Triangles

SAS Congruence Rule

Congruency of Triangles

Congruency of triangles helps us find solutions to many problems in real life. For example, the distance travelled by a ball in a golf course is easy to measure when the ball is on land; however, when the land is separated by a water body like a pond or any other thing, the task of measurement becomes difficult. In such cases, we use the concept of congruency.

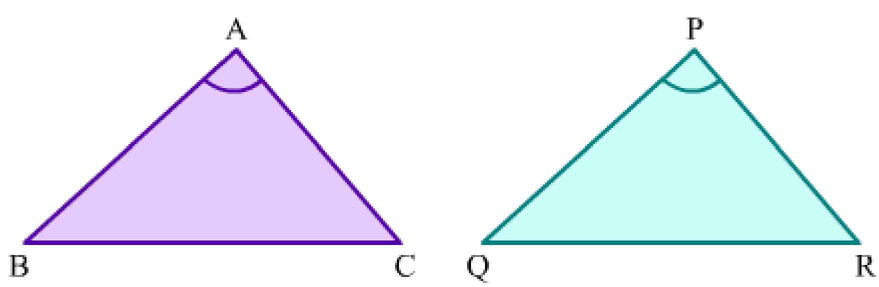
There are certain rules to check the congruency of triangles. One of these is the SAS (Side-Angle-Side) congruence rule. In this lesson, we will learn this rule and its applications.

SAS Congruence Rule

Consider a triangle two of whose sides and the included angle are known. We can check for the congruency of this triangle with respect to another triangle if we know the corresponding sides and angle of that triangle. Two triangles can, thus, be termed 'congruent' or 'incongruent' by using the SAS congruence rule. This rule states that:

If two sides of a triangle and the angle between them are equal to the corresponding sides and angle of another triangle, then the two triangles are congruent.

Look at the given ΔABC and ΔPQR.



Let us consider sides AB and AC and the included ∠BAC in ΔABC, and the corresponding sides and angle in ΔPQR, i.e., PQ, PR and ∠QPR.

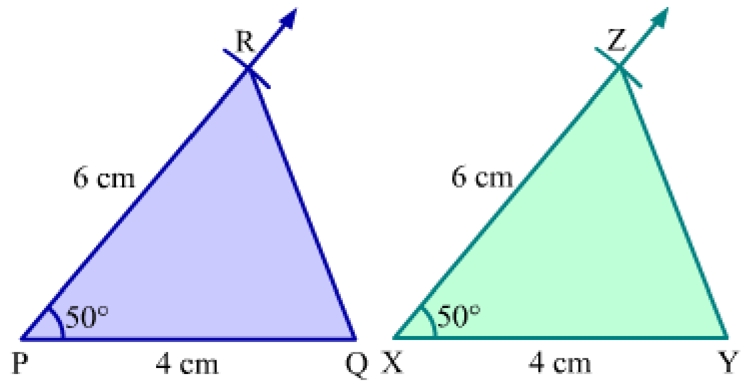
By the SAS congruence rule, the two triangles will be congruent if AB = PQ, AC = PR and ∠BAC = ∠QPR.

Similarly, we can check for congruency by taking other pairs of sides and included angles in these triangles.

Verification of SAS Congruence Rule

The SAS congruence rule for triangles is taken as a postulate, so there is no proof for the same but we we can verify it by doing an activity.

The steps of the activity are as follows:



i) Construct ΔPQR such that PQ = 4 cm, PR = 6 cm and ∠QPR = 50°.

ii) Construct ΔXYZ with the same measures such that XY = 4 cm, XZ = 6 cm and ∠YXZ = 50°.

iii) Cut both the triangles along their boundaries.

iv) Try to superpose one triangle by the other. One triangle can be placed on the other in six different ways such that vertex lie on vertex.

v) In one of the the trials, you will get P falling over X, Q falling over Y and R falling over Z. In this case, you will see that both the triangles cover each other exactly.

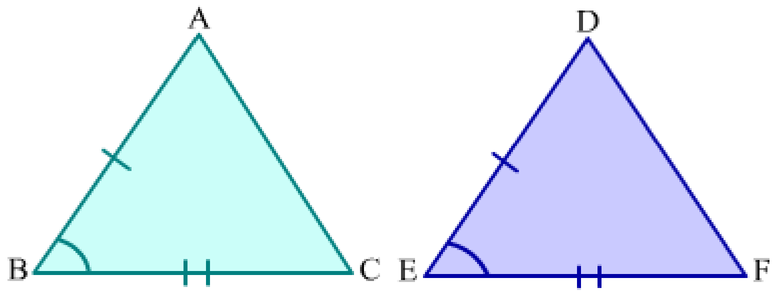
vi) Thus, under the correspondence PQR↔XYZ, the triangles are congruent.

This verifies the SAS congruence rule.

CPCT

CPCT stands for 'corresponding parts of congruent triangles'. 'Corresponding parts' means corresponding sides and angles of triangles. According to CPCT, if two or more triangles are congruent to one another, then all of their corresponding parts are equal.

For example, in the given ΔABC and ΔDEF, AB = DE, ∠B = ∠E and BC = EF. So, according to the SAS congruence criterion, we have ΔABC ≅ΔDEF.



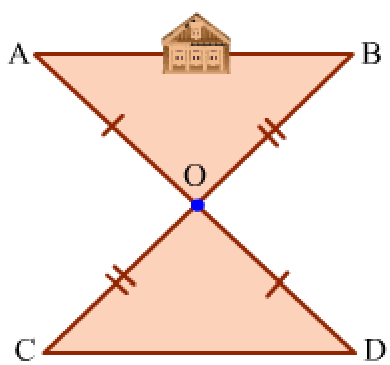
Now, by CPCT, we can say that the remaining corresponding parts of the two congruent triangles are also equal. This means that AC = DF, ∠A = ∠D and ∠C = ∠F.

Similarly, we can apply CPCT in case of other congruent triangles.

Solved Examples

Easy

Example 1: **Observe the following figure.**



Ajay wishes to determine the distance between two objects A and B, but there is a house in between. So, he devises an ingenious way to fix the problem. First, he fixes a pole at any point O so that both A and B are visible from O. He then fixes another pole at point D which is collinear to point O and object A, and is at the same distance from O as A, i.e., DO = AO. Similarly, he fixes a pole at point C which is collinear to point O and object B, and is at the same distance from O as B, i.e., CO = BO. Finally, he measures CD to find the distance between A and B. How can Ajay be sure that CD = AB?

Solution:

We have two triangles in the given figure, i.e., ΔAOB and ΔDOC.

In these two triangles, we have:

AO = DO (Given)

∠AOB = ∠DOC (Vertically opposite angles)

BO = CO (Given)

Therefore, by the SAS congruence rule, we can say that:

ΔAOB ≅ ΔDOC

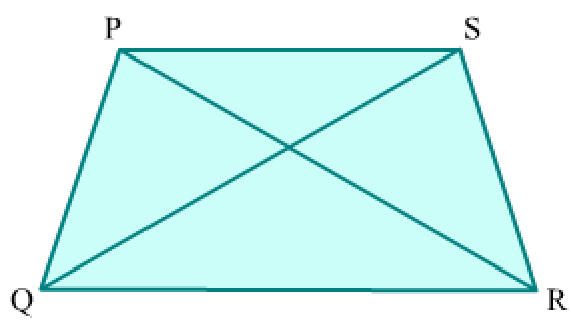
⇒ AB = CD (By CPCT)

This is the reason why Ajay measures CD to find the distance between objects A and B.

Example 2: In the given quadrilateral PQRS, PR bisects ∠QPS and PQ = PS. Prove that:

i) ΔPQR ≅ΔPSR

ii) QR = SR



Solution:

i) In ΔPQR and ΔPSR, we have:

PQ = PS(Given)

PR = PR(Common side)

∠QPR = ∠SPR(&because PR bisects ∠QPS)

So, by the SAS congruence rule, we obtain:

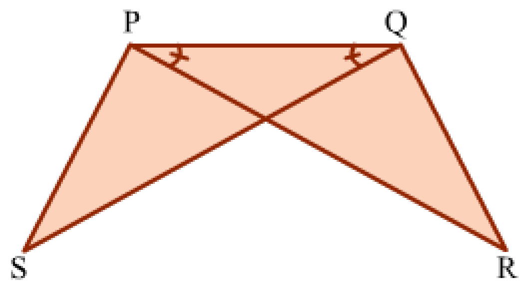
ΔPQR ≅ ΔPSR

ii) We have proved that ΔPQR ≅ ΔPSR.

∴ QR = SR (&because Corresponding parts of congruent triangles are equal)

Medium

Example 1: In the shown figure, PR = QS and ∠QPR = ∠PQS. Prove that ΔPQR ≅ ΔQPS. Also, show that PS = QR and ∠QPS = ∠PQR.



Solution:

In ΔPQR and ΔQPS, we have:

PR = QS(Given)

∠QPR = ∠PQS(Given)

PQ = PQ(Common side)

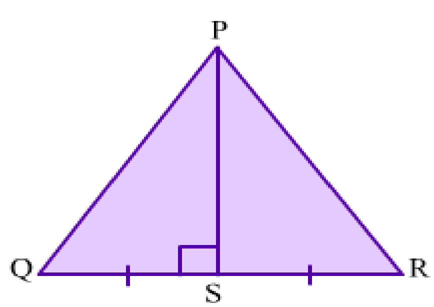
∴ ΔPQR ≅ ΔQPS(By the SAS congruence criterion)

⇒ PS = QR and ∠QPS = ∠PQR (By CPCT)

Example 2: Prove that ΔPQR is isosceles if the altitude drawn from a vertex bisects the opposite side.

Solution:

The given figure shows the ΔPQR having PS as an altitude that bisects the opposite side QR.



In ΔPSQ and ΔPSR, we have:

QS = SR (&because Altitude PS bisects QR)

PS = PS(Common side)

∠PSQ = ∠PSR = 90°(&because PS is the altitude to QR)

∴ ΔPSQ ≅ ΔPSR(By the SAS congruence rule)

⇒ PQ = PR(By CPCT)

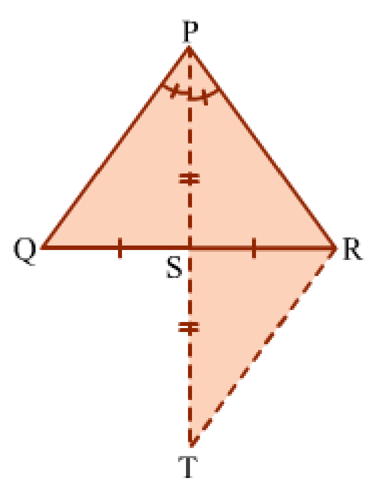
Therefore, ΔPQR is isosceles.

Example 3: If the angle bisector of any angle of a triangle bisects the opposite side then show that the triangle is isosceles.

Solution:

Let ΔPQR be the given triangle and PS is the angle bisector of ∠QPR such that it bisects the side QR.

Let us extend the segment PS to point T such that PS = TS.



In ΔPQS and ΔTRS, we have

QS = RS (Given)

∠PSQ = ∠TSR (Vertically opposite angles)

PS = TS (By construction)

So, by the SAS congruence criterion, we have:

$\Delta PQS \cong \Delta TRS$

By CPCT, we obtain

PQ = TR ...(1)

And $\angle QPS = \angle RTS$ ...(2)

But $\angle QPS = \angle RPS$ ...(3) (PS bisects $\angle QPR$)

$\therefore \angle RTS = \angle RPS$ [From (2) and (3)]

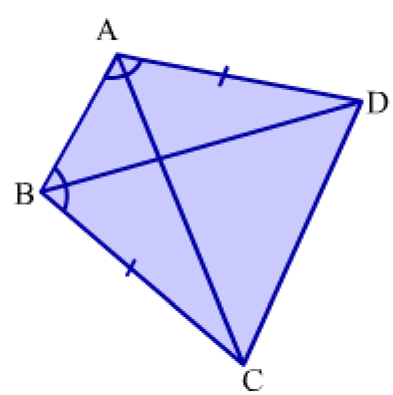
$\Rightarrow$ PR = TR (Sides opposite to equal angles)

$\therefore$ PQ = PR [From (1)]

Thus, ΔPQR is an isosceles triangle.

Hard

Example 1: ABCD is a quadrilateral in which AD = BC and $\angle DAB = \angle CBA$.



Prove that:

i) $\Delta ABD \cong \Delta BAC$

ii) BD = AC

iii) $\angle ABD = \angle BAC$

Solution:

i) In ΔABD and ΔBAC, we have:

AD = BC (Given)

∠DAB = ∠CBA (Given)

AB = BA (Common side)

So, by the SAS congruence criterion, we have:

ΔABD ≅ ΔBAC

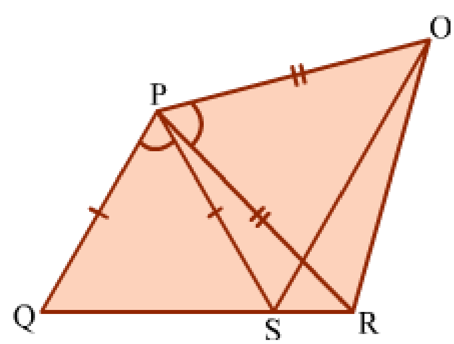
ii) We have proved that ΔABD ≅ ΔBAC.

∴ BD = AC (By CPCT)

iii) Since ΔABD ≅ ΔBAC, we have:

∠ABD = ∠BAC(By CPCT)

Example 2: In the given figure, PR = PO, PQ = PS and ∠QPS = ∠OPR. Show that QR = SO.



Solution:

It is given that ∠QPS = ∠OPR.

∴ ∠QPS + ∠SPR = ∠OPR + ∠SPR

⇒ ∠QPR = ∠SPO… (1)

In ΔQPR and ΔSPO, we have:

PQ = PS (Given)

∠QPR = ∠SPO (From equation 1)

PR = PO (Given)

So, by the SAS congruence rule, we have:

ΔQPR ≅ ΔSPO

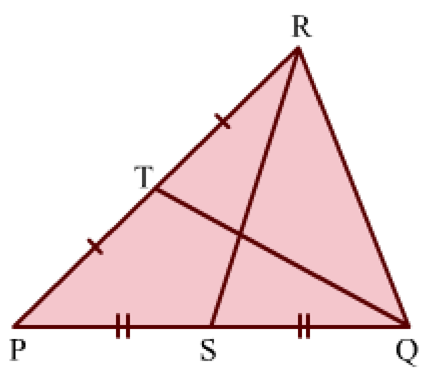
⇒ QR = SO (By CPCT)

Example 3: In an isosceles triangle, prove that the medians on the equal sides are equal.

Solution:

Let ΔPQR be an isosceles triangle such that PQ = PR.

Also, let RS and QT be the medians to the sides PQ and PR respectively.



In ΔPQR, we have

PS = SQ = $\frac{1}{2}PQ$ (RS is the median)

And PT = TR = $\frac{1}{2}PR$ (QT is the median)

But PQ = PR

∴ PS = SQ = PT = TR ...(1)

In ΔPRS and ΔPQT, we have

PQ = PR (Given)

∠RPS = ∠QPT (Common angle)

PS = PT [From (1)]

So, by the SAS congruence rule, we have:

ΔPRS ≅ ΔPQT

∴ RS = QT (By CPCT)

Thus, the medians on the equal sides of an isosceles triangle are equal.

