

Inverse Trigonometric Functions

Question1

Considering only the principal values of inverse trigonometric functions, the number of positive real values of x satisfying

$$\tan^{-1}(x) + \tan^{-1}(2x) = \frac{\pi}{4} \text{ is :}$$

[27-Jan-2024 Shift 2]

Options:

A.

More than 2

B.

1

C.

2

D.

0

Answer: B

Solution:

$$\tan^{-1}x + \tan^{-1}2x = \frac{\pi}{4}; x > 0$$

$$\Rightarrow \tan^{-1}2x = \frac{\pi}{4} - \tan^{-1}x$$

Taking tan both sides

$$\Rightarrow 2x = \frac{1-x}{1+x}$$

$$\Rightarrow 2x^2 + 3x - 1 = 0$$

$$x = \frac{-3 \pm \sqrt{9+8}}{8} = \frac{-3 \pm \sqrt{17}}{8}$$

$$\text{Only possible } x = \frac{-3 + \sqrt{17}}{8}$$

Question2

Let $x = m/n$ (m, n are co-prime natural numbers) be a solution of the equation $\cos(2\sin^{-1}x) = 1/9$ and let α, β ($\alpha > \beta$) be the roots of the equation $mx^2 - nx - m + n = 0$. Then the point (α, β) lies on the line

[29-Jan-2024 Shift 2]

Options:

A.

$$3x + 2y = 2$$

B.

$$5x - 8y = -9$$

C.

$$3x - 2y = -2$$

D.

$$5x + 8y = 9$$

Answer: D

Solution:

Assume $\sin^{-1}x = \theta$

$$\cos(2\theta) = \frac{1}{9}$$

$$\sin \theta = \pm \frac{2}{3}$$

as m and n are co-prime natural numbers,

$$x = \frac{2}{3}$$

i.e. $m = 2, n = 3$

So, the quadratic equation becomes $2x^2 - 3x + 1 = 0$ whose roots are $\alpha = 1, \beta = \frac{1}{2}$

$\left(1, \frac{1}{2}\right)$ lies on $5x + 8y = 9$

Question3

For $\alpha, \beta, \gamma \neq 0$. If $\sin^{-1}\alpha + \sin^{-1}\beta + \sin^{-1}\gamma = \pi$ and $(\alpha + \beta + \gamma)(\alpha - \gamma + \beta) = 3\alpha\beta$, then γ equal to

[31-Jan-2024 Shift 1]

Options:

A.

$$\sqrt{3}/2$$

B.

$$1/\sqrt{2}$$

C.

$$\frac{\sqrt{3}-1}{2\sqrt{2}}$$

D.

$$\sqrt{3}$$

Answer: A

Solution:

$$\text{Let } \sin^{-1}\alpha = A, \sin^{-1}\beta = B, \sin^{-1}\gamma = C$$

$$A + B + C = \pi$$

$$(\alpha + \beta)^2 - \gamma^2 = 3\alpha\beta$$

$$\alpha^2 + \beta^2 - \gamma^2 = \alpha\beta$$

$$\frac{\alpha^2 + \beta^2 - \gamma^2}{2\alpha\beta} = \frac{1}{2}$$

$$\Rightarrow \cos C = \frac{1}{2}$$

$$\sin C = \gamma$$

$$\cos C = \sqrt{1 - \gamma^2} = \frac{1}{2}$$

$$\gamma = \frac{\sqrt{3}}{2}$$

Question4

If $a = \sin^{-1}(\sin(5))$ and $b = \cos^{-1}(\cos(5))$, then $a^2 + b^2$ is equal to

[31-Jan-2024 Shift 2]

Options:

A.

$$4\pi^2 + 25$$

B.

$$8\pi^2 - 40\pi + 50$$

C.

$$4\pi^2 - 20\pi + 50$$

D.

$$25$$

Answer: B

Solution:

$$a = \sin^{-1}(\sin 5) = 5 - 2\pi$$

$$\text{and } b = \cos^{-1}(\cos 5) = 2\pi - 5$$

$$\begin{aligned}\therefore a^2 + b^2 &= (5 - 2\pi)^2 + (2\pi - 5)^2 \\ &= 8\pi^2 - 40\pi + 50\end{aligned}$$

Question5

$\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right) + \sec^{-1}\left(\sqrt{\frac{8+4\sqrt{3}}{6+3\sqrt{3}}}\right)$ is equal to

[24-Jan-2023 Shift 1]

Options:

A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{6}$

Answer: C

Solution:

Solution:

$$\begin{aligned}\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right) + \sec^{-1}\left(\sqrt{\frac{8+4\sqrt{3}}{6+3\sqrt{3}}}\right) \\ = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{3}\end{aligned}$$

Question6

If the sum of all the solutions of

$$\tan^{-1} \left(\frac{2x}{1-x^2} \right) + \cot^{-1} \left(\frac{1-x^2}{2x} \right) = \frac{\pi}{3}$$

$-1 < x < 1, x \neq 0$, is $\alpha - \frac{4}{\sqrt{3}}$, then α is equal to _____.

[25-Jan-2023 Shift 1]

Answer: 2

Solution:

Solution:

Case I : $x > 0$

$$\tan^{-1} \frac{2x}{1-x^2} + \tan^{-1} \frac{2x}{1-x^2} = \frac{\pi}{3}$$

$$x = 2 - \sqrt{3}$$

Case II : $x < 0$

$$\tan^{-1} \frac{2x}{1-x^2} + \tan^{-1} \frac{2x}{1-x^2} + \pi = \frac{\pi}{3}$$

$$x = \frac{-1}{\sqrt{3}} \Rightarrow \alpha = 2$$

Question7

Let $a_1 = 1, a_2, a_3, a_4, \dots$ be consecutive natural numbers. Then

$\tan^{-1} \left(\frac{1}{1+a_1a_2} \right) + \tan^{-1} \left(\frac{1}{1+a_2a_3} \right) + \dots + \tan^{-1} \left(\frac{1}{1+a_{2021}a_{2022}} \right)$ is equal to

[30-Jan-2023 Shift 2]

Options:

A. $\frac{\pi}{4} - \cot^{-1}(2022)$

B. $\cot^{-1}(2022) - \frac{\pi}{4}$

C. $\tan^{-1}(2022) - \frac{\pi}{4}$

D. $\frac{\pi}{4} - \tan^{-1}(2022)$

Answer: 0

Solution:

Sol. $a_2 - a_1 = a_3 - a_2 = \dots = a_{2022} - a_{2021} = 1$.

$$\begin{aligned} \therefore \tan^{-1} \left(\frac{a_2 - a_1}{1 + a_1 a_2} \right) + \tan^{-1} \left(\frac{a_3 - a_2}{1 + a_2 a_3} \right) + \dots + \tan^{-1} \left(\frac{a_{2022} - a_{2021}}{1 + a_{2021} a_{2022}} \right) \\ = [(\tan^{-1} a_2) - \tan^{-1} a_1] + [\tan^{-1} a_3 - \tan^{-1} a_2] + \dots \\ + [\tan^{-1} a_{2022} - \tan^{-1} a_{2021}] \\ = \tan^{-1} a_{2022} - \tan^{-1} a_1 \\ = \tan^{-1}(2022) - \tan^{-1} 1 = \tan^{-1} 2022 - \frac{\pi}{4} \text{ (option 3)} \\ = \left(\frac{\pi}{2} - \cot^{-1}(2022) \right) - \frac{\pi}{4} \\ = \frac{\pi}{4} - \cot^{-1}(2022) \text{ (option 1)} \end{aligned}$$

Question8

If $\sin^{-1} \frac{\alpha}{17} + \cos^{-1} \frac{4}{5} - \tan^{-1} \frac{77}{36} = 0$, $0 < \alpha < 13$, then $\sin^{-1}(\sin \alpha) + \cos^{-1}(\cos \alpha)$ is equal to
[31-Jan-2023 Shift 1]

Options:

- A. π
- B. 16
- C. 0
- D. $16 - 5\pi$

Answer: A

Solution:

Solution:

$$\cos^{-1} \frac{4}{5} = \tan^{-1} \frac{3}{4}$$

$$\therefore \sin^{-1} \frac{\alpha}{17} = \tan^{-1} \frac{77}{36} - \tan^{-1} \frac{3}{4} = \tan^{-1} \left(\frac{\frac{77}{36} - \frac{3}{4}}{1 + \frac{77}{36} \cdot \frac{3}{4}} \right)$$

$$\sin^{-1} \frac{\alpha}{17} = \tan^{-1} \frac{8}{15} = \sin^{-1} \frac{8}{17}$$

$$\Rightarrow \frac{\alpha}{17} = \frac{8}{17} \Rightarrow \alpha = 8$$

$$\begin{aligned} \therefore \sin^{-1}(\sin 8) + \cos^{-1}(\cos 8) \\ = 3\pi - 8 + 8 - 2\pi \\ = \pi \end{aligned}$$

Question9

If the sum and product of four positive consecutive terms of a G.P., are 126 and 1296, respectively, then the sum of common ratios of all such GPs is

[31-Jan-2023 Shift 1]

Options:

A. 7

B. $\frac{9}{2}$

C. 3

D. 14

Answer: A

Solution:

$a, ar, ar^2, ar^3 (a, r > 0)$

$$a^4 r^6 = 1296$$

$$a^2 r^3 = 36$$

$$a = \frac{6}{r^{3/2}}$$

$$a + ar + ar^2 + ar^3 = 126$$

$$\frac{1}{r^{3/2}} + \frac{r}{r^{3/2}} + \frac{r^2}{r^{3/2}} + \frac{r^3}{r^{3/2}} = \frac{126}{6} = 21$$

$$(r^{-3/2} + r^{3/2}) + (r^{1/2} + r^{-1/2}) = 21$$

$$r^{1/2} + r^{-1/2} = A$$

$$r^{-3/2} + r^{3/2} + 3A = A^3$$

$$A^3 - 3A + A = 21$$

$$A^3 - 2A = 21$$

$$A = 3$$

$$\sqrt{r} + \frac{1}{\sqrt{r}} = 3$$

$$r_r + 1 = 3\sqrt{r}$$

$$r^2 + 2r + 1 = 9r$$

$$r^2 - 7r + 1 = 0$$

Question 10

Let $(a, b) \subset (0, 2\pi)$ be the largest interval for which

$\sin^{-1}(\sin \theta) - \cos^{-1}(\sin \theta) > 0$, $\theta \in (0, 2\pi)$ holds. If

$$\alpha x^2 + \beta x + \sin^{-1}(x^2 - 6x + 10) + \cos^{-1}(x^2 - 6x + 10) = 0$$

and $\alpha - \beta = b - a$, then α is equal to :

[31-Jan-2023 Shift 2]

Options:

A. $\frac{\pi}{48}$

B. $\frac{\pi}{16}$

C. $\frac{\pi}{8}$

D. $\frac{\pi}{12}$

Answer: D

Solution:

