CHAPTER THREE

Quadratic Equations

QUADRATIC EQUATIONS

An equation of the form

$$ax^2 + bx + c = 0 \qquad (1)$$

where $a \neq 0$, a, b, $c \in \mathbf{C}$, the set of complex numbers, is called a *quadratic equation*.

A *root* of the quadratic equation (1) is a complex number α such that

$$a\alpha^2 + b\alpha + c = 0$$

The quantity $D = b^2 - 4ac$ is known as the *discriminant* of the equation (1).

The roots of equation (1) are given by the *quadratic formula*

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

NATURE OF ROOTS

1. If a, b, $c \in \mathbf{R}$ and $a \neq 0$. Then the followings hold good:
   (a) The equation (1) has real and distinct roots if and only if $D > 0$.
   (b) The equation (1) has real and equal roots if and only if $D = 0$.
   (c) The equation (1) has complex roots with non-zero imaginary parts if and only if $D < 0$.
   (d) $p + iq$ ($p, q \in \mathbf{R}$, $q \neq 0$) is a root of equation (1) if and only if $p - iq$ is a root of equation (1).
2. If a, b, $c \in \mathbf{Q}$ and D is a perfect square of a rational number, then equation (1) has rational roots.
3. If a, b, $c \in \mathbf{Q}$ and $p + \sqrt{q}$ ($p, q \in \mathbf{Q}$) is an irrational root of equation (1) then $p - \sqrt{q}$ is also a root of equation (1).
4. If $a = 1$, b, $c \in \mathbf{I}$, the set of integers, and the roots of equation (1) are rational numbers, then these roots must be integers.
5. If equation (1) is satisfied by more than two distinct complex numbers, then equation (1) becomes an identity, that is $a = b = c = 0$.

RELATION BETWEEN ROOTS AND COEFFICIENTS

If α and β are the roots of the quadratic equation (1), then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Note that a quadratic equation whose roots are α and β is given by

$$(x - \alpha)(x - \beta) = 0$$

QUADRATIC EXPRESSION AND ITS GRAPH

Let $\quad f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{R}$ and $a \neq 0$.

We have

$$\begin{aligned} f(x) &= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] \\ &= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \qquad (2) \end{aligned}$$

When is a Quadratic Expression always positive Or always negative?

It follows from (2) that $f(x) > 0$ (< 0) $\forall x \in \mathbf{R}$ if and only if $a > 0$ (< 0) and $D = b^2 - 4ac < 0$. See Fig. 3.1 and (Fig. 3.2). Also, it follows from (2) that $f(x) \geq 0$ (≤ 0) $\forall x \in \mathbf{R}$ if and only if $a > 0$ (< 0) and $D = b^2 - 4ac = 0$. In this case $f(x) > 0$ (< 0) for each $x \in \mathbf{R}$, $x \neq -b/2a$. Also, in this case the graph of $y = f(x)$ will touch the x-axis at $x = -b/2a$.

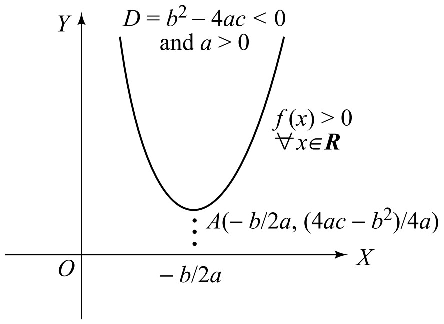


Fig. 3.1

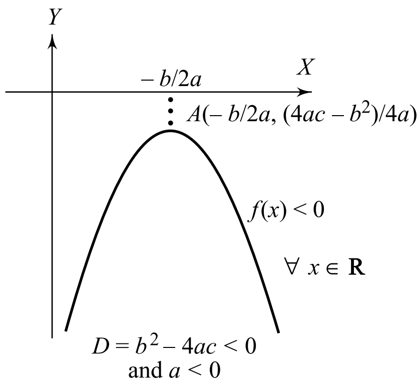


Fig. 3.2

Sign of a Quadratic Expression

If $D = b^2 - 4ac > 0$, then (2) can be written as

$$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right)\right]$$

$$= a\,(x - \alpha)\,(x - \beta)$$

where $\alpha = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

If $D = b^2 - 4ac > 0$ and $a > 0$ then

$$f(x) \begin{cases} > 0 & \text{for } x < \alpha \text{ or } x > \beta \\ < 0 & \text{for } \alpha < x < \beta \\ = 0 & \text{for } x = \alpha, \beta \end{cases}$$

See Fig. 3.3

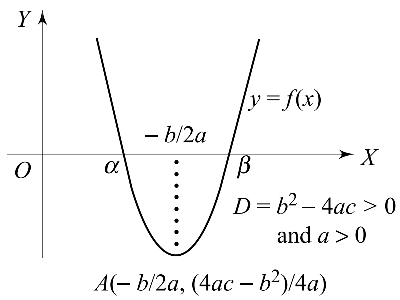


Fig. 3.3

If $D = b^2 - 4ac > 0$ and $a < 0$ then

$$f(x) \begin{cases} < 0 & \text{for } x < \beta \text{ or } x > \alpha \\ > 0 & \text{for } \beta < x < \alpha \\ = 0 & \text{for } x = \alpha, \beta \end{cases}$$

See Fig. 3.4

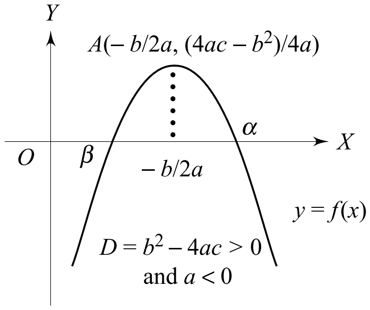


Fig. 3.4

Note that if $a > 0$, then $f(x)$ attains the least value at $x = -b/2a$. This least value is given by

$$f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}$$

If $a < 0$, then $f(x)$ attains the a greatest value at $x = -b/2a$. This greatest value is given by

$$f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}$$

POSITION OF ROOTS OF A QUADRATIC EQUATION

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{R}$ be a quadratic expression and let k be a real number.