Proving Theorem of Right Angled Triangle

There is a theorem of right angled triangles which states that:

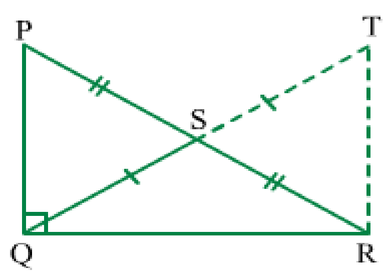
In a right angled triangle, median drawn to the hypotenuse from the opposite vertex is equal to the half of the hypotenuse.

Let us prove the theorem.

Given: Right angled ΔPQR, ∠PQR = 90° and median QS to hypotenuse PR.

To prove: QS = $\frac{1}{2}PR$

Construction: Extend QS to T such that QS = ST and join T to R.



Proof:

In ΔPSQ and ΔRST, we have

QS = TS (By construction)

∠PSQ = ∠RST (Vertically opposite angles)

PS = RS (Given)

So, by SAS congruence criterion, we have

$\Delta PSQ \cong \Delta RST$

$\therefore$ PQ = RT ...(1) (By CPCT)

And $\angle QPS = \angle TRS$ (By CPCT)

Thus, PQ || RT ($\angle QPS$ and $\angle TRS$ alternate interior angles formed by transversal PR)

Now, QR is also a transversal to parallel line segments PQ and RT.

$\angle PQR + \angle TRQ = 180°$ (Sum of interior angles on the same side of transversal)

But $\angle PQR = 90°$

$\therefore \angle TRQ = 90°$ (2)

In ΔPQR and ΔTRQ, we have

PQ = RT [From (1)]

$\angle PQR = \angle TRQ = 90°$

QR = QR (Common side)

So, by SAS congruence criterion, we have

$\Delta PQR \cong \Delta TRQ$

$\therefore$ PR = QT (By CPCT)

$\Rightarrow \frac{1}{2}PR = \frac{1}{2}QT$

But $\frac{1}{2}QT = QS$

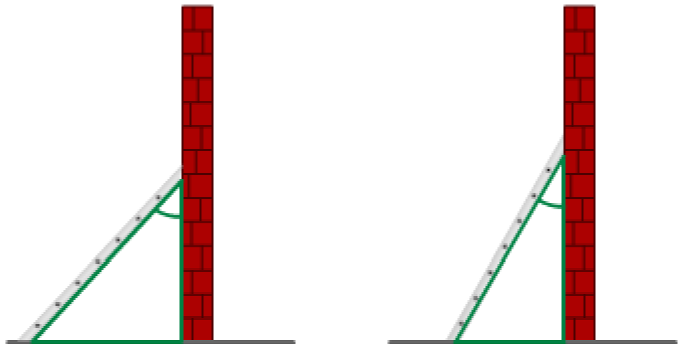
$\therefore QS = \frac{1}{2}PR$

Hence, proved.

ASA Congruence Rule

ASA Congruence Rule

Look at the given figure.



Observe how the ladder, the wall and the horizontal together make triangles in the figure. It can be seen that the angle marked between the ladder and the wall on the left is greater than the same angle marked on the right. Clearly, the triangles are not congruent although it is the same ladder on both sides.

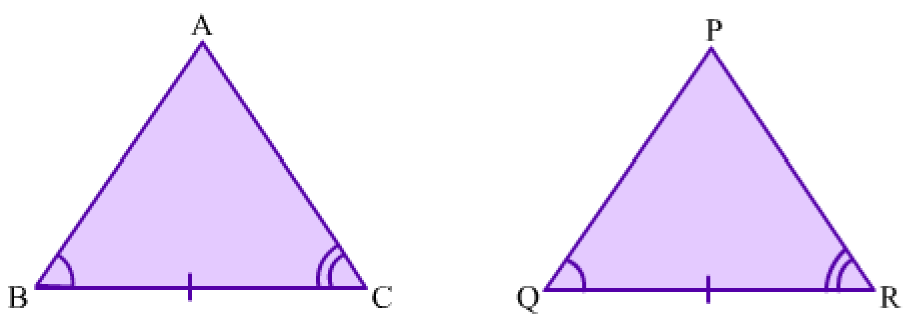
Both angles and sides play an important role in determining the congruency of triangles. In this lesson, we will discuss the ASA (Angle-Side-Angle) congruence rule and solve some problems based on it.

ASA Congruence Rule

The ASA congruence rule for triangles states that:

If two angles of a triangle and the side between them are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

Consider the given ΔABC and ΔPQR.



Observe how corresponding components of the two triangles are marked.

Now, by the ASA congruence rule, the two triangles will be congruent if these corresponding components are equal, i.e., if ∠ABC = ∠PQR, BC = QR and ∠ACB = ∠PRQ, then ΔABC ≅ ΔPQR.

Note that, under the above condition of congruence, we cannot write ΔABC ≅ ΔQRP. The order of the vertices matters in any congruency.

Did You Know?

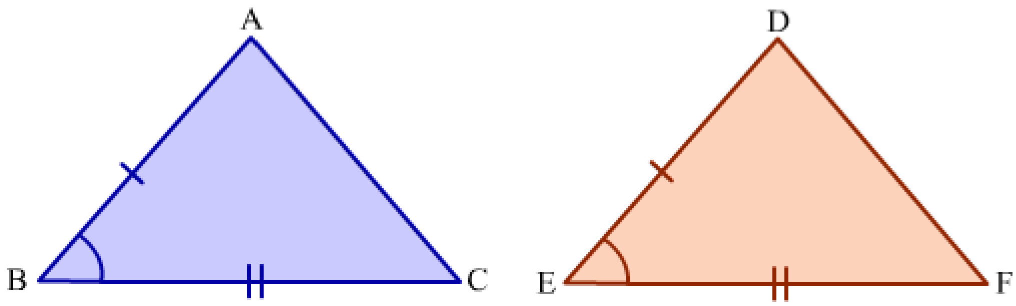
A bright meteor was seen in the sky above Greenland on December 9, 1997. In an attempt to find the fragments of the meteorite, scientists collected data from eyewitnesses who observed the meteor passing through the sky. As is shown in the figure below, the scientists considered sightlines of observers in different towns.

One such sightline was of observers in the town called Paamiut (Point P). Another was of observers in the town called Narsarsuaq (Point N). Using the ASA congruence rule, the scientists were able to gather enough information to successfully locate the fragments of the meteorite (Point M).

Proof of the ASA Congruence Rule: Case 1

Let us consider ΔABC and ΔDEF such that ∠ABC = ∠DEF, ∠ACB = ∠DFE and BC = EF. By the ASA congruence rule, ΔABC and ΔDEF are congruent. By CPCT, we have AB = DE.

Case 1: Let us prove ΔABC ≅ ΔDEF by taking AB = DE.



In this case, we have

AB = DE (Given)

∠ABC = ∠DEF (Given)

BC = EF (Given)

So, by the SAS congruence rule, we have:

ΔABC ≅ ΔDEF

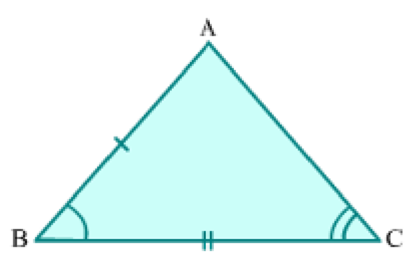
Proof of the ASA Congruence Rule: Case 2

Let us consider ΔABC and ΔDEF such that ∠ABC = ∠DEF, ∠ACB = ∠DFE and BC = EF. By the ASA congruence rule, ΔABC and ΔDEF are congruent. By CPCT, we have AB = DE. Let us assume AB ≠ DE.

Case 2: Let us prove ΔABC ≅ ΔDEF by taking AB < DE.

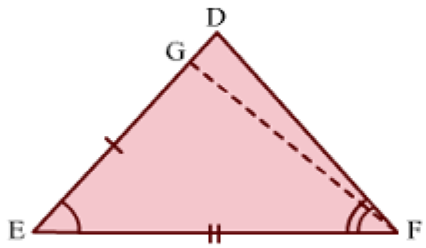
Construction: Mark a point G on DE such that GE = AB. Join G to F.

Now, in ΔABC and ΔGEF, we have:



AB = GE (By construction)

∠ABC = ∠GEF (∵∠ABC = ∠DEF and ∠DEF = ∠GEF)



BC = EF (Given)

So, by the SAS congruence rule, we obtain:

ΔABC ≅ ΔGEF

⇒ ∠ACB = ∠GFE (By CPCT)

But ∠ACB = ∠DFE (Given)

∴ ∠GFE = ∠DFE

This can be possible only when line segment GF coincides with line segment DF or point G coincides with point D. Therefore, AB must be equal to DE and ΔGEF must be ΔDEF.

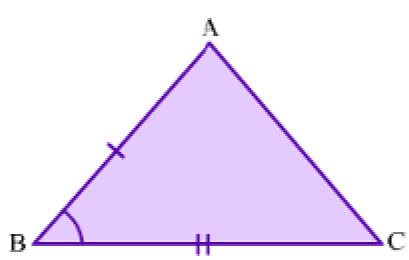
∴ ΔABC ≅ ΔDEF

Proof of the ASA Congruence Rule: Case 3

Let us consider ΔABC and ΔDEF such that ∠ABC = ∠DEF, ∠ACB = ∠DFE and BC = EF. By the ASA congruence rule, ΔABC and ΔDEF are congruent. By CPCT, we have AB = DE. Let us assume AB ≠ DE.

Case 3: Let us prove ΔABC ≅ ΔDEF by taking AB > DE.

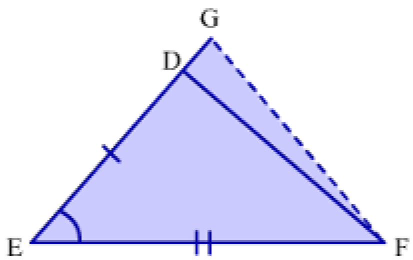
Construction: Extend ED to point G such that GE = AB. Join G to F.



Now, in ΔABC and ΔGEF, we have:

AB = GE (By construction)

∠ABC = ∠DEF (∵∠ABC = ∠DEF and ∠DEF = ∠GEF)



BC = EF (Given)

So, by the SAS congruence rule, we obtain:

ΔABC ≅ ΔGEF

⇒ ∠ACB = ∠GFE (By CPCT)

But ∠ACB = ∠DFE (Given)

∴ ∠GFE = ∠DFE

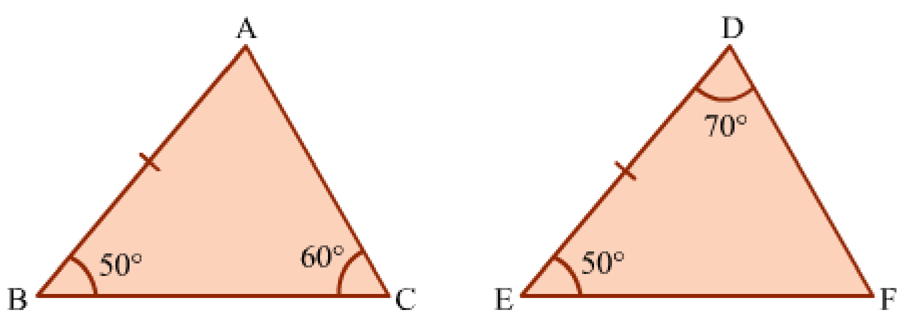
This can be possible only when line segment GF coincides with line segment DF or point G coincides with point D. Therefore, AB must be equal to DE and ΔGEF must be ΔDEF.

∴ ΔABC ≅ ΔDEF

Solved Examples

Easy

Example 1: Check whether the given triangles are congruent or not.



Solution:

In ΔABC, we have:

∠ABC + ∠BCA + ∠BAC = 180° (By the angle sum property)

⇒ 50° + 60° + ∠BAC = 180°

⇒ 110° + ∠BAC = 180°

⇒ ∠BAC = 180° − 110°

⇒ ∠BAC = 70°

In ΔABC and ΔDEF, we have:

∠BAC = ∠EDF = 70°

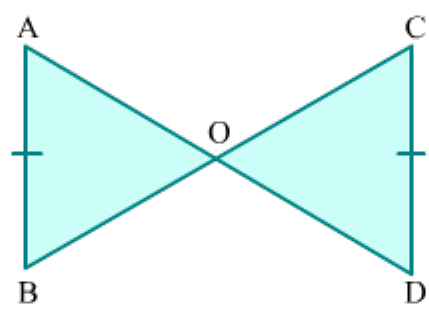
AB = DE (Given)

∠ABC = ∠DEF = 50°

Therefore, by the ASA congruence rule, we have:

ΔABC ≅ ΔDEF

Example 2: In the given figure, AB and CD are two equal and parallel lines. Prove that ΔABO ≅ ΔCDO.



Solution:

It is given that AB||CD. AD and BC are transversals lying on lines AB and CD.

So, by the alternate angles axiom, we obtain:

∠OAB = ∠ODC… (1)

∠OBA = ∠OCD… (2)

In ΔABO and ΔCDO, we have:

∠OAB = ∠ODC (By equation 1)

AB = CD(Given)

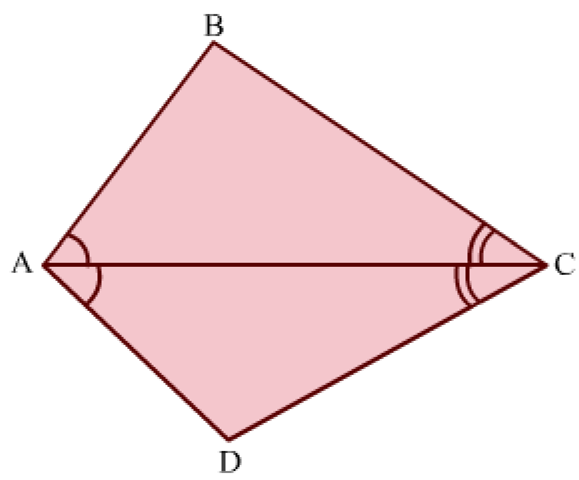
∠OBA = ∠OCD (By equation 2)

Thus, by the ASA congruence rule, we obtain:

ΔABO ≅ ΔCDO

Medium

Example 1: In the given quadrilateral ABCD, diagonal AC bisects ∠BAD and ∠BCD. Prove that
AB = AD and CB = CD.



Solution:

Since diagonal AC bisects ∠BAD and ∠CAD, we have:

∠BAC = ∠DAC and ∠BCA = ∠DCA

In ΔACB and ΔACD, we have:

∠BAC **=** ∠DAC (Given)

∠BCA = ∠DCA (Given)

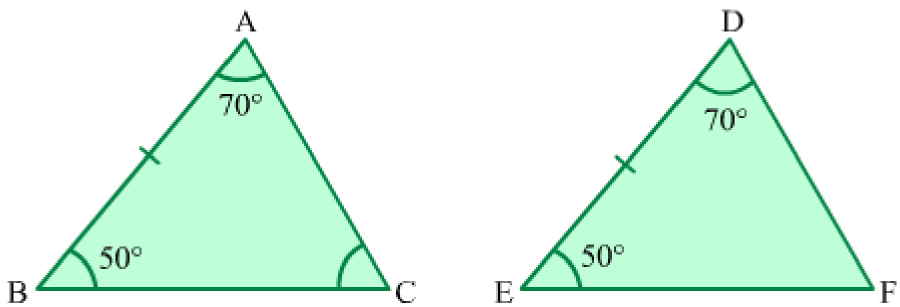
AC = AC (Common side)

So, by the ASA congruence rule, we have:

ΔACB ≅ ΔACD

⇒ AB = AD and CB = CD (By CPCT)

Example 2: Consider the two triangular parks ABC and DEF shown below.



