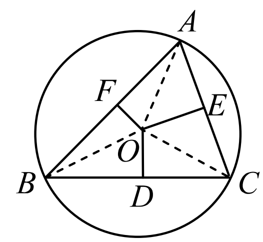
Properties & Solution of Triangle, Height & Distance

Chapter 14

THE LAW OF SINES OR SINE RULE

The sides of a triangle are proportional to the sines of the angles opposite to them

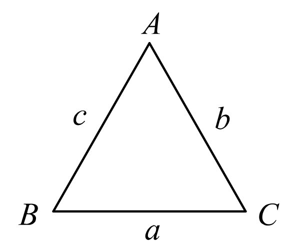
i.e., $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k, \text{(say)}$



More generally, if R be the radius of the circumcircle of the triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

THE LAW OF COSINES OR COSINE RULE

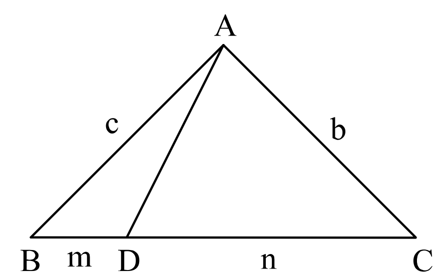


1. $a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$
2. $b^2 = c^2 + a^2 - 2ca\cos B \Rightarrow \cos B = \frac{c^2 + a^2 - b^2}{2ca}$
3. $c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$

PROJECTION FORMULAE

1. $a = b\cos C + c\cos B$
2. $b = c\cos A + a\cos C$
3. $c = a\cos B + b\cos A$

APOLLONIUS THEOREM

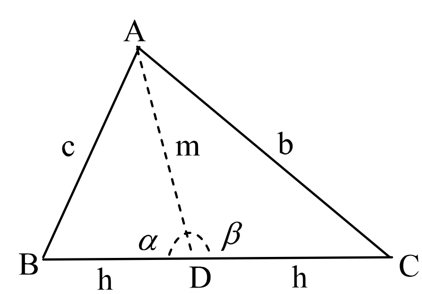


(a) $(m+n)AD^2 = mb^2 + nc^2 - mCD^2 - nBD^2$

(b) $(m+n)^2 AD^2 = (m+n)(mb^2 + nc^2) - a^2mn$

Apollonius theorem for medians

In every triangle the sum of the squares of any two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.



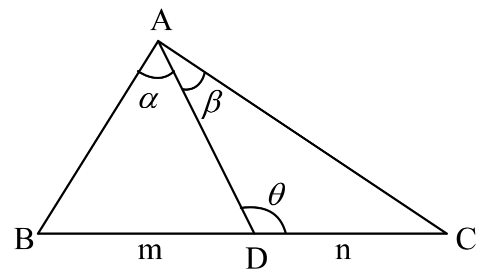
For any triangle ABC, $b^2 + c^2 = 2(h^2 + m^2) = 2\left\{m^2 + \left(a/2\right)^2\right\}$ by use of cosine rule.

If $\therefore \Delta$ be right angled , the mid point of hypotenuse is equidistant from the three vertices so that $DA = DB = DC$.

$\therefore\ b^2 + c^2 = a^2$ which is Pythagoras theorem. This theorem is very useful for solving problems of height and distance.

ANGLE BETWEEN MEDIAN AND THE SIDE OF A TRIANGLE

$$\sin\theta = \frac{2b}{\sqrt{2b^2+2c^2-a^2}}\sin C$$



THE $m-n$ Rule

If the triangle ABC, point D divides BC in the ratio $m:n$, and $\angle ADC = \theta$, then

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$;

(ii) $(m+n)\cot\theta = n\cot B - m\cot C$.

NAPIER'S ANALOGY (LAW OF TANGENTS)

For any triangle ABC,

(i) $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$ (ii) $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$

(iii) $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

MOLLWEIDE'S FORMULA

$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}, \qquad \frac{a-b}{c} = \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$$

AREA OF TRIANGLE

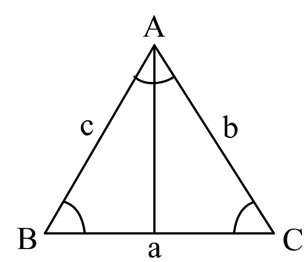
Let three angles of $a\Delta$ be denoted by A, B, C and the sides opposite to these angles by letters a, b, and c respectively

1. **When two sides and the included angle be given:**

   The area of triangle ABC is given by

   $$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

   i.e. $\Delta = \frac{1}{2}(\text{Product of two side}) \times \text{sine of included angle}$



2. **When three sides are given**

   Area of $\Delta ABC = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

   where semi perimeter of triangle $s = \frac{a+b+c}{2}$.

3. **When three sides and the circum – radius be given** :

   Area of triangle $\Delta = \frac{abc}{4R}$, where R be the circum-radius of the triangle.