Solution:

$$\sin^{-1} \sin \theta - \left(\frac{\pi}{2} - \sin^{-1} \sin \theta \right) > 0$$

$$\Rightarrow \sin^{-1} \sin \theta > \frac{\pi}{4}$$

$$\Rightarrow \sin \theta > \frac{1}{\sqrt{2}}$$

$$\text{So, } \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4} \right)$$

$$\theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4} \right) = (a, b)$$

$$\Rightarrow \alpha - a = \frac{\pi}{2} = \alpha - \beta$$

$$\Rightarrow \beta = \alpha - \frac{\pi}{2}$$

$$x + \beta x + \sin^2[(x-3)^2 + 1] + \cos^{-1}[(x-3)^2 + 1] = 0$$

$$x = 3, 9\alpha + 3\beta + \frac{\pi}{2} + 0 = 0$$

$$\Rightarrow 9\alpha + 3\left(\alpha - \frac{\pi}{2}\right) + \frac{\pi}{2} = 0$$

$$\Rightarrow 12\alpha - \pi = 0$$

$$\alpha = \frac{\pi}{12}$$

Question11

Let S be the set of all solutions of the equation

$$\cos^{-1}(2x) - 2\cos^{-1}(\sqrt{1-x^2}) = \pi, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Then $\sum_{x \in S} 2\sin^{-1}(x^2 - 1)$ is equal to

[1-Feb-2023 Shift 1]

Options:

A. 0

B. $\frac{-2\pi}{3}$

C. $\pi - \sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

D. $\pi - 2\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

Answer: B

Solution:

Solution:

$$\cos^{-1}(2x) = \pi + 2\cos^{-1}(\sqrt{1-x^2})$$

$$\text{LHS} = [0, \pi]$$

For equation to be meaningful $\cos^{-1} 2x = \pi$ and $\cos^{-1}(\sqrt{1-x^2}) = 0 \Rightarrow x = \frac{-1}{2}$ and $x = 0$

which is not possible

$\therefore x \in \emptyset$

Now $\Sigma(x) = 0$

$\therefore$ Sum over empty set is always 0

Question 12

Let $S = \{x \in \mathbb{R} : 0 < x < 1 \text{ and } 2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\}$. If $n(S)$ denotes the number of elements in S then :
[1-Feb-2023 Shift 2]

Options:

- A. $n(S) = 2$ and only one element in S is less than $\frac{1}{2}$.
- B. $n(S) = 1$ and the element in S is more than $\frac{1}{2}$.
- C. $n(S) = 1$ and the element in S is less than $\frac{1}{2}$.
- D. $n(S) = 0$

Answer: C

Solution:

Solution:

$$0 < x < 1$$

$$2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\tan^{-1}x = \theta \in \left(0, \frac{\pi}{4}\right) \therefore x = \tan \theta$$

$$2\tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right) = \cos^{-1}(\cos 2\theta)$$

$$2\left(\frac{\pi}{4} - \theta\right) = 2\theta \therefore 4\theta = \frac{\pi}{2} \therefore \theta = \frac{\pi}{8}$$

$$x = \tan \frac{\pi}{8} \therefore x = \sqrt{2} - 1 \approx 0.414$$

Question 13

If $S = \left\{x \in \mathbb{R} : \sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4}\right\}$, then is equal to _____.
[13-Apr-2023 shift 1]

Answer: 4

Solution:

$$\sin^{-1}\left(\frac{(x+1)}{\sqrt{(x+1)^2+1}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4}$$

$$\because \frac{t}{\sqrt{t^2+1}} \in (-1, 1)$$

$$\sin^{-1}\left(\frac{(x+1)}{\sqrt{(x+1)^2+1}}\right) = \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) + \frac{\pi}{4}$$

$$\frac{(x+1)}{\sqrt{(x+1)^2+1}} = \left(\frac{1}{\sqrt{2}}\right) \cos\left(\sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right)\right) + \frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}\right)$$

$$\frac{(x+1)}{\sqrt{(x+1)^2+1}} = \frac{1}{\sqrt{2}}\left(\frac{1+x}{\sqrt{x^2+1}}\right)$$

After solving this equation, we get

$$x = -1 \text{ or } x = 0$$

$$S = \{-1, 0\}$$

$$\sum_{x \in \mathbb{R}} \left(\sin\left((x^2+x+5)\frac{\pi}{2}\right) - \cos((x^2+x+5)\pi) \right)$$

$$= \left[\sin\left(\frac{5\pi}{2}\right) - \cos(5\pi) \right] + \left[\sin\left(\frac{5\pi}{2}\right) - \cos(5\pi) \right] = 4$$

Question14

For $x \in (-1, 1]$, the number of solutions of the equation

$\sin^{-1}x = 2\tan^{-1}x$ is equal to _____.

[13-Apr-2023 shift 2]

Answer: 2

Solution:

$$\sin^{-1}x = 2\tan^{-1}x$$

$$\sin^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow x = \frac{2x}{1+x^2}$$

$$\Rightarrow x\left(\frac{2}{1+x^2} - 1\right) = 0$$

$$\Rightarrow x = 0, 1, -1 \text{ but } -1 \text{ is not included.}$$

Answer 2 solutions

Question15

Let $x * y = x^2 + y^3$ and $(x * 1) * 1 = x * (1 * 1)$. Then a value of

$2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$ is

[24-Jun-2022-Shift-2]

Options:

A. $\frac{\pi}{4}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{2}$

D. $\frac{\pi}{6}$

Answer: B

Solution:

Given,

$$x * y = x^2 + y^3$$

$$\therefore x * 1 = x^2 + 1^3 = x^2 + 1$$

$$\text{Now, } (x * 1) * 1 = (x^2 + 1) * 1$$

$$\Rightarrow (x * 1) * 1 = (x^2 + 1)^2 + 1^3$$

$$\Rightarrow (x * 1) * 1 = x^4 + 1 + 2x^2 + 1$$

$$\text{Also, } x * (1 * 1)$$

$$= x * (1^2 + 1^3)$$

$$= x * 2$$

$$= x^2 + 2^3$$

$$= x^2 + 8$$

Given that,

$$(x * 1) * 1 = x * (1 * 1)$$

$$\therefore x^4 + 1 + 2x^2 + 1 = x^2 + 8$$

$$\Rightarrow x^4 + x^2 - 6 = 0$$

$$\Rightarrow x^4 + 3x^2 - 2x^2 - 6 = 0$$

$$\Rightarrow x^2(x^2 + 3) - 2(x^2 + 3) = 0$$

$$\Rightarrow (x^2 + 3)(x^2 - 2) = 0$$

$$\Rightarrow x^2 = 2, -3$$

[$x^2 = -3$. not possible as square of anything should be always possible]

$$\therefore x^2 = 2$$

$\therefore$ Now,

$$\begin{aligned}
 & 2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right) \\
 &= 2\sin^{-1}\left(\frac{2^2+2-2}{2^2+2+2}\right) \\
 &= 2\sin^{-1}\left(\frac{4}{8}\right) \\
 &= 2\sin^{-1}\left(\frac{1}{2}\right) \\
 &= 2 \times \frac{\pi}{6} \\
 &= \frac{\pi}{3}
 \end{aligned}$$

Question16

The value of $\tan^{-1}\left(\frac{\cos\left(\frac{15\pi}{4}\right)-1}{\sin\left(\frac{\pi}{4}\right)}\right)$ is equal to :

[25-Jun-2022-Shift-2]

Options:

- A. $-\frac{\pi}{4}$
- B. $-\frac{\pi}{8}$
- C. $-\frac{5\pi}{12}$
- D. $-\frac{4\pi}{9}$

Answer: B

Solution:

Solution:

$$\begin{aligned}
 & \tan^{-1}\left(\frac{\cos\left(\frac{15\pi}{4}\right)-1}{\sin\frac{\pi}{4}}\right) \\
 &= \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}-1}{\frac{1}{\sqrt{2}}}\right) \\
 &= \tan^{-1}(1-\sqrt{2}) = -\tan^{-1}(\sqrt{2}-1) \\
 &= -\frac{\pi}{8}
 \end{aligned}$$

Question17

If the inverse trigonometric functions take principal values then

$\cos^{-1}\left(\frac{3}{10}\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) + \frac{2}{5}\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right)$ is equal to:

[26-Jun-2022-Shift-2]

Options:

A. 0

B. $\frac{\pi}{4}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{6}$

Answer: C

Solution:

Solution:

$$\begin{aligned} & \cos^{-1}\left(\frac{3}{10}\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) + \frac{2}{5}\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right) \\ &= \cos^{-1}\left(\frac{3}{10} \cdot \frac{3}{5} + \frac{2}{5} \cdot \frac{4}{5}\right) \\ &= \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \end{aligned}$$

Question18

$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) + \cos^{-1}\left(\cos\frac{7\pi}{6}\right) + \tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ is equal to:

[27-Jun-2022-Shift-1]

Options:

A. $\frac{11\pi}{12}$

B. $\frac{17\pi}{12}$

C. $\frac{31\pi}{12}$

D. $-\frac{3\pi}{4}$

Answer: A

Solution:

Solution:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) + \cos^{-1}\left(\cos\frac{7\pi}{6}\right) + \tan^{-1}\tan\left(\frac{3\pi}{4}\right)$$

$$\sin^{-1} \sin \left(\frac{2\pi}{3} \right) = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$$

$$\cos^{-1} \left(\cos \frac{2\pi}{6} \right) = 2\pi - \frac{7\pi}{6} = \frac{5\pi}{6}$$

$$\tan^{-1} \tan \left(\frac{3\pi}{4} \right) = \frac{3\pi}{4} - \pi = \frac{-\pi}{4}$$

$$\begin{aligned} \sin^{-1} \left(\sin \frac{2\pi}{3} \right) + \cos^{-1} \cos \frac{7\pi}{6} + \tan^{-1} \tan \frac{3\pi}{4} \\ = \frac{11\pi}{12} \end{aligned}$$

Question19

The value of $\cot \left(\sum_{n=1}^{50} \tan^{-1} \left(\frac{1}{1+n+n^2} \right) \right)$ is

[27-Jun-2022-Shift-2]

Options:

A. $\frac{26}{25}$

B. $\frac{25}{26}$

C. $\frac{50}{51}$

D. $\frac{52}{51}$

Answer: A

Solution:

Solution:

$$\begin{aligned} & \cot \left(\sum_{n=1}^{50} \tan^{-1} \left(\frac{1}{1+n+n^2} \right) \right) \\ &= \cot \left(\sum_{n=1}^{50} \tan^{-1} \left(\frac{(n+1)-n}{1+(n+1)n} \right) \right) \\ &= \cot \left(\sum_{n=1}^{50} (\tan^{-1}(n+1) - \tan^{-1}n) \right) \\ &= \cot(\tan^{-1}51 - \tan^{-1}1) \\ &= \cot \left(\tan^{-1} \left(\frac{51-1}{1+51} \right) \right) \\ &= \cot \left(\cot^{-1} \left(\frac{52}{50} \right) \right) \\ &= \frac{26}{25} \end{aligned}$$

Question20

$50 \tan \left(3 \tan^{-1} \left(\frac{1}{2} \right) + 2 \cos^{-1} \left(\frac{1}{\sqrt{5}} \right) \right) + 4\sqrt{2} \tan \left(\frac{1}{2} \tan^{-1}(2\sqrt{2}) \right)$ is equal to

[29-Jun-2022-Shift-1]