Tina jogs around park ABC and Aliya jogs around park DEF daily. Paths AB and DE are equal in length. If both girls jog an equal number of rounds daily, then check whether or not they cover the same distance while jogging?

Solution:

In ΔABC and ΔDEF, we have:

∠BAC = ∠EDF = 70°(Given)

AB = DE(Given)

∠ABC = ∠DEF = 50°(Given)

Therefore, by the ASA congruency rule, we obtain:

ΔABC ≅ ΔDEF

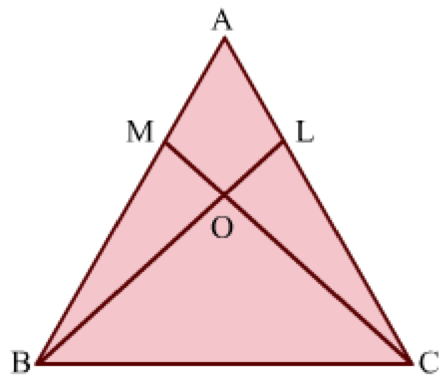
⇒ AC = DF and BC = EF(By CPCT)

∴ AB + BC + CA = DE + EF + FD

Hence, both Tina and Aliya cover the same distance daily while jogging.

Hard

Example 1: The given ΔABC is isosceles with AB = AC. ∠LOC = 2∠OBC and ∠MOB = 2∠OCB. Prove that ΔBCM ≅ ΔCBL.



Solution:

It is given that:

∠LOC = 2∠OBC… (1)

∠MOB = 2∠OCB… (2)

Now, ∠LOC = ∠MOB   (Vertically opposite angles)

Using equations (1) and (2), we obtain:

∠OCB = ∠OBC

⇒ ∠MCB = ∠LBC… (3)

In ΔBCM and ΔCBL, we have:

∠MBC = ∠LCB   (∵ ΔABC is isosceles with AB = AC)

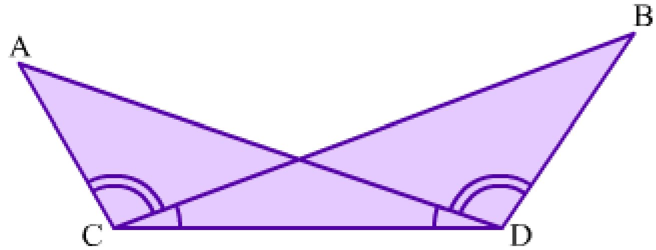
BC = CB (Common side)

∠MCB = ∠LBC (Using equation 3)

Thus, by the ASA congruence rule, we obtain:

ΔBCM ≅ ΔCBL

Example 2: In the given figure, ∠BCD = ∠ADC and ∠ACB = ∠BDA. Prove that AD = BC and ∠CAD = ∠DBC.



Solution:

It is given that:

∠BCD = ∠ADC … (1)

∠ACB = ∠BDA … (2)

On adding equations (1) and (2), we get:

∠BCD + ∠ACB = ∠ADC + ∠BDA

⇒ ∠ACD = ∠BDC… (3)

In ΔACD and ΔBDC, we have:

∠ADC = ∠BCD(Given)

CD = DC (Common side)

∠ACD = ∠BDC (By equation 3)

So, by the ASA congruence rule, we have:

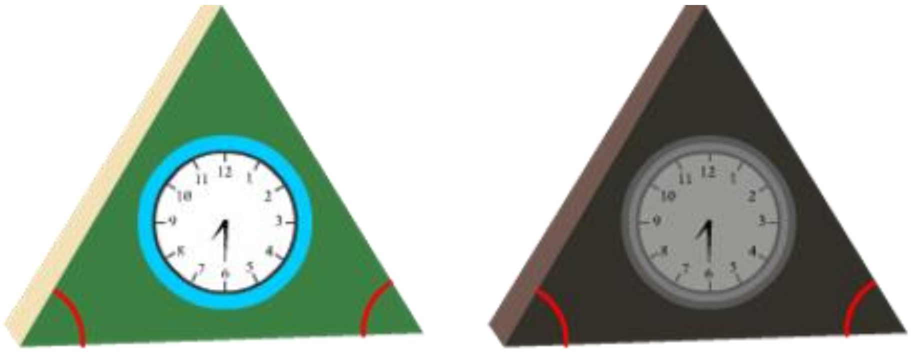
ΔACD ≅ ΔBDC

⇒ AD = BC and ∠CAD = ∠DBC (By CPCT)

AAS Congruence Rule

AAS Congruence Rule

Consider the following triangular clocks.



In each clock, two angles on the base are marked. We can check the congruency of the clocks if we know the marked angles and the included side (i.e., the base) by using the ASA congruence rule.

Now, can we check the congruency of these clocks if instead of the base we are given any of the other two sides? Yes, we can do so and this is why we have the AAS (Angle-Angle-Side) congruence rule. In this lesson, we will discuss this rule and solve some examples related to it.

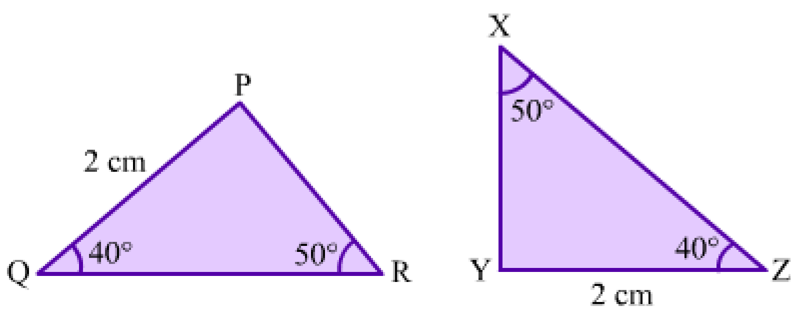
AAS Congruence Rule

The AAS congruence rule for triangles states that:

If any two angles and one side of a triangle are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

This is nothing but the generalized form of the ASA congruence rule.

Consider the given ΔPQR and ΔXYZ.



In each triangle, the given side is not between the given angles. So, we cannot apply the ASA congruence rule here. Instead, we will use the AAS congruence rule.

In ΔPQR and ΔYZX, we have:

PQ = YZ = 2 cm

∠PQR = ∠YZX = 40°

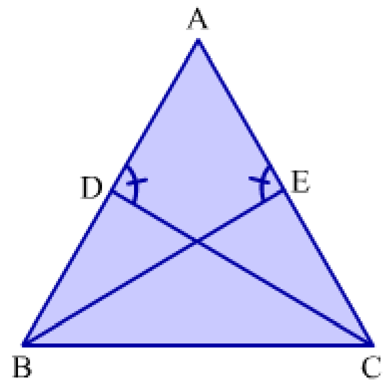
∠PRQ = ∠YXZ = 50°

Thus, by using the AAS congruence criterion, we obtain: ΔPQR ≅ ΔYZX

Solved Examples

Easy

Example 1: The given ΔABC is isosceles with AB = AC and ∠ADC = ∠AEB. Prove that ΔABE ≅ ΔACD.



Solution:

In ΔABE and ΔACD, we have:

AB = AC (Given)

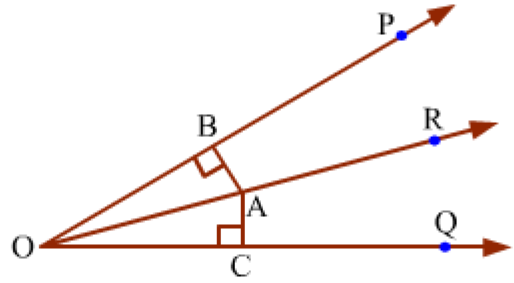
∠AEB = ∠ADC (Given)

∠BAE = ∠CAD (Common angle)

Thus, by the AAS congruence rule, we obtain:

ΔABE ≅ ΔACD

Example 2: In the given figure, OR bisects ∠POQ and A is any point on OR. AB and AC are the perpendiculars drawn from A to the arms OP and OQ respectively. Prove that ΔAOB ≅ ΔAOC.



Solution:

It is given that OR bisects ∠POQ.

∴ ∠POR = ∠QOR

⇒ ∠BOA = ∠COA… (1)

In ΔAOB and ΔAOC, we have:

∠ABO = ∠ACO = 90° (∵ AB and AC are perpendiculars)

∠BOA = ∠COA (By equation 1)

AO = AO (Common side)

Thus, by the AAS congruence rule, we obtain:

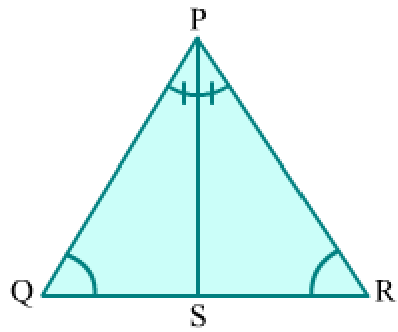
ΔAOB ≅ ΔAOC

Medium

Example 1: If two angles of a triangle are equal, then prove that the sides opposite them are also equal.

Solution:

Consider a ΔPQR with ∠PQR = ∠PRQ.



We have to prove that PQ = PR.

Construction: Draw the bisector of ∠QPR and let it meet side QR at point S.

In ΔPSQ and ΔPSR, we have:

∠PQS = ∠PRS (Given)

PS = PS (Common side)

∠QPS **=** ∠RPS (∵ PS bisects ∠QPR)

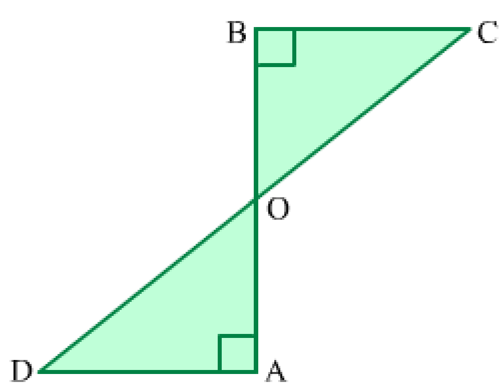
So, by the AAS congruence rule, we obtain:

ΔPSQ ≅ ΔPSR

⇒ PQ = PR (By CPCT)

Hence, we have proved that the sides opposite equal angles of a triangle are also equal.

Example 2: DA and CB are equal perpendiculars to a line segment AB. Show that line segment CD bisects AB at point O.



Solution:

In ΔDAO and ΔCBO, we have:

∠AOD = ∠BOC (Vertically opposite angles)

∠OAD = ∠OBC = 90° (∵ DA and CB are perpendiculars)

DA = CB (Given)

So, by the AAS congruence rule, we obtain:

ΔDAO ≅ ΔCBO

⇒ AO = BO (By CPCT)

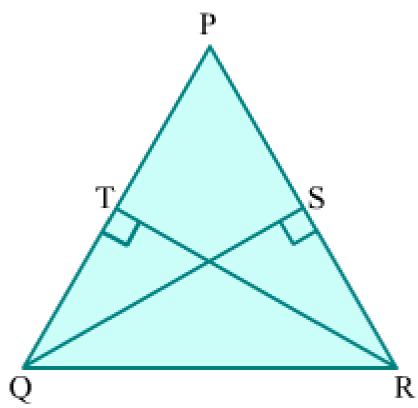
Now, AO + BO = AB

Thus, line segment CD bisects line segment AB at point O.

Example 3: If two altitudes of a triangle are equal then show that the triangle is isosceles.

Solution:

Let us consider ΔPQR such that RT and QS are the equal altitudes drawn to the sides PQ and PR respectively.



In ΔPTR and ΔPSQ, we have

∠PTR = ∠PSQ = 90°

∠TPR = ∠SPQ   (Common angle)

RT = QS (Given)

So, by the AAS congruence rule, we obtain:

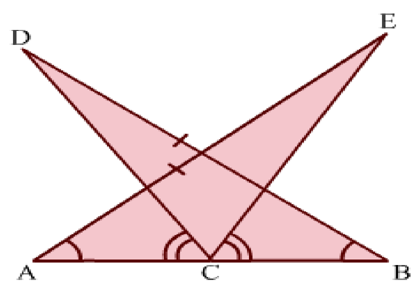
ΔPTR ≅ ΔPSQ

∴ PR = PQ (By CPCT)

Thus, ΔPQR is an isosceles triangle.

Hard

Example 1: In the given figure, ∠DBC = ∠EAC, ∠DCA = ∠ECB and BD = AE. Prove that BC = AC.



Solution:

It is given that ∠DCA = ∠ECB.

On adding ∠ECD to both sides of the above equation, we get:

∠DCA + ∠ECD = ∠ECB + ∠ECD

⇒ ∠ECA = ∠DCB… (1)

In ΔBDC and ΔAEC, we have:

∠DCB = ∠ECA (From equation 1)

∠DBC = ∠EAC (Given)

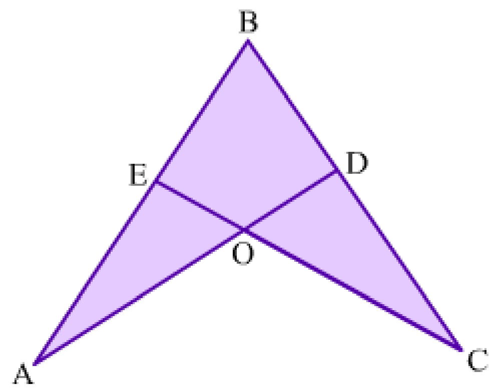
BD = AE (Given)

So, by the AAS congruence rule, we have:

ΔBDC ≅ ΔAEC

⇒ BC = AC (By CPCT)

Example 2: In the figure, ∠BAD = ∠BCE and AB = CB. Prove that ΔABD ≅ΔCBE.



Solution:

In ΔAOE and ΔCOD, we have:

∠EAO = ∠DCO (Given)

∠AOE = ∠COD (Vertically opposite angles)

∴ ∠AEO = ∠CDO … (1)   [By the angle sum property]

Now,

∠AEO + ∠OEB = 180° (Linear pair)

∠CDO + ∠ODB = 180° (Linear pair)

∴ ∠AEO + ∠OEB = ∠CDO + ∠ODB

⇒ ∠AEO + ∠OEB = ∠AEO + ∠ODB (Using equation (1))

⇒ ∠OEB = ∠ODB

⇒ ∠CEB = ∠ADB … (2)

In ΔABD and ΔCBE, we have:

∠BAD = ∠BCE (Given)

∠ADB = ∠CEB (From equation 2)

AB = CB (Given)

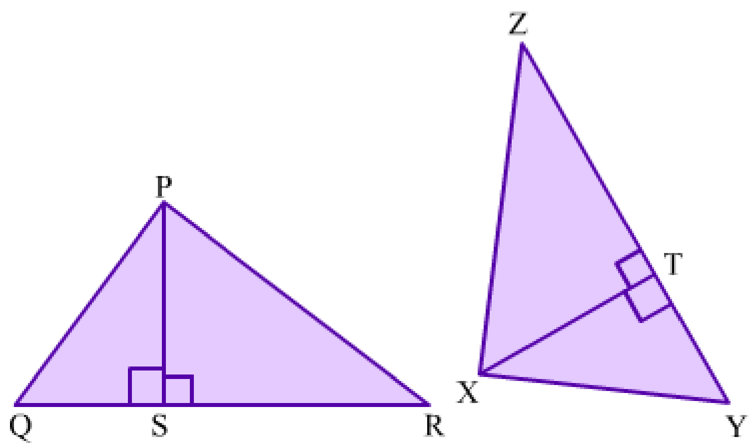
So, by the AAS congruence rule, we have:

ΔABD ≅ ΔCBE

Example 3: Prove that if two triangles are congruent then their corresponding altitudes are equal.