Conditions for Both the Roots to be more than a Real Number *k*.

If $a > 0$ and $b^2 - 4ac > 0$ then the parabola $y = ax^2 + bx + c$ opens upwards and intersect the x-axis in α and β where

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case both the roots α and β will be more than k if k lies to left of both α and β. See Fig. 3.5

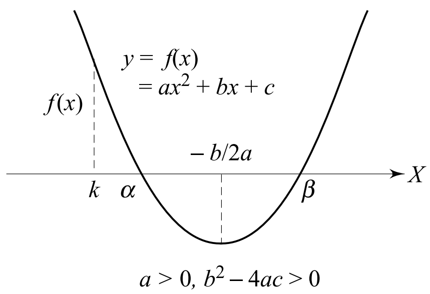


Fig. 3.5

From the Fig. 3.5, we note that both the roots are more than k if and only if

(i) $D > 0$ (ii) $k < \dfrac{-b}{2a}$ (iii) $f(k) > 0$

In case $a < 0$, both the roots will be more than k (see Fig. 3.6) if and only if

(i) $D > 0$ (ii) $k < \dfrac{-b}{2a}$ (iii) $f(k) < 0$

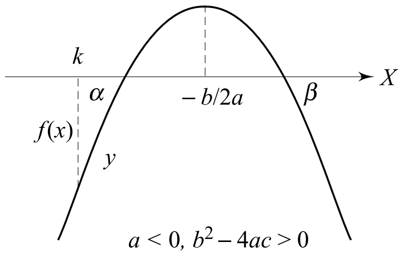


Fig. 3.6

Combining the above two sets, we get both the roots of $ax^2 + bx + c = 0$ are more than a real number k if only if

(i) $D > 0$ (ii) $k < \dfrac{-b}{2a}$ (iii) $af(k) > 0$

Conditions for Both the Roots to be less than a Real Number *k*

Both the roots of $ax^2 + bx + c = 0$ are less than a real number k if and only if

(i) $D > 0$ (ii) $k > \dfrac{-b}{2a}$ (iii) $af(k) > 0$

Note

1. Both the roots of $ax^2 + bx + c = 0$ are **positive** if and only if

   $D \geq 0,\ -\dfrac{b}{a} > 0$ and $\dfrac{c}{a} > 0$

2. Both the roots of $ax^2 + bx + c = 0$ are negative if and only if

   $D \geq 0,\ -\dfrac{b}{a} < 0$ and $\dfrac{c}{a} > 0$

CONDITIONS FOR A NUMBER *k* TO LIE BETWEEN THE ROOTS OF A QUADRATIC EQUATION

The real number k lies between the roots of the quadratic equation $f(x) = ax^2 + bx + c = 0$ if and only if a and $f(k)$ are of opposite signs, that is, if and only if

(i) $a > 0$ (ii) $D > 0$ (iii) $f(k) < 0$ [See Fig. 3.7]

or

(i) $a < 0$ (ii) $D > 0$ (iii) $f(k) > 0$ [See Fig. 3.8]

Combining, we may say k lies between the roots of $f(x) = ax^2 + bx + c = 0$ if and only if

(i) $D > 0$ (ii) $af(k) < 0$

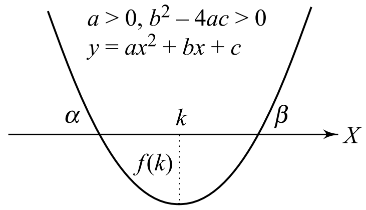


Fig. 3.7

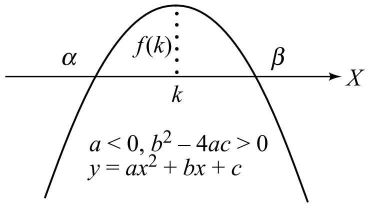


Fig. 3.8

CONDITIONS FOR EXACTLY ONE ROOT OF A QUADRATIC EQUATION TO LIE IN THE INTERVAL (k_1, k_2) WHERE $k_1 < k_2$

If $a > 0$, then exactly one root of $f(x) = ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if and only if $f(k_1) > 0$ and $f(k_2) < 0$. Also, exactly one root lies in the interval (k_1, k_2) if and only if $f(k_1) < 0$ and $f(k_2) > 0$. See Fig. 3.9. Thus, if $a > 0$, exactly one root of $f(x) = ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if and only if $f(k_1)\,f(k_2) < 0$.

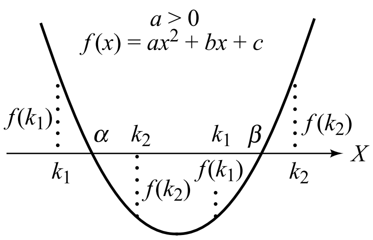


Fig. 3.9

Similarly, if $a < 0$, exactly one of the roots of $f(x) = ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if $f(k_1)\,f(k_2) < 0$.

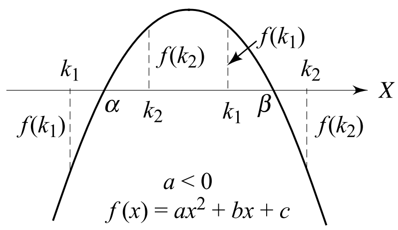


Fig. 3.10

Hence, if $f(k_1)\,f(k_2) < 0$, then exactly one root of $f(x) = 0$ lies in the interval (k_1, k_2).

CONDITIONS FOR BOTH THE ROOTS OF A QUADRATIC EQUATION TO LIE IN THE INTERVAL (k_1, k_2) WHERE $k_1 < k_2$.

