4. If $\sin^{-1}\left(\frac{x^5-y^5}{x^5+y^5}\right)=\frac{\pi}{6}$, show that $\frac{dy}{dx}=\frac{x^4}{3y^4}$.

ANSWERS

1. (1) $-\sqrt{\frac{y}{x}}$

(2) $\frac{-\sqrt{y}\,(2\sqrt{x}+\sqrt{y})}{\sqrt{x}\,(\sqrt{x}+2\sqrt{y})}$

(3) $\frac{-(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$

(4) $\frac{1}{(x+y)\cos y-1}$

(5) $-\left(\dfrac{e^y + ye^x}{e^x + xe^y}\right)$

(6) $\dfrac{\cos(x+y)}{1-\cos(x+y)}$

(7) $-\left[\dfrac{1+y\sin(xy)}{1+x\sin(xy)}\right]$

(8) $-y^2\left[1+\dfrac{1}{x\sqrt{x^2-1}}\right]$.

Solved Examples	**3 marks each**

Ex. 16. If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, show that $\dfrac{dy}{dx}=\sqrt{\dfrac{1-y^2}{1-x^2}}$.

Solution : $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$ ... (1)

Put $x=\sin\theta$, $y=\sin\phi$

$\therefore\ \theta=\sin^{-1}x,\ \phi=\sin^{-1}y$

$\therefore$ (1) becomes, $\sqrt{1-\sin^2\theta}+\sqrt{1-\sin^2\phi}=a(\sin\theta-\sin\phi)$

$\therefore\ \cos\theta+\cos\phi=a(\sin\theta-\sin\phi)$

$$\therefore\ 2\cos\left(\frac{\theta+\phi}{2}\right)\cdot\cos\left(\frac{\theta-\phi}{2}\right)=a\times 2\cos\left(\frac{\theta+\phi}{2}\right)\cdot\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\therefore\ \cos\left(\frac{\theta-\phi}{2}\right)=a\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\therefore\ \frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\sin\left(\frac{\theta-\phi}{2}\right)}=a$$

$$\therefore\ \cot\left(\frac{\theta-\phi}{2}\right)=a$$

$$\therefore\ \frac{\theta-\phi}{2}=\cot^{-1}a$$

$\therefore\ \theta-\phi=2\cot^{-1}a$

$\therefore\ \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx}=0$$

$$\therefore\ \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

$$\therefore\ \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}.$$

Ex. 17. If $x^y = e^{x-y}$, show that $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$.

Solution : $x^y = e^{x-y}$ $\therefore$ $\log x^y = \log e^{x-y}$

$\therefore$ $y \log x = (x-y) \log e$

$\therefore$ $y \log x = x - y$ ... $[\because \log e = 1]$

$\therefore$ $y + y \log x = x$ $\therefore$ $y(1+\log x) = x$

$\therefore$ $y = \dfrac{x}{1+\log x}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$= \frac{(1+\log x)(1) - x\left(\frac{1}{x}\right)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}.$$

Ex. 18. If $\sec^{-1}\left(\dfrac{x^3+y^3}{x^3-y^3}\right) = 2a$, then show that $\dfrac{dy}{dx} = \dfrac{x^2 \tan^2 a}{y^2}$, where a is a constant.

Solution : $\sec^{-1}\left(\dfrac{x^3+y^3}{x^3-y^3}\right) = 2a$

$\therefore$ $\cos^{-1}\left(\dfrac{x^3-y^3}{x^3+y^3}\right) = 2a$ ... $\left[\because \sec^{-1} x = \cos^{-1}\left(\dfrac{1}{x}\right)\right]$

$\therefore$ $\dfrac{x^3-y^3}{x^3+y^3} = \cos 2a$

$\therefore$ $x^3 - y^3 = x^3 \cos 2a + y^3 \cos 2a$

$\therefore$ $x^3 - x^3 \cos 2a = y^3 + y^3 \cos 2a$

$\therefore$ $x^3(1-\cos 2a) = y^3(1+\cos 2a)$

$\therefore$ $y^3 = \left(\dfrac{1-\cos 2a}{1+\cos 2a}\right)x^3$

$\therefore$ $y^3 = \left(\dfrac{2\sin^2 a}{2\cos^2 a}\right)x^3 = (\tan^2 a)\,x^3$

Differentiating w.r.t. x, we get

$3y^2 \dfrac{dy}{dx} = (\tan^2 a) \times 3x^2$ $\therefore$ $\dfrac{dy}{dx} = \dfrac{x^2 \tan^2 a}{y^2}$.