Solution:

Let PS and XT be corresponding altitudes of congruent triangles ΔPQR and ΔXYZ.



We have

ΔPQR ≅ ΔXYZ

∴ PQ = XY   ...(1)   (By CPCT)

∠PQR = ∠XYZ   ...(2)   (By CPCT)

In ΔPQS and ΔXYT, we have

∠PSQ = ∠XTY = 90°

∠PQR = ∠XYZ   [From (2)]

PQ = XY   [From (1)]

So, by the AAS congruence rule, we have:

ΔPQS ≅ ΔXYT

∴ PS = XT (By CPCT)

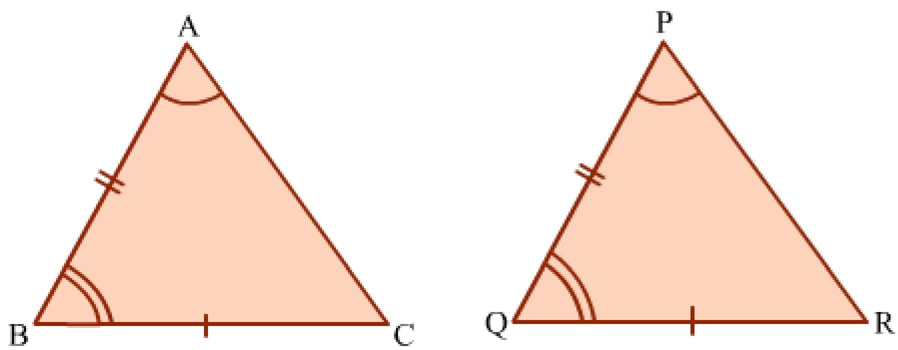
Thus, the corresponding altitudes of congruent triangles are equal.

Difference between AAS and ASA

The AAS and ASA criterions are both used for checking or proving congruency of triangles, but there is a subtle difference between the two.

- The ASA rule is applied when two interior angles of a triangle and the side between them are considered.
- The AAS rule is applied when two interior angles of a triangle and any side other than the one between them are considered.

Take, for example, the given ΔABC and ΔPQR.



To prove ΔABC ≅ ΔPQR by the ASA rule, we need:

∠BAC = ∠QPR, ∠ABC = ∠PQR and AB = PQ

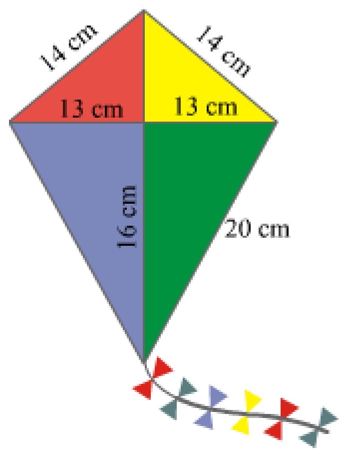
To prove ΔABC ≅ ΔPQR by the AAS rule, we need:

∠BAC = ∠QPR, ∠ABC = ∠PQR and BC = QR

SSS Congruence Rule

Relation between the Congruency of Triangles and Their Sides

Consider the **kite** shown below.



It can be seen that the red and yellow coloured triangles have equal sides. On the basis of this information, can we say that the two triangles are congruent? Or, to rephrase the question, do the sides of triangles determine the congruency of the triangles? Yes, they do, and this is why we have the SSS (Side-Side-Side) congruence rule.

In this lesson, we will discuss the SSS congruence rule and its proof. We will also crack some problems based on it.

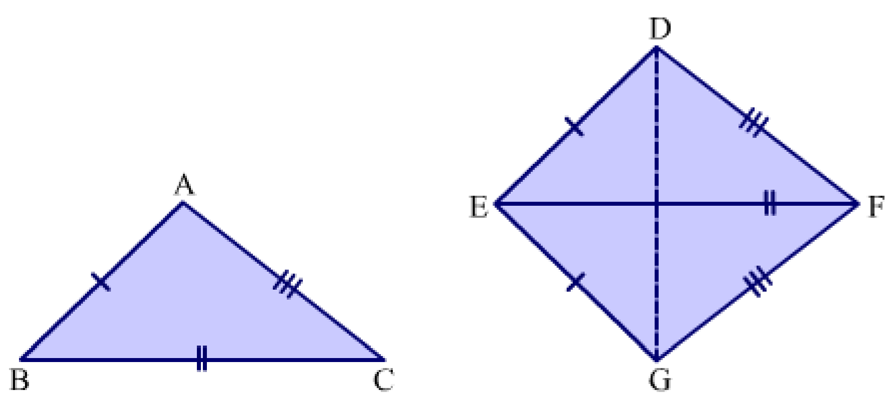
SSS Congruence Rule

Statement: Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

Given: ΔABC and ΔDEF such that AB = DE, BC = EF and AC = DF

To prove: ΔABC ≅ ΔDEF

***Construction*:** Suppose BC and EF are the longest sides of the two triangles. Draw EG such that ∠GEF = ∠ABC and GE = AB. Join point G to points F and D.



(i) **(ii)**

Proof: In ΔABC and ΔGEF, we have:

BC = EF (Given)

AB = GE (By construction)

∠ABC = ∠GEF (By construction)

So, by the SAS congruence rule, we have:

ΔABC ≅ ΔGEF

⇒ ∠BAC = ∠EGF and AC = GF (By CPCT)

Now, AB = DE and AB = GE

⇒ DE = GE … (1)

Similarly, AC = DF and AC = GF

⇒ DF = GF … (2)

In ΔDEG, we have:

DE = GE (From equation 1)

⇒ ∠EDG = ∠EGD … (3)

In ΔDFG, we have:

DF = GF (From equation 2)

⇒ ∠FDG = ∠FGD… (4)

On adding equations 3 and 4, we get:

∠EDG + ∠FDG = ∠EGD + ∠FGD

⇒ ∠EDF = ∠EGF

We know that ∠EGF = ∠BAC (Proved above)

∴ ∠BAC = ∠EDF … (5)

Thus, in ΔABC and ΔDEF, we have:

AB = DE (Given)

∠BAC = ∠EDF (From equation 5)

AC = DF (Given)

So, by the SAS congruence rule, we obtain:

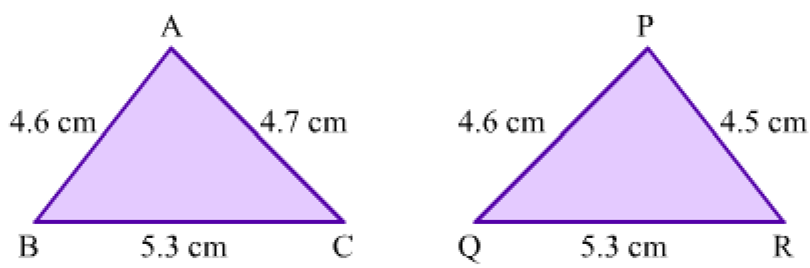
ΔABC ≅ ΔDEF

Hence, the SSS congruence rule holds true.

Solved Examples

Easy

Example 1: Are the following triangles congruent?



Solution:

In ΔABC and ΔPQR, we have:

AB = PQ = 4.6 cm

BC = QR = 5.3 cm

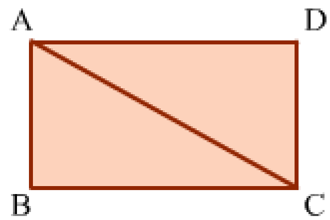
But AC ≠ PR

Therefore, ΔABC and ΔPQR are not congruent.

Example 2: ABCD is a rectangle with AC as one of its diagonals. Prove that the triangles formed on the two sides of diagonal AC are congruent.

Solution:

The required rectangle ABCD with AC as its diagonal can be drawn as is shown.



In ΔABC and ΔCDA, we have:

AB = CD (∵ Opposite sides of a rectangle are equal)

BC = DA (∵ Opposite sides of a rectangle are equal)

CA = AC (Common side)

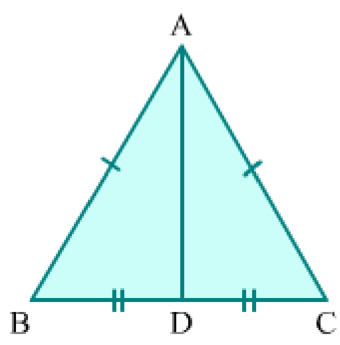
Therefore, by the SSS congruence rule, we have:

ΔABC ≅ ΔCDA

Thus, the triangles formed on the two sides of diagonal AC are congruent.

Medium

Example 1: The given ΔABC is isosceles with AB = AC. AD is a median of the triangle. Prove that AD is perpendicular to BC.



Solution:

In ΔABD and ΔACD, we have:

AB = AC (Given)

BD = DC (∵ D is the midpoint of BC)

AD = AD (Common side)

Therefore, by the SSS congruence rule, we obtain:

$\Delta ABD \cong \Delta ACD$

⇒ ∠ADB = ∠ADC (By CPCT)

Also, ∠ADB and ∠ADC form a linear pair.

So, ∠ADB + ∠ADC = 180°

⇒ ∠ADB + ∠ADB = 180° (∵ ∠ADB = ∠ADC)

⇒ 2∠ADB = 180°

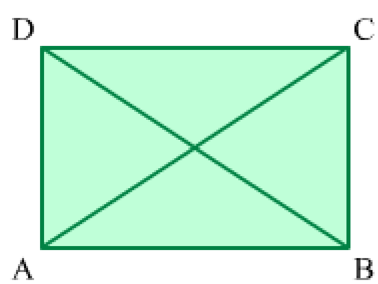
⇒ ∠ADB = 90°

Thus, ∠ADB = ∠ADC = 90°, which means that AD is perpendicular to BC.

Example 2: ABCD is a parallelogram. If the diagonals of ABCD are equal, then find the measure of ∠ABC.

Solution:

The given parallelogram ABCD with equal diagonals AC and BD is shown below.



In parallelogram ABCD, we have:

AB = CD and AD = BC   (∵ Opposite sides of a parallelogram are equal)

In ΔADB and ΔBCA, we have:

AD = BC (Proved above)

BD = AC (Given)

BA = AB (Common side)

So, by the SSS congruence rule, we have:

ΔADB ≅ ΔBCA

⇒ ∠BAD **=** ∠ABC ... (1)   [By CPCT]

Now, AD is parallel to BC and the transversal AB intersects them at A and B respectively.

We know that the sum of the interior angles on the same side of a transversal is supplementary.

∴ ∠BAD + ∠ABC = 180°

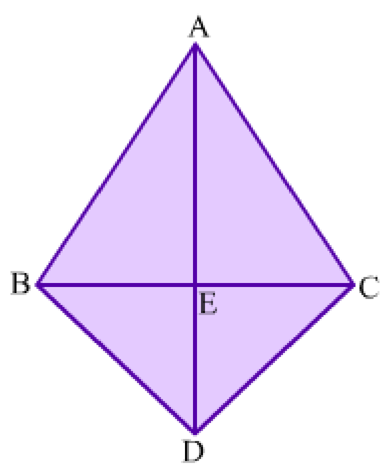
⇒ ∠ABC + ∠ABC = 180°   (By equation 1)

⇒ 2∠ABC = 180°

⇒ ∠ABC = 90°

Hard

Example 1: In the given figure, ΔABC and ΔDBC are isosceles with AB = AC and DB = DC. Prove that AD is the perpendicular bisector of BC.



Solution:

In ΔABD and ΔACD, we have:

AB = AC (Given)

DB = DC (Given)

AD = AD (Common side)

So, by the SSS congruence rule, we have:

$\Delta ABD \cong \Delta ACD$

$\Rightarrow \angle BAE = \angle CAE$ ... (1) [By CPCT]

In ΔBAE and ΔCAE, we have:

AB = AC (Given)

$\angle BAE = \angle CAE$ (From equation 1)

AE = AE (Common side)

So, by the SAS congruence rule, we have:

$\Delta BAE \cong \Delta CAE$

$\Rightarrow$ BE = CE and $\angle BEA = \angle CEA$ (By CPCT)

We know that $\angle BEA + \angle CEA = 180°$ as they form a linear pair.

So, $2\angle BEA = 180°$ (Proved above that $\angle BEA = \angle CEA$)

$\Rightarrow \angle BEA = 90°$

Therefore, $\angle BEA = \angle CEA = 90°$

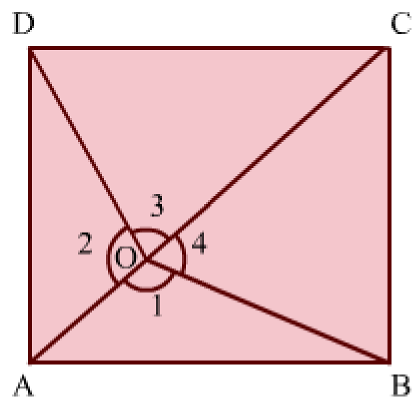
Since BE = CE and $\angle BEA = \angle CEA = 90°$, AD is the perpendicular bisector of BC.

Example 2: O is a point inside a square ABCD such that it is at an equal distance from points B and D. Prove that points A, O and C are collinear.

Solution:

The square with the given specifications is drawn as is shown.

Construction**:** Join point O to the vertices of the square.



In ΔAOD and ΔAOB, we have:

AD = AB   (Sides of a square)

AO = AO (Common side)

OD = OB (Given)

So, by the SSS congruence rule, we have:

ΔAOD ≅ ΔAOB

⇒ ∠1 = ∠2… (1)   [By CPCT]

Similarly, ΔDOC ≅ ΔBOC

⇒ ∠3 = ∠4 … (2)   [By CPCT]

We know that:

∠1 + ∠2 + ∠ 3 + ∠4 = 360°

⇒ 2∠2 + 2∠3 = 360° (From equations 1 and 2)

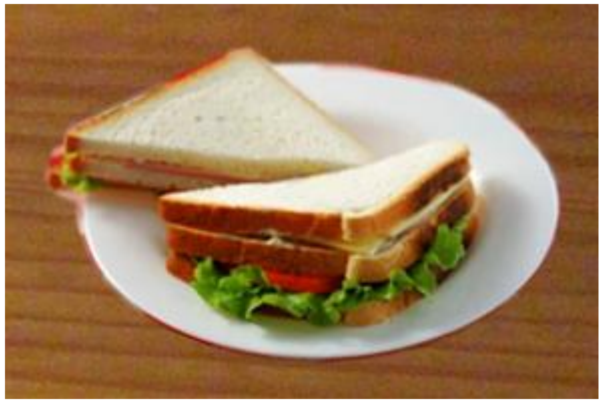
⇒ ∠2 + ∠3 = 180°

Thus, ∠2 and ∠3 form a linear pair. Therefore, AOC is a line; in other words, points A, O and C are collinear.