4. **When two angles and included side be given** :

   $$\Delta = \frac{1}{2}a^2\frac{\sin B\sin C}{\sin(B+C)} = \frac{1}{2}b^2\frac{\sin A\sin C}{\sin(A+C)} = \frac{1}{2}c^2\frac{\sin A\sin B}{\sin(A+B)}$$

HALF ANGLE FORMULAE

If $2s$ shows the perimeter of a triangle ABC then,

$$2s = a+b+c$$

1. **Formulae for** $\sin\frac{A}{2},\quad \sin\frac{B}{2},\quad \sin\frac{C}{2}$

(i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ca}}$ (iii) $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

2. **Formulae for** $\cos\frac{A}{2},\quad \cos\frac{B}{2},\quad \cos\frac{C}{2}$

(i) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

3. **Formulae for** $\tan\frac{A}{2},\quad \tan\frac{B}{2},\quad \tan\frac{C}{2}$

(i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{\Delta}$ (ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{(s-c)(s-a)}{\Delta}$

(iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{(s-a)(s-b)}{\Delta}$ (iv) $\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = \frac{s(s-a)}{\Delta}$

(v) $\cot\frac{B}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \frac{s(s-b)}{\Delta}$ (vi) $\cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \frac{s(s-c)}{\Delta}$.

Note : $(a+b-c)(b+c-a)(c+a-b) = a^2b+b^2a+a^2c+ac^2+b^2c+bc^2-a^3-b^3-c^3$

OBLIQUE TRIANGLE

The triangle which are not right-angled is called **oblique triangle**. We can solve a triangle if we know three of its parts at least one of which is a side. Different cases are as follows.

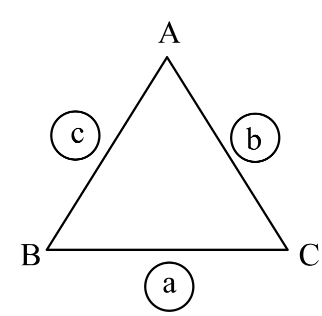
Case I. The three sides are given.
Case II. Two sides and included angles are given.
Case III. Two sides and the angle opposite to one of them are given.
Case IV. One side and two angles are given.

CASE I. Given the three sides, to solve the triangle.

Proof : Let ABC be the triangle in which all the tree sides a, b, c are given. The three angle A, B, C are to be determined.

Since, $2s=a+b+c$

$\therefore$ values of $s,\ s-a,\ s-b,\ s-c$ are known.



(i) *To find A*

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\therefore \quad \log\tan\frac{A}{2}=\frac{1}{2}\left[\log(s-b)+\log(s-c)-\log s-\log(s-a)\right]$$

$\therefore$ $\log\tan\frac{A}{2}$ and $\therefore$ $\frac{A}{2}$ can be determined with the help of tables.

$\therefore$ A is determined.

(ii) *To find B*

$$\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$\therefore \quad \log \tan \frac{B}{2} = \frac{1}{2}\left[\log(s-a) + \log(s-c) - \log s - \log(s-b)\right]$

$\therefore \quad \log \tan \frac{B}{2}$ and $\quad \therefore \quad \frac{B}{2}$ can be determined with the help of tables.

$\therefore \quad B$ is determined.

(iii) *To find C*

$A + B + C = 180°$

$\therefore \quad C = 180° - \overline{A+B}$, is determined.

Thus, A, B, C being known, the triangle is solve.

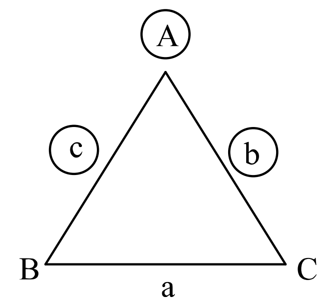
CASE II : Given two sides and the included angle ; to solve the triangle.

Proof : Let *ABC* be a triangle, in which sides *b*, *c* (*b* > *c*) and the included angle *A* are given. The side *a* and angles *B*, *C* are to be determined.

(i) *To find B and C.*

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} \quad [\textit{Napier's Analogy}]$$

$$= \frac{b-c}{b+c} \tan \frac{B+C}{2} \qquad \left[\because \cot \frac{A}{B} = \cot\left(90° - \frac{B+C}{2}\right) = \tan \frac{B+C}{2}\right]$$



$\therefore \quad \log \tan \frac{B-C}{2} = \log(b-c) + \log \tan \frac{B+C}{2} - \log(b+c)$

$\therefore \quad \log \tan \frac{B-C}{2}$ and

$\therefore \quad \frac{B-C}{2}$ can be obtained with the help of tables.

$\therefore \quad \frac{B-C}{2}$ is known. …(1)

Also $\frac{B+C}{2} = 90° - \frac{A}{2}$ $\quad [\because A + B + C = 180°]$ …(2)

$\therefore$ from (1) and (2), by addition and subtraction *B* and *C* are known.

(ii) *To find a*

since $\quad \frac{a}{\sin A} = \frac{b}{\sin B} \quad$ [*Sine formula*]

$\therefore \quad a = \frac{b \sin A}{\sin B}$

$\therefore \quad \log a = \log b + \log \sin A - \log \sin B$

$\therefore \quad \log a$ and $\quad \therefore \ a$ can be determined with the help of the table.

Thus, *B*, *C* and *a* are known, the triangle is solved.