Answer: 29

Solution:

$$\begin{aligned} & 50 \tan \left(3 \tan^{-1} \frac{1}{2} + 2 \cos^{-1} \frac{1}{\sqrt{5}} \right) \\ & + 4\sqrt{2} \tan \left(\frac{1}{2} \tan^{-1} 2\sqrt{2} \right) \\ & = 50 \tan \left(\tan^{-1} \frac{1}{2} + 2 \left(\tan^{-1} \frac{1}{2} + \tan^{-1} 2 \right) \right) \\ & + 4\sqrt{2} \tan \left(\frac{1}{2} \tan^{-1} 2\sqrt{2} \right) \\ & = 50 \tan \left(\tan^{-1} \frac{1}{2} + 2 \cdot \frac{\pi}{2} \right) + 4\sqrt{2} \times \frac{1}{\sqrt{2}} \\ & = 50 \left(\tan \tan^{-1} \frac{1}{2} \right) + 4 \\ & = 25 + 4 = 29 \end{aligned}$$

Question 21

The set of all values of k for which $(\tan^{-1}x)^3 + (\cot^{-1}x)^3 - k\pi^3$, $x \in \mathbb{R}_u$ is the interval :
[24-Jun-2022-Shift-1]

Options:

- A. $\left[\frac{1}{32}, \frac{7}{8} \right)$
- B. $\left(\frac{1}{24}, \frac{13}{16} \right)$
- C. $\left[\frac{1}{48}, \frac{13}{16} \right]$
- D. $\left[\frac{1}{32}, \frac{9}{8} \right)$

Answer: A

Solution:

Solution:

$$(\tan^{-1}x)^3 + (\cot^{-1}x)^3 = k\pi^3$$

$$\text{Let } f(t) = t^3 + \left(\frac{\pi}{2} - t\right)^3$$

$$\text{Where } t = \tan^{-1}x; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$= t^3 + \left(\frac{\pi}{2}\right)^3 - \frac{3\pi^2 t}{4} + \frac{3\pi}{2}t^2 - t^3$$

$$f(t) = \frac{3\pi}{2}t^2 - \frac{3\pi^2}{4} \cdot t + \frac{\pi^3}{8}$$

This is a quadratic equation of t.

Here, coefficient of t^2 term is $\frac{3\pi}{2}$ which is >0 .

$\therefore$ It is a upward parabola.

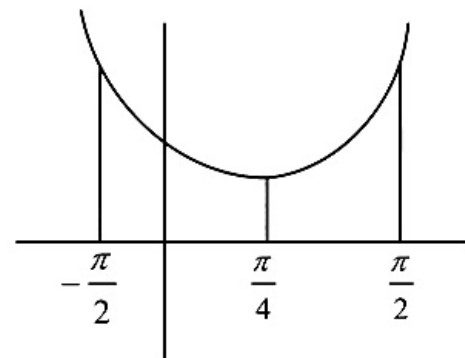
$$\text{Now, } f'(t) = 3\pi t - \frac{3\pi^2}{4}$$

$$f''(t) = 3\pi > 0$$

$$\therefore 3\pi t - \frac{3\pi^2}{4} = 0$$

$$\Rightarrow t = \frac{\pi}{4} \text{ (minima)}$$

$\therefore$ vertex of graph at $\frac{\pi}{4}$



$\therefore$ Minimum value at $\frac{\pi}{4}$ and maximum value at $-\frac{\pi}{2}$.

$$\therefore f\left(\frac{\pi}{4}\right) = \frac{\pi^3}{64} + \left(\frac{\pi}{2} - \frac{\pi}{4}\right)^3 = \frac{\pi^3}{32}$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi^3}{8} + \pi^3$$

$$= \frac{7\pi^3}{8}$$

$$\therefore k\pi^3 \in \left[\frac{\pi^3}{32}, \frac{7\pi^3}{8}\right)$$

$$\Rightarrow k \in \left[\frac{1}{32}, \frac{7}{8}\right)$$

Question22

Let $x = \sin(2\tan^{-1}\alpha)$ and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$. If $S = \{a \in \mathbb{R} : y^2 = 1 - x\}$, then $\sum_{\alpha \in S} 16\alpha^3$ is equal to

[25-Jul-2022-Shift-2]

Answer: 130

Solution:

Solution:

$$\therefore x = \sin(2\tan^{-1}\alpha) = \frac{2\alpha}{1+\alpha^2} \dots\dots (i)$$

$$\text{and } y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right) = \sin\left(\sin^{-1}\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$$

$$\text{Now, } y^2 = 1 - x$$

$$\frac{1}{5} = 1 - \frac{2\alpha}{1+\alpha^2}$$

$$\Rightarrow 1 + \alpha^2 = 5 + 5\alpha^2 - 10\alpha$$

$$\Rightarrow 2\alpha^2 - 5\alpha + 2 = 0$$

$$\therefore \alpha = 2, \frac{1}{2}$$

$$\therefore \sum_{\alpha \in S} 16\alpha^3 = 16 \times 2^3 + 16 \times \frac{1}{2^3} = 130$$

Question23

$\tan\left(2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{\sqrt{5}}{2} + 2\tan^{-1}\frac{1}{8}\right)$ is equal to :

[26-Jul-2022-Shift-1]

Options:

A. 1

B. 2

C. $\frac{1}{4}$

D. $\frac{5}{4}$

Answer: B

Solution:

Solution:

$$\tan\left(2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{\sqrt{5}}{2} + 2\tan^{-1}\frac{1}{8}\right)$$

$$= \tan\left(2\tan^{-1}\left(\frac{\frac{1}{5} + \frac{1}{8}}{1 - \frac{1}{5} \cdot \frac{1}{8}}\right) + \sec^{-1}\frac{\sqrt{5}}{2}\right)$$

$$= \tan\left[2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right]$$

$$= \tan\left[\tan^{-1}\frac{\frac{2}{3}}{1 - \frac{1}{9}} + \tan^{-1}\frac{1}{2}\right]$$

$$\begin{aligned}
&= \tan \left[\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{2} \right] \\
&= \tan \left[\frac{\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{2}}{1 - \frac{3}{4} \cdot \frac{1}{2}} \right] = \tan \left[\tan^{-1} \frac{\frac{3}{4} + \frac{1}{2}}{\frac{5}{8}} \right] \\
&= \tan[\tan^{-1} 2] = 2
\end{aligned}$$

Question24

If $0 < x < \frac{1}{\sqrt{2}}$ and $\frac{\sin^{-1}x}{\alpha} = \frac{\cos^{-1}x}{\beta}$, then the value of $\sin \left(\frac{2\pi\alpha}{\alpha + \beta} \right)$ is
[26-Jul-2022-Shift-2]

Options:

- A. $4\sqrt{(1-x^2)}(1-2x^2)$
- B. $4x\sqrt{(1-x^2)}(1-2x^2)$
- C. $2x\sqrt{(1-x^2)}(1-4x^2)$
- D. $4\sqrt{(1-x^2)}(1-4x^2)$

Answer: B

Solution:

Solution:

Let $\frac{\sin^{-1}x}{\alpha} = \frac{\cos^{-1}x}{\beta} = k \Rightarrow \sin^{-1}x + \cos^{-1}x = k(\alpha + \beta) \Rightarrow \alpha + \beta = \frac{\pi}{2k}$

Now, $\frac{2\pi\alpha}{\alpha + \beta} = \frac{2\pi\alpha}{\frac{\pi}{2k}} = 4k\alpha = 4\sin^{-1}x$

Here $\sin \left(\frac{2\pi\alpha}{\alpha + \beta} \right) = \sin(4\sin^{-1}x)$

Let $\sin^{-1}x = \theta$

$\therefore x \in \left(0, \frac{1}{\sqrt{2}} \right) \Rightarrow \theta \in \left(0, \frac{\pi}{4} \right)$

$\Rightarrow x = \sin \theta$

$\Rightarrow \cos \theta = \sqrt{1-x^2}$

$\Rightarrow \sin 2\theta = 2x \cdot \sqrt{1-x^2}$

$\Rightarrow \cos 2\theta = \sqrt{1-4x^2(1-x^2)} = \sqrt{(2x^2-1)^2} = 1-2x^2$

$\therefore (\cos 2\theta > 0, \text{ as } 2\theta \in \left(0, \frac{\pi}{2} \right))$

$\Rightarrow \sin 4\theta = 2 \cdot 2x\sqrt{1-x^2}(1-2x^2)$

$= 4x\sqrt{1-x^2}(1-2x^2)$

Question25

For $k \in \mathbb{R}$, let the solutions of the equation $\cos(\sin^{-1}(x \cot(\tan^{-1}(\cos(\sin^{-1}x)))) = k$, $0 < \left| x \right| < \frac{1}{\sqrt{2}}$ be α and β , where

the inverse trigonometric functions take only principal values. If the solutions of the equation $x^2 - bx - 5 = 0$ are $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ and $\frac{\alpha}{\beta}$, then $\frac{b}{k^2}$ is equal to _____.
[27-Jul-2022-Shift-1]

Answer: 12

Solution:

Solution:

$$\cos(\sin^{-1}(x \cot(\tan^{-1}(\cos(\sin^{-1})))) = k.$$

$$\Rightarrow \cos(\sin^{-1}(x \cot(\tan^{-1}(\sqrt{1-x^2}))) = k$$

$$\Rightarrow \cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) = k$$

$$\Rightarrow \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}} = k$$

$$\Rightarrow \frac{1-2x^2}{1-x^2} = k^2$$

$$\Rightarrow 1-2x^2 = k^2 - k^2x^2$$

$$\therefore x^2 - \left(\frac{k^2-1}{k^2-2}\right) = 0 \quad \begin{matrix} / \alpha \\ \backslash \beta \end{matrix}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 2\left(\frac{k^2-2}{k^2-1}\right) \dots\dots (1)$$

$$\text{and } \frac{\alpha}{\beta} = -1 \dots\dots (2)$$

$$\therefore 2\left(\frac{k^2-2}{k^2-1}\right)(-1) = -5$$

$$\Rightarrow k^2 = \frac{1}{3}$$

$$\text{and } b = S \cdot R = 2\left(\frac{k^2-2}{k^2-1}\right) - 1 = 4$$

$$\therefore \frac{b}{k^2} = \frac{4}{\frac{1}{3}} = 12$$

Question26

Considering the principal values of the inverse trigonometric functions, the sum of all the solutions of the equation $\cos^{-1}(x) - 2\sin^{-1}(x) - \cos^{-1}(2x)$ is equal to :
[28-Jul-2022-Shift-1]

Options:

A. 0

B. 1

C. $\frac{1}{2}$

D. $-\frac{1}{2}$

Answer: A

Solution:

Solution:

$$\cos^{-1}x - 2 \sin^{-1}x = \cos^{-1}2x$$

$$\text{For Domain : } x \in \left[-\frac{1}{2}, \frac{1}{2} \right]$$

$$\cos^{-1}x - 2 \left(\frac{\pi}{2} - \cos^{-1}x \right) = \cos^{-1}(2x)$$

$$\Rightarrow \cos^{-1}x + 2\cos^{-1}x = \pi + \cos^{-1}2x$$

$$\Rightarrow \cos(3\cos^{-1}x) = -\cos(\cos^{-1}2x)$$

$$\Rightarrow 4x^3 = x$$

$$\Rightarrow x = 0, \pm \frac{1}{2}$$

Question 27

If $\frac{\sin^{-1}(x)}{a} = \frac{\cos^{-1}x}{b} = \frac{\tan^{-1}y}{c}$ $0 < x < 1$, then the value of $\cos\left(\frac{\pi c}{a+b}\right)$ is