RHS Congruence Rule

RHS Congruence Rule

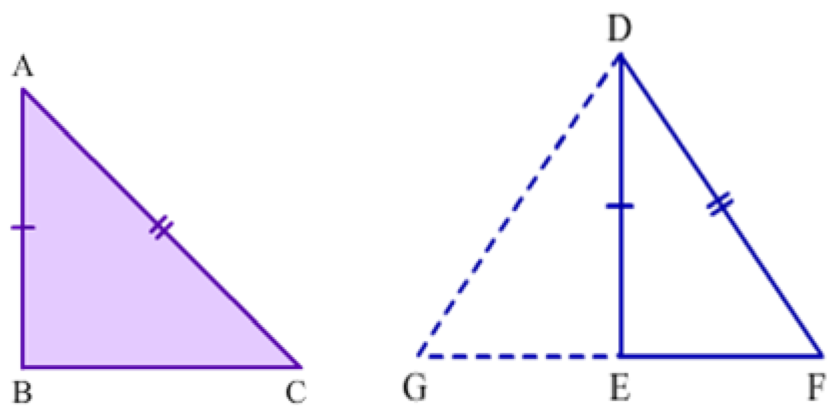
Right angles are all around us. For example, while building houses, the walls are kept at a right angle to the horizontal. Different square and rectangular figures surround us and all of them consist of right angles. The following figure also shows right angles.



In the figure, the pieces of bread resemble right-angled triangles. They also seem to be congruent. Right-angled triangles are special and their congruency is checked by a special congruence rule known as the RHS (Right-Hypotenuse-Side) rule.

We will study the RHS congruence rule in this lesson and solve some examples to familiarize ourselves with the concept.

RHS congruence theorem: **Two right-angled triangles are congruent if the hypotenuse and a side of one triangle are equal to the hypotenuse and the corresponding side of the other triangle.**



Given: Two right-angled triangle ABC and DEF such that ∠B = ∠E = 90°; Hypotenuse AC = Hypotenuse DF and AB = DE.

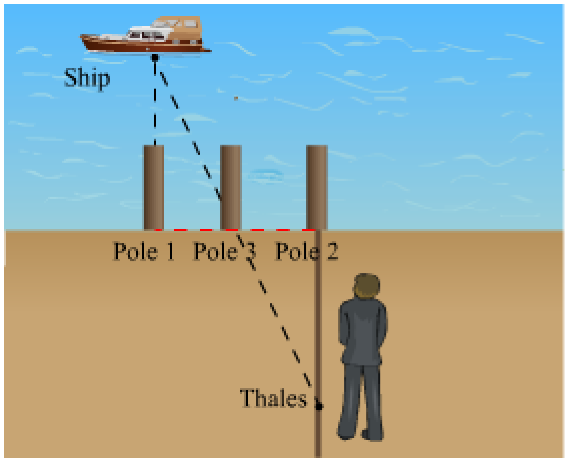
To prove: ΔABC ≅ ΔDEF.

Construction: Produce FE to G so that EG = BC and join DG.

Proof:

In triangles ABC and DEF,
AB = DE (Given)
BC = EG (By construction)
∠ABC= ∠DEF (Each equal to 90°)
Thus, by SAS congruence criterion,
ΔABC ≅ ΔDEG
⇒ ∠ACB = ∠ DGE and AC = DG (CPCT)
Given, AC = DF
∴ DG = AC = DF
In ΔDGF, we have
DG = DF
∠G = ∠F (Angles opposite to equal sides are equal)
In ΔDEF and ΔDEG,
∠G = ∠F (Proved)
∠DEG = ∠DEF (Both equal to 90°)
Thus, ∠GDE = 180° – (∠G + ∠DEG) = 180° – (∠F +∠DEF) = ∠FDE
In ΔDEG and ΔDEF,
DG = DF (Proved)
DE = DE (Common)
∠GDE = ∠FDE (Proved)
Thus, by SAS congruence criterion
ΔDEG ≅ ΔDEF
But, we have ΔABC ≅ ΔDEG
Thus, ΔABC ≅ ΔDEF
Whiz Kid

RHS congruence rule was used by the famous ancient Greek mathematician Thales to calculate the distance of a ship anchored at sea from the shore. For this, he stuck three poles on the shore such that the first one was directly in front of the ship, the second was at some distance from the first pole and the third was exactly between the other two poles. He then walked backward along a line from the second pole perpendicular to the shore until the middle pole and the ship were in the same line of sight. Then, he marked his position. This is shown in the following figure.



It can be seen that the triangle formed on the sea is congruent to the triangle formed on the shore by the RHS rule. So, the distance between the ship and the shore is equal to the distance between the second pole and the spot where Thales stands.

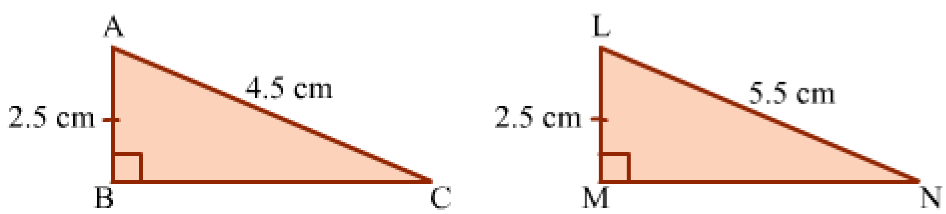
Solved Examples

Easy

Example 1: ΔABC and ΔLMN are right-angled at ∠ABC and ∠LMN respectively. In ΔABC, AB = 2.5 cm and AC = 4.5 cm. In ΔLMN, LN = 5.5 cm and LM = 2.5 cm. Examine whether the two triangles are congruent.

Solution:

On the basis of the given information, the two triangles can be drawn as is shown.



In ΔABC and ΔLMN, we have:

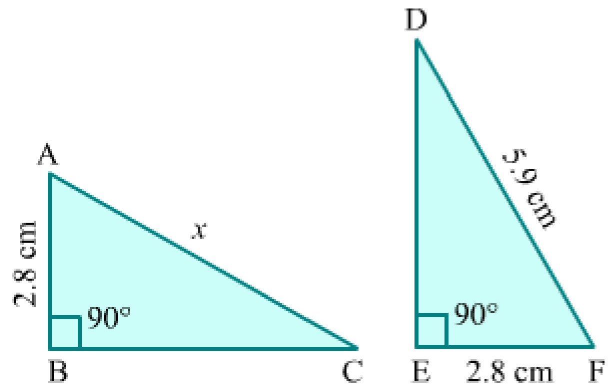
∠ABC = ∠LMN (Right angles)

AB = LM = 2.5 cm (Given)

But AC ≠ LN

Hence, ΔABC and ΔLMN are not congruent.

Example 2: Find the value of *x* if the shown triangles ABC and DEF are congruent.



Solution:

It is given that ΔABC ≅ ΔDEF.

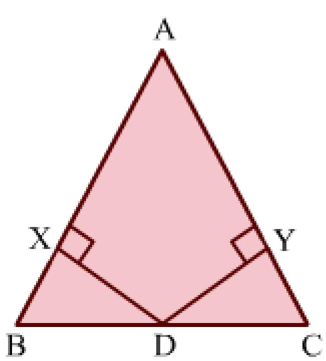
When two triangles are congruent, their corresponding sides are equal.

∴ AC = DF = 5.9 cm

Thus, the value of *x* is 5.9 cm.

Medium

Example 1: In the given ΔABC, D is the midpoint of side BC. The perpendiculars DX and DY drawn from point D to sides AB and BC respectively are of the same length. Prove that DX and DY make the same angle with BC.



Solution:

On comparing ΔDXB and ΔDYC, we get:

DX = DY (Given)

∠DXB = ∠DYC = 90° (∵ DX and DY are perpendiculars)

BD = CD (∵ D is the midpoint of BC)

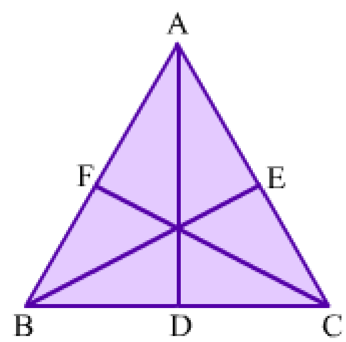
Thus, by the RHS congruence rule, we have:

ΔDXB ≅ ΔDYC

⇒ ∠BDX = ∠CDY (By CPCT)

Thus, the perpendiculars DX and DY make the same angle with side BC.

Example 2: In the given ΔABC, AD, BE and CF are the altitudes. If the three altitudes are equal, then prove that the triangle is equilateral.



Solution:

In ΔBEC and ΔCFB, we have:

BC = CB (Common side)

BE = CF (Given)

∠BEC = ∠CFB = 90° (∵ BE and CF are altitudes)

So, by the RHS congruence rule, we obtain:

ΔBEC ≅ ΔCFB

⇒ ∠BCE = ∠CBF (By CPCT)

⇒ ∠CBA = ∠BCA

⇒ AC = AB … (1) [∵ Sides opposite equal angles are equal]

Similarly, we can prove that ΔADB ≅ ΔBEA.

⇒ ∠DBA = ∠BAE (By CPCT)

⇒ ∠CBA = ∠BAC

⇒ AC = BC … (2) [∵ Sides opposite equal angles are equal]

From equations (1) and (2), we get:

AB = BC = AC

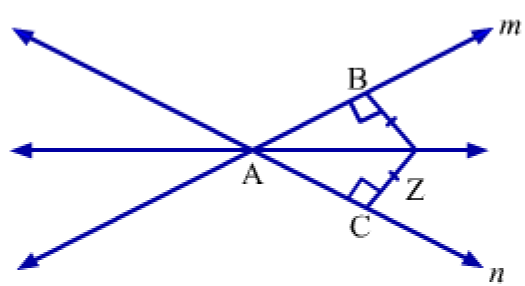
Hence, ΔABC is equilateral.

Hard

Example 1: If Z is a point equidistant from two lines *m* and *n* intersecting at point A, then prove that AZ bisects the angle between *m* and *n*.

Solution:

The following figure can be drawn as per the given information.



***Construction*:** ZB and ZC are perpendiculars drawn from point Z to lines *m* and *n* respectively.

Since Z is equidistant from *m* and *n*, we have:

ZB = ZC

In ΔZBA and ΔZCA, we have:

ZB = ZC (Proved above)

∠ZBA = ∠ZCA = 90° (∵ ZB and ZC are perpendiculars)

ZA = ZA (Common side)

So, by the RHS congruence rule, we have:

ΔZBA ≅ ΔZCA

⇒ ∠ZAB = ∠ZAC (By CPCT)

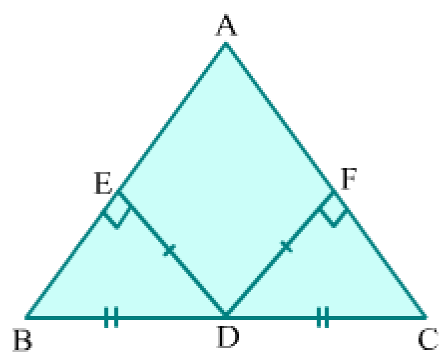
Now, ∠ZAB + ∠ZAC = ∠BAC

Therefore, AZ bisects the angle between lines *m* and *n*.

Example 2: In a ΔABC, BD = DC. If the perpendiculars from point D to sides AB and AC are equal, then prove that AB = AC.

Solution:

The triangle with the given specifications is drawn below.



In ΔABC, D is the midpoint of BC. Also, DE and DF are the perpendiculars from D to AB and AC respectively.

In ΔDEB and ΔDFC, we have:

∠DEB = ∠DFC = 90° (∵ DE and DF are perpendiculars)

DB = DC (Given)

DE = DF (Given)

So, by the RHS congruence rule, we obtain:

ΔDEB ≅ ΔDFC

⇒ ∠DBE = ∠DCF (By CPCT)

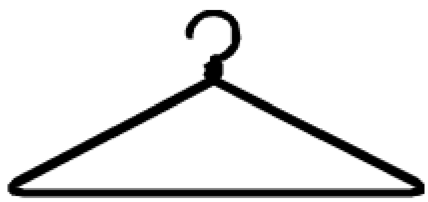
⇒ ∠CBA = ∠BCA

⇒ AC = AB (As sides opposite to equal angles are equal)

Angles Opposite to Equal Sides of an Isosceles Triangle are Equal

Isosceles Triangles around Us

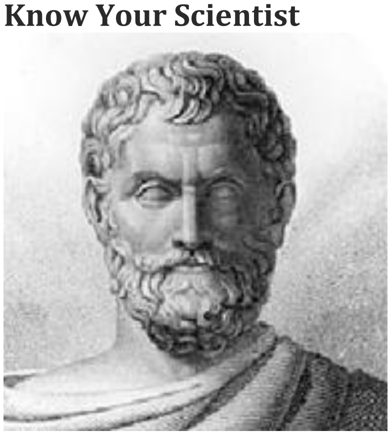
You must have seen triangular clothes hangers or coat hangers such as the one shown.



What do you observe about the sides of the hanger? The base is clearly the longest side, while the other two sides are equal. We know that a triangle with any two sides equal is called an isosceles triangle. So, the hanger is a real-life example of an isosceles triangle. Another example of an isosceles triangle is the roof of a hut. Yet another example is a diagonally cut slice of bread.

You can see that the angles opposite the equal sides of the clothes hanger are equal. In this lesson, we will discuss about the same, i.e., angles opposite equal sides of an isosceles triangle. We will also solve some problems based on this concept.

Know Your Scientist



Thales (624 BC−546 BC) is believed to be the first philosopher in the Greek tradition. A great mathematician and scientist, he is regarded as 'the father of science'. Thales is credited for giving the first known mathematical proof, i.e., 'a circle is bisected by its diameter'. He was also the first to prove the theorem: 'any angle inscribed in a semicircle is a right angle'. For this reason, this theorem is called Thales Theorem.

Thales also generalized the concept of equality of the base angles of an isosceles triangle. He also propounded a few general notions such as 'all straight angles are equal', 'equals subtracted from equals are equal' and 'equals added to equals are equal'. These theorems have helped in the evolution of modern mathematics.

Did You Know?

The word ‘isosceles’ is a combination of the Greek words ‘isos’meaning‘equal’ and ‘skelos’ meaning‘leg’. So, a triangle having two equal legs is called an isosceles triangle.

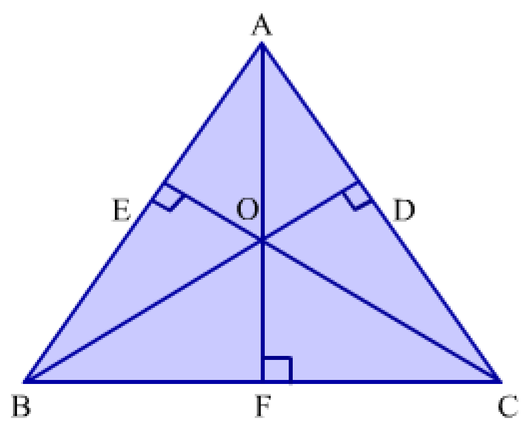
Know More

- The altitude to the base of an isosceles triangle bisects the vertex angle.
- The altitude to the base of an isosceles triangle bisects the base.
- When the equal sides are at right angle, the triangle is called a ‘right isosceles triangle’.