Ex. 19. If $x^7 \cdot y^5 = (x+y)^{12}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$. *(Sept. '21)*

Solution : $x^7 \cdot y^5 = (x+y)^{12}$

$\therefore$ $(\log x^7 \cdot y^5) = \log (x+y)^{12}$

$\therefore$ $\log x^7 + \log y^5 = \log (x+y)^{12}$

$\therefore$ $7 \log x + 5 \log y = 12 \log (x+y)$

Differentiating both sides w.r.t. x, we get

$$7 \times \frac{1}{x} + 5 \times \frac{1}{y} \cdot \frac{dy}{dx} = 12 \times \frac{1}{x+y} \cdot \frac{d}{dx}(x+y)$$

$$\therefore \frac{7}{x} + \frac{5}{y} \cdot \frac{dy}{dx} = \frac{12}{x+y} \cdot \left(1 + \frac{dy}{dx}\right)$$

$$\therefore \frac{7}{x} + \frac{5}{y} \cdot \frac{dy}{dx} = \frac{12}{x+y} + \frac{12}{x+y} \cdot \frac{dy}{dx}$$

$$\therefore \left(\frac{5}{y} - \frac{12}{x+y}\right) \frac{dy}{dx} = \frac{12}{x+y} - \frac{7}{x}$$

$$\therefore \left[\frac{5x + 5y - 12y}{y(x+y)}\right] \frac{dy}{dx} = \frac{12x - 7x - 7y}{x(x+y)}$$

$$\therefore \left[\frac{5x - 7y}{y(x+y)}\right] \frac{dy}{dx} = \frac{5x - 7y}{x(x+y)}$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \qquad \therefore \frac{dy}{dx} = \frac{y}{x}.$$

Ex. 20. If $\log\left(\frac{x^{20} - y^{20}}{x^{20} + y^{20}}\right) = 20$, then show that $\frac{dy}{dx} = \frac{y}{x}$.

Solution : $\log\left(\frac{x^{20} - y^{20}}{x^{20} + y^{20}}\right) = 20$

$$\therefore \frac{x^{20} - y^{20}}{x^{20} + y^{20}} = e^{20} = k \qquad \text{... (Say)}$$

$\therefore$ $x^{20} - y^{20} = kx^{20} + ky^{20}$

$\therefore$ $(1+k)y^{20} = (1-k)x^{20}$

$$\therefore \frac{y^{20}}{x^{20}} = \frac{1-k}{1+k}$$

$$\therefore \frac{y}{x} = \left(\frac{1-k}{1+k}\right)^{1/20}, \text{ a constant}$$

Differentiating both sides w.r.t. x, we get $\dfrac{d}{dx}\left(\dfrac{y}{x}\right) = 0$

$$\therefore \quad \frac{x\dfrac{dy}{dx} - y \cdot \dfrac{d}{dx}(x)}{x^2} = 0$$

$$\therefore \quad x\frac{dy}{dx} - y \times 1 = 0$$

$$\therefore \quad x\frac{dy}{dx} = y$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x}.$$

Examples for Practice	3 marks each

1. Find $\dfrac{dy}{dx}$ in each of the following cases :

(a) (1) $\sqrt{x^3 + y^3} = 2axy$ (2) $x^5 + xy^3 + x^2y + y^4 = 4.$

(b) (1) $x^p \cdot y^4 = (x+y)^{p+4}, p \in N$ (2) $(x^2 + y)^{17} = x^8 y^{13}.$

(c) (1) $x^y = 2^{x-y}$ (2) $2^x + 2^y = 2^{x+y}$

(3) $y^y = x \sin y$ (4) $y = xe^{xy}.$

2. If $\tan^{-1}\left(\dfrac{x^2 - 2y^2}{x^2 + 2y^2}\right) = a$, show that $\dfrac{dy}{dx} = \dfrac{x\,(1 - \tan a)}{2y\,(1 + \tan a)}.$

3. If $y\sqrt{1 - x^2} + x\sqrt{1 - y^2} = 1$, show that $\dfrac{dy}{dx} = -\sqrt{\dfrac{1 - y^2}{1 - x^2}}.$

4. If $e^x + e^y = e^{x+y}$, show that $\dfrac{dy}{dx} = -e^{y-x}.$

5. If $x \sin(a + y) + \sin a \cdot \cos(a + y) = 0$, then show that $\dfrac{dy}{dx} = \dfrac{\sin^2(a + y)}{\sin a}.$

6. If $x^m \cdot y^n = (x + y)^{m+n}$, then prove that $\dfrac{dy}{dx} = \dfrac{y}{x}.$

ANSWERS

1. (a) (1) $\dfrac{8a^2xy^2 - 3x^2}{3y^2 - 8a^2x^2y}$ (2) $-\left(\dfrac{5x^4 + 2xy + y^3}{x^2 + 3xy^2 + 4y^3}\right)$

(b) (1) $\dfrac{y}{x}$ (2) $\dfrac{2y}{x}$

(c) (1) $\dfrac{x \log 2 - y}{x \log 2x}$ (2) -2^{y-x}

(3) $\dfrac{1}{x(1 + \log y - \cot y)}$ (4) $\dfrac{y(1 + xy)}{x(1 - xy)}.$

8.4 DERIVATIVES OF PARAMETRIC FUNCTIONS

Theory Question	3 or 4 marks

Q. 3. If $x = f(t)$ and $y = g(t)$ are differentiable functions of t, so that y is differentiable function of x and $\frac{dx}{dt} \neq 0$, then prove that $\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)}$.

Hence, find $\frac{dy}{dx}$, *if* $x = \sin t$ and $y = \cos t$. *(March '22)*

Proof : Given : $x = f(t)$ and $y = g(t)$

Let δx and δy be the increments in x and y respectively corresponding to the increment δt in t.