[2021, 26 Feb. Shift-1]

Options:

A. $\frac{1-y^2}{y\sqrt{y}}$

B. $1-y^2$

C. $\frac{1-y^2}{1+y^2}$

D. $\frac{1-y^2}{2y}$

Answer: C

Solution:

Solution:

$$\frac{\sin^{-1}x}{a} = \frac{\cos^{-1}x}{b} = \frac{\tan^{-1}y}{c} \dots (i)$$

Take first two terms of Eq. (i)

$$\frac{\sin^{-1}x}{a} = \frac{\cos^{-1}x}{b}$$

$$\Rightarrow \frac{\sin^{-1}x}{a} = \frac{\cos^{-1}x}{b} = \frac{\sin^{-1}x + \cos^{-1}x}{a+b}$$

$$\Rightarrow \frac{\sin^{-1}x}{a} = \frac{\cos^{-1}x}{b} = \frac{\pi/2}{a+b}$$

$$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \right]$$

$$\Rightarrow \frac{\sin^{-1}x}{a} = \frac{\cos^{-1}x}{b} = \frac{\pi/2}{a+b} = \frac{\tan^{-1}y}{c}$$

Using last two terms,

$$\frac{\tan^{-1}y}{c} = \frac{\pi/2}{a+b}$$

$$\Rightarrow \tan^{-1}y = \frac{\pi c}{2(a+b)}$$

$$\Rightarrow 2\tan^{-1}y = \frac{\pi c}{(a+b)}$$

$$\Rightarrow \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right) = \frac{\pi c}{a+b}$$

$$\left[\because 2\tan^{-1}y = \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right) \right]$$

$$\Rightarrow \frac{1-y^2}{1+y^2} = \cos\left(\frac{\pi c}{a+b}\right)$$

$$\therefore \cos\left(\frac{\pi c}{a+b}\right) = \frac{1-y^2}{1+y^2}$$

Question 28

cosec $\left[2\cot^{-1}(5) + \cos^{-1}\left(\frac{4}{5}\right) \right]$ is equal to [2021, 25 Feb. Shift-II]

Options:

- A. $\frac{56}{33}$
- B. $\frac{65}{33}$
- C. $\frac{65}{56}$
- D. $\frac{75}{56}$

Answer: C

Solution:

Solution:

$$\begin{aligned} & \text{cosec}[2\cot^{-1}(5) + \cos^{-1}(4/5)] \\ &= \text{cosec}\left[2\tan^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right)\right] \\ & \left[\because \tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right) \right] \\ &= \text{cosec}\left[\tan^{-1}\left(\frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\right) + \cos^{-1}\left(\frac{4}{5}\right)\right] \\ & \left[\because 2\tan^{-1}\theta = \tan^{-1}\left(\frac{2\theta}{1-\theta^2}\right) \right] \\ &= \text{cosec}\left(\tan^{-1}\frac{5}{12} + \cos^{-1}\frac{4}{5}\right) \\ & \text{Let } \tan^{-1}\left(\frac{5}{12}\right) = x, \text{ then } \tan x = \frac{5}{12} \text{ gives} \\ & \sin x = \frac{5}{13}, \cos x = \frac{12}{13} \\ & \text{Let } \cos^{-1}\left(\frac{4}{5}\right) = y, \text{ then } \cos y = \frac{4}{5} \text{ gives,} \\ & \sin y = \frac{3}{5} \\ & \text{Now, } \text{cosec}(x+y) = \frac{1}{\sin(x+y)} \end{aligned}$$

$$= \frac{1}{\sin x \cos y + \cos x \sin y}$$

$$= \frac{1}{\left(\frac{5}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right)} = \frac{65}{56}$$

Question29

A possible value of $\tan \left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8} \right)$ is
[2021, 24 Feb. Shift-11]

Options:

- A. $\frac{1}{\sqrt{7}}$
- B. $2\sqrt{2} - 1$
- C. $\sqrt{7} - 1$
- D. $\frac{1}{2\sqrt{2}}$

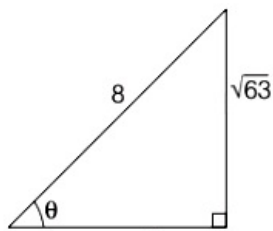
Answer: A

Solution:

Solution:

Given, $\tan \left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8} \right)$

Let $\sin^{-1} \frac{\sqrt{63}}{8} = \theta$



$$\Rightarrow \sin \theta = \frac{\sqrt{63}}{8}$$

$$\Rightarrow \cos \theta = \frac{1}{8}$$

$$\text{Also, } \cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}}$$

$$\frac{1 + \frac{1}{8}}{2} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$$\therefore \tan \left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8} \right) = \tan \left(\frac{\theta}{4} \right)$$

$$= \sqrt{\frac{1 - \cos \frac{\theta}{2}}{1 + \cos \frac{\theta}{2}}}$$

$$= \sqrt{\frac{1 - \frac{3}{4}}{1 + \frac{3}{4}}} = \frac{1}{\sqrt{7}}$$

Question30

If $\cot^{-1}(\alpha) = \cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + \dots$ upto 100 terms, then α is
[2021, 17 March Shift-I]

Options:

- A. 1.01
- B. 1.00
- C. 1.02
- D. 1.03

Answer: A

Solution:

Solution:

$\cot^{-1}\alpha = \cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 \dots$ upto 100 terms Let T_n be the n th term of $\cot^{-1}\alpha$.

$$T_n = \cot^{-1}(2n^2) = \tan^{-1}\left(\frac{1}{2n^2}\right)$$

$$= \tan^{-1}\left[\frac{(2n+1) - (2n-1)}{1 + (2n+1)(2n-1)}\right]$$

$$\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\therefore \tan^{-1}\left\{\frac{(2n+1) - (2n-1)}{1 + (2n+1)(2n-1)}\right\}$$

$$= \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

$$T_1 = \tan^{-1}3 - \tan^{-1}1$$

$$T_2 = \tan^{-1}5 - \tan^{-1}3$$

$$T_3 = \tan^{-1}7 - \tan^{-1}5$$

$\vdots$

$$T_{99} = \tan^{-1}199 - \tan^{-1}197$$

$$T_{100} = \tan^{-1}201 - \tan^{-1}199$$

$$\Sigma T_r = \tan^{-1}201 - \tan^{-1}1$$

$$= \tan^{-1}\left(\frac{201-1}{1+201 \cdot 1}\right) = \tan^{-1}\left(\frac{200}{202}\right)$$

$$\Rightarrow \cot^{-1}(\alpha) = \tan^{-1}\left(\frac{200}{202}\right)$$

$$\Rightarrow \cot^{-1}(\alpha) = \cot^{-1}\left(\frac{202}{200}\right)$$

$$\therefore \alpha = \frac{202}{200} = \frac{101}{100} = 1.01$$

Question31

The sum of possible values of x for

$$\tan^{-1}(x+1) + \cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1}\left(\frac{8}{31}\right) \text{ is}$$

[2021, 17 March Shift-1]

Options:

A. $\frac{-32}{4}$

B. $-\frac{31}{4}$

C. $-\frac{30}{4}$

D. $-\frac{33}{4}$

Answer: A

Solution:

Solution:

$$\tan^{-1}(x+1) + \cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$$

$$\Rightarrow \tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}(8/31)$$

$$\Rightarrow \tan^{-1}\left[\frac{(x+1) + (x-1)}{1 - (x+1)(x-1)}\right] = \tan^{-1}\left(\frac{8}{31}\right)$$

$$\Rightarrow \tan^{-1}\left[\frac{2x}{1 - (x^2 - 1)}\right] = \tan^{-1}\left(\frac{8}{31}\right)$$

$$\Rightarrow \frac{2x}{2 - x^2} = \frac{8}{31}$$

$$\Rightarrow 62x = 16 - 8x^2$$

$$\Rightarrow 8x^2 + 62x - 16 = 0$$

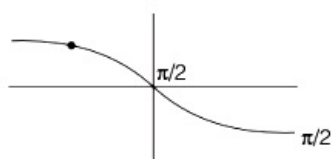
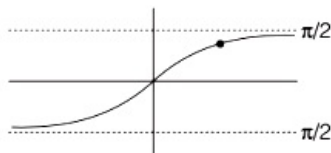
$$\Rightarrow 2(x+8)(4x-1) = 0$$

$$\Rightarrow x = -8, 1/4$$

But at $x = 1/4$

$$\text{H S S} \Rightarrow \tan^{-1}\left(1 + \frac{1}{4}\right) + \cot^{-1}\left(\frac{1}{\frac{1}{4} - 1}\right)$$

$$= \tan^{-1}\frac{5}{4} + \cot^{-1}\left(\frac{-4}{3}\right)$$



$$= \tan^{-1}\frac{5}{4} + \cot^{-1}\left(\frac{-4}{3}\right)_{>\pi/2 >\pi/2}$$

$$\therefore \text{LH S} > \pi/2$$

$$\text{RH S} = \tan^{-1}\left(\frac{8}{31}\right)_{<\pi/2}$$

As, LH S $> \pi/2$ and RH S $< \pi/2$.

So, $x = -8$ is the only solution.

Question32

The number of solutions of the equation

$\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$, for $x \in [-1, 1]$, and $[x]$ denotes the greatest integer less than or equal to x , is

[2021,17 March Shift-II]

Options:

A. 2

B. 0

C. 4

D. infinite

Answer: B

Solution:

Solution:

$$\text{Given, } \sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$$

$$\sin^{-1}\left[\left(x^2 - \frac{2}{3} + 1\right)\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$$

$$\Rightarrow \sin^{-1}\left[\left(x^2 - \frac{2}{3}\right) + 1\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] \dots (i)$$

$$(\because [x + n] = [x] + n, n \in \mathbb{I})$$

$$\therefore \left[x^2 - \frac{2}{3}\right] \text{ gives always integral values.}$$

$$\therefore \left[x^2 - \frac{2}{3}\right] = 0, -1 \text{ are possible values for } x \in [-1, 1]$$

$$[\because -1 \leq x \leq 1]$$

$$\therefore 0 \leq x^2 \leq 1$$

$$\Rightarrow -\frac{2}{3} \leq x^2 - \frac{2}{3} \leq 1 - \frac{2}{3}$$

$$\Rightarrow -\frac{2}{3} \leq x^2 - \frac{2}{3} \leq \frac{1}{3}$$

$$\Rightarrow -0.66 \leq x^2 - \frac{2}{3} \leq 0.33$$

$$\therefore \left[x^2 - \frac{2}{3}\right] = -1, 0 \text{ are possibilities. }]$$

$$\text{Case I If } \left[x^2 - \frac{2}{3}\right] = 0$$

Then, Eq. (i) becomes,

$$\sin^{-1}(1) + \cos^{-1}(0) = x^2$$

$$\Rightarrow x^2 = \pi$$

$$\Rightarrow x = \pm\sqrt{\pi}$$

But at this value of

$$x^2, \left[x^2 - \frac{2}{3}\right] \neq 0$$

$$\text{Case II If } \left[x^2 - \frac{2}{3}\right] = -1$$

Then, Eq. (i) becomes,

$$\sin^{-1}(0) + \cos^{-1}(-1) = x^2$$

$$\Rightarrow x^2 = \pi$$

$$\Rightarrow x = \pm\sqrt{\pi}$$

$$\text{But at this value of } x^2, \left[x^2 - \frac{2}{3}\right] \neq -1$$

$$\therefore x^2 = \pi$$

(Rejected also)

Hence, there is no solution for Eq. (i).

$$\therefore \text{Total number of solution (s)} = 0$$

Question33

Let $S_k = \sum_{r=1}^k \tan^{-1} \left(\frac{6^r}{2^{2r+1} + 3^{2r+1}} \right)$.