If $a > 0$, both the roots of $f(x) = ax^2 + bx + c = 0$ lie in the interval (k_1, k_2) if and only if

(i) $D > 0$ (ii) $k_1 < -\dfrac{b}{2a} < k_2$

(iii) $f(k_1) > 0$ and $f(k_2) > 0$

See Fig. 3.11

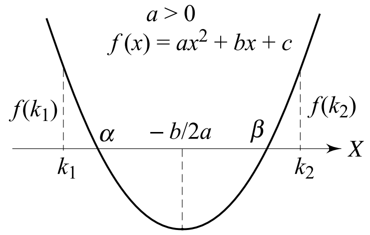


Fig. 3.11

In case $a < 0$, the conditions read as

(i) $D > 0$ (ii) $k_1 < -\dfrac{b}{2a} < k_2$

(iii) $f(k_1) < 0$ and $f(k_2) < 0$

See Fig. 3.12

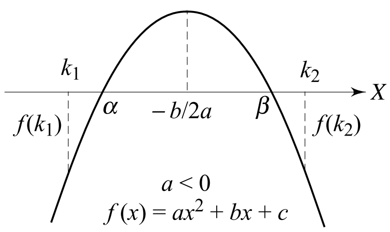


Fig. 3.12

Some Useful Tips

Let quadratic equation be

$$f(x) = ax^2 + bx + c = 0 \tag{1}$$

where $a, b, c \in \mathbf{R}$, $a \neq 0$.

Let roots of (1) be α, β, and $D = b^2 - 4ac$.

1. Both the roots are positive

   $\Leftrightarrow D \geq 0,\ -\dfrac{b}{2a} = \alpha + \beta > 0,\ \dfrac{c}{a} = \alpha\beta > 0$
2. Both the roots are negative

   $\Leftrightarrow D \geq 0,\ -\dfrac{b}{2a} = \alpha + \beta < 0,\ \dfrac{c}{a} = \alpha\beta > 0$
3. Both the roots lie in the interval (p, q).

   $\Leftrightarrow D \geq 0, p < -\dfrac{b}{2a} < q, af(p) > 0, af(q) > 0$
4. If one of α, β is real, then other must be real.
5. If sum of the coefficients of a quadratic equation is 0, then 1 is a root of the quadratic equation.

Illustration 1

1 is a root of $(b - c)x^2 + (c - a)x + a - b = 0$ as $(b - c) + (c - a) + (a - b) = 0$.

Finding Range of a Rational Function

Let $f(x) = ax^2 + bx + c$, $g(x) = px^2 + qx + r$

where $a, b, c, p, q, r \in \mathbf{R}$ and one of $a, p \neq 0$.

To find range of

$$R(x) = \frac{f(x)}{g(x)},\ x \in \mathbf{R},\text{ put}$$

$$y = \frac{ax^2 + bx + c}{px^2 + qx + r},$$

$\Rightarrow$ $(a - py)x^2 + (b - qy)x + (c - ry) = 0$.

Since x is real,

$(b - qy)^2 - 4(a - py)(c - ry) \geq 0$

We use this inequality to obtain range of $R(x)$.

Illustration 2

Find range of

$$y = \frac{x^2 - 2x + 5}{x^2 + 2x + 7}$$

$\Rightarrow$ $(x^2 + 2x + 7)y = x^2 - 2x + 5$

$\Rightarrow$ $(y - 1)x^2 + 2x(y + 1) + 7y - 5 = 0$

As x is real,

$4(y + 1)^2 - 4(y - 1)(7y - 5) \geq 0$

$\Rightarrow$ $(y^2 + 2y + 1) - (7y^2 - 12y + 5) \geq 0$

$\Rightarrow$ $3y^2 - 7y + 2 \leq 0$

$\Rightarrow$ $(3y - 1)(y - 2) \leq 0$

$\Rightarrow$ $1/3 \leq y \leq 2$

Some Useful Tips

Suppose range of $\dfrac{ax^2 + bx + c}{px^2 + qx + r}$ is the set S and $h(x)$ takes all real values, then range of $\dfrac{a(h(x))^2 + b(h(x)) + c}{p(h(x))^2 + q(h(x)) + r}$ is also S.

Illustration 3

Range of $\dfrac{\tan^2 x - 2\tan x + 5}{\tan^2 x + 2\tan x + 7}$ is $\dfrac{1}{3} \leq y \leq 2$.

However, range of $\dfrac{\sin^2 x - 2\sin x + 5}{\sin^2 x + 2\sin x + 7}$ is **not**

[1/3, 2] as $-1 \leq \sin x < 1$ will curtail its range. For instance try $y = 2$.

CONDITIONS FOR A QUADRATIC EQUATION TO HAVE A REPEATED ROOT

The quadratic equation $f(x) = ax^2 + bx + c = 0$, $a \neq 0$ has α as a repeated if and only if $f(\alpha) = 0$ and $f'(\alpha) = 0$. In this case $f(x) = a(x - \alpha)^2$. In fact $\alpha = -b/2a$. See Fig. 3.13 and Fig. 3.14.

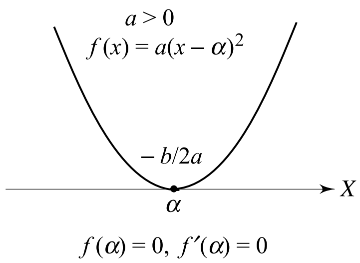


Fig. 3.13

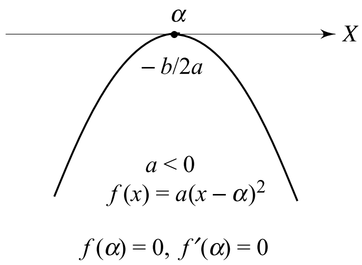


Fig. 3.14

CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE A COMMON ROOT

Suppose that the quadratic equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ (where a, $a' \neq 0$ and $ab' - a'b \neq 0$) have a common root. Let this common root be α. Then

$$a\alpha^2 + b\alpha + c = 0 \quad \text{and} \quad a'\alpha^2 + b'\alpha + c' = 0$$

Solving the above equations, we get

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

$$\Rightarrow \qquad \alpha^2 = \frac{bc' - b'c}{ab' - a'b} \quad \text{and } \alpha = \frac{a'c - ac'}{ab' - a'b}$$