Since x and y are differentiable functions of t,

$$\frac{dx}{dt} = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} \text{ and } \frac{dy}{dt} = \lim_{\delta t \to 0} \frac{\delta y}{\delta t} \quad \text{... (1)}$$

Also, as $\delta t \to 0$, $\delta x \to 0$ ... (2)

Now, $\frac{\delta y}{\delta x} = \frac{(\delta y/\delta t)}{(\delta x/\delta t)}$ ... $[\delta t \neq 0]$

Taking limits as $\delta t \to 0$, we get

$$\lim_{\delta t \to 0} \frac{\delta y}{\delta x} = \lim_{\delta t \to 0} \frac{(\delta y/\delta t)}{(\delta x/\delta t)}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{\lim_{\delta t \to 0} (\delta y/\delta t)}{\lim_{\delta t \to 0} (\delta x/\delta t)} = \frac{(dy/dt)}{(dx/dt)} \quad \text{... [By (1) and (2)]}$$

$\because$ the limits in R.H.S. exist

$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ exists and is equal to $\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, if $\frac{dx}{dt} \neq 0$.

To find $\frac{dy}{dx}$, if $x = \sin t$ and $y = \cos t$:

$x = \sin t$, $y = \cos t$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t \quad \text{and} \quad \frac{dy}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-\sin t}{\cos t} = -\tan t.$$

Solved Examples	2 marks each

Ex. 21. Find $\frac{dy}{dx}$, if :

(1) $x = at^4, y = 2at^2$ **(2) $x = \sqrt{a^2 + m^2}, y = \log (a^2 + m^2)$**

Solution :

(1) $x = at^4, y = 2at^2$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(at^4) = a\frac{d}{dt}(t^4)$$

$$= a \times 4t^3 = 4at^3$$

and $$\frac{dy}{dt} = \frac{d}{dt}(2at^2) = 2a\frac{d}{dt}(t^2)$$

$$= 2a \times 2t = 4at$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{4at}{4at^3} = \frac{1}{t^2}.$$

---

(2) $x = \sqrt{a^2 + m^2}, y = \log (a^2 + m^2)$

Differentiating x and y w.r.t. m, we get

$$\frac{dx}{dm} = \frac{d}{dm}(\sqrt{a^2 + m^2})$$

$$= \frac{1}{2\sqrt{a^2 + m^2}} \cdot \frac{d}{dm}(a^2 + m^2)$$

$$= \frac{1}{2\sqrt{a^2 + m^2}} \times (0 + 2m) = \frac{m}{\sqrt{a^2 + m^2}}$$

and $$\frac{dy}{dm} = \frac{d}{dm}[\log (a^2 + m^2)]$$

$$= \frac{1}{a^2 + m^2} \cdot \frac{d}{dm}(a^2 + m^2)$$

$$= \frac{1}{a^2 + m^2} \times (0 + 2m) = \frac{2m}{a^2 + m^2}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dm)}{(dx/dm)} = \frac{\left(\frac{2m}{a^2 + m^2}\right)}{\left(\frac{m}{\sqrt{a^2 + m^2}}\right)} = \frac{2}{\sqrt{a^2 + m^2}}.$$

Ex. 22. If $x = \operatorname{cosec}^2\theta$, $y = \cot^3\theta$, find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{6}$.

Solution : $x = \operatorname{cosec}^2\theta$, $y = \cot^3\theta$

Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\operatorname{cosec}\ \theta)^2 = 2\ \operatorname{cosec}\ \theta \cdot \frac{d}{d\theta}(\operatorname{cosec}\ \theta)$$

$$= 2\ \operatorname{cosec}\ \theta\ (-\operatorname{cosec}\ \theta\ \cot\ \theta)$$

$$= -2\ \operatorname{cosec}^2\theta\ \cot\ \theta$$

and $$\frac{dy}{d\theta} = \frac{d}{d\theta}(\cot\ \theta)^3 = 3\ \cot^2\theta \cdot \frac{d}{d\theta}(\cot\ \theta)$$

$$= 3\ \cot^2\theta \cdot (-\operatorname{cosec}^2\theta)$$

$$= -3\ \cot^2\theta \cdot \operatorname{cosec}^2\theta$$

$$\therefore\ \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{-3\ \cot^2\theta \cdot \operatorname{cosec}^2\theta}{-2\ \operatorname{cosec}^2\theta \cdot \cot\ \theta}$$

$$= \frac{3}{2}\cot\ \theta$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at } \theta = \frac{\pi}{6}} = \frac{3}{2}\cot\ \frac{\pi}{6} = \frac{3\sqrt{3}}{2}.$$

Examples for Practice	**2 marks each**

1. Find $\dfrac{dy}{dx}$, if :

(1) $x = at^2$, $y = 2at$

(2) $x = \sin\sqrt{t}$, $y = e^{\sqrt{t}}$

(3) $x = t - \sqrt{t}$, $y = t + \sqrt{t}$

(4) $x = a(1 - \cos\theta)$, $y = b(\theta - \sin\theta)$

(5) $x = \sqrt{1 - t^2}$, $y = \sin^{-1} t$.