Note: If $C > B$, then use the formula

$$\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2}.$$

CASE III : Given one sides and two angles ; to solve the triangle.

Proof : Let *ABC* be a triangle in which '*a*' be the given side and *B*, *C* be the given angles. Sides b, c and angle A are to be determined.

(i) *To find A*

$A + B + C = 180°$

$\therefore \quad A = 180° - (B + C)$

$\therefore$ A is known.

(ii) *To find b*

Since $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

$$b = \frac{a \sin B}{\sin A} \qquad [\textit{Sine formula}]$$

$\therefore \quad \log b = \log a + \log \sin B - \log \sin A$

$\therefore$ With the help of tables, $\log b$ and therefore, b is determined.

(iii) *To find c*

Again $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$ [*Sine formula*]

$$\therefore \quad c = \frac{a \sin C}{\sin A}$$

$\therefore \quad \log c = \log a + \log \sin C - \log \sin A$

$\therefore$ With the help of the tables, $\log c$ and therefore c is determined.

Thus, A, b, c being known, the triangle is solved.

CASE IV : When two sides and an angle opposite to one of them is given. (Ambiguous case)

Let the two sides say a and b and an angle A opposite to a be given.

Here we use $a / \sin A = b / \sin B$.

$\therefore \quad \sin B = b \sin A / a \qquad \ldots(1)$

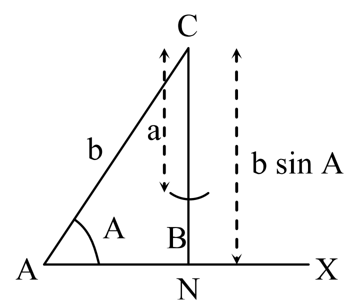
We calculate angle B from (1) and then angle C is obtained by using

$\angle C = 180° - (\angle A + \angle B)$.

Also, to find side c, we use

$a / \sin A = c / \sin C$

$\therefore \quad c = (a \sin C) / \sin A \qquad \ldots(2)$



From relation (1), the following possibilities will arise :

Case I : *When A is an acute angle.*

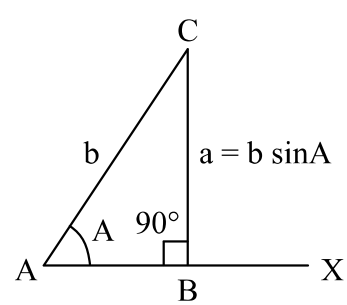
(a) If $a < b \sin A$, there is no triangle. When $a < b \sin A$, from (1), $\sin B > 1$, which is impossible. Hence no triangle is possible in this case.

From the following fig., if

$AC = b$; $\angle CAX = A$,

then perpendicular $CN = b \sin A$. Now taking c as centre, if we draw an arc of radius a then if the line AX and hence no triangle ABC can be constructed in this case.

(b) If $a = b \sin A$, then only one triangle is possible which is right angled at B.



When $a = b \sin A$, then from (1),

$\sin B = 1, \qquad \therefore \angle B = 90°$

From fig. it is clear that

$CB = a = b \sin A$.

Thus, in this case, only one triangle is possible which is right angled at B.

(c) If $a > b\sin A$, then further three possibilities will arise :

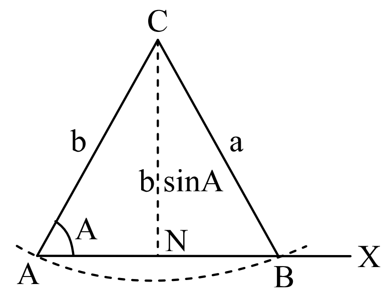
(i) $a = b$. In this case, from (1),

$$\sin B = \sin A$$

$\therefore \quad B = A$ or $B = 180° - A$

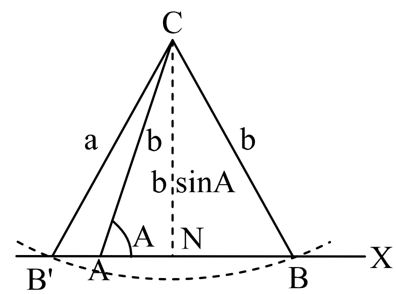
But $B = 180 - A \Rightarrow A + B - 180°$, which is not possible in *a*.

In this case we get $\angle A = \angle B$. Hence, if $b = a > b\sin A$ then only isosceles triangle ABC is possible in which $\angle A = \angle B$.



(ii) $a > b$. In the following fig., let $AC = b, \angle CAX = A$, and $a > b$, also $a > b\sin A$. Now taking C as centre, if we draw an arc of radius *a*, it will intersect *AX* at one point *B* and hence only one ΔABC is constructed. Also this arc will intersect *XA* produced at B' and $\Delta AB'C$ is also formed but this Δ is in admissible (because $\angle CAB$ is an obtuse angle in this triangle).

Hence, if $a > b$ and $a > b\sin A$, then only one triangle is possible.