Then, $\lim_{k \rightarrow \infty} S_k$ is equal to

[2021, 16 March Shift-1]

Options:

A. $\tan^{-1} \left(\frac{3}{2} \right)$

B. $\frac{\pi}{2}$

C. $\cot^{-1} \left(\frac{3}{2} \right)$

D. $\tan^{-1}(3)$

Answer: C

Solution:

Solution:

$$\begin{aligned} S_k &= \sum_{r=1}^k \tan^{-1} \left(\frac{6^r}{2^{2r+1} + 3^{2r+1}} \right) \\ &= \sum \tan^{-1} \frac{2^r 3^{r+1} - 3^r 2^{r+1}}{2^{2r+1} \left[1 + \left(\frac{3}{2} \right)^{2r+1} \right]} \\ &= \sum \tan^{-1} \frac{2^{2r+1} \left[\left(\frac{3}{2} \right)^{r+1} - \left(\frac{3}{2} \right)^r \right]}{2^{2r+1} \left[1 + \left(\frac{3}{2} \right)^{2r+1} \right]} \\ &= \sum_{r=1}^k \tan^{-1} \frac{\left(\frac{3}{2} \right)^{r+1} - \left(\frac{3}{2} \right)^r}{1 + \left(\frac{3}{2} \right)^{2r+1}} \\ &= \sum_{r=1}^k \tan^{-1} \left(\frac{3}{2} \right)^{r+1} - \tan^{-1} \left(\frac{3}{2} \right)^r \\ &= \tan^{-1} \left(\frac{3}{2} \right)^2 - \tan^{-1} \left(\frac{3}{2} \right) \\ &\quad \tan^{-1} \left(\frac{3}{2} \right)^3 - \tan^{-1} \left(\frac{3}{2} \right)^2 \\ &\quad \vdots \\ &\quad \tan^{-1} \left(\frac{3}{2} \right)^{k+1} - \tan^{-1} \left(\frac{3}{2} \right)^k \\ S_k &= \tan^{-1} \left(\frac{3}{2} \right)^{k+1} - \tan^{-1} \left(\frac{3}{2} \right) \end{aligned}$$

When $k \rightarrow \infty$, $\tan^{-1} \left(\frac{3}{2} \right)^{k+1} \rightarrow \pi / 2$

$\lim_{k \rightarrow \infty} S_k = \frac{\pi}{2} - \tan^{-1} \left(\frac{3}{2} \right) = \cot^{-1} \left(\frac{3}{2} \right)$

Question34

Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy

$\sin^{-1}\left(\frac{3x}{2}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1}x$ is equal to
[2021, 16 March Shift-II]

Options:

- A. 2
- B. 1
- C. 3
- D. 0

Answer: C

Solution:

Solution:

$$\text{Given, } \sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1}x$$

$$\Rightarrow \sin^{-1}\left(\frac{3x}{5}\sqrt{1 - \frac{16x^2}{25}} + \frac{4x}{5}\sqrt{1 - \frac{9x^2}{25}}\right) = \sin^{-1}x$$

$$\Rightarrow 3x\sqrt{25 - 16x^2} + 4x\sqrt{25 - 9x^2} = 25x$$

$$\Rightarrow \sqrt{225x^2 - 114x^4} + \sqrt{400x^2 - 144x^4}$$

$$= 25x$$

$$= 25x \text{ (i)}$$

$$(400x^2 - 144x^4) - (225x^2 - 144x^4) = 175x^2 \dots \text{(ii)}$$

On dividing Eq. (ii) by Eq. (i),

$$\sqrt{400x^2 - 144x^4} - \sqrt{225x^2 - 144x^4}$$

$$= 7x \dots \text{(iii)}$$

Now, adding Eqs. (i) and (iii),

$$2\sqrt{400x^2 - 144x^4} = 32x$$

$$\Rightarrow 400x^2 - 144x^4 = 256x^2$$

$$\Rightarrow 144x^2 - 144x^4 = 0$$

$$\Rightarrow 144x^2(1 - x^2) = 0$$

$$x = 0, -1, 1$$

Hence, 3 real values for x satisfies the equation.

Question35

The number of real roots of the equation
 $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{4}$ is
[2021, 20 July Shift-1]

Options:

- A. 1
- B. 4
- C. 3
- D. 0

Answer: D

Solution:

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{4}$$

Domain, $x(x+1) \geq 0$

$$0 \leq x^2 + x + 1 \leq 1$$

So, only when $x^2 + x = 0$, equation will be define(d)

$$x = 0, -1$$

$$\text{At } x = 0, \tan^{-1}0 + \sin^{-1}1 = \frac{\pi}{2}$$

$$x = -1, \tan^{-1}0 + \sin^{-1}1 = \frac{\pi}{2}$$

$\therefore$ No solution.

Question36

The value of $\tan\left(2\tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)\right)$ is equal to
[2021, 20 July Shift-II]

Options:

A. $\frac{-181}{69}$

B. $\frac{220}{21}$

C. $\frac{-291}{76}$

D. $\frac{151}{63}$

Answer: B

Solution:

Solution:

Let

$$A = \tan\left(2\tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)\right) \dots (i)$$

$$\text{Now, using } 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$$

$$2\tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{2\left(\frac{3}{5}\right)}{1-\left(\frac{3}{5}\right)^2}\right)$$

$$= \tan^{-1}\left(\frac{\frac{6}{5}}{1-\frac{9}{25}}\right)$$

$$= \tan^{-1}\left(\frac{30}{16}\right) = \tan^{-1}\left(\frac{15}{8}\right)$$

$$\text{Let } \sin^{-1}\left(\frac{5}{13}\right) = \theta, \text{ then } \sin\theta = \frac{5}{13}$$

$$\therefore \tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{\frac{5}{13}}{\sqrt{1-\left(\frac{5}{13}\right)^2}} = \frac{\frac{5}{13}}{\frac{12}{13}}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{5}{12}\right) = \sin^{-1}\left(\frac{5}{13}\right)$$

From Eq. (i),

$$A = \tan\left(\tan^{-1}\left(\frac{15}{8}\right) + \tan^{-1}\left(\frac{5}{12}\right)\right)$$

$$= \tan\left[\tan^{-1}\left(\frac{\frac{15}{8} + \frac{5}{12}}{1 - \left(\frac{15}{8}\right)\left(\frac{5}{12}\right)}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{\frac{55}{24}}{\frac{21}{96}}\right)\right]$$

$$= \tan\left[\tan^{-1}\frac{55 \times 4}{21}\right]$$

$$= \tan\left(\tan^{-1}\frac{220}{21}\right) = \frac{220}{21}$$

Question37

If $(\sin^{-1}x)^2 - (\cos^{-1}x)^2 = a$, $0 < x < 1$, $a \neq 0$, then the value of $2x^2 - 1$ is
[2021, 27 Aug. Shift-1]

Options:

A. $\cos\left(\frac{4a}{\pi}\right)$

B. $\sin\left(\frac{2a}{\pi}\right)$

C. $\cos\left(\frac{2a}{\pi}\right)$

D. $\sin\left(\frac{4a}{\pi}\right)$

Answer: B

Solution:

Solution:

$$\text{Given, } (\sin^{-1}x)^2 - (\cos^{-1}x)^2 = a$$

$$\Rightarrow (\sin^{-1}x + \cos^{-1}x)(\sin^{-1}x - \cos^{-1}x) = a$$

$$\Rightarrow \frac{\pi}{2}(\sin^{-1}x - \cos^{-1}x) = a$$

$$\Rightarrow \frac{\pi}{2} - 2\cos^{-1}x = \frac{2a}{\pi}$$

$$\Rightarrow 2\cos^{-1}x = \frac{\pi}{2} - \frac{2a}{\pi}$$

$$\Rightarrow \cos^{-1}(2x^2 - 1) = \frac{\pi}{2} - \frac{2a}{\pi}$$

$$\Rightarrow 2x^2 - 1 = \cos\left(\frac{\pi}{2} - \frac{2a}{\pi}\right)$$

$$\Rightarrow 2x^2 - 1 = \sin\left(\frac{2a}{\pi}\right)$$

Question38

Let M and m respectively be the maximum and minimum values of the function $f(x) = \tan^{-1}(\sin x + \cos x)$ in $\left[0, \frac{\pi}{2}\right]$, then the value of $\tan(M - m)$ is
[2021, 27 Aug. Shift-II]

Options:

- A. $2 + \sqrt{3}$
- B. $2 - \sqrt{3}$
- C. $3 + 2\sqrt{2}$
- D. $3 - 2\sqrt{2}$

Answer: D

Solution:

Solution:

We have, $f(x) = \tan^{-1}(\sin x + \cos x)$

$$\because x \in \left[0, \frac{\pi}{2}\right]$$

$$\Rightarrow 1 \leq \sin x + \cos x \leq \sqrt{2}$$

$$[\because -\sqrt{A^2 + B^2} \leq A \sin x + B \cos x \leq \sqrt{A^2 + B^2}]$$

$$\Rightarrow \tan^{-1}(1) \leq \tan^{-1}(\sin x + \cos x) \leq \tan^{-1}(\sqrt{2})$$

$$\therefore m = \tan^{-1}(1) \text{ and } M = \tan^{-1}(\sqrt{2})$$

$$\therefore M - m = \tan^{-1}\sqrt{2} - \tan^{-1}(\sqrt{1})$$

$$= \tan^{-1}\left(\frac{\sqrt{2} - 1}{1 + \sqrt{2}}\right)$$

$$= \tan^{-1}(3 - 2\sqrt{2})$$

Question 39

If $y(x) = \cot^{-1}\left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right)$

• $x \in \left(\frac{\pi}{2}, \pi\right)$, then $\frac{dy}{dx}$ at $x = \frac{5\pi}{6}$ is

[2021, 27 Aug. Shift-II]

Options:

- A. $-\frac{1}{2}$
- B. -1
- C. $\frac{1}{2}$
- D. 0

Answer: A

Solution:

Solution:

$$y(x) = \cot^{-1} \left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} \right)$$

$$x \in \left(\frac{\pi}{2}, \pi \right)$$

$$= \cot^{-1} \frac{\left| \cos \frac{x}{2} + \sin \frac{x}{2} \right| + \left| \cos \frac{x}{2} - \sin \frac{x}{2} \right|}{\left| \cos \frac{x}{2} + \sin \frac{x}{2} \right| - \left| \cos \frac{x}{2} - \sin \frac{x}{2} \right|}$$

$$= \cot^{-1} \left(\frac{\cos \frac{x}{2} + \sin \frac{x}{2} + \sin \frac{x}{2} - \cos \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2} - \sin \frac{x}{2} + \cos \frac{x}{2}} \right)$$

$$\left[\because \frac{x}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right) \right]$$

$$= \cot^{-1} \left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right) = \cot^{-1} \left(\tan \frac{x}{2} \right)$$