2. (1) If $x = 2\cos t + \cos 2t$, $y = 2\sin t - \sin 2t$, find $\dfrac{dy}{dx}$ at $t = \dfrac{\pi}{4}$.

(2) If $x = \sec^2\theta$, $y = \tan^3\theta$, find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$.

(3) If $x = t + \dfrac{1}{t}$, $y = \dfrac{1}{t^2}$, find $\dfrac{dy}{dx}$ at $t = \dfrac{1}{2}$.

ANSWERS

1. (1) $\frac{1}{t}$ (2) $\frac{2\sqrt{t}+1}{2\sqrt{t}-1}$ (3) $\frac{t \tan t}{\sin(\log t)}$ (4) $\left(\frac{b}{a}\right)\tan\left(\frac{\theta}{2}\right)$

(5) $-\frac{1}{t}$

2. (1) $1-\sqrt{2}$ (2) $\frac{3\sqrt{3}}{2}$ (3) $\frac{16}{3}$.

Solved Examples	3 or 4 marks each

Ex. 23. If $x = e^{\sin 3t}$, $y = e^{\cos 3t}$, then show that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$.

Solution : $x = e^{\sin 3t}$, $y = e^{\cos 3t}$

$\therefore \log x = \log e^{\sin 3t}$, $\log y = \log e^{\cos 3t}$

$\therefore \log x = (\sin 3t)(\log e)$, $\log y = (\cos 3t)(\log e)$

$\therefore \log x = \sin 3t$, $\log y = \cos 3t$ ... (1) ... [$\because \log e = 1$]

Differentiating both sides w.r.t. t, we get

$$\frac{1}{x}\cdot\frac{dx}{dt} = \frac{d}{dt}(\sin 3t) = \cos 3t \cdot \frac{d}{dt}(3t)$$

$$= \cos 3t \times 3 = 3\cos 3t$$

and $\frac{1}{y}\cdot\frac{dy}{dt} = \frac{d}{dt}(\cos 3t) = -\sin 3t \cdot \frac{d}{dx}(3t)$

$$= -\sin 3t \times 3 = -3\sin 3t$$

$\therefore \frac{dx}{dt} = 3x\cos 3t$ and $\frac{dy}{dt} = -3y\sin 3t$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-3y\sin 3t}{3x\cos 3t}$$

$$= \frac{-y\sin 3t}{x\cos 3t} = -\frac{y\log x}{x\log y}. \quad \text{... [By (1)]}$$

Ex. 24. If $x = \log(1+t^2)$, $y = t - \tan^{-1} t$, show that $\frac{dy}{dx} = \frac{\sqrt{e^x - 1}}{2}$.

Solution : $x = \log(1+t^2)$, $y = t - \tan^{-1} t$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}[\log(1+t^2)] = \frac{1}{1+t^2} \cdot \frac{d}{dt}(1+t^2)$$

$$= \frac{1}{1+t^2} \times (0+2t) = \frac{2t}{1+t^2}$$

and $\dfrac{dy}{dt} = \dfrac{d}{dt}(t) - \dfrac{d}{dt}(\tan^{-1} t)$

$$= 1 - \frac{1}{1+t^2} = \frac{1+t^2-1}{1+t^2} = \frac{t^2}{1+t^2}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\left(\dfrac{t^2}{1+t^2}\right)}{\left(\dfrac{2t}{1+t^2}\right)} = \frac{t}{2}$$

Now, $x = \log(1+t^2)$

$\therefore\ 1+t^2 = e^x$

$\therefore\ t^2 = e^x - 1$

$\therefore\ t = \sqrt{e^x - 1}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{\sqrt{e^x-1}}{2}$.

Ex. 25. If $x^2 + y^2 = t - \dfrac{1}{t}$ and $x^4 + y^4 = t^2 + \dfrac{1}{t^2}$, then show that $x^3 y \dfrac{dy}{dx} = 1$.

Solution : Given : $x^2 + y^2 = t - \dfrac{1}{t}$

and $x^4 + y^4 = t^2 + \dfrac{1}{t^2}$ ... (1)

$$\therefore\ (x^2+y^2)^2 = \left(t - \frac{1}{t}\right)^2$$

$$\therefore\ x^4 + y^4 + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$$

$\therefore\ x^4 + y^4 + 2x^2y^2 = x^4 + y^4 - 2$ ... [By (1)]

$\therefore\ 2x^2y^2 = -2$

$\therefore\ x^2y^2 = -1$ ... (2)

Differentiating both sides w.r.t. x, we get

$$x^2 \cdot \frac{d}{dx}(y^2) + y^2 \cdot \frac{d}{dx}(x^2) = 0$$

$$\therefore\ x^2 \times 2y\frac{dy}{dx} + y^2 \times 2x = 0$$

$\therefore\ 2x^2y\dfrac{dy}{dx} = -2xy^2$

$$\therefore\ \frac{dy}{dx} = \frac{-xy^2}{x^2y} = \frac{-x\left(-\frac{1}{x^2}\right)}{x^2y} \qquad \text{... [By (2)]}$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{x^3y} \qquad \therefore\ x^3y\frac{dy}{dx} = 1.$$

Ex. 26. Find the derivative of

(1) $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ **w.r.t.** $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$

(2) 3^x **w.r.t.** $\log_x 3$.