Eliminating α we get

$$\frac{(a'c - ac')^2}{(ab' - a'b)^2} = \frac{bc' - b'c}{ab' - a'b}$$

$$\Rightarrow \qquad (a'c - ac')^2 = (bc' - b'c)(ab' - a'b)$$

This is the required condition for two quadratic equation to have a common root. See Fig. 3.15

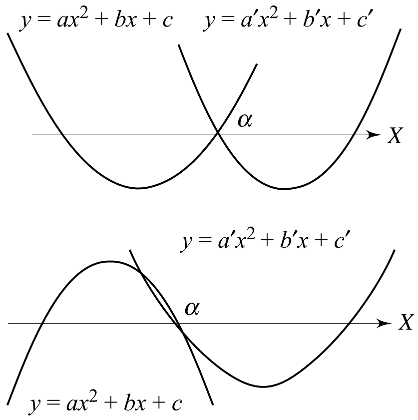


Fig. 3.15

TIP

How to Obtain the Common Root?
Make coefficients of x^2 in both the equations same and subtract one equation from the other to obtain a linear equation in x. Solve it for x to obtain the common root.

CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE THE SAME ROOTS

Two quadratic equations $ax^2 + bx + c = 0$. and $a'x^2 + b'x + c' = 0$ have the same roots if and only if

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

EQUATIONS OF HIGHER DEGREE

The equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

where $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbf{C}$, the set of complex numbers, and $a_0 \neq 0$, is said to be an equation of degree n. An equation of degree n has exactly n roots. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{C}$ be the n roots of (1). Then

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$$

Also $\qquad \alpha_1 + \alpha_2 + \ldots + \alpha_n = -\dfrac{a_1}{a_0}$

and $\qquad \alpha_1\alpha_2 \ldots \alpha_n = (-1)^n \dfrac{a_n}{a_0}$.

CUBIC AND BIQUADRATIC EQUATIONS

If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}, \ \beta\gamma + \gamma\alpha + \alpha\beta = \frac{c}{a} \text{ and } \alpha\beta\gamma = -\frac{d}{a}.$$

Also, if $\alpha, \beta, \gamma, \delta$ are the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$\alpha + \beta + \gamma + \delta = \frac{-b}{a}, (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a}$$

$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = \frac{-d}{a}, \alpha\beta\gamma\delta = \frac{e}{a}.$$

TRANSFORMATION OF EQUATIONS

We now list some of the rules to form an equation whose roots are given in terms of the roots of another equation. Let given equation be

$$a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0 \qquad (1)$$

Rule 1: To form an equation whose roots are $k(\neq 0)$ times roots of the equations in (1), replace x by $\frac{x}{k}$ in (1).

Rule 2: To form an equation whose roots are the negatives of the roots in equation (1), replace x by $-x$ in (1). Alternatively, change the sign of the coefficients of $x^{n-1}, x^{n-3}, x^{n-5}, \dots$ etc. in (1).

Rule 3: To form an equation whose roots are k more than the roots of equation in (1), replace x by $x - k$ in (1).

Rule 4: To form an equation whose roots are reciprocals of the roots in equation (1), replace x by $\frac{1}{x}$ in (1) and then multiply both the sides by x^n.

Rule 5: To form an equation whose roots are square of the roots of the equation in (1) proceed as follows:

Step 1 Replace x by $\sqrt{x}$ in (1)

Step 2 Collect all the terms involving $\sqrt{x}$ on one side.

Step 3 Square both the sides and simplify.

For instance, to form an equation whose roots are squares of the roots of $x^3 + 2x^2 - x + 2 = 0$, replace x by $\sqrt{x}$ to obtain

$$x\sqrt{x} + 2x - \sqrt{x} + 2 = 0$$

$$\Rightarrow \qquad \sqrt{x}(x - 1) = -2(x + 1)$$

Squaring we get

$$x(x - 1)^2 = 4(x + 1)^2$$

or

$$x^3 - 6x^2 - 7x - 4 = 0$$

Rule 6: To form an equation whose roots are cubes of the roots of the equation in (1) proceed as follows:

Step 1 Replace x by $x^{1/3}$.

Step 2 Collect all the terms involving $x^{1/3}$ and $x^{2/3}$ on one side.

Step 3 Cube both the sides and simplify.

DESCARTES RULE OF SIGNS FOR THE ROOTS OF A POLYNOMIAL

Rule 1: The maximum number of positive real roots of a polynomial equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

is the number of changes of the signs of coefficients from positive to negative and negative to positive. For instance, in the equation $x^3 + 3x^2 + 7x - 11 = 0$ the signs of coefficients are

$$+ + + -$$

As there is just one change of sign, the number of positive roots of $x^3 + 3x^2 + 7x - 11 = 0$ is at most 1.

Rule 2: The maximum number of negative roots of the polynomial equation $f(x) = 0$ is the number of changes from positive to negative and negative to positive in the signs of coefficients of the equation $f(-x) = 0$.

SOME HINTS FOR SOLVING POLYNOMIAL EQUATIONS

1. To solve an equation of the form $(x - a)^4 + (x - b)^4 = A$

   put $\quad y = x - \frac{a + b}{2}$

   In general to solve an equation of the form

   $$(x - a)^{2n} + (x - b)^{2n} = A$$

   where n is a positive integer, we put $y = x - \frac{a + b}{2}$.
2. To solve an equation of the form

   $$a_0 f(x)^{2n} + a_1(f(x))^n + a_2 = 0 \qquad (1)$$

   we put $(f(x))^n = y$ and solve $a_0 y^2 + a_1 y + a_2 = 0$ to obtain its roots y_1 and y_2.