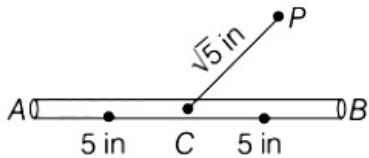
$$= \cot^{-1} \left(\cot \left(\frac{\pi}{2} - \frac{x}{2} \right) \right)$$

$$\Rightarrow y(x) = \frac{\pi}{2} - \frac{x}{2}$$

$$\therefore y'(x) = \frac{-1}{2}$$

Question40

A 10 inches long pencil AB with mid-point C and a small eraser P are placed on the horizontal top of a table such that $PC = \sqrt{5}$ inches and $\angle PCB = \tan^{-1}(2)$. The acute angle through which the pencil must be rotated about C so that the perpendicular distance between eraser and pencil becomes exactly 1 inch is



[2021, 26 Aug. Shift-II]

Options:

A. $\tan^{-1} \left(\frac{3}{4} \right)$

B. $\tan^{-1}(1)$

C. $\tan^{-1} \left(\frac{4}{3} \right)$

D. $\tan^{-1} \left(\frac{1}{2} \right)$

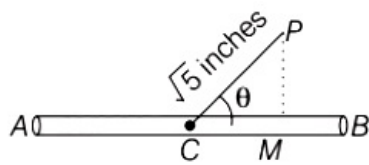
Answer: A

Solution:

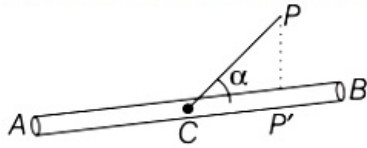
Solution:

Let the angle before rotation be θ , then

$$\theta = \angle PCB = \tan^{-1}(2)$$



Now, after rotation let angle become α .



In $\triangle PCM$

$$PC = \sqrt{5}$$

$$PM = \sqrt{5} \sin \theta = \sqrt{5} \left(\frac{2}{\sqrt{5}} \right) = 2$$

After rotation perpendicular distance becomes $PP' = 1$

$$PC \sin \alpha = 1$$

$$\sqrt{5} \sin \alpha = 1$$

$$\Rightarrow \sqrt{5} \sin \alpha = 1$$

$$\Rightarrow \sin \alpha = \frac{1}{\sqrt{5}}$$

$$\Rightarrow \tan \alpha = \frac{\sin \alpha}{\sqrt{1 - \sin^2 \alpha}} = \frac{1}{2}$$

$\therefore$ Rotated angle

$$= \theta - \alpha = \tan^{-1}(2) - \tan^{-1}\left(\frac{1}{2}\right)$$

$$= \tan^{-1}\left(\frac{2 - 1/2}{1 + 2 \times 1/2}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

Question41

$\cos^{-1}(\cos(-5)) + \sin^{-1}(\sin(6)) - \tan^{-1}(\tan(12))$ is equal to (The inverse trigonometric functions take the principal values)
[2021, 01 Sep. Shift-2]

Options:

A. $3\pi - 11$

B. $4\pi - 9$

C. $4\pi - 11$

D. $3\pi + 1$

Answer: C

Solution:

Solution:

$$\cos^{-1}(\cos(-5)) + \sin^{-1}(\sin(6))$$

$$- \tan^{-1}(\tan(12))$$

$$= 2\pi - 5 + (-2\pi + 6) - (12 - 4\pi)$$

$$= 4\pi - 11$$

Question42

$2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} \right)$ is equal to:

[Sep. 03, 2020 (I)]

Options:

A. $\frac{\pi}{2}$

B. $\frac{5\pi}{4}$

C. $\frac{3\pi}{2}$

D. $\frac{7\pi}{4}$

Answer: C

Solution:

Solution:

$$\begin{aligned} & 2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} \right) \\ &= 2\pi - \left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{16}{63} \right) \left[\because \sin^{-1}\frac{4}{5} = \tan^{-1}\frac{4}{3} \right] \\ &= 2\pi - \left\{ \tan^{-1} \left(\frac{\frac{4}{3} + \frac{5}{12}}{1 - \frac{4}{3} \cdot \frac{5}{12}} \right) + \tan^{-1}\frac{16}{63} \right\} \\ &= 2\pi - \left(\tan^{-1}\frac{63}{16} + \tan^{-1}\frac{16}{63} \right) \\ &= 2\pi - \left(\tan^{-1}\frac{63}{16} + \cot^{-1}\frac{63}{16} \right) \\ &= 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}. \end{aligned}$$

Question43

If S is the sum of the first 10 terms of the

series $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{21}\right) + \dots$ then $\tan(S)$ is equal to:

[Sep. 05, 2020 (I)]

Options:

A. $\frac{5}{6}$

B. $\frac{5}{11}$

C. $-\frac{6}{5}$

D. $\frac{10}{11}$

Answer: A

Solution:

Solution:

$$S = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \dots \text{ upto 10 terms}$$

$$= \tan^{-1}(2 - 11 + 2 \cdot 1) + \tan^{-1}\left(\frac{3 - 2}{1 + 3 \cdot 2}\right)$$

$$+ \tan^{-1}\left(\frac{4 - 3}{1 + 3 \cdot 4}\right) + \dots + \tan^{-1}\left(\frac{11 - 10}{1 + 11 \cdot 10}\right) = (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) +$$

$$(\tan^{-1}4 - \tan^{-1}3) + \dots + (\tan^{-1}11 - \tan^{-1}10)$$

$$= \tan^{-1}11 - \tan^{-1}1 = \tan^{-1}\left(\frac{11 - 1}{1 + 11 \cdot 1}\right) = \tan^{-1}\left(\frac{5}{6}\right)$$

$$\therefore \tan(S) = \frac{5}{6}$$

Question44

Considering only the principal values of inverse functions, the set

$$A = \left\{ x \geq 0 : \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4} \right\}$$

[Jan. 12, 2019 (I)]

Options:

A. contains two elements

B. contains more than two elements

C. is a singleton

D. is an empty set

Answer: C

Solution:

Solution:

$$\text{Consider, } \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1 - 6x^2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{1 - 6x^2} = 1 \Rightarrow 5x = 1 - 6x^2$$

$$\Rightarrow 6x^2 + 5x - 1 = 0$$

$$\Rightarrow (6x - 1)(x + 1) = 0$$

$$\Rightarrow x = \frac{1}{6} \text{ (as } x \geq 0)$$

Therefore, A is a singleton set.

Question45

All x satisfying the inequality $(\cot^{-1}x)^2 - 7(\cot^{-1}x) + 10 > 0$, lie in the interval :

[Jan. 11, 2019 (II)]

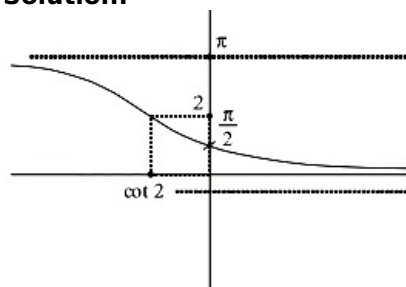
Options:

- A. $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$
- B. $(\cot 2, \infty)$
- C. $(-\infty, \cot 5) \cup (\cot 2, \infty)$
- D. $(\cot 5, \cot 4)$

Answer: B

Solution:

Solution:



$$(\cot^{-1}x)^2 - 7(\cot^{-1}x) + 10 > 0$$

$$(\cot^{-1}x - 5)(\cot^{-1}x - 2) > 0$$

$$\cot^{-1}x \in (-\infty, 2) \cup (5, \infty) \dots\dots(1)$$

But $\cot^{-1}x$ lies in $(0, \pi)$

Now, from equation (1)

$$\cot^{-1}x \in (0, 2)$$

Now, it is clear from the graph $x \in (\cot 2, \infty)$

Question46

The value of $\cot \left(\sum_{n=1}^{19} \cot^{-1} \left(1 + \sum_{p=1}^n 2p \right) \right)$ is:

[Jan. 10, 2019 (II)]

Options:

- A. $\frac{21}{19}$
- B. $\frac{19}{21}$
- C. $\frac{22}{23}$
- D. $\frac{23}{22}$

Answer: A

Solution:

Solution:

$$\cot \left(\sum_{n=1}^{19} \cot^{-1} \left(1 + \sum_{p=1}^n 2p \right) \right)$$

$$\begin{aligned}
&= \cot \left(\sum_{n=1}^{19} \cot^{-1}(1 + n(n+1)) \right) \\
&= \cot \left(\sum_{n=1}^{19} \tan^{-1} \left(\frac{(n+1) - n}{1 + (n+1)n} \right) \right) \left[\cot^{-1}x = \tan^{-1} \left(\frac{1}{x} \right) : \text{for } x > 0 \right] \\
&= \cot \left(\sum_{n=1}^{19} (\tan^{-1}(n+1) - \tan^{-1}n) \right) \\
&= \cot(\tan^{-1}20 - \tan^{-1}1) \\
&= \cot \left(\tan^{-1} \left(\frac{20-1}{1+20 \times 1} \right) \right) \\
&= \cot \left(\tan^{-1} \left(\frac{19}{21} \right) \right) = \cot \cot^{-1} \left(\frac{21}{19} \right) = \frac{21}{19}
\end{aligned}$$

Question47

If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to:
[Jan. 09, 2019 (II)]

Options:

- A. 0
- B. 10
- C. 7π
- D. π

Answer: D

Solution:

Solution:
 $x = \sin^{-1}(\sin 10)$
 $\Rightarrow x = 3\pi - 10 \quad \left\{ \begin{array}{l} 3\pi - \frac{\pi}{2} < 10 < 3\pi + \frac{\pi}{2} \\ \Rightarrow 3\pi - x = 10 \end{array} \right.$
and $y = \cos^{-1}(\cos 10) \quad \left\{ \begin{array}{l} 3\pi < 10 < 4\pi \\ \Rightarrow 4\pi - x = 10 \end{array} \right.$
 $\Rightarrow y = 4\pi - 10$
 $\therefore y - x = (4\pi - 10) - (3\pi - 10) = \pi$

Question48

If $\cos^{-1} \left(\frac{2}{3x} \right) + \cos^{-1} \left(\frac{3}{4x} \right) = \frac{\pi}{2} \left(x > \frac{3}{4} \right)$, then x is equal to:
[Jan. 09, 2019 (I)]

Options:

- A. $\frac{\sqrt{145}}{12}$
- B. $\frac{\sqrt{145}}{10}$

C. $\frac{\sqrt{146}}{12}$

D. $\frac{\sqrt{145}}{11}$

Answer: A

Solution:

Solution:

$$\begin{aligned}\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) &= \frac{\pi}{2}; \left(x > \frac{3}{4}\right) \\ \Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) &= \frac{\pi}{2} - \cos^{-1}\left(\frac{3}{4x}\right) \\ \Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) &= \sin^{-1}\left(\frac{3}{4x}\right) \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right] \\ \sin^{-1}\left(\frac{3}{4x}\right) &= \theta \Rightarrow \sin \theta = \frac{3}{4x} \\ \Rightarrow \cos \theta &= \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \frac{9}{16x^2}} \\ \Rightarrow \theta &= \cos^{-1}(\sqrt{16x^2 - 9}/4x) \\ \therefore \cos^{-1}\left(\frac{2}{3x}\right) &= \cos^{-1}\left(\frac{\sqrt{16x^2 - 9}}{4x}\right) \\ \Rightarrow \frac{2}{3x} &= \frac{\sqrt{16x^2 - 9}}{4x} \Rightarrow x^2 = \frac{64 + 81}{9 \times 16} \Rightarrow x = \pm \sqrt{\frac{145}{144}} \\ \Rightarrow x &= \frac{\sqrt{145}}{12} \quad \left(\because x > \frac{3}{4}\right)\end{aligned}$$