Solution : (1) Let $u = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ and

$v = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$. Then we want to find $\dfrac{du}{dv}$.

Put $x = \cos\theta$. Then $\theta = \cos^{-1}x$.

$$\therefore\ u = \tan^{-1}\left(\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right) = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}(\cot\theta) = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right]$$

$$= \frac{\pi}{2} - \theta = \frac{\pi}{2} - \cos^{-1}x$$

$$\therefore\ \frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(\cos^{-1}x)$$

$$= 0 - \frac{-1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$v = \sec^{-1}\left(\frac{1}{2x^2-1}\right) = \cos^{-1}(2x^2-1)$$

$$= \cos^{-1}(2\cos^2\theta - 1) = \cos^{-1}(\cos 2\theta)$$

$$= 2\theta = 2\cos^{-1}x$$

$$\therefore\ \frac{dv}{dx} = 2\cdot\frac{d}{dx}(\cos^{-1}x) = \frac{-2}{\sqrt{1-x^2}}$$

$$\therefore\ \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{-2} = -\frac{1}{2}.$$

---

(2) Let $u = 3^x$ and $v = \log_x 3$.

Then we want to find $\dfrac{du}{dv}$.

Differentiating u and v w.r.t. x, we get

$$\frac{du}{dx} = \frac{d}{dx}(3^x) = 3^x \cdot \log 3$$

$$\text{and } \frac{dv}{dx} = \frac{d}{dx}(\log_x 3) = \frac{d}{dx}\left(\frac{\log 3}{\log x}\right)$$

$$= \log 3 \cdot \frac{d}{dx}(\log x)^{-1}$$

$$= (\log 3)(-1)(\log x)^{-2} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{-\log 3}{(\log x)^2} \times \frac{1}{x} = \frac{-\log 3}{x(\log x)^2}$$

$$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{3^x \cdot \log 3}{\left[\dfrac{-\log 3}{x(\log x)^2}\right]}$$

$$= -x(\log x)^2 \cdot 3^x.$$

Examples for Practice	**3 or 4 marks each**

1. Find $\dfrac{dy}{dx}$, if :

(1) $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

(2) $x = \sin^{-1}(3t - 4t^3)$, $y = \cos^{-1}(\sqrt{1 - t^2})$

(3) $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$

(4) $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$ at $\theta = \dfrac{\pi}{4}$.

2. If $x = \sin^{-1}\left(\dfrac{t}{\sqrt{1+t^2}}\right)$, $y = \cos^{-1}\left(\dfrac{1}{\sqrt{1+t^2}}\right)$, show that $\dfrac{dy}{dx} = 1$.

3. (1) If $x = 2\cos^4(t+3)$, $y = 3\sin^4(t+3)$, show that $\dfrac{dy}{dx} = -\sqrt{\dfrac{3y}{2x}}$.

(2) If $x = \sin^{-1}(e^t)$, $y = \sqrt{1 - e^{2t}}$, show that $\sin x + \dfrac{dy}{dx} = 0$.

(3) If $x = \dfrac{2bt}{1+t^2}$, $y = a\left(\dfrac{1-t^2}{1+t^2}\right)$, show that $\dfrac{dx}{dy} = -\dfrac{b^2y}{a^2x}$.

(4) If $x = a\cos^3 t$, $y = a\sin^3 t$, show that $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$.

4. Differentiate :

(1) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$.

(2) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ w.r.t. $\tan^{-1}x$.

(3) $x\sin x$ w.r.t. $\tan x$.

(4) $\cos^{-1}x$ w.r.t. $\sqrt{1-x^2}$.

ANSWERS

1. (1) $\tan\theta$ (2) $\dfrac{1}{3}$ (3) $-\dfrac{y}{x}$ (4) 1.

4. (1) $\dfrac{1}{4}$ (2) 2 (3) $\dfrac{x\cos x + \sin x}{\sec^2 x}$

(4) $\dfrac{1}{x}$.

8.5 HIGHER ORDER DERIVATIVES

Solved Examples	3 or 4 marks each

Ex. 27. If $x = at^2$ and $y = 2at$, then show that $xy\dfrac{d^2y}{dx^2} + a = 0$.

Solution : $x = at^2$, $y = 2at$ ... (1)

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(at^2) = a\frac{d}{dt}(t^2)$$

$$= a \times 2t = 2at \quad \text{... (2)}$$

and $\dfrac{dy}{dt} = \dfrac{d}{dt}(2at) = 2a\dfrac{d}{dt}(t)$

$$= 2a \times 1 = 2a$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2a}{2at} = \frac{1}{t}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}(t^{-1}) \cdot \frac{dt}{dx}$$

$$= (-1)\ t^{-2} \cdot \frac{1}{\left(\frac{dx}{dt}\right)} = \frac{-1}{t^2} \times \frac{1}{2at} \quad \text{... [By (2)]}$$

$$= -\frac{1}{2at^3}.$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{(at^2)(2at)} \times a$$

$$= -\frac{a}{xy} \quad \text{... [By (1)]}$$

$$\therefore xy\frac{d^2y}{dx^2} = -a$$

$$\therefore xy\frac{d^2y}{dx^2} + a = 0.$$

Ex. 28. If $ax^2 + 2hxy + by^2 = 0$, show that $\frac{d^2y}{dx^2} = 0$.