   Finally, to obtain solutions of (1) we solve,

   $$(f(x))^n = y_1$$

   and $\quad (f(x))^n = y_2$
3. An equation of the form

   $$(ax^2 + bx + c_1)(ax^2 + bx + c_2) \dots (ax^2 + bx + c_n) = A$$

   where $c_1, c_2, \dots, c_n, A \in \mathbf{R}$, can be solved by putting $ax^2 + bx = y$.
4. An equation of the form

   $$(x - a)(x - b)(x - c)(x - d) = Ax^2$$

   where $ab = cd$, can be reduced to a product of two quadratic polynomials by putting $y = x + \frac{ab}{2}$.
5. An equation of the form

   $$(x - a)(x - b)(x - c)(x - d) = A$$

   where $a < b < c < d$, $b - a = d - c$ can be solved by change of variable

   $$y = \frac{(x - a) + (x - b) + (x - c) + (x - d)}{4}$$

   $$= x - \frac{1}{4}(a + b + c + d)$$
6. A polynomial $f(x, y)$ is said to be symmetric if $f(x, y) = f(y, x) \;\forall\; x, y$.

   A symmetric polynomials can be represented as a function of $x + y$ and xy.

Equations Reducible to Quadratic

1. To solve an equation of the type $ax^4 + bx^2 + c = 0$, put $x^2 = y$.
2. To solve an equation of the type $a\,(p(x))^2 + bp(x) + c = 0$ (where $p(x)$ is an expression in x) put $p(x) = y$.
3. To solve an equation of the type
$$ap(x) + \frac{b}{p(x)} + c = 0$$
where $p(x)$ is an expression in x, put $p(x) = y$. This reduces the equation to
$$ay^2 + cy + b = 0.$$
4. To solve an equation of the form
$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$$
put $\quad x + \dfrac{1}{x} = y$
and to solve
$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x - \frac{1}{x}\right) + c = 0$$
put $\quad x - \dfrac{1}{x} = y$
5. To solve reciprocal equation of the type
$$ax^4 + bx^3 + cx^2 + bx + a = 0, a \neq 0$$
we divide the equation by x^2, to obtain
$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$$
and then put $\quad x + \dfrac{1}{x} = y.$
6. To solve equation of the type
$$(x + a)\,(x + b)\,(x + c)\,(x + d) + k = 0$$
where $a + b = c + d$, put $x^2 + (a + b)x = y$
7. To solve equation of the type
$$\sqrt{ax+b} = cx + d$$
or
$$\sqrt{ax^2+bx+c} = dx + e$$
square both the sides and solve for x.
8. To solve equation of the type
$$\sqrt{ax+b} \pm \sqrt{cx+d} = e$$
proceed as follows.

Step 1 Transfer one of the radical to the other side and square both the sides.
Step 2 Keep the expression with radical sign on one side and transfer the remaining expression on the other side.
Step 3 Now solve as in 7 above.

USE OF CONTINUITY AND DIFFERENTIABILITY TO FIND ROOTS OF A POLYNOMIAL EQUATION

Let $f(x) = a_0\, x^n + a_1\, x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1}\, x + a_n$, then f is continuous on **R**.

Since f is continuous on **R**, we may use the intermediate value theorem to find whether or not f has a real root in an interval (a, b).

If there exist a and b such that $a < b$ and $f(a)\,f(b) < 0$, then there exists at least one $c \in (a, b)$ such that $f(c) = 0$.

Also, if $\lim\limits_{x\to-\infty} f(x)$ and $f(a)$ are of opposite signs, then at least one root of $f(x) = 0$ lies in the interval $(-\infty, a)$. Also, if $f(a)$ and $\lim\limits_{x\to\infty} f(x)$ are of opposite signs, then at least one root of $f(x) = 0$ lies in the interval (a, ∞).

Result 1 If $f(x) = 0$ has a root α of multiplicity r (where $r > 1$), then we can write
$$f(x) = (x - \alpha)^r\, g(x)$$
where $g(\alpha) \neq 0$.
Also, $f'(x) = 0$ has α as a root with multiplicity $r - 1$.

Result 2 If $f(x) = 0$ has n real roots, then $f'(x) = 0$ has $(n - 1)$ real roots.

It follows immediately using Result 1 and Rolle's Theorem.

Result 3 If $f(x) = 0$ has n distinct real roots, we can write
$$f(x) = a_0\,(x - \alpha_1)\,(x - \alpha_2)\ldots(x - \alpha_n)$$
where $\alpha_1, \alpha_2, \ldots \alpha_n$ are n distinct roots of $f(x) = 0$.

We also have
$$\frac{f'(x)}{f(x)} = \sum_{k=1}^{n} \frac{1}{x - \alpha_k}$$

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: Solution set of
$\sqrt{3-x} = -x^2 - x - 1, x \in$ **R** is

(a) $(-1, \infty)$ (b) $(0, 1)$
(c) $(-1, 3/4)$ (d) ϕ

Ans. (d)

Solution: Note that LHS $= \sqrt{3-x} \geq 0 \;\forall\; x \leq 3.$

and RHS $= -\left(x - \dfrac{1}{2}\right)^2 - \dfrac{5}{4} < 0 \;\forall\; x.$

Thus, the given equation has no solution.

Example 2: Number of solutions of

$$x - \frac{5}{x-2} = 2 - \frac{5}{x-2} \qquad (1)$$

is

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (a)

Solution: The equation (1) is defined when $x \neq 2$. But when $x \neq 2$. We can cancel $5/(x-2)$ from both the sides of (1) to obtain $x = 2$
Thus, (1) has no solution.

Example 3: The number of real solutions of $x^2 + 5|x| + 4 = 0$ is

(a) 0 (b) 1
(c) 2 (d) 4

Ans. (a)

Solution: If $x \in \mathbf{R}$, then $x^2 + 5|x| + 4 \geq 4$.
Thus, the given equation has no solution.

Example 4: If $\frac{1}{3-4i}$ is a root of $ax^2 + bx + c = 0$, ($a, b, c \in \mathbf{R}$, $a \neq 0$), then

(a) $b + 6c = 0$ (b) $b = 6c$
(c) $a + 25c = 0$ (d) $b^2 = 6c$

Ans. (a)

Solution: TIP: As $a, b, c \in \mathbf{R}$, the other root must $\frac{1}{3+4i}$.