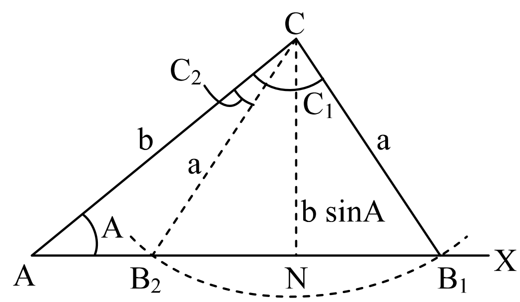


(iii) $b > a\left(i.e., b > a > b\sin A\right)$.

In the following fig., let

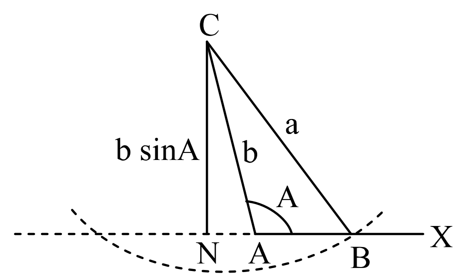
$$AC = b, \angle CAX = A.$$

Now taking C as centre, if we draw an arc of radius a, then it will intersect AX at two points B_1 and B_2. Thus two triangles AB_1C and AB_2C are formed.



Hence, if $b > a > b\sin A$, then there are two triangles.

Case II : *When A is an obtuse angle.*
In this case, there is only one triangle, if $a > b$.



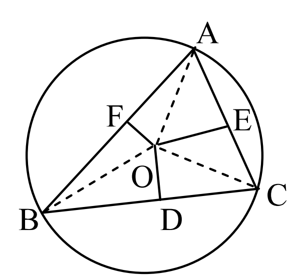
CIRCLE CONNECTED WITH TRIANGLE

1. **Circumcircle of a triangle its radius**

The circum-radius of a ΔABC is given by

(i) $\dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C} = R$

(ii) $R = \dfrac{abc}{4\Delta}$ $\left[\Delta = area\, of\ \Delta ABC\right]$



2. **Inscribed circle or in-circle of a triangle and its radius**

The radius r of the inscribed circle of a triangle ABC is given by

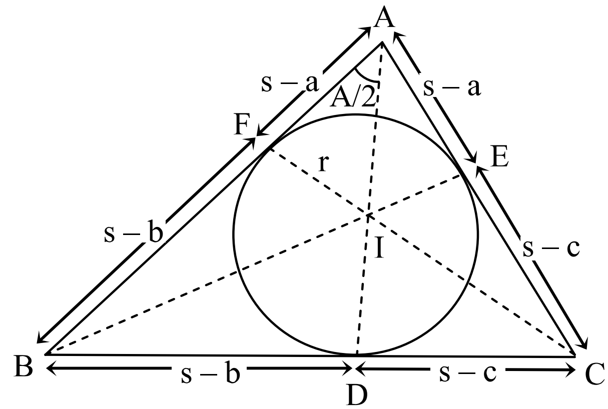
(i) $r = \dfrac{\Delta}{s}$

(ii) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iii) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

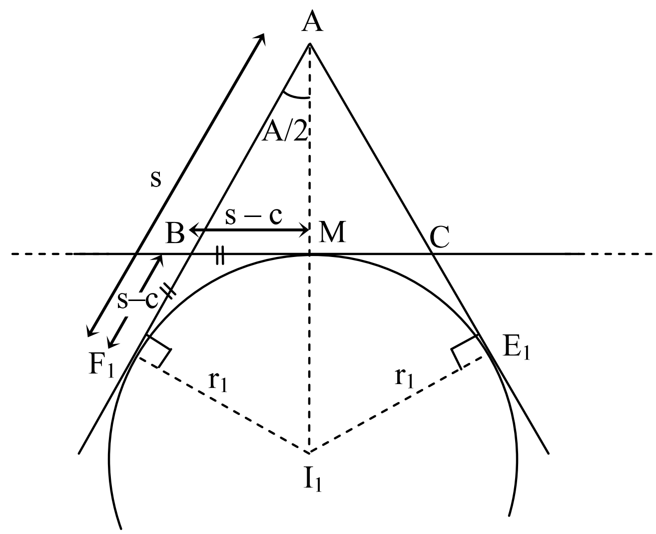
(iv) $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \dfrac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = \dfrac{c\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$

(v) $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$



3. **Escribed circle of a triangle and their radii**

In any ΔABC, we have



(i) $r_1 = \dfrac{\Delta}{s-a} = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

$= s\tan\frac{A}{2} = (s-c)\cot\frac{B}{2}.$

$= (s-b)\cot\frac{C}{2} = \dfrac{a}{\tan\frac{B}{2} + \tan\frac{C}{2}}$

(ii) $r_2 = \dfrac{\Delta}{s-b} = \dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}} = s\tan\frac{B}{2} = (s-c)\cot\frac{A}{2} = (s-a)\cot\frac{C}{2} = \dfrac{b}{\tan\frac{C}{2} + \tan\frac{A}{2}}$

(iii) $r_3 = \dfrac{\Delta}{s-c} = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}} = s\tan\frac{C}{2} = (s-b)\cot\frac{A}{2} = (s-a)\cot\frac{B}{2} = \dfrac{c}{\tan\frac{A}{2} + \tan\frac{B}{2}}$