Whiz Kid

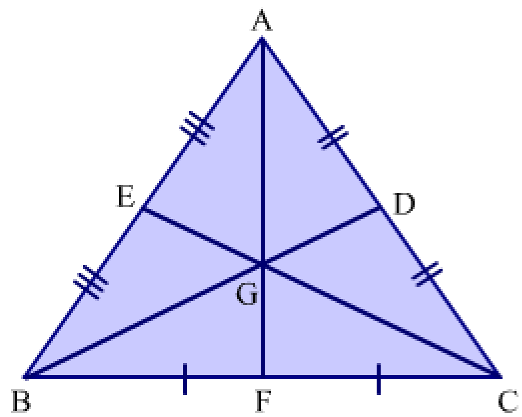
In an isosceles triangle, the orthocentre, the centroid, the incentre and the circumcentre, all lie on the median to the base.

Orthocentre: It is the point where the altitudes of the three sides of a triangle intersect.



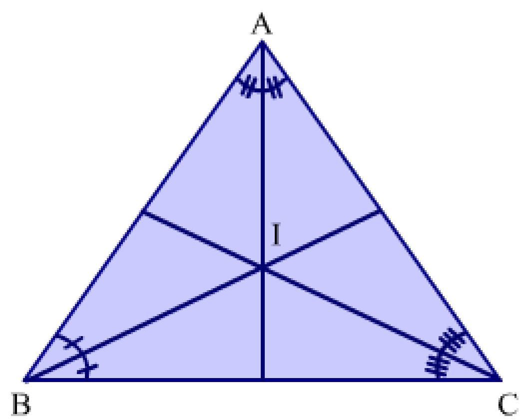
In ΔABC, O is the orthocentre.

Centroid: It is the point where the medians of the three sides of a triangle intersect.



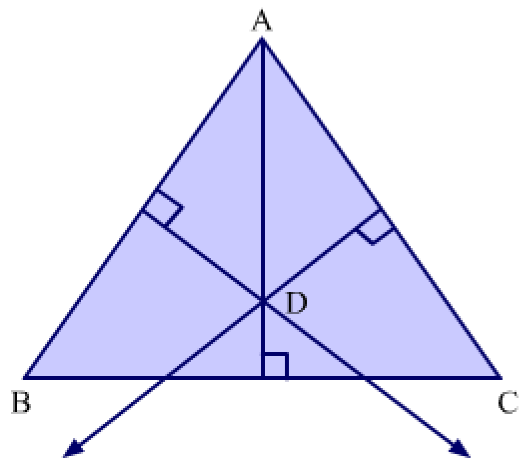
In ΔABC, G is the centroid.

Incentre: It is the point where the interior angle bisectors of a triangle intersect.



In ΔABC, I is the incentre.

Circumcentre: It is the point where the perpendicular bisectors of the three sides of a triangle intersect.

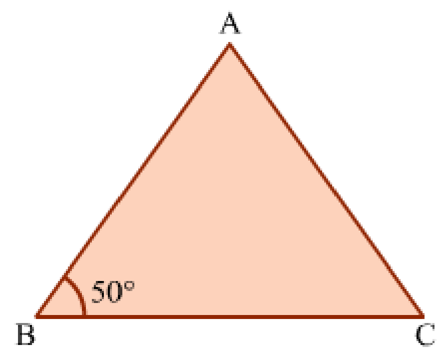


In ΔABC, D is the circumcentre.

Solved Examples

Easy

Example 1: In the given ΔABC, AB = AC. What is the measure of ∠BAC?



Solution:

ΔABC is an isosceles triangle with AB = AC.

By the property of isosceles triangles, we obtain:

∠ABC = ∠ACB

⇒ ∠ACB = 50°

By the angle sum property, we have:

∠ABC + ∠ACB + ∠BAC = 180°

⇒ 50° + 50° + ∠BAC = 180°

⇒ ∠BAC = 180° − 100°

⇒ ∠BAC = 80°

Example 2: Ram lives in a triangular tree house. The measure of each base angle is the same and the length of the base is 4 m. He wants to cover the slant sides with metal sheets. If the boundary of the triangular structure is 10 m, then what length of sheet does Ram need to cover the slant sides (irrespective of the width of the sheet).

Solution:

It is given that the base angles are equal; so, the triangular structure is isosceles. Hence, the slant sides are equal.

Now, we know that:

Base = 4 m

Boundary of the triangular structure = Perimeter of the isosceles triangle = 10 m

⇒ 10 m = 4 m + 2 × (Measure of each slant side)

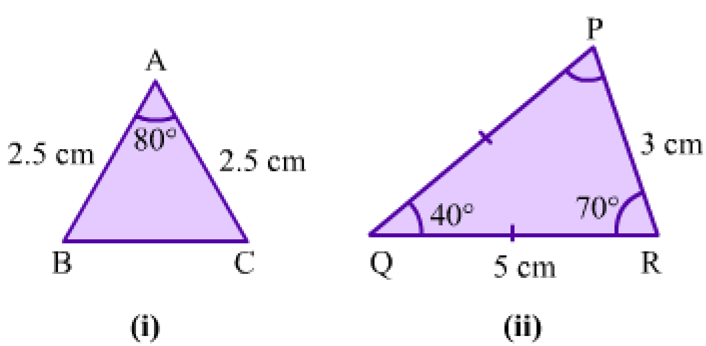
⇒ 2 × (Measure of each slant side) = 10 m − 4 m

⇒ 2 × (Measure of each slant side) = 6 m

Hence, Ram needs 6 m of metal sheet to cover the slant sides of the triangular structure.

Medium

Example 1: Find the missing angles in the following triangles.



Solution:

1. In ΔABC, AB = AC = 2.5 cm

   Since the angles opposite equal sides of a triangle are equal, we obtain:

   ∠ABC = ∠ACB

   Let ∠ABC = ∠ACB = x

   By the angle sum property of triangles, we have:

   ∠ABC + ∠ACB + ∠BAC = 180°

   ⇒ $x + x + 80° = 180°$

   ⇒ $2x = 100°$

   ⇒ $x = 50°$

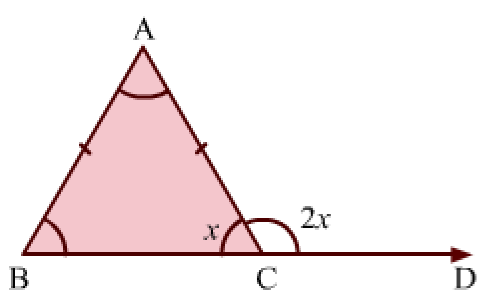
   Thus, ∠ABC = ∠ACB = 50°

2. In ΔPQR, PQ = QR = 5 cm and PR = 3 cm

   Since PQ = QR, we obtain

   ∠QPR = ∠PRQ = 70°

Example 2: The shown ΔABC is isosceles with AB = AC. Find the measures of ∠BAC, ∠ABC and ∠ACB.



Solution:

∠ACB and exterior angle ∠ACD form a linear pair.

∴ ∠ACB + ∠ACD = 180°

⇒ $x + 2x = 180°$

⇒ $3x = 180°$

⇒ ∴ $x = 60°$

So, ∠ACB = 60° and ∠ACD = 120°

ΔABC is isosceles.

∴ ∠ABC = ∠ACB = 60° (∵ Angles opposite equal sides are equal)

Now, by the exterior angle property, we have:

∠ACD = ∠BAC + ∠ABC

⇒ 120° = ∠BAC + 60°

⇒ ∠BAC = 60°

Thus, ∠BAC = ∠ABC = ∠ACB = 60°.

Example 3: Prove that in an isosceles triangle, the angle bisector of the apex angle is the perpendicular bisector of the base.

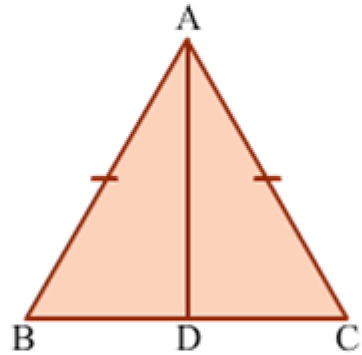
Solution:

Given: ABC is a triangle in which AB = AC and apex angle ∠A.

To Prove: AD is perpendicular bisector of BC and BD = DC.

Construction: Draw an angle bisector AD from A on BC.

Proof:



In ΔABD and ΔACD,

AB = *AC* (Given)

AD = *AD* (Common side)

∠BAD = ∠CAD (AD is a bisector of ∠A)

So, by SAS congruence criterion,

Δ ADB ≅ Δ ADC

⇒BD = DC and ∠ADB =∠ADC (CPCT)

Now,

∠ ADB + ∠ADC= 180° (Linear Pair)

⇒∠ ADB + ∠ ADB = 180°

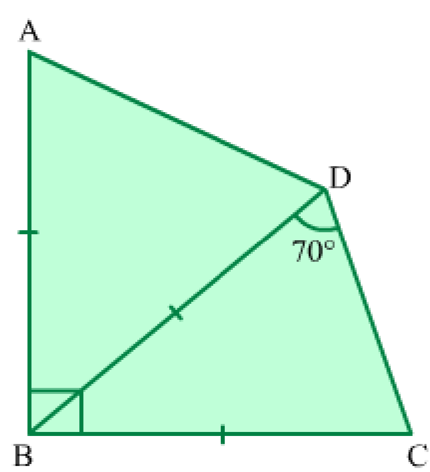
⇒ 2∠ ADB = 180°

$\Rightarrow \angle ADB = \frac{180^\circ}{2} = 90^\circ$

Thus, AD is the perpendicular bisector of BC.

Hard

Example 1: ΔABD is isosceles (figure not drawn to scale) with AB = BD, while ΔBCD is isosceles with BC = BD. Also, ∠BDC measures 70° and ∠ABC measures 90°. What is the measure of ∠ADC?



Solution:

In ΔBCD, BC = BD.

∴ ∠BCD = ∠BDC = 70º (∵ Angles opposite equal sides are equal)

In ΔBCD, by the angle sum property, we obtain:

∠BCD + ∠BDC + ∠CBD = 180º

⇒ 70º + 70º + ∠CBD = 180º

⇒ 140º + ∠CBD = 180º

⇒ ∠CBD = 40º

It is given that ∠ABC = 90º.

⇒ ∠ABD + ∠CBD = 90º

⇒ ∠ABD + 40º = 90º

⇒ ∠ABD = 50º

In ΔABD, we have:

AB = BD

∴ ∠BAD = ∠BDA = x (∵ Angles opposite equal sides are equal)

In ΔABD, by the angle sum property, we obtain:

∠ABD + ∠BAD + ∠BDA = 180º

⇒ 50º + x + x = 180º

⇒ $2x$ = 130º

⇒ x = 65º

So, ∠BAD = ∠BDA = 65º

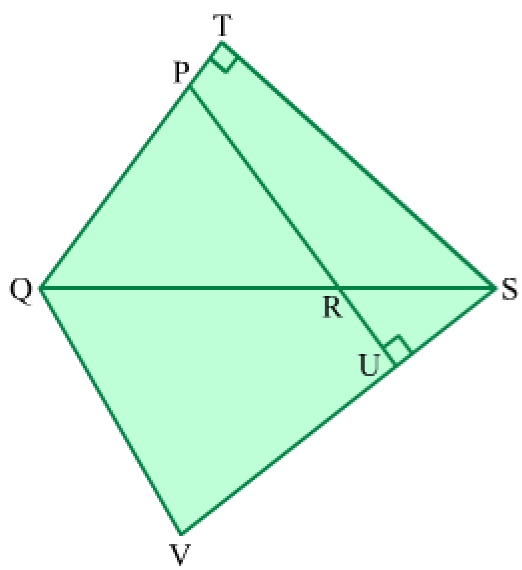
Now,

$\angle ADC = \angle BDA + \angle CDB$

$\Rightarrow \angle ADC = 65^{\circ} + 70^{\circ}$

$\Rightarrow \angle ADC = 135^{\circ}$

Example 2: In the given figure, ΔPQR is isosceles with PQ = PR. Side SU is extended up to point V such that ST = SV. If ∠QTS = ∠RUS = 90°, prove that QS bisects ∠TSU and hence show that ΔQTS ≅ΔQVS.



Solution:

In ΔPQR, we have:

PQ = PR (Given)

$\therefore \angle PQR = \angle PRQ$ (∵ Angles opposite equal sides are equal)

It is given that ∠QTS = 90°.

Using the angle sum property in ΔQTS, we obtain:

$\angle TQS + \angle TSQ + \angle QTS = 180^{\circ}$

$\Rightarrow \angle TQS + \angle TSQ + 90^{\circ} = 180^{\circ}$

$\Rightarrow \angle TSQ = 90^{\circ} - \angle TQS$ ... (1)

It is given that ∠RUS = 90°.

Using the angle sum property in ΔRUS, we obtain:

∠SRU + ∠RUS + ∠RSU = 180°

⇒ ∠SRU + 90° + ∠RSU = 180°

⇒ ∠RSU = 90° − ∠SRU … (2)

∠SRU = ∠PRQ (Vertically opposite angles)

Also, ∠PQR = ∠PRQ

∴ ∠SRU = ∠PQR … (3)

Now, from equations (2) and (3), we get:

∠RSU = 90° − ∠PQR

⇒ ∠RSU = 90° − ∠TQS … (4) [∵ ∠PQR = ∠TQS]

From equations (1) and (4), we get:

∠TSQ = ∠RSU

Hence, QS bisects ∠TSU.

Now, consider ΔQTS and ΔQVS.

QS = QS (Common side)

∠TSQ = ∠QSV (∵ ∠TSQ = ∠RSU and ∠RSU = ∠QSV)

ST = SV (Given)

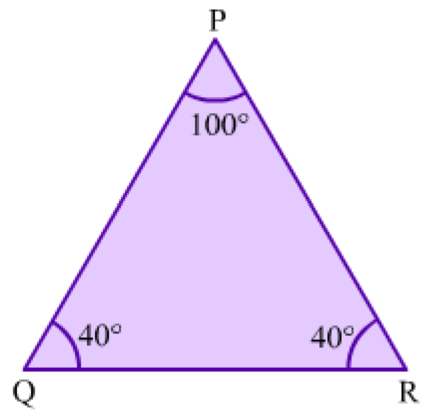
Thus, by the SAS congruency rule, we get:

ΔQTS ≅ ΔQVS

Sides Opposite to Equal Angles of a Triangle are Equal

Observing the Equal Angles and the Sides Opposite to Them in an Isosceles Triangle

Consider the following ΔPQR.



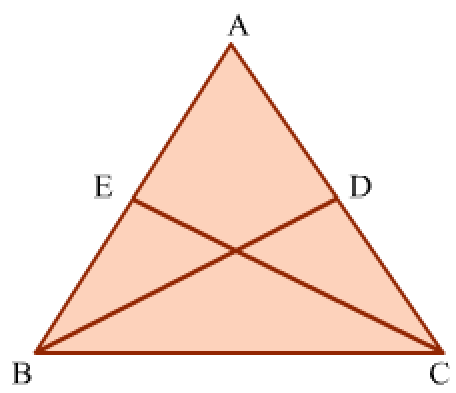
Is ΔPQR isosceles? We know that if two sides of a triangle are equal (or congruent), then the triangle is isosceles. However, in ΔPQR, two angles are equal (or congruent). We have studied that the angles opposite to equal (or congruent) sides of an isosceles triangle are equal (or congruent). Is the converse of this property also true?