Thus, $\frac{1}{3-4i} + \frac{1}{3+4i} = -\frac{b}{a}$,

and $\frac{1}{3-4i} \cdot \frac{1}{3+4i} = \frac{c}{a}$

$\Rightarrow \quad \frac{6}{25} = -\frac{b}{a}$ and $\frac{1}{25} = \frac{c}{a}$

Now, $6\left(\frac{c}{a}\right) = -\frac{b}{a} \Rightarrow b + 6c = 0$.

Also, $a - 25c = 0$

Example 5: If $1 - p$ is a root of the quadratic equation $x^2 + px + 1 - p = 0$, then its roots are

(a) 0, 1 (b) –1, 1
(c) 0, – 1 (d) –2, 1

Ans. (c)

Solution: $(1-p)^2 + p(1-p) + (1-p) = 0$
$\Rightarrow \quad (1-p)[1 - p + p + 1] = 0 \Rightarrow 2(1-p) = 0$
$\Rightarrow \quad p = 1$
Thus, equation becomes $x^2 + x = 0$
$\Rightarrow \quad x = 0$ or -1

Example 6: Suppose $p \in \mathbf{R}$. If the roots of $3x^2 + 2x + p(1-p) = 0$ are of opposite signs, then p must lie in the interval

(a) $(-\infty, 0)$ (b) $(0, 1)$
(c) $(1, \infty)$ (d) $(2, \infty)$

Ans. (b)

Solution: We must have $\frac{1}{3}p(1-p) < 0$

$\Rightarrow \quad p(1-p) < 0 \Rightarrow 0 < p < 1$.

Example 7: If α, β are the roots of $(x-a)(x-b) + c = 0$, $c \neq 0$, then roots of $(\alpha\beta - c)x^2 + (\alpha + \beta)x + 1 = 0$ are

(a) $1/a, 1/b$ (b) $-1/a, -1/b$
(c) $1/a, -1/b$ (d) $-1/a, 1/b$

Ans. (b)

Solution: α, β are roots of $x^2 - (a+b)x + ab + c = 0$.
$\therefore \quad \alpha + \beta = a + b, \alpha\beta = ab + c$
The equation

$$(\alpha\beta - c)x^2 + (\alpha + \beta)x + 1 = 0$$

becomes

$$abx^2 + (a+b)x + 1 = 0$$

$\Rightarrow \quad (ax + 1)(bx + 1) = 0$
$\Rightarrow \quad x = -1/a, -1/b$

Example 8: If α, β are roots $x^2 + px + q = 0$, then value of $\alpha^3 + \beta^3$ is

(a) $3pq + p^3$ (b) $3pq - p^3$
(c) $3pq$ (d) $p^3 - 3pq$

Ans. (b)

Solution: $\alpha + \beta = -p$, $\alpha\beta = q$.
Now, $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
$= -p^3 + 3pq = 3pq - p^3$

Example 9: If $a \in \mathbf{R}$ and both the roots of $x^2 - 6ax + 9a^2 + 2a - 2 = 0$ exceed 3, then a lies in the interval

(a) $(1, \infty)$ (b) $(2, \infty)$
(c) $(11/9, \infty)$ (d) ϕ

Ans. (d)

Solution: Given equation can be written as $(x - 3a)^2 = 2(1-a)$
We must have $1 - a \geq 0$ or $a \leq 1$.
Also, $x = 3a \pm \sqrt{2}\sqrt{1-a}$.
Both the roots will exceed 3, if smaller of the two roots viz $3a - \sqrt{2}\sqrt{1-a}$ exceeds 3, that is, $3a - \sqrt{2}\sqrt{1-a} > 3$

$\Rightarrow \quad 3(a-1) > \sqrt{2}\sqrt{1-a}. \qquad (1)$

For $a = 1$, this is not possible. For $a < 1$, we can write (1) as

$$-3\left(\sqrt{1-a}\right)^2 > \sqrt{2}\sqrt{1-a}$$

which is no possible

Thus, there is not real value of a.

Example 10: If $x = \sqrt{7-4\sqrt{3}}$, then $x + \dfrac{1}{x}$ is equal to:

(a) 2 (b) $3\sqrt{7}$

(c) 4 (d) $4\sqrt{7}$

Ans. (c)

Solution: $7 - 4\sqrt{3} = 2^2 + (\sqrt{3})^2 - 4\sqrt{3}$

$$= (2-\sqrt{3})^2$$

$$\therefore \quad x = \sqrt{7-4\sqrt{3}} = 2-\sqrt{3}$$

and $$\frac{1}{x} = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}$$

Thus, $x + \dfrac{1}{x} = 4$

Example 11: Suppose α, β are roots of $ax^2 + bx + c = 0$ then roots of $a(x-2)^2 - b(x-2)(x-3) + c(x-3)^2 = 0$ are

(a) $\dfrac{2+\alpha}{1+\beta}, \dfrac{2+\beta}{1+\alpha}$ (b) $\dfrac{2+\alpha}{3+\beta}, \dfrac{3+\alpha}{2+\beta}$

(c) $\dfrac{2+3\alpha}{2+\alpha}, \dfrac{2+3\beta}{2+\beta}$ (d) $\dfrac{2+3\alpha}{1+\alpha}, \dfrac{2+3\beta}{1+\beta}$

Ans. (d)

Solution: Write the equation as

$$a\left(-\frac{x-2}{x-3}\right)^2 + b\left(-\frac{x-2}{x-3}\right) + c = 0$$

Now, $-\dfrac{x-2}{x-3} = \alpha, \beta$

$$\Rightarrow \quad x = \frac{2+3\alpha}{1+\alpha}, \frac{2+3\beta}{1+\beta}$$

Example 12: If $x, y, z \in \mathbf{R}$ then the least value of the expression $E = 3x^2 + 5y^2 + 4z^2 - 6x + 20y - 8z - 3$ is:

(a) – 15 (b) – 30

(c) – 45 (d) – 60

Ans. (b)

Solution: We can write

$$E = 3(x^2 - 2x + 1) + 5(y^2 + 4y + 4) + 4(z^2 - 2z + 1) - 30$$
$$= 3(x-1)^2 + 5(y+2)^2 + 4(z-1)^2 - 30 \geq -30$$

The least value is attained when

$$x = 1, y = -2, z = 1.$$

Example 13: The number real of roots of $\sqrt{x-3}\,(x^2 + 7x + 10) = 0$, where $x \in \mathbf{R}$ is

(a) 0 (b) 1

(c) 2 (d) 3

Ans. (b)

Solution: For the equation to be defined, $x \geq 3$, and $x^2 + 7x + 10 \geq 40 \not\to x \geq 3$. Therefore, the given equation has exactly one root, *viz*. 3.