(iv) $r_1 + r_2 + r_3 - r = 4R$

(v) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$

(vi) $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}$

(vii) $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}$

(viii) $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$

(ix) $\Delta = 2R^2 \sin A.\sin B.\sin C = 4Rr\cos\frac{A}{2}.\cos\frac{B}{2}\cos\frac{C}{2}$

(x) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$; $r_2 = 4R\cos\frac{A}{2}.\sin\frac{B}{2}.\cos\frac{C}{2}$, $r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

DISTANCE OF CIRCUMCENTRE (*O*) FROM THE ORTHOCENTRE (*H*), INCENTRE (*I*) AND EXCENTRES (I_1, I_2, I_3)

1. Distance between circumcentre (*O*) and orthocentre (*H*)

$$OH = R\sqrt{1-8\cos A\cos B\cos C}$$

2. Distance between circumcentre (*O*) and incentre (*I*)

$$OI = R\sqrt{1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R}\sqrt{R-2r}$$

3. (a) Distance between circumcentre (*O*) and excentre (I_1) of the escribed circle having opposite angle *A*

$$OI_1 = R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R}\sqrt{R+2r_1}$$

(b) Distance between circucentre (*O*) and excentre (I_2) of the escribed circle having opposite angle *B*

$$OI_2 = R\sqrt{1+8\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R}\sqrt{R+2r_2}$$

(c) Distance between circumcentre (*O*) and excentre (I_3) of the escribed circle having opposite angle *C*

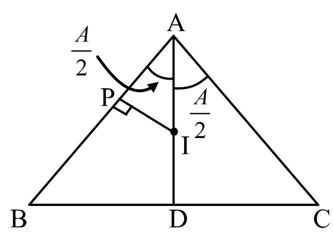
$$OI_3 = R\sqrt{1+8\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R}\sqrt{R+2r_3}$$

DISTANCE OF INCENTRE FROM THE VERTICES OF THE TRIANGLE

Let *I* be the In-centre. Let $IP. \perp AB.$ Clearly, $IP = r.\angle PAI = \frac{A}{2}$

From right angled triangle IPA,

$$\sin\frac{A}{2} = \frac{r}{AI} \Rightarrow AI = r\,\text{cosec}\frac{A}{2}$$



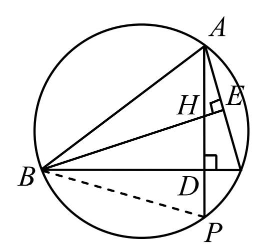
Similarly $BI = r\ \text{cosec}\frac{B}{2}$ and $CI = r\ \text{cosec}\frac{C}{2}$

Thus, $AI = r\ \text{cosec}\frac{A}{2}$, $BI = r\ \text{cosec}\frac{B}{2}$ and $CI = r\ \text{cosec}\frac{C}{2}$.

Note :

(i) The centroid of any triangle divides the join of circumcentre and orthocentre internally in the ratio 1 : 2.

(ii) If H is the orthocentre of ΔABC and AH produced meets BC at D and the circumcircle of ΔABC at P, then $HD = DP$.

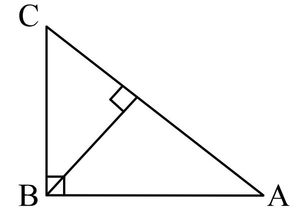


$\because BD = c\cos B$ and $DP = BD\cot C$

$\therefore DP = c\cos B.\cot C = 2R\cos B\cos C$

(iii) The orthocentre of an acute angled triangle is the incentre of the Pedal triangle.

(iv) The centre of the circum circle falls inside the triangle if triangle is acute angled but outside when it is obtuse angled. If the triangle is right angled the centre lies on mid-point of the hypotenuse.

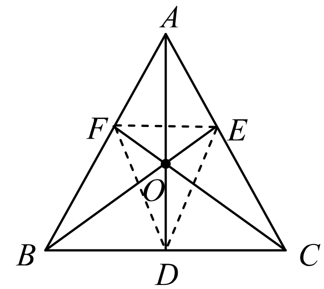


(v) The orthocentre falls inside the triangle if triangle is acute angled and outside when it is obtuse angled.
If the triangle is right angled the orthocentre (*B*) lies on the triangle.

PEDAL TRIANGLE

Let the perpendicular AD, BE and CF from the vertices A, B and C on the opposite sides BC, CA and AB of $\Delta\ ABC$ respectively, meet at O. Then O is the orthocentre of the ΔABC. The triangle DEF is called the pedal triangle of ΔABC.