In this lesson, we will study about the equality of sides opposite equal angles in an isosceles triangle. We will also solve some examples related to this concept.

Whiz Kid

In an isosceles triangle, the medians drawn from the base vertices to the opposite sides are of equal length**.**

For example:



The shown ΔABC is isosceles such that AB = AC. BD and CE are the respective medians from vertices B and C to sides AC and AB. Therefore, BD = CE.

Solved Examples

Easy

Example 1: In a ΔABC, $\angle BAC = 2x$ and $\angle ABC = \angle ACB = x$. Find the value of x and hence show that AB = AC.

Solution:

It is given that $\angle BAC = 2x$ and $\angle ABC = \angle ACB = x$.

By applying the angle sum property, we obtain:

$\angle BAC + \angle ABC + \angle ACB = 180°$

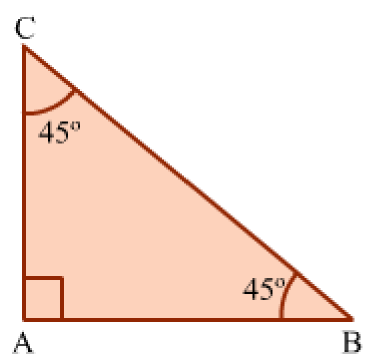
$\Rightarrow 2x + x + x = 180°$

$\Rightarrow 4x = 180°$

$\Rightarrow x = 45°$

So, $\angle BAC = 2x = 2 \times 45° = 90°$ and $\angle ABC = \angle ACB = x = 45°$

The given triangle can be drawn as is shown.

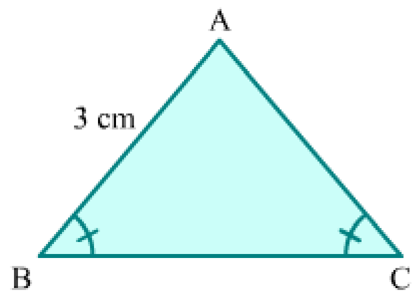


We know that the sides opposite equal angles of a triangle are equal.

$\therefore$ AB = AC

Hence, ΔABC is isosceles with AB = AC.

Example 2: In the given ΔABC, $\angle ABC = \angle ACB$ and the perimeter is 11 cm. Find the length of the base of the triangle.



Solution: It is given that ∠ABC = ∠ACB and AB = 3 cm.

We know that the sides opposite equal angles of a triangle are equal.

∴ AC = AB = 3 cm

Perimeter of ΔABC = AB + BC + AC

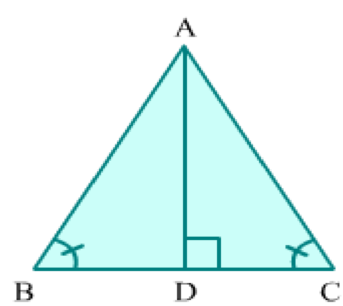
⇒ 11 cm = 3 cm + BC + 3 cm

⇒ BC = 5 cm

Thus, the length of the base of ΔABC is 5 cm.

Medium

Example 1: In the given ΔABC, ∠ABD = ∠ACD and AD is perpendicular to BC. Prove that AD bisects ∠BAC.



Solution:

In ΔABD and ΔACD, we have:

∠ABD = ∠ACD (Given)

∠ADB = ∠ADC = 90°

AB = AC (∵ Sides opposite equal angles are equal)

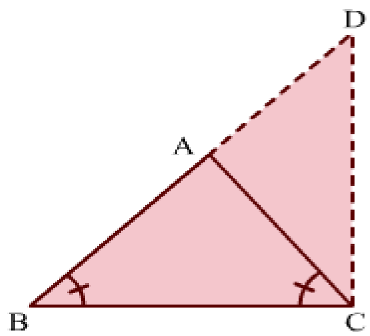
Thus, by the AAS congruence rule, we have:

ΔABD ≅ ΔACD

∴ ∠BAD = ∠CAD (By CPCT)

Thus, AD bisects ∠BAC.

Example 2: In the given ΔABC, ∠ABC = ∠ACB. Side BA is produced up to point D such that AB = AD. Prove that ∠BCD is a right angle.



Solution:

In ΔABC, we have:

∠ABC = ∠ACB … (1)   [Given]

∴ AB = AC   (∵ Sides opposite equal angles are equal)

Now,

AB = AD   (Given)

AD = AC   (∵ AB = AC)

Thus, in ΔADC, we have:

∠ACD = ∠ADC … (2)   (∵ Angles opposite equal sides are equal)

On adding equations (1) and (2), we get:

∠ACB + ∠ACD = ∠ABC + ∠ADC

⇒ ∠BCD = ∠ABC + ∠BDC   (∵ ∠ADC = ∠BDC)

On adding ∠BCD to both sides of the equation, we get:

∠BCD + ∠BCD = ∠ABC + ∠BDC + ∠BCD

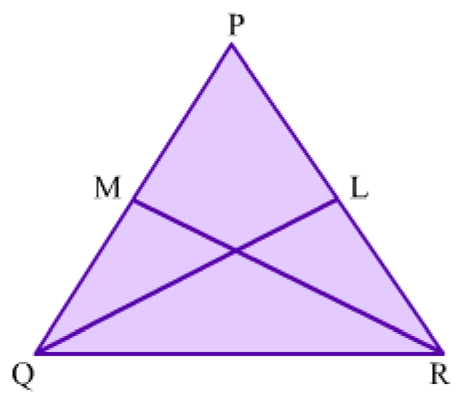
⇒ 2∠BCD = 180°   (By the angle sum property)

⇒ ∠BCD = 90°

Hence, ∠BCD is a right angle.

Hard

Example 1: The shown ΔPQR is isosceles with PQ = PR. QL and RM are the respective medians from vertices Q and R to sides PR and PQ. Prove that the medians have the same length.



Solution:

It is given that PQ = PR.

Also, QL and RM are medians.

∴ PL = LR and PM = MQ

So,

PL + LR = PR

LR + LR = PR

2LR = PR

LR = $\frac{PR}{2}$ ... (1)

Similarly, MQ = $\frac{PQ}{2}$ ... (2)

Since PQ = PR, using equations (1) and (2), we obtain:

LR = MQ ... (3)

In ΔQRL and ΔRQM, we have:

LR = MQ (From (3))

∠LRQ = ∠MQR (∵ Angles opposite equal sides are equal)

QR = RQ (Common side)

∴ ΔQRL ≅ ΔRQM (By the SAS congruence criterion)

⇒ QL = RM (By CPCT)

Thus, the medians QL and RM have the same length.

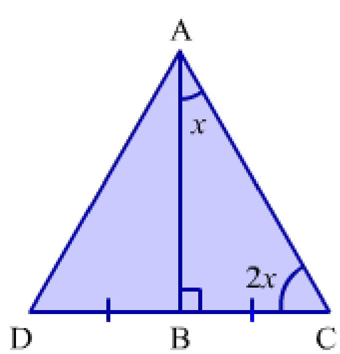
Example 2:

In a right-angled triangle ABC, ∠ACB = 2∠BAC. Prove that AC = 2BC.

Solution:

The given ΔABC can be drawn as is shown.

Construction: Produce CB up to point D such that BD = BC. Join point A to point D.



In ΔABD and ΔABC, we have:

BD = BC (By construction)

∠ABD = ∠ABC = 90°

AB = AB (Common side)

So, by the SAS congruence rule, we obtain:

ΔABD ≅ ΔABC

⇒ AD = AC and ∠DAB = ∠BAC (By CPCT)

Let ∠BAC be x. Then, ∠DAB will also be x.

Now, ∠DAC = ∠DAB + ∠BAC

⇒ ∠DAC = $x + x$

⇒ ∠DAC = $2x$

⇒ ∠DAC = ∠ACB (∵ ∠ACB = 2∠BAC = $2x$)

⇒ DC = AD (∵ Sides opposite equal angles are equal)

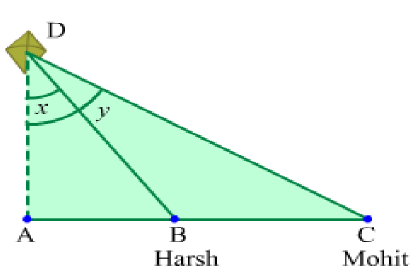
Since BC = DB, we have:

DC = 2BC

⇒ 2BC = AD

⇒ 2BC = AC (∵ We have proved AD = AC)

Observing the Relation between the Angles and Sides of a Triangle



Harsh is flying a kite. The thread of the kite makes angle *x* with the vertical, as is shown in the figure. Sometime later, he gives the thread to his friend Mohit who is standing some distance away from him. At that position, the thread makes angle *y* with the vertical.

The figure clearly shows that angle *y* of ΔACD is greater than angle *x* of ΔABD. Also, side AC is greater than side AB. This shows us that if we increase the length of any side of a triangle, then the angle facing that side also increases.

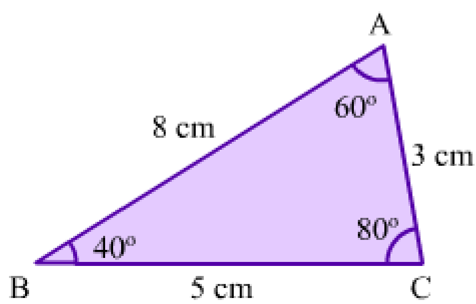
In this lesson, we will discuss the relation between the angles and sides of a triangle. We will then solve some examples relating to the same.

Triangle Inequality Theorem

We know that if two sides of a triangle are equal, then the angles opposite these sides are also equal. Now, what **if all the sides of a triangle are unequal? What can be said about its angles?**The **triangle inequality theorem** describes such a triangle. It states that:

If two sides of a triangle are unequal, then the longer side has the greater angle opposite it.

Consider the given ΔABC.



Let us apply the stated theorem in this triangle.

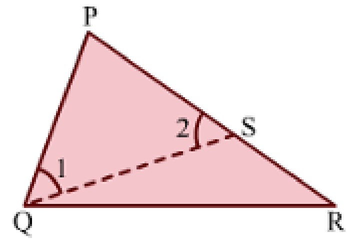
AB = 8 cm is the longest side in ΔABC. Therefore, the angle opposite AB, i.e., ∠BCA is the greatest angle of the triangle. Also, AC = 3 cm is the shortest side in ΔABC. Therefore, the angle opposite AC, i.e., ∠ABC is the smallest angle of the triangle.

Thus, we can conclude that the angle opposite the longer side is greater. Therefore, the theorem holds true.

Proving the Triangle Inequality Theorem

Given: ΔPQR in which PR > PQ

To prove: ∠Q > ∠R



Construction: Mark a point S on PR such that PQ = PS. Join Q to S.
PP

Proof: In ΔPQS, we have:

PQ = PS (By construction)

⇒ ∠1 = ∠2 ... (1) [∵ Angles opposite equal sides are equal]

In ΔQRS, ∠2 is the exterior angle; so, it is greater than the interior opposite angles of ΔQRS.

∴ ∠2 > ∠SRQ

⇒ ∠2 > ∠PRQ … (2) [∵ ∠PRQ = ∠SRQ]

From (1) and (2), we have:

∠1 > ∠PRQ ... (3)

Now, ∠1 is a part of ∠PQR.

So, ∠PQR > ∠1 … (4)

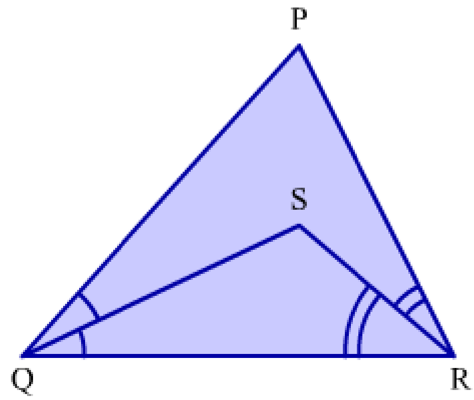
Thus, from (3) and (4), we can conclude that:

∠PQR > ∠PRQ

Solved Examples

Easy

Example 1: In the given ΔPQR, PQ is greater than PR. Also, QS and RS are the respective bisectors of ∠PQR and ∠PRQ. Prove that ∠SRQ > ∠SQR.



Solution:

In ΔPQR, we have:

PQ > PR (Given)

⇒ ∠PRQ > ∠PQR (∵ Angle opposite longer side is greater)

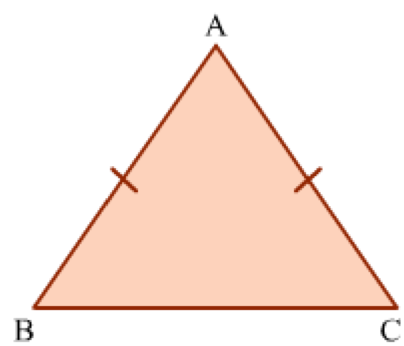
$$\Rightarrow \frac{\angle PRQ}{2} > \frac{\angle PQR}{2}$$

⇒ ∠SRQ > ∠SQR (∵ RS bisects ∠PRQ and QS bisects ∠PQR)

Example 2: ABC is an isosceles triangle with AB = AC and AB < BC. Prove that ∠BAC > ∠ABC.

Solution:

Consider the following ΔABC in which AB = AC and AB < BC.



In ΔABC, we have:

∠ABC = ∠ACB … (1)   [∵ Angles opposite equal sides AB and AC are equal]

By the triangle inequality theorem, we have:

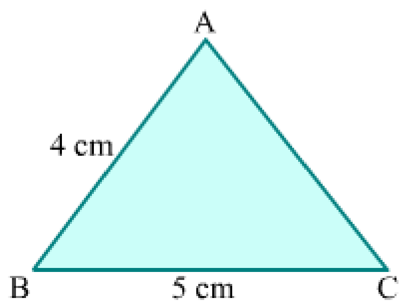
∠ACB < ∠BAC … (2)   [∵ Angle opposite longer side BC is greater]

Thus, from (1) and (2), we obtain:

∠BAC > ∠ABC

Medium

Example 1: In the shown ΔABC, AB = 4 cm, BC = 5 cm and the perimeter is 16 cm. Determine the smallest and greatest angles of the triangle.



Solution:

In ΔABC, we have:

AB = 4 cm and BC = 5 cm (Given)

Perimeter = 16 cm (Also given)

⇒ AB + BC + CA = 16 cm (∵ Perimeter is the sum of all sides)

⇒ 4 cm + 5 cm + CA = 16 cm

⇒ CA = 7 cm

Now, CA = 7 cm is the longest side of ΔABC. Thus, the angle opposite it, i.e., ∠ABC is the greatest angle of the triangle.

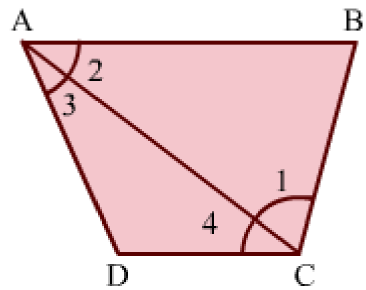
∴ ∠ABC > ∠BAC and ∠ABC > ∠ACB … (1)

Also, BC > AB

∴ ∠BAC > ∠ACB … (2)

Thus, from (1) and (2), we can conclude that ∠ABC is the greatest angle and ∠ACB is the smallest angle in ΔABC.