Example 14: Suppose $a \in \mathbf{R}$. If $3x^2 + 2(a^2 + 1)x + (a^2 - 3a + 2) = 0$ possesses roots of opposite signs, then a lies in the interval:

(a) $(-\infty, -1)$ (b) $(-1, 1)$

(c) $(1, 2)$ (d) $(2, 3)$

Ans. (c)

Solution: As the roots are of opposite signs, the product of roots must be negative, that is,

$$\frac{a^2 - 3a + 2}{3} < 0 \Rightarrow (a-1)(a-2) < 0$$

$$\Rightarrow \quad a \in (1, 2)$$

Example 15: Suppose $a^2 = 5a - 8$ and $b^2 = 5b - 8$, then equation whose roots are $\dfrac{a}{b}$ and $\dfrac{b}{a}$ is

(a) $6x^2 - 5x + 6 = 0$ (b) $8x^2 - 9x + 8 = 0$

(c) $9x^2 - 8x + 9 = 0$ (d) $8x^2 + 9x + 8 = 0$

Ans. (b)

Solution: a, b are roots of $x^2 - 5x + 8 = 0$.

$$\therefore \quad a + b = 5, ab = 8$$

Now, $$\frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab} = \frac{(a+b)^2 - 2ab}{ab}$$

$$= \frac{25}{8} - 2 = \frac{9}{8}$$

$\therefore$ required equation is

$$x^2 - (9/8)x + 1 = 0 \text{ or } 8x^2 - 9x + 8 = 0$$

Example 16: Sum of the roots of the equation $x^2 + |2x - 3| - 4 = 0$ is

(a) 2 (b) –2

(c) $\sqrt{2}$ (d) $-\sqrt{2}$

Ans. (c)

Solution: **Case 1:** When $x < 3/2$

In this case the equation becomes

$$x^2 - (2x - 3) - 4 = 0$$

$$\Rightarrow \quad x^2 - 2x + 1 = 2$$

$$\Rightarrow \quad (x-1)^2 = 2 \Rightarrow x - 1 = \pm\sqrt{2}$$

$$\Rightarrow \quad x = 1 \pm \sqrt{2}$$

As $x < 3/2$, we take, $x = 1 - \sqrt{2}$

Case 2: When $x \geq 3/2$

In this case the equation becomes

$$x^2 + (2x - 3) - 4 = 0$$

$$\Rightarrow \quad (x+1)^2 = 8 \Rightarrow x + 1 \pm 2\sqrt{2}$$

As $x \geq 3/2$, $x = -1 + 2\sqrt{2}$

Sum of the roots is

$$(1 - \sqrt{2}) + (-1 + 2\sqrt{2}) = \sqrt{2}$$

Example 17: If the quadratic equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$, where a, b, c are distinct real numbers and $abc \neq 0$, has equal roots, then a, b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

Ans. (c)

Solution:

TIP

If sum of the coefficients of a polynomial equation is zero, then 1 is a root of the equaiton.

As 1 is a root of the equation and equation has equal roots, the other root must be 1. Thus we have

$$1 = \text{product of roots} = \frac{c(a-b)}{a(b-c)}$$

$$\Rightarrow \quad a(b - c) = c(a - b)$$

$$\Rightarrow \quad b = \frac{2ac}{a+c}$$

$\Rightarrow$ a, b, c are in H.P.

Example 18: Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$, and $(a + c)^2 < b^2$. Then the quadratic equation $ax^2 + bx + c = 0$ has

(a) real and distinct roots (b) real and equal roots
(c) purely imaginary roots (d) none of these

Ans. (a)

Solution:

TIP

Suppose $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbf{R}$, $a \neq 0$. If $f(\alpha) f(\beta) < 0$, then exactly one root of the equation $f(x) = 0$ lies between α and β and the equation $f(x) = 0$ has real roots.

Let $f(x) = ax^2 + bx + c$.
Then $f(1) = a + b + c$, $f(-1) = a - b + c$.
We have $f(1) f(-1) = (a + c)^2 - b^2 < 0$

Thus, $f(x) = 0$ has exactly one root in $(-1, 1)$. Hence, $ax^2 + bx + c = 0$ has real and distinct roots.

Example 19: If both the roots of the quadratic equation $x^2 + 2(a + 2)x + 9a - 1 = 0$ are negative, then a lies in the set

(a) $[1, \infty)$ (b) $(1/9, 1] \cup [4, \infty)$
(c) $(-2, 4) \cup (6, \infty)$ (d) ϕ

Ans. (b)

Solution: TIP: Both the roots are negative, if and only if $D \geq 0$, sum of roots < 0 and product of roots > 0

Thus, $4(a + 2)^2 - 4(9a - 1) \geq 0$,

$$-2(a + 2) < 0, \; 9a - 1 > 0$$

$$\Rightarrow \quad a^2 - 5a + 4 \geq 0, \; a > -2, \; a > 1/9$$

$$\Rightarrow \quad (a - 1)(a - 4) \geq 0, \; a > 1/9$$

$$\Rightarrow \quad a \leq 1 \text{ or } a \geq 4 \text{ and } a > 1/9$$

$$\Rightarrow \quad a \in (1/9, 1] \cup [4, \infty)$$

Example 20: Suppose $p \in \mathbf{R}$ and α, β are roots of $x^2 - px + \dfrac{1}{2p^2} = 0$, then the minimum value of $\alpha^4 + \beta^4$ is