Orthocentre of the triangle is the incentre of the pedal triangle. If O is the orthocentre and DEF the pedal triangle of the ΔABC, where AD, BE, CF are the perpendiculars drawn from A, B, C on the opposite side BC, CA, AB respectively, then



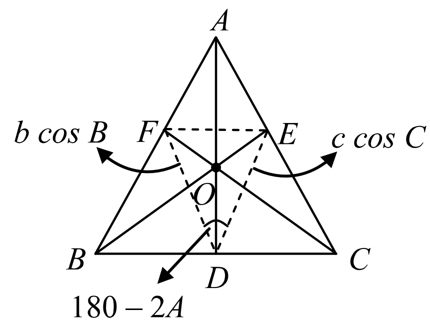
(i) $OA = 2R\cos A,\ OB = 2R\cos B$ and $OC = 2R\cos C$

(ii) $OD = 2R\cos B\cos C,\ OE = 2R\cos C\cos A$ and $OF = 2R\cos A\cos B$

1. **Sides and angles of a pedal triangle**

The angles of pedal triangle DEF are : $180-2A, 180-2B$, $180-2C$ and sides of pedal triangle are : $EF = a\cos A$ or $R\sin 2A$; $FD = b\cos B$ or $R\sin 2B$; $DE = c\cos C$ or $R\sin 2C$.

If given $\Delta\ ABC$ is obtuse, then angles are represented by $2A,\ 2B,\ 2C-180°$ and the sides are $a\cos A,\ b\cos B,\ -c\cos C$.



2. **Area and circum-radius and in-radius of pedal triangle**

Area of pedal triangle $= \frac{1}{2}$ (Product of sides) × (sine of included angle)

$$\Delta = \frac{1}{2}R^2\cdot\sin 2A\sin 2B\sin 2C$$

Circum-radius of pedal triangle $= \dfrac{EF}{2\sin FDE} = \dfrac{R\sin 2A}{2\sin(180°-2A)} = \dfrac{R}{2}$

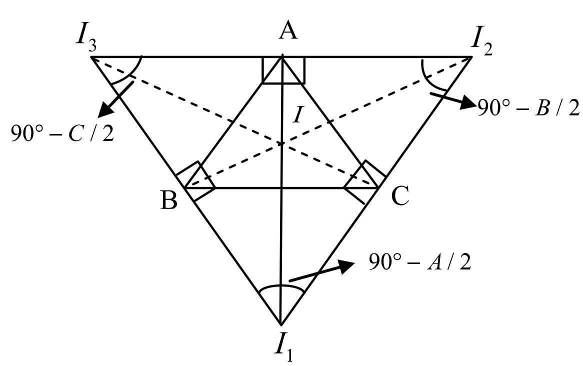
In-radius of pedal triangle $= \dfrac{\text{area of } \Delta\ DEF}{\text{semi} - \text{perimeter of } \Delta\ DEF} = \dfrac{\frac{1}{2}R^2\sin 2A.\sin 2B\sin 2C}{2R\sin A.\sin B.\sin C}$

$= 2R\cos A.\cos B.\cos C$.

EX-CENTRAL TRIANGLE

Let ABC be triangle and I be the centre of incircle. Let I_1, I_2 and I_3 be the centres of the escribed circle which are opposite to A, B, C respectively then $I_1I_2I_3$ is called the Ex-central triangle of ΔABC.

$I_1I_2I_3$ is a triangle, thus the triangle ABC is the pedal triangle of its ex-central triangle $I_1I_2I_3$.



The angle of ex-central triangle $I_1I_2I_3$ are $90^0-\frac{A}{2}, 90^0-\frac{B}{2}, 90^0-\frac{C}{2}$

$II_1 = 4R\sin\frac{A}{2},\quad II_2 = 4R\sin\frac{B}{2},\quad II_3 = 4R\sin\frac{C}{2}$

and sides are

$I_1I_3 = 4R\cos\frac{B}{2}; I_1I_2 = 4R\cos\frac{C}{2}; I_2I_3 = 4R\cos\frac{A}{2}$.

Area and circum-radius of the ex-central triangle

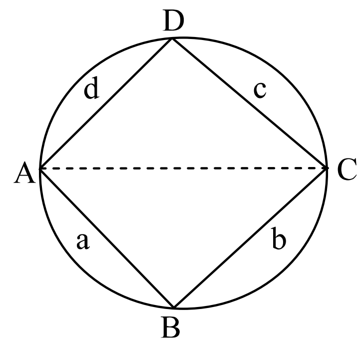
Area of triangle

$= \frac{1}{2}$(Product of two sides) $\times$(sine of included angles) $\Rightarrow \Delta = \frac{1}{2}\left(4R\cos\frac{B}{2}\right).\left(4R\cos\frac{C}{2}\right)\times\sin\left(90^0 - \frac{A}{2}\right)$

$\Delta = 8R^2\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$ Circum-radius $= \frac{I_2I_3}{2\sin I_2I_1I_3} = \frac{4R\cos\frac{A}{2}}{2\sin\left(90^0 - \frac{A}{2}\right)} = 2R.$

CYCLIC QUADRILATERAL

A quadrilateral ABCD is said to be cyclic quadrilateral if there exists a circle passing through all its four vertices A, B, C and D. Let a cyclic quadrilateral be such that]



$AB = a,\ BC = b,\ CD = c$ and $DA = d$.

Then $\angle B + \angle D = 180°,\ \angle A + \angle C = 180°$

Let $2s = a + b + c + d$,

Area of cyclic quadrilateral $= \frac{1}{2}(ab + cd)\sin B$

Also, area of cyclic quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$

Where 2s = a + b + c + d and $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$.