Example 2: Suppose AB is the longest side and CD the shortest side of the given quadrilateral ABCD. Then, prove that ∠BCD > ∠BAD.



Solution:

In ΔABC, AB is the longest side.

So, AB > BC

⇒ ∠1 > ∠2 … (1)   [∵ Angle opposite longer side is greater]

In ΔADC, CD is the shortest side.

So, AD > CD

⇒ ∠4 > ∠3 … (2)   [∵ Angle opposite longer side is greater]

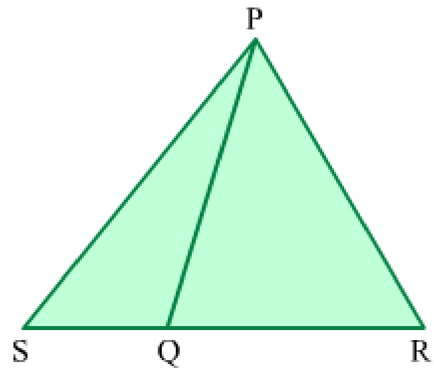
On adding (1) and (2), we get:

∠1 + ∠4 > ∠2 + ∠3

⇒ ∠BCD > ∠BAD

Hard

Example 1: In the given figure, PQ = PR. Show that ∠PRS > ∠PSR.



Solution:

In ΔPQR, we have:

PQ = PR

⇒ ∠PRQ = ∠PQR … (1)   [∵ Angles opposite equal sides are equal]

In ΔPSQ, SQ is produced to R.

So, ∠PQR = ∠PSQ + ∠SPQ   (By the exterior angle property)

⇒ ∠PQR > ∠PSQ … (2)

From (1) and (2), we obtain:

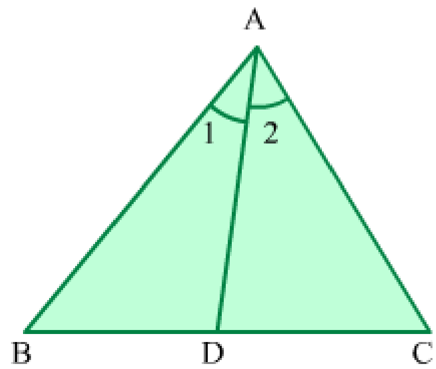
∠PRQ >∠PSQ

⇒ ∠PRS > ∠PSR (∵ ∠PRQ = ∠PRS and ∠PSQ = ∠PSR)

Example 2: In a ΔABC, AB > AC and AD is the bisector of ∠BAC. Show that ∠ADB > ∠ADC.

Solution:

Let the following ΔABC be the given triangle such that AB > AC. Also, AD is the bisector of ∠BAC.



In ΔABC, we have:

AB > AC (Given)

⇒ ∠ACB > ∠ABC   (∵ Angle opposite longer side is greater)

On adding ∠1 to both sides, we get:

∠ACB + ∠1 > ∠ABC + ∠1

⇒ ∠ACB + ∠2 > ∠ABC + ∠1 … (1)   [∵ AD bisects ∠BAC; ∠1 = ∠2]

By the exterior angle property, we have:

∠ADB = ∠ACB + ∠2 … (2)

Similarly, ∠ADC = ∠ABC + ∠1 … (3)

Thus, by using (1), (2) and (3), we can conclude that:

∠ADB > ∠ADC

Observation of the Sides of a Triangle by Seeing the Angles

The red car has to travel at an angle of 65°, while the blue car has to travel at an angle of 50° with respect to line AB to reach the finish point C. Do you think any one car has an advantage over the other?

On observing the triangular track, it seems that path BC is longer than path AC. So, clearly, the red car has an advantage over the blue car. Also, the angle opposite BC is greater than the angle opposite AC. So, what does this tell us about the relation between the sides and angles of the triangular racetrack?

Let us go through this lesson to learn about the relation between the sides and angles of a triangle. We will also solve some problems based on this relation.

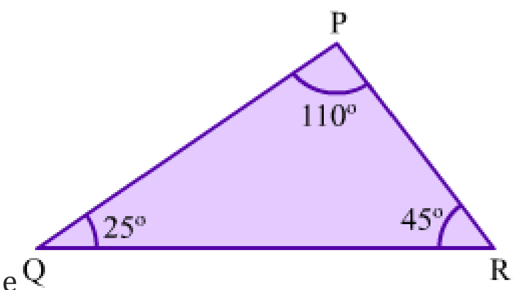
In a Triangle, the Side Opposite the Greater Angle is

Longer

We have studied that in a triangle having two unequal sides, the angle opposite the longer side is greater. The converse of this property is also true. It states that:

If two angles of a triangle are unequal, then the greater angle has the longer side opposite it. In other words, the smaller angle has the shorter side opposite it.

Consider the following ΔPQR.

Let us apply the stated property in this triangle 

.

∠QPR = 110° is the greatest angle in ΔPQR.

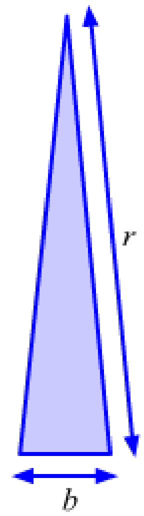
Therefore, the side opposite ∠QPR, i.e., QR is the longest side of the triangle. Also, ∠PQR = 25° is the smallest angle in ΔPQR.

Therefore, the side opposite ∠PQR, i.e., PR is the shortest side of the triangle.

We can apply this property to other triangles as well. Using this property, we can say that in a right triangle, the hypotenuse is the longest side.

Whiz Kid

In trigonometry, a skinny triangle is one whose height is much greater than its base.



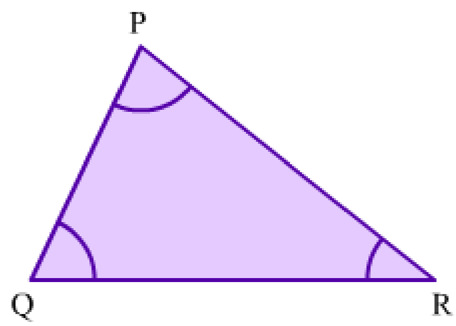
The given triangle is an example of a skinny triangle.

Proving the Property

Given: ΔPQR in which ∠Q > ∠R

To prove: PR > PQ.

Proof: In ΔPQR, we can have three possible cases: (1) PQ > PR, (2) PQ = PR and (3) PQ < PR.



CASE 1:

When PQ > PR, we have:

∠R > ∠Q (∵ Angle opposite longer side is greater)

But this contradicts the given hypothesis that ∠R < ∠Q. Thus, PQ > PR is not true.

CASE 2:

When PQ = PR, we have:

∠R = ∠Q (∵ Angles opposite equal sides are equal)

But this too contradicts the given hypothesis that ∠R < ∠Q. Thus, PQ = PR is also not true.

So, we are left with the third possibility, i.e., PQ < PR (or PR > PQ), which must be true.

Thus, we have proven that if two angles of a triangle are unequal, then the greater angle has the longer side opposite it.

Solved Examples

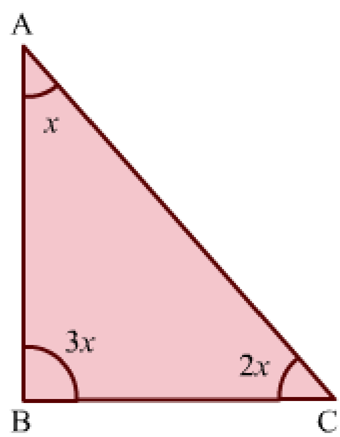
Easy

Example 1: The angles of a triangle are in the ratio 1 : 2 : 3. Find the greatest angle and identify the longest side of the triangle.

Solution:

It is given that the angles of the triangle, say ΔABC, are in ratio 1 : 2 : 3.

Let the angles be *x*, 2*x* and 3*x*, as is shown in the figure.



By the angle sum property of triangles, we have:

∠BAC + ∠ABC + ∠ACB = 180°

⇒ $x + 3x + 2x = 180°$

⇒ $6x = 180°$

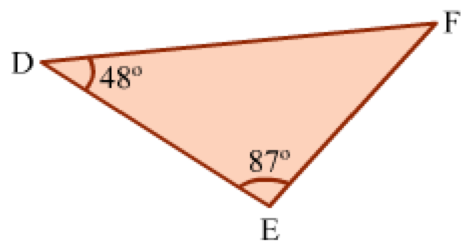
⇒ $x = 30°$

So, $2x = 2 \times 30° = 60°$ and $3x = 3 \times 30° = 90°$

Thus, the greatest angle in the given triangle is ∠ABC, i.e., 90°.

Now, we know that the side opposite the greater angle is longer. In ΔABC, side AC is opposite the greatest angle; hence, it is the longest.

Example 2: Which side of the given triangle is the shortest?



Solution:

In order to find the shortest side of ΔDEF, we need to figure out the smallest angle of the triangle. This is because the smallest side is opposite the smallest angle.

By the angle sum property of triangles, we have:

∠EDF + ∠DEF + ∠EFD = 180°

⇒ 48° + 87° + ∠EFD = 180°

⇒ 135° + ∠EFD = 180°

⇒ ∠EFD = 45°

Clearly, ∠EFD has the smallest measure in the given triangle. So, the side opposite it, i.e., DE is the shortest side of the triangle.

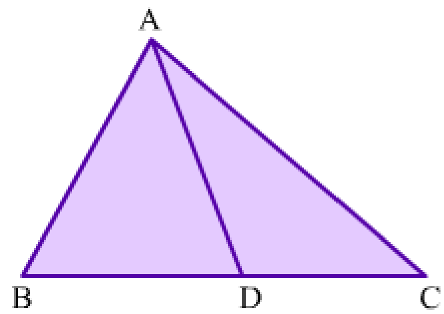
Medium

Example 1: In the given ΔABC, ∠ABD > ∠ACD and AD is the bisector of ∠BAC.

Prove that:

1. AB < AC

2. ∠ADB < ∠ADC



Solution:

1. It is given that ∠ABD > ∠ACD.

   ∴ AC > AB (∵ Side opposite greater angle is longer)

   Or, AB < AC

2. In ΔABD, we have:

   ∠ABD + ∠BAD + ∠ADB = 180° (By the angle sum property)

   ⇒ ∠ACD + ∠BAD + ∠ADB < 180° (∵ ∠ABD > ∠ACD)

   ⇒ ∠ACD + ∠CAD + ∠ADB < 180° (∵ AD bisects ∠BAC; ∠BAD = ∠CAD)

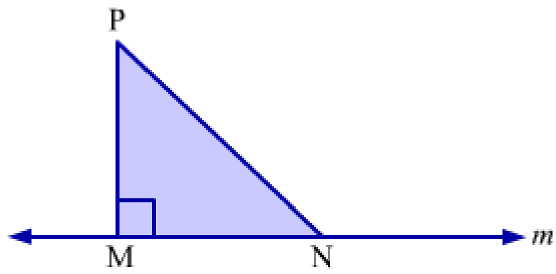
   ⇒ ∠ADB < 180° − (∠ACD +∠CAD)

   ⇒ ∠ADB < ∠ADC (By the angle sum property in ΔADC)

Example 2: Prove that of all the line segments that can be drawn to a given line from a point lying outside the line, the perpendicular line segment is the shortest.

Solution:

Let there be a straight line *m*, a point P lying outside the line and a point M lying on the line. Also, PM ⊥ *m* and N is any point other than M on *m*.



Since PM is perpendicular to *m*, ∠PMN = 90°.

In ΔPMN, we have:

⇒ ∠PMN + ∠MPN + ∠PNM = 180° (By the angle sum property)

⇒ 90° + ∠MPN + ∠PNM = 180°

⇒ ∠MPN + ∠PNM = 90°

⇒ ∠PNM < 90°

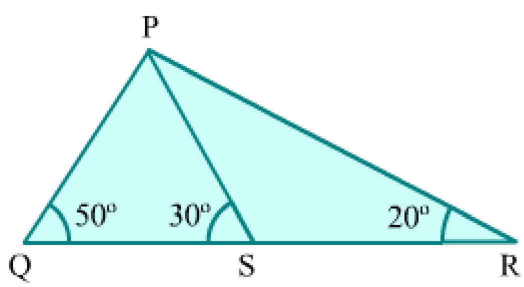
⇒ ∠PNM < ∠PMN

⇒ PM < PN (∵ Side opposite greater angle is longer)

Thus, PM is the shortest of all line segments from point P to line *m*.

Hard

Example 1: In the given ΔPQR, ∠PQR = 50°, ∠PRQ = 20° and ∠PSQ = 30°. Prove that QS > SR.



Solution:

In ΔPQS, we have:

$\angle PQR = 50°$ and $\angle PSQ = 30°$ (Given)

By using the angle sum property in ΔPQS, we obtain:

$\angle PQR + \angle PSQ + \angle QPS = 180°$

$\Rightarrow 50° + 30° + \angle QPS = 180°$

$\Rightarrow \angle QPS = 180° - (50° + 30°)$

$\Rightarrow \angle QPS = 100°$

So, $\angle QPS > \angle PQR > \angle PSQ$

Since the side opposite the greater angle is longer, we get:

$QS > PS > PQ$ ... (1)

$\angle PSQ$ and $\angle PSR$ form a linear pair.

So, $\angle PSQ + \angle PSR = 180°$

$\Rightarrow 30° + \angle PSR = 180°$

$\Rightarrow \angle PSR = 180° - 30°$

$\Rightarrow \angle PSR = 150°$

In ΔPSR, we have:

$\angle PSR + \angle SPR + \angle PRQ = 180°$ (By the angle sum property)

$\Rightarrow 150° + \angle SPR + 20° = 180°$

$\Rightarrow \angle SPR = 180° - (150° + 20°)$

$\Rightarrow \angle SPR = 10°$

So, $\angle PSR > \angle PRQ > \angle SPR$

Since the side opposite the greater angle is longer, we get:

$PR > PS > SR$ ... (2)

By using (1) and (2), we can conclude that:

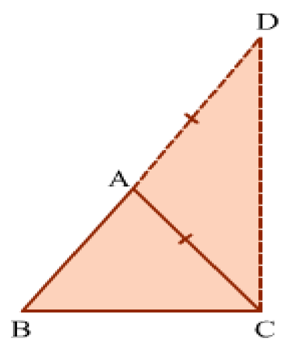
QS > SR

Example 2: Prove that the difference between any two sides of a triangle is less than the third side.

Solution:

We have to prove that the difference between any two sides of a triangle (let us say ΔABC) is less than the third side of the triangle. We will do so by showing that in ΔABC, BC − AC < AB.

Construction: Extend side BA up to point D such that AD = AC. Join C to D.



In ΔACD, we have:

AD = AC … (1)

We know that in an isosceles triangle, the angles opposite equal sides are equal.

∴ ∠ACD = ∠ADC

⇒ ∠ACD + ∠ACB > ∠ADC

⇒ ∠BCD > ∠ADC

We know that the side opposite the greater angle is longer. So, we obtain, BD > BC

⇒ AB + AD > BC (∵ BD = AB + AD)

⇒ AB + AC > BC (Using equation 1)

⇒ BC − AC < AB

Similarly, we can prove that AB − BC < AC and AC − AB < BC.

Thus, we have proved that the difference between any two sides of a triangle is less than the third side.