Question49

If $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$, $\beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to:

[April 8, 2019 (I)]

Options:

A. $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

B. $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

C. $\tan^{-1}\left(\frac{9}{14}\right)$

D. $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

Answer: D

Solution:

Solution:

$$\because \cos \alpha = \frac{3}{5}, \text{ then } \sin \alpha = \frac{4}{5}$$

$$\Rightarrow \tan \alpha = \frac{4}{3}$$

$$\text{and } \tan \beta = \frac{1}{3}$$

$$\because \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \cdot \tan \beta}$$

$$= \frac{\frac{4}{3} - \frac{1}{3}}{1 + \frac{4}{9}} = \frac{\frac{3}{3}}{\frac{13}{9}} = \frac{9}{13}$$

$$\therefore \alpha - \beta = \tan^{-1}\left(\frac{9}{13}\right) = \sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$$

$$= \cos^{-1}\left(\frac{13}{5\sqrt{10}}\right)$$

Question 50

The value of $\sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right)$ is equal to
[April 12, 2019 (I)]

Options:

A. $\pi - \sin^{-1}\left(\frac{63}{65}\right)$

B. $\frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$

C. $\frac{\pi}{2} - \cos^{-1}\left(\frac{9}{65}\right)$

D. $\pi - \cos^{-1}\left(\frac{33}{65}\right)$

Answer: B

Solution:

Solution:

$$-\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right) = -\sin^{-1}\left(\frac{3}{5} \times \frac{5}{13} - \frac{12}{13} \times \frac{4}{5}\right)$$

$$(\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\} \text{ if } x^2 + y^2 \leq 1)$$

$$= \sin^{-1}\left(\frac{-33}{65}\right) = \sin^{-1}\left(\frac{33}{65}\right)$$

$$= \cos^{-1}\left(\frac{56}{65}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$$

Question 51

If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, where $-1 \leq x \leq 1$, $-2 \leq y \leq 2$, $x \leq \frac{y}{2}$, then for all x, y , $4x^2 - 4xy\cos\alpha + y^2$ is equal to:
[April 10, 2019 (II)]

Options:

A. $4\sin^2\alpha$

B. $2\sin^2\alpha$

C. $4\sin^2\alpha - 2x^2y^2$

D. $4\cos^2\alpha + 2x^2y^2$

Answer: A

Solution:

Solution:

Given, $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$

$$\Rightarrow \cos^{-1}\left(\frac{xy}{2} + \sqrt{1-x^2} \cdot \sqrt{1-\frac{y^2}{4}}\right) = \alpha$$

$$\Rightarrow \frac{xy}{2} + \frac{\sqrt{1-x^2}\sqrt{4-y^2}}{2} = \cos\theta$$

$$\Rightarrow xy + \sqrt{1-x^2}\sqrt{4-y^2} = 2\cos\alpha$$

$$\Rightarrow (xy - 2\cos\alpha)^2 = (1-x^2)(4-y^2)$$

$$\Rightarrow x^2y^2 + 4\cos^2\alpha - 4xy\cos\alpha = 4 - y^2 - 4x^2 + x^2y^2$$

$$\Rightarrow 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$$

Question52

**A value of x satisfying the equation $\sin[\cot^{-1}(1+x)] = \cos[\tan^{-1}x]$, is :
[Online April 9, 2017]**

Options:

A. $-\frac{1}{2}$

B. -1

C. 0

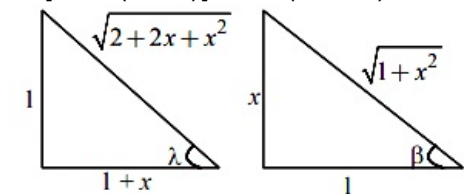
D. $\frac{1}{2}$

Answer: A

Solution:

Solution:

$$\sin[\cot^{-1}(1+x)] = \cos(\tan^{-1}x)$$



$$\text{Let; } \cot\lambda = 1+x$$

$$\tan\beta = x$$

$$\Rightarrow \sin\lambda = \cos\beta$$

$$\Rightarrow \frac{1}{\sqrt{x^2 + 2x + 2}} = \frac{1}{1\sqrt{1 + x^2}}$$

$$\Rightarrow x^2 + 2x + 2 = x^2 + 1$$

$$\Rightarrow x = -1/2$$

Question53

The value of $\tan^{-1} \left[\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right]$, $|x| < \frac{1}{2}$, $x \neq 0$, is equal to
[Online April 8, 2017]

Options:

A. $\frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$

B. $\frac{\pi}{4} + \cos^{-1}x^2$

C. $\frac{\pi}{4} - \frac{1}{2}\cos^{-1}x^2$

D. $\frac{\pi}{4} - \cos^{-1}x^2$

Answer: A

Solution:

Solution:

$$\text{Let } x^2 = \cos 2\theta; \Rightarrow \theta = \frac{1}{2}\cos^{-1}x^2$$

$$\Rightarrow \tan^{-1} \left[\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} \right] = \left[\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}} \right]$$

$$\Rightarrow \tan^{-1} \left[\frac{1+\tan\theta}{1-\tan\theta} \right] = \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \theta \right) \right]$$

$$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$$

Question54

Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1} \left(\frac{2x}{1-x^2} \right)$,

where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is :
[2015]

Options:

A. $\frac{3x-x^3}{1+3x^2}$

B. $\frac{3x+x^3}{1+3x^2}$

C. $\frac{3x - x^3}{1 - 3x^2}$

D. $\frac{3x + x^3}{1 - 3x^2}$

Answer: C

Solution:

Solution:

$$\text{Given that, } \tan^{-1}y = \tan^{-1}x + \tan^{-1}\left[\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}x + 2\tan^{-1}x = 3\tan^{-1}x$$

$$\tan^{-1}y = \tan^{-1}\left[\frac{3x - x^3}{1 - 3x^2}\right]$$

$$\Rightarrow y = \frac{3x - x^3}{1 - 3x^2}$$

Question55

If $f(x) = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $x > 1$ then $f(5)$ is equal to :

[Online April 10, 2015]

Options:

A. $\tan^{-1}\left(\frac{65}{156}\right)$

B. $\frac{\pi}{2}$

C. π

D. $4\tan^{-1}(5)$

Answer: C

Solution:

Solution:

$$f(x) = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow f(x) = 2\tan^{-1}x + \pi - 2\tan^{-1}x$$

$$\Rightarrow f(x) = \pi$$

$$\Rightarrow f(5) = \pi$$

Question56

The principal value of $\tan^{-1}\left(\cot\frac{43\pi}{4}\right)$ is:

[Online April 19, 2014]

Options:

A. $-\frac{3\pi}{4}$

B. $\frac{3\pi}{4}$

C. $-\frac{\pi}{4}$

D. $\frac{\pi}{4}$

Answer: C

Solution:

Solution:

Consider

$$\tan^{-1}\left[\cot\frac{43\pi}{4}\right] = \tan^{-1}\left[\cot\left(10\pi + \frac{3\pi}{4}\right)\right]$$

$$= \tan^{-1}\left[\cot\frac{3\pi}{4}\right] \quad [\because \cot(2n\pi + \theta) = \cot\theta]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{3\pi}{4}\right)\right]$$

$$= \frac{\pi}{2} - \frac{3\pi}{4} = \frac{2\pi - 3\pi}{4} = \frac{-\pi}{4}$$

Question 57

Statement I: The equation $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 - a\pi^3 = 0$ has a solution for all $a \geq \frac{1}{32}$.

Statement II: For any $x \in \mathbb{R}$, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ and

$$0 \leq \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 \leq \frac{9\pi^2}{16}$$

[Online April 12, 2014]

Options:

A. Both statements I and II are true.

B. Both statements I and II are false.

C. Statement I is true and statement II is false.

D. Statement I is false and statement II is true

Answer: A

Solution:

Solution:

$$\sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow -\frac{3\pi}{4} \leq \left(\sin^{-1}x - \frac{\pi}{4}\right) \leq \frac{\pi}{4}$$

$$0 \leq \left(\sin^{-1}x - \frac{\pi}{4} \right)^2 \leq \frac{9}{16}\pi^2$$

Statement II is true

$$(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = a\pi^3$$

$$\Rightarrow (\sin^{-1}x + \cos^{-1}x)[(\sin^{-1}x + \cos^{-1}x)^2 - 3\sin^{-1}x\cos^{-1}x] = a\pi^3$$

$$\Rightarrow \frac{\pi^2}{4} - 3\sin^{-1}x\cos^{-1}x = 2a\pi^2$$

$$\Rightarrow \sin^{-1}x \left(\frac{\pi}{2} - \sin^{-1}x \right) = \frac{\pi^2}{12}(1 - 8a)$$

$$\Rightarrow \left(\sin^{-1}x - \frac{\pi}{4} \right)^2 = \frac{\pi^2}{12}(8a - 1) + \frac{\pi^2}{16}$$

$$\Rightarrow \left(\sin^{-1}x - \frac{\pi}{4} \right)^2 = \frac{\pi^2}{48}(32a - 1)$$

Putting this value in equation

$$0 \leq \frac{\pi^2}{48}(32a - 1) \leq \frac{9}{16}\pi^2$$

$$\Rightarrow 0 \leq 32a - 1 \leq 27$$

$$\frac{1}{32} \leq a \leq \frac{7}{8}$$

Statement-I is also true

Question58

The number of solutions of the equation, $\sin^{-1}x = 2\tan^{-1}x$ (in principal values) is:

[Online April 22, 2013]

Options:

A. 1

B. 4

C. 2

D. 3

Answer: A

Solution:

Solution:

Given equation is

$$\sin^{-1}x = 2\tan^{-1}x$$

Now, this equation has only one solution.

$$\text{LH S} = \sin^{-1}1 = \frac{\pi}{2}$$

$$\text{and RHS} = 2\tan^{-1}1 = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$

Also, $x = 1$ gives angle value as $\frac{\pi}{4}$ and $\frac{5\pi}{4}$

$\frac{5\pi}{4}$ is outside the principal value.