Solution :

$$ax^2 + 2hxy + by^2 = 0 \quad \text{... (1)}$$

$$\therefore ax^2 + hxy + hxy + by^2 = 0$$

$$\therefore x(ax + hy) + y(hx + by) = 0$$

$$\therefore \frac{ax + hy}{hx + by} = \frac{-y}{x} \quad \text{... (2)}$$

Differentiating (1) w.r.t x, we get

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0$$

$$\therefore 2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} = 0$$

$$\therefore 2(hx + by)\frac{dy}{dx} = -2(ax + hy)$$

$$\therefore \frac{dy}{dx} = -\left(\frac{ax + hy}{hx + by}\right) = \frac{y}{x} \quad \text{... [By (2)]}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{x\frac{dy}{dx} - y\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left(\frac{y}{x}\right) - y(1)}{x^2} = \frac{y-y}{x^2} = \frac{0}{x^2} = 0.$$

Ex. 29. If $y = e^{m \tan^{-1} x}$, show that $(1+x^2)\frac{d^2y}{dx^2} + (2x-m)\frac{dy}{dx} = 0$.

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Solution : $y = e^{m \tan^{-1} x}$ ... (1)

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(e^{m \tan^{-1} x}\right)$$

$$= e^{m \tan^{-1} x} \cdot \frac{d}{dx}(m \tan^{-1} x)$$

$$= e^{m \tan^{-1} x} \times m \times \frac{1}{1+x^2}$$

$$\therefore (1+x^2)\frac{dy}{dx} = my \quad \text{... [By (1)]}$$

Differentiating again w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2) = m\frac{dy}{dx}$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(0+2x) = m\frac{dy}{dx}$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = m\frac{dy}{dx}$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2} + (2x-m)\frac{dy}{dx} = 0.$$

Ex. 30. If $x = \sin t$, $y = e^{mt}$, then show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$.

Solution : $x = \sin t,\ y = e^{mt}$

$\therefore\ t = \sin^{-1}x$ and $y = e^{m\sin^{-1}x}$ ... (1)

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(e^{m\sin^{-1}x}\right)$$

$$= e^{m\sin^{-1}x} \cdot \frac{d}{dx}(m\sin^{-1}x)$$

$$= e^{m\sin^{-1}x} \times m \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \sqrt{1-x^2}\,\frac{dy}{dx} = my \qquad \text{... [By (1)]}$$

$$\therefore (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$

Differentiating again w.r.t. x, we get

$$(1-x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\cdot\frac{d}{dx}(1-x^2) = m^2\cdot\frac{d}{dx}(y^2)$$

$$\therefore (1-x^2)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(0-2x) = m^2\times 2y\frac{dy}{dx}$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = m^2y$$

$$\therefore (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0.$$

Ex. 31. Find the n^{th} order derivatives of the following :

(1) $\log(ax+b)$ **(2) $\sin(ax+b)$**

(3) $\frac{1}{3x-5}$.

Solution :

(1) Let $y = \log(ax+b)$

Then $\frac{dy}{dx} = \frac{d}{dx}[\log(ax+b)]$

$$= \frac{1}{ax+b}\cdot\frac{d}{dx}(ax+b)$$

$$= \frac{1}{ax+b}\times(a\times 1+0) = \frac{a}{ax+b}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\right) = a\,\frac{d}{dx}(ax+b)^{-1}$$

$$= a(-1)\,(ax+b)^{-2}\cdot\frac{d}{dx}(ax+b)$$

$$= \frac{(-1)\, a}{(ax+b)^2} \times (a \times 1 + 0) = \frac{(-1)\ a^2}{(ax+b)^2}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{(-1)^1 a^2}{(ax+b)^2} \right] = (-1)^1\ a^2 \cdot \frac{d}{dx}\,(ax+b)^{-2}$$

$$= (-1)^1\ a^2 \cdot (-2)(ax+b)^{-3} \cdot \frac{d}{dx}\,(ax+b)$$

$$= \frac{(-1)^2 \cdot 1 \cdot 2 \cdot a^2}{(ax+b)^3} \times (a \times 1 + 0)$$

$$= \frac{(-1)^2 \cdot 2!\ a^3}{(ax+b)^3}$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = \frac{(-1)^{n-1} \cdot (n-1)!\, a^n}{(ax+b)^n}.$$