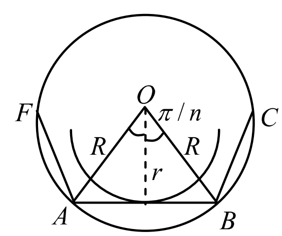
Circumradius of cyclic quadrilateral

Circum circle of quadrilateral ABCD is also the circumcircle of ΔABC. $R = \frac{1}{4\Delta}\sqrt{(ac+bd)(ad+bc)(ab+cd)} = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$.

REGULAR POLYGON

A regular polygon is a polygon which has all its sides equal and all its angles equal.

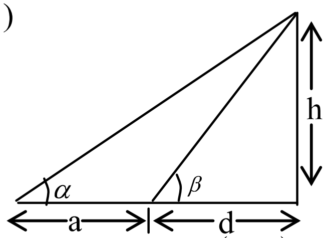
1. Each interior angle of a regular polygon of n sides is $\left(\frac{2n-4}{n}\right)\times$ right angles $= \left[\frac{2n-4}{n}\right]\times\frac{\pi}{2}$ radians.
2. The circle passing through all the vertices of a regular polygon is called its *circumscribed* circle.
   If a is the length of each side of a regular polygon of n sides, then the radius R of the circumscribed circle, is given by $R = \frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{n}\right)$.
3. The circle which can be inscribed within the regular polygon so as to touch all its sides is called its inscribed circle.
   Again if a is the length of each side of a regular polygon of n sides, then the radius r of the inscribed circle is given by $r = \frac{a}{2}\cot\left(\frac{\pi}{n}\right)$



4. The area of a regular polygon is given by $\Delta = n\times$area of triangle OAB
   $= \frac{1}{4}na^2\cot\left(\frac{\pi}{n}\right)$, (in terms of side)
   $= nr^2.\tan\left(\frac{\pi}{n}\right)$, (in terms of in-radius) $= \frac{n}{2}.R^2\sin\left(\frac{2\pi}{n}\right)$, (in terms of circum-radius).

SOME IMPORTANT RESULTS

(1)

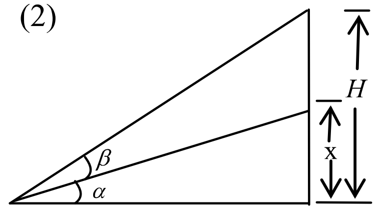


$a = h(\cot\alpha - \cot\beta) = \dfrac{h\sin(\beta-\alpha)}{\sin\alpha.\sin\beta}$

$\therefore h = \sin\alpha + \sin\beta \operatorname{cosec}(\beta\text{-}\alpha)$ *and*

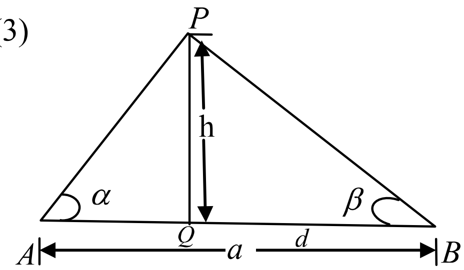
$d = h\cot\beta = a\sin\alpha.\cos\beta.\cos ec(\beta-\alpha)$

(2)



$H = x\cot\alpha\tan(\alpha+\beta)$

(3)

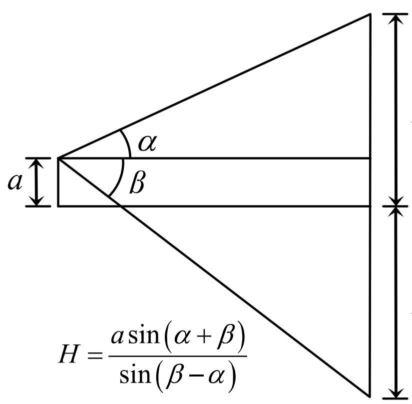


$a = h(\cot\alpha + \cot\beta)$, where by

$h = a\sin\alpha\sin\beta.\operatorname{cosec}(\alpha+\beta)$ and

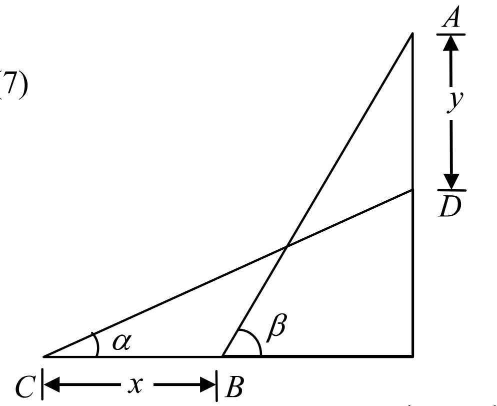
$d = h\cot\beta = a\sin\alpha.\cos\beta.\operatorname{cosec}(\alpha+\beta)$

(6)