Question59

If x, y, z are in A.P. and $\tan^{-1}x, \tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then

[2013]

Options:

A. $x = y = z$

B. $2x = 3y = 6z$

C. $6x = 3y = 2z$

D. $6x = 4y = 3z$

Answer: A

Solution:

Solution:

Since, x, y, z are in A.P.

$$\therefore 2y = x + z$$

Also, we have

$$2\tan^{-1}y = \tan^{-1}x + \tan^{-1}(z)$$

$$\Rightarrow \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$$\Rightarrow \frac{x+z}{1-y^2} = \frac{x+z}{1-xz} (\because 2y = x+z)$$

$$\Rightarrow y^2 = xz \text{ or } x+z=0 \Rightarrow x=y=z=0$$

Question60

**Let $x \in (0, 1)$. The set of all x such that $\sin^{-1}x > \cos^{-1}x$, is the interval:
[Online April 25, 2013]**

Options:

A. $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

B. $\left(\frac{1}{\sqrt{2}}, 1\right)$

C. $(0,1)$

D. $\left(0, \frac{\sqrt{3}}{2}\right)$

Answer: B

Solution:

Solution:

Given $\sin^{-1}x > \cos^{-1}x$ where $x \in (0, 1)$

$$\Rightarrow \sin^{-1}x > \frac{\pi}{2} - \sin^{-1}x$$

$$\Rightarrow 2\sin^{-1}x > \frac{\pi}{2} \Rightarrow \sin^{-1}x > \frac{\pi}{4}$$

$$\Rightarrow x > \sin \frac{\pi}{4} \Rightarrow x > \frac{1}{\sqrt{2}}$$

Maximum value of $\sin^{-1}x$ is $\frac{\pi}{2}$

So, maximum value of x is 1. So, $x \in \left(\frac{1}{\sqrt{2}}, 1 \right)$.

Question61

$S = \tan^{-1} \left(\frac{1}{n^2 + n + 1} \right) + \tan^{-1} \left(\frac{1}{n^2 + 3n + 3} \right) + \dots + \tan^{-1} \left(\frac{1}{1 + (n + 19)(n + 20)} \right)$, then $\tan S$ is equal to:
[Online April 23, 2013]

Options:

A. $\frac{20}{401 + 20n}$

B. $\frac{n}{n^2 + 20n + 1}$

C. $\frac{20}{n^2 + 20n + 1}$

D. $\frac{n}{401 + 20n}$

Answer: C

Solution:

Solution:

We know that,

$$\tan^{-1} \frac{1}{1+2} + \tan^{-1} \frac{1}{1+2 \times 3} + \tan^{-1} \frac{1}{1+3 \times 4} + \dots + \tan^{-1} \frac{1}{1+(n-1)n} + \tan^{-1} \frac{1}{1+n(n+1)} + \dots +$$

$$\tan^{-1} \frac{1}{1+(n+19)(n+20)} = \tan^{-1} \frac{n+19}{n+21}$$

$$\Rightarrow \tan^{-1} \frac{n-1}{n+1} + \tan^{-1} \frac{1}{1+n(n+1)} + \tan^{-1} \frac{1}{1+(n+1)(n+2)} + \dots + \frac{1}{1+(n+19)(n+20)} = \tan^{-1} \frac{n+19}{n+21}$$

$$\Rightarrow \tan^{-1} \frac{1}{1+n(n+1)} + \tan^{-1} \frac{1}{1+(n+1)(n+2)} + \dots + \frac{1}{1+(n+19)(n+20)} = \tan^{-1} \frac{n+19}{n+21} - \tan^{-1} \frac{n-1}{n+1}$$

$$\Rightarrow \tan^{-1} \left(\frac{1}{n^2 + n + 1} \right) + \tan^{-1} \left(\frac{1}{n^2 + 3n + 3} \right) + \dots + \tan^{-1} \frac{1}{1+(n+19)(n+20)}$$

$$= \tan^{-1} \left(\frac{\frac{n+19}{n+21} - \frac{n-1}{n+1}}{1 + \frac{n+19}{n+21} \times \frac{n-1}{n+1}} \right) = \tan^{-1} \frac{20}{n^2 + 20n + 1} = S$$

$$\therefore \tan^{-1} S = \frac{20}{n^2 + 20n + 1}$$

Question62

A value of x for which $\sin(\cot^{-1}(1+x)) = \cos(\tan^{-1}x)$, is
[Online April 9, 2013]

Options:

A. $-\frac{1}{2}$

B. 1

C. 0

D. $\frac{1}{2}$

Answer: A

Solution:

Solution:

$$\sin(\cot^{-1}(1+x)) = \cos(\tan^{-1}x)$$

$$\Rightarrow \operatorname{cosec}^2(\cot^{-1}(1+x)) = \sec^2(\tan^{-1}x)$$

$$\Rightarrow 1 + [\cot(\cot^{-1}(1+x))]^2 = 1 + [\tan(\tan^{-1}x)]^2$$

$$\Rightarrow (1+x)^2 = x^2 \Rightarrow x = -\frac{1}{2}$$

Question63

A value of $\tan^{-1}\left(\sin\left(\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)\right)\right)$ is

[Online May 19, 2012]

Options:

A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{6}$

Answer: D

Solution:

Solution:

$$\text{Consider } \tan^{-1}\left[\sin\left(\cos^{-1}\sqrt{\frac{2}{3}}\right)\right]$$

$$\text{Let } \cos^{-1}\sqrt{\frac{2}{3}} = \theta \Rightarrow \cos\theta = \sqrt{\frac{2}{3}}$$

$$\Rightarrow \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{2}{3}} = \sqrt{\frac{1}{3}}$$

$$\therefore \tan^{-1}\left[\sin\left(\cos^{-1}\sqrt{\frac{2}{3}}\right)\right] = \tan^{-1}[\sin\theta]$$

$$= \tan^{-1}\left[\sqrt{\frac{1}{3}}\right] = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Question64

The largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which the function, $f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$, is defined, is [2007]

Options:

A. $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$

B. $\left[0, \frac{\pi}{2}\right)$

C. $[0, \pi]$

D. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Answer: B

Solution:

Solution:

Given that

$$f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$$

$$f(x) \text{ is defined if } -1 \leq \left(\frac{x}{2} - 1\right) \leq 1 \text{ and } \cos x > 0$$

$$\Rightarrow 0 \leq \frac{x}{2} \leq 2 \text{ and } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\Rightarrow 0 \leq x \leq 4 \text{ and } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\therefore x \in \left[0, \frac{\pi}{2}\right)$$

Question65

If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the values of x is [2007]

Options:

A. 4

B. 5

C. 1

D. 3

Answer: D

Solution:

Solution:

$$\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \operatorname{cosec}^{-1}\left(\frac{5}{4}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right) \quad [\because \sin^{-1}x + \cos^{-1}x = \pi/2]$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \cos^{-1}\left(\frac{4}{5}\right)$$

$$\sin^{-1}\frac{x}{5} = \sin^{-1}\sqrt{1 - \left(\frac{4}{5}\right)^2} \quad [\because \cos^{-1}x = \sin^{-1}\sqrt{1 - x^2}]$$

$$\Rightarrow \sin^{-1}\frac{x}{5} = \sin^{-1}\frac{3}{5} \Rightarrow \frac{x}{5} = \frac{3}{5}$$

$$\Rightarrow x = 3$$

Question66

If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy \cos \alpha + y^2$ is equal to
[2005]

Options:

A. $2 \sin 2 \alpha$

B. 4

C. $4 \sin^2 \alpha$

D. $-4 \sin^2 \alpha$

Answer: C

Solution:

Solution:

$$\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$$

$$\Rightarrow \cos^{-1}\left(\frac{xy}{2} + \sqrt{(1-x^2)\left(1-\frac{y^2}{4}\right)}\right) = \alpha$$

$$\Rightarrow \cos^{-1}\left(\frac{xy + \sqrt{4-y^2-4x^2+x^2y^2}}{2}\right) = \alpha$$

$$\Rightarrow xy + \sqrt{4-y^2-4x^2+x^2y^2} = 2 \cos \alpha$$

$$\Rightarrow \sqrt{4-y^2-4x^2+x^2y^2} = 2 \cos \alpha - xy$$

Squaring both sides, we get

$$\Rightarrow 4 - y^2 - 4x^2 + x^2y^2 = 4\cos^2\alpha + x^2y^2 - 4xy \cos \alpha$$

$$\Rightarrow 4x^2 + y^2 - 4xy \cos \alpha = 4\sin^2\alpha$$

Question67

The domain of the function $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$
[2004]

Options:

A. [1, 2]

B. [2, 3)

C. [1, 2]

D. [2, 3]

Answer: B

Solution:

Solution:

$f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is defined

When $-1 \leq x-3 \leq 1 \Rightarrow 2 \leq x \leq 4$ (i)

and $9-x^2 > 0 \Rightarrow -3 < x < 3$ (ii)

from (i) and (ii),

we get $2 \leq x < 3 \therefore \text{Domain} = [2, 3)$

Question68

The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for [2003]

Options:

A. $|a| \leq \frac{1}{\sqrt{2}}$

B. $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$

C. all real values of a

D. $|a| < \frac{1}{2}$

Answer: A

Solution:

Solution:

Given that $\sin^{-1}x = 2\sin^{-1}a$

We know that $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} \leq 2\sin^{-1}a \leq \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4} \leq \sin^{-1}a \leq \frac{\pi}{4} \Rightarrow \frac{-1}{\sqrt{2}} \leq a \leq \frac{1}{\sqrt{2}}$

$\therefore |a| \leq \frac{1}{\sqrt{2}}$

Question69

$\cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x$, then $\sin x =$
[2002]

Options:

A. $\tan^2\left(\frac{\alpha}{2}\right)$

B. $\cot^2\left(\frac{\alpha}{2}\right)$

C. $\tan \alpha$

D. $\cot\left(\frac{\alpha}{2}\right)$

Answer: A

Solution:

Solution:

Given that, $\cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x$

$$\tan^{-1}\left(\frac{1}{\sqrt{\cos \alpha}}\right) - \tan^{-1}(\sqrt{\cos \alpha}) = x$$

$$\Rightarrow \tan^{-1} \frac{\frac{1}{\sqrt{\cos \alpha}} - \sqrt{\cos \alpha}}{1 + \frac{1}{\sqrt{\cos \alpha}} \cdot \sqrt{\cos \alpha}} = x$$

$$\Rightarrow \tan^{-1} \frac{1 - \cos \alpha}{2\sqrt{\cos \alpha}} = x$$

$$\Rightarrow \tan x = \frac{1 - \cos \alpha}{2\sqrt{\cos \alpha}}$$

$$\Rightarrow \cot x = \frac{2\sqrt{\cos \alpha}}{1 - \cos \alpha} = \frac{B}{P}$$

$$P = (1 - \cos \alpha) \text{ and } B = 2\sqrt{\cos \alpha}$$

$$H = \sqrt{P^2 + B^2} = 1 + \cos \alpha$$

$$\Rightarrow \sin x = \frac{1 - \cos \alpha}{1 + \cos \alpha} = \frac{1 - (1 - 2\sin^2 \alpha / 2)}{1 + 2\cos^2 \alpha / 2 - 1}$$

$$\text{or } \sin x = \tan^2 \frac{\alpha}{2}$$

Question70

The domain of $\sin^{-1}[\log_3(x / 3)]$ is
[2002]

Options:

A. [1,9]

B. [-1,9]

C. [-9,1]

D. [-9,-1]

Answer: A

Solution:

$$f(x) = \sin^{-1}\left(\log_3\left(\frac{x}{3}\right)\right)$$

We know that domain of $\sin^{-1}x$ is $x \in [-1, 1]$

$$\therefore -1 \leq \log_3\left(\frac{x}{3}\right) \leq 1 \Rightarrow 3^{-1} \leq \frac{x}{3} \leq 3^1$$

$$\Rightarrow 1 \leq x \leq 9 \text{ or } x \in [1, 9]$$