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(2) Let $y = \sin\,(ax+b)$

Then $\dfrac{dy}{dx} = \dfrac{d}{dx}\,[\sin\,(ax+b)]$

$$= \cos\,(ax+b) \cdot \frac{d}{dx}\,(ax+b)$$

$$= \cos\,(ax+b) \times (a \times 1 + 0)$$

$$= a\,\cos\,(ax+b)$$

$$= a\,\sin\left[\frac{\pi}{2} + (ax+b)\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\,[a\,\cos\,(ax+b)]$$

$$= a\,\frac{d}{dx}\,[\cos\,(ax+b)]$$

$$= a\,[-\sin\,(ax+b)] \cdot \frac{d}{dx}\,(ax+b)$$

$$= a\,[-\sin\,(ax+b)] \times (a \times 1 + 0)$$

$$= a^2[-\sin\,(ax+b)]$$

$$= a^2 \cdot \sin\,[\pi + (ax+b)]$$

$$= a^2 \cdot \sin\left[\frac{2\pi}{2} + (ax+b)\right]$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\,[-a^2 \sin(ax+b)]$$

$$= -a^2 \frac{d}{dx}\,[\sin(ax+b)]$$

$$= -a^2 \cdot \cos(ax+b) \cdot \frac{d}{dx}\,(ax+b)$$

$$= -a^2 \cdot \cos(ax+b) \times (a \times 1 + 0)$$

$$= a^3[-\cos(ax+b)]$$

$$= a^3 \cdot \sin\left[\frac{3\pi}{2} + (ax+b)\right]$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = a^n \cdot \sin\left[\frac{n\pi}{2} + (ax+b)\right].$$

---

(3) Let $y = \dfrac{1}{3x-5}$

Then $\dfrac{dy}{dx} = \dfrac{d}{dx}\,(3x-5)^{-1}$

$$= -1\,(3x-5)^{-2} \cdot \frac{d}{dx}\,(3x-5)$$

$$= \frac{-1}{(3x-5)^2} \times (3 \times 1 - 0)$$

$$= \frac{(-1)^1 \cdot 3}{(3x-5)^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{(-1)^1 \cdot 3}{(3x-5)^2}\right]$$

$$= (-1)^1 \cdot 3\, \frac{d}{dx}\,(3x-5)^{-2}$$

$$= (-1)^1 \cdot 3 \cdot (-2)(3x-5)^{-3} \cdot \frac{d}{dx}(3x-5)$$

$$= \frac{(-1)^2 \cdot 3 \cdot 2}{(3x-5)^3} \times (3 \times 1 - 0) = \frac{(-1)^2 \cdot 2! \cdot 3^2}{(3x-5)^3}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{(-1)^2 \cdot 2! \cdot 3^2}{(3x-5)^3}\right]$$

$$= (-1)^2 \cdot 2! \cdot 3^2 \cdot \frac{d}{dx}\,(3x-5)^{-3}$$

$$= (-1)^2 \cdot 2! \cdot 3^2 \cdot (-3)\ (3x-5)^{-4} \cdot \frac{d}{dx}(3x-5)$$

$$= \frac{(-1)^3 \times 3 \cdot 2! \times 3^2}{(3x-5)^4} \times (3 \times 1 - 0)$$

$$= \frac{(-1)^3 \cdot 3! \cdot 3^3}{(3x-5)^4}$$

In general, the n^{th} order derivative is given by

$$\frac{d^n y}{dx^n} = \frac{(-1)^n \cdot n! \cdot 3^n}{(3x-5)^{n+1}}.$$

Examples for Practice	**3 or 4 marks each**

1. **Find $\frac{d^2y}{dx^2}$, if :**

(1) $x = 2at^2,\ y = 4at$

(2) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$

(3) $x = a\cos^3\theta,\ y = b\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

2. (1) If $2y = \sqrt{x+1} + \sqrt{x-1}$, show that $4(x^2-1)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - y = 0$.

(2) If $x = \cos t,\ y = e^{mt}$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$.

(3) If $y = \cos(m\cos^{-1}x)$, then show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$.

(4) If $y = (\tan^{-1}x)^2$, show that $(x^2+1)\frac{d^2y}{dx^2} + 2x\,(x^2+1)\frac{dy}{dx} = 2$.

(5) If $y = \log(x + \sqrt{x^2+a^2})^m$, show that $(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.

3. If $x^m \cdot y^n = (x+y)^{m+n}$, show that $\frac{d^2y}{dx^2} = 0$.

4. Find the n^{th} order derivatives of the following :

(1) $\frac{1}{x}$

(2) $\cos x$

(3) $\log(2x+3)$

(4) $\frac{1}{ax+b}$.

ANSWERS

1. (1) $-\frac{1}{4at^3}$ (2) $-\frac{1}{4a}\cdot \text{cosec}^4\left(\frac{\theta}{2}\right)$ (3) $\frac{4\sqrt{2}b}{3a^2}$.

4. (1) $\frac{(-1)^n \cdot n!}{x^{n+1}}$ (2) $\cos\left(\frac{n\pi}{2}+x\right)$

(3) $\frac{(-1)^{n-1}\cdot(n-1)!\,2^n}{(2x+3)^n}$ (4) $\frac{(-1)^n \cdot n! \cdot a^n}{(ax+b)^{n+1}}$

MULTIPLE CHOICE QUESTIONS	2 marks each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. Let $f(1)=3$, $f'(1)=-\frac{1}{3}$, $g(1)=-4$ and $g'(1)=-\frac{8}{3}$.