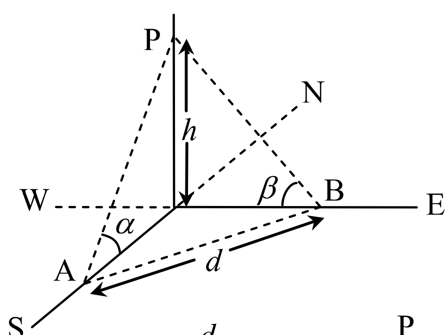
$H = \dfrac{a\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$

(7)



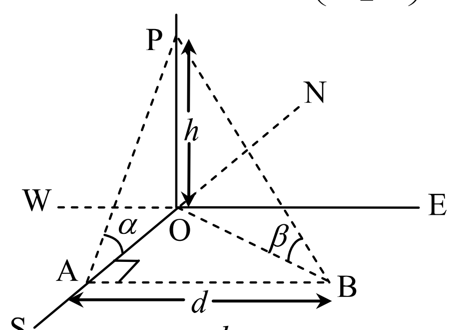
If $AB = CD$. Then, $x = y\tan\left(\dfrac{\alpha+\beta}{2}\right)$

(8)



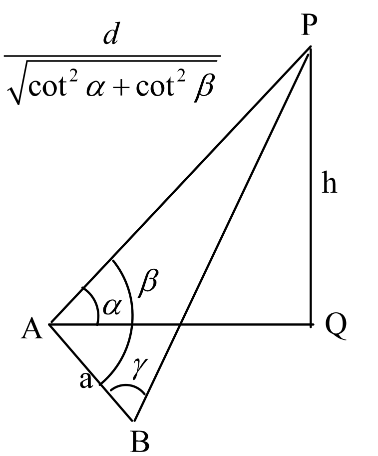
$h = \dfrac{d}{\sqrt{\cot^2\alpha + \cot^2\beta}}$

(9)



$h = \dfrac{d}{\sqrt{\cot^2\alpha - \cot^2\beta}}$

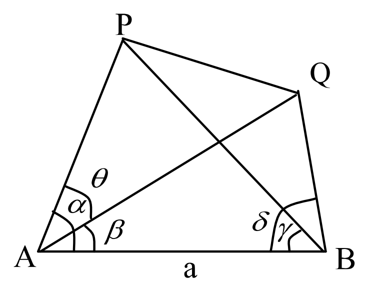
(10)



$h = AP\sin\alpha = \alpha\sin\alpha.\sin\gamma.\cos ec(\beta-\gamma)$

If $AQ = d = AP\cos\alpha = \alpha\cos\alpha.\sin\gamma.\operatorname{cosec}(\beta-\gamma)$

(11)



$AP = a\sin\gamma.\cos ec(\alpha-\gamma)$,

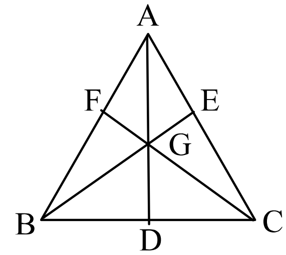
$AQ = a\sin\delta.\cos ec(\beta-\delta)$

and apply,

$PQ^2 = AP^2 + AQ^2 - 2AP.AQ\cos\theta$

TIPS & TRICKS

The length of the medians AD, BE, CF of ΔABC are given by



$$AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}, \qquad = \frac{1}{2}\sqrt{b^2 + c^2 + 2bc\cos A}$$

$$BE = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2} = \frac{1}{2}\sqrt{c^2 + a^2 + 2ca.\cos B}$$

$$CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2} = \frac{1}{2}\sqrt{a^2 + b^2 + 2ab.\cos C}$$

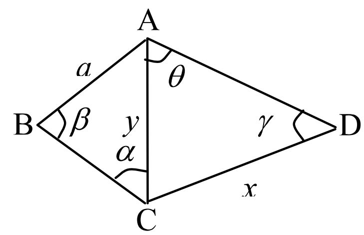
The distance between the circumcentre O and the in centre

I of ΔABC given by $OI = R\sqrt{1 - 8\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}$

If I_1 is the centre of the escribed circle opposite to the

Angle A, then $OI_1 = R\sqrt{1 + 8\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}$, $\qquad OI_3 = R\sqrt{1 + 8\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}}$

Similarly $OI_2 = R\sqrt{1 + 8\cos A/2.\sin B/2.\cos C/2}$



In the application of sine rule, the following point be noted. We are given one side a and some other side x is to be found. Both these are in different triangles.

We choose a common side y of these triangles. Then apply sine rule for a and y in one triangle and for x and y for the other triangle and eliminate y.

Thus, we will get unknown side x in terms of a. In the adjoining figure a is known side of ΔABC and x is unknown is side of triangle ACD. The common side of these triangle is $AC = y$ (say).

Now apply sine rule

$$\therefore \quad \frac{a}{\sin\alpha} = \frac{y}{\sin\beta} \quad ...(i) \quad \text{and} \quad \frac{x}{\sin\theta} = \frac{y}{\sin\gamma} \quad ...(ii)$$

Dividing (ii) by (i) we get,

$$\frac{x\sin\alpha}{a\sin\theta} = \frac{\sin\beta}{\sin\gamma}; \qquad \therefore x = \frac{a\sin\beta\sin\theta}{\sin\alpha\sin\gamma}.$$