The derivative of $\sqrt{[f(x)]^2+[g(x)]^2}$ w.r.t. x at $x=1$ is

(a) $-\frac{29}{15}$ (b) $\frac{7}{3}$ (c) $\frac{31}{15}$ (d) $\frac{29}{15}$

2. If $f(x)=x^5+2x-3$, then $(f^{-1})'(-3)=$ ....

(a) 0 (b) -3 (c) $-\frac{1}{3}$ (d) $\frac{1}{2}$ ***(March '22)***

3. If $y=\sin(2\sin^{-1}x)$, then $\frac{dy}{dx}=$ ....

(a) $\frac{2-4x^2}{\sqrt{1-x^2}}$ (b) $\frac{2+4x^2}{\sqrt{1-x^2}}$ (c) $\frac{4x^2-1}{\sqrt{1-x^2}}$ (d) $\frac{1-2x^2}{\sqrt{1-x^2}}$

4. If $y=\log_x a$, $\frac{dy}{dx}=$ ....

(a) $\frac{-\log a}{x(\log x)^2}$ (b) $x\log a$ (c) $\frac{1}{\log_x a}$ (d) $\frac{1}{x\log a}$

5. If $y=x^{\sqrt{x}}$, $\frac{dy}{dx}=$ ....

(a) $\frac{2+\log x}{2\sqrt{x}}$ (b) $x^{\sqrt{x}}\left[\frac{2+\log x}{\sqrt{x}}\right]$

(c) $x^{\sqrt{x}}\left[\frac{2+\log x}{2\sqrt{x}}\right]$ (d) $x^{\sqrt{x}}\left[\frac{1+\log x}{2\sqrt{x}}\right]$

6. The derivative of $\log_{10}x$ w.r.t. $\log_x 10$ is

(a) $-\dfrac{(\log x)^2}{(\log 10)^2}$ (b) $\dfrac{(\log_x 10)^2}{(\log 10)^2}$ (c) 1 (d) $\dfrac{(\log x)^2}{\log 10^2}$

7. If $x^2+y^2=t+\dfrac{1}{t}$, $x^4+y^4=t^2+\dfrac{1}{t^2}$, then $x\dfrac{dy}{dx}=$ ....

(a) $\dfrac{1}{x^2}$ (b) $-\dfrac{1}{x^3}$ (c) $\dfrac{1}{x}$ (d) $-\dfrac{1}{x}$

8. Let $f(x)-2f\left(\dfrac{1}{x}\right)=x$, then $f'(2)=$ ....

(a) $\dfrac{2}{7}$ (b) $\dfrac{1}{2}$ (c) 2 (d) $\dfrac{7}{2}$

9. If $x^3y^4=(x+y)^{n+1}$ and $\dfrac{dy}{dx}=\dfrac{y}{x}$, then $n=$ ....

(a) 3 (b) 4 (c) 6 (d) 7

10. Derivative of $\tan^3\theta$ with respect to $\sec^3\theta$ at $\theta=\dfrac{\pi}{3}$ is

(a) $\dfrac{3}{2}$ (b) $\dfrac{\sqrt{3}}{2}$ (c) $\dfrac{1}{2}$ (d) $-\dfrac{\sqrt{3}}{2}$

11. If $x=e^{\log(\cos 4\theta)}$, $y=e^{\log(\sin 4\theta)}$, then $\dfrac{dy}{dx}$ is

(a) $-\dfrac{x}{y}$ (b) $\dfrac{x}{y}$ (c) $\dfrac{y}{x}$ (d) $\sqrt{\dfrac{y}{x}}$

12. If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots\infty}}}$, then $\dfrac{dy}{dx}$ is

(a) $\dfrac{1}{y-1}$ (b) $\dfrac{1}{x(2y-1)}$ (c) $\dfrac{1}{2\log y}$ (d) $\dfrac{1}{y}$

13. If $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$, then $\left(\dfrac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{4}}=$ ....

(a) $\dfrac{8\sqrt{2}}{a\pi}$ (b) $-\dfrac{8\sqrt{2}}{a\pi}$ (c) $\dfrac{a\pi}{8\sqrt{2}}$ (d) $\dfrac{4\sqrt{2}}{a\pi}$

14. If g is the inverse of function f and $f'(x)=\dfrac{1}{1+x^7}$, then the value of $g'(x)$ is equal to :

(a) $1+x^7$ (b) $\dfrac{1}{1+[g(x)]^7}$ (c) $1+[g(x)]^7$ (d) $7x^6$

ANSWERS

1. (d) $\dfrac{29}{15}$
2. (d) $\dfrac{1}{2}$
3. (a) $\dfrac{2-4x^2}{\sqrt{1-x^2}}$
4. (a) $\dfrac{-\log a}{x\,(\log x)^2}$
5. (c) $x^{\sqrt{x}}\left[\dfrac{2+\log x}{2\sqrt{x}}\right]$
6. (a) $-\dfrac{(\log x)^2}{(\log 10)^2}$
7. (b) $-\dfrac{1}{x^3}$
8. (b) $\dfrac{1}{2}$
9. (c) 6
10. (b) $\dfrac{\sqrt{3}}{2}$
11. (a) $-\dfrac{x}{y}$
12. (b) $\dfrac{1}{x\,(2y-1)}$
13. (a) $\dfrac{8\sqrt{2}}{a\pi}$
14. (c) $1+[g\,(x)]^7$.