(a) $\sqrt{2}$ (b) $2 - \sqrt{2}$
(c) $\sqrt{2} - 2$ (d) -2

Ans. (c)

Solution: $\alpha + \beta = p$, $\alpha\beta = 1/2p^2$

$$\Rightarrow \quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 1/p^2$$

$$\therefore \quad \alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = \left(p^2 - \frac{1}{p^2}\right)^2 - \frac{1}{2p^4}$$

$$= p^4 + \frac{1}{p^4} - 2 - \frac{1}{2p^4}$$

$$= p^4 + \frac{1}{2p^4} - 2$$

$$\geq 2\left(p^4 \cdot \frac{1}{2p^4}\right)^{1/2} - 2 = \sqrt{2} - 2$$

[$\because$ AM $\geq$ GM]

Thus, minimum possible value of $\alpha^4 + \beta^4$ is $\sqrt{2} - 2$, which is attained when $p = 1$

LEVEL 1

Straight Objective Type Questions

Example 21: The number of real solutions of

$$x + \sqrt{x} + \sqrt{x - 2} = 3 \qquad (1)$$

is:

(a) 0 (b) 1
(c) 2 (d) 4

Ans. (a)

Solution: For $\sqrt{x}$, $\sqrt{x-2}$ to be defined $x \geq 0$, $x - 2 \geq 0$

$\Rightarrow$ $x \geq 2$. But for $x \geq 2$

$$x + \sqrt{x} + \sqrt{x-2} \geq 2 + \sqrt{2} > 3$$

Thus, (1) has no real solution.

Example 22: In Fig. 3.16 graph of $y = ax^2 + bx + c$ is given. Which one of the following is not true?

(a) $a < 0$ (b) $c > 0$
(c) $b^2 - 4ac > 0$ (d) $b < 0$

Ans. (d)

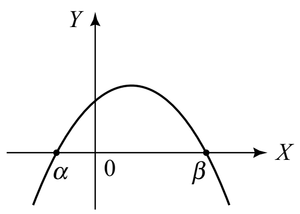


Fig. 3.16

Solution: The given parabola opens downwards, $a < 0$. Also, $y(0) = c > 0$.
As roots of $ax^2 + bx + c = 0$ are distinct, $b^2 - 4ac > 0$.
However nothing can be said about b. It can be positive, zero or negative. For instance, consider equations $-x^2 - 5x + 6 = 0$, $-x^2 + 3x + 4 = 0$ and $-x^2 + 9 = 0$.

Example 23: Let α and β be the roots of $px^2 + qx + r = 0$, $p \neq 0$. If p, q, r are in A.P. and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then value of $|\alpha - \beta|$ is

(a) $\frac{1}{9}\sqrt{61}$ (b) $\frac{2}{9}\sqrt{17}$
(c) $\frac{1}{9}\sqrt{34}$ (d) $\frac{2}{9}\sqrt{13}$

Ans. (d)

Solution: $4 = \frac{\alpha+\beta}{\alpha\beta} = \frac{-q/p}{r/p} = -\frac{q}{r}$

$\Rightarrow \quad q + 4r = 0$

Also, $\quad 2q = p + r \qquad [\because p, q, r \text{ are in A.P.}]$

Thus, $\quad p = -9r,\ q = -4r$

$$|\alpha - \beta|^2 = (\alpha + \beta)^2 - 4\alpha\beta$$
$$= \left(\frac{-q}{p}\right)^2 - \frac{4r}{p}$$
$$= \frac{16}{81} - 4\left(-\frac{1}{9}\right) = \frac{52}{81}$$

$\Rightarrow \quad |\alpha - \beta| = \frac{2}{9}\sqrt{13}$

Example 24: If tan 25° and tan 20° are roots of the quadratic equation $x^2 + 2px + q = 0$, then $2p - q$ is equal to

(a) -2 (b) -1
(c) 0 (d) 1

Ans. (b)

Solution: We have
$\tan 25° + \tan 20° = -2p$
and $\tan 25° \tan 20° = q$

Now, $\quad 1 = \tan 45° = \tan(25° + 20°)$
$$= \frac{\tan(25°) + \tan(20°)}{1 - \tan(25°)\tan(20°)} = \frac{-2p}{1-q}$$

$\Rightarrow \quad 1 - q = -2p \Rightarrow 2p - q = -1$

Example 25: Suppose $0 < \alpha < \beta$, and $2\alpha + \beta = \pi/2$. If $\tan\alpha$, $\tan\beta$ are roots of $ax^2 + bx + c = 0$, then

(a) $\tan\alpha = \frac{c-a}{b}$ (b) $\tan\beta = \frac{c+a}{b}$
(c) $\tan\alpha = \frac{b}{c-a}$ (d) $\tan\beta = -\frac{b}{c+a}$

Ans. (a)

Solution: $\tan\alpha + \tan\beta = -b/a$ and $\tan\alpha\tan\beta = c/a$.

Now, $\quad \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\Rightarrow \quad \tan(\pi/2 - \alpha) = \frac{-b/a}{1 - c/a}$

$\Rightarrow \quad \cot\alpha = \frac{-b}{a-c}$

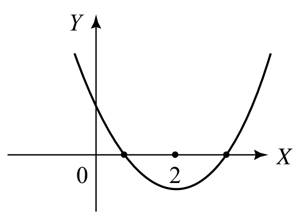
Thus, $\quad \tan\alpha = \frac{c-a}{b}$.

Example 26: The value of a for which one root of the equation $x^2 - (a+1)x + a^2 + a - 8 = 0$ exceeds 2 and other is less than 2, are given by

(a) $3 < a < 10$ (b) $a \geq 10$
(c) $-2 < a < 3$ (d) $a \leq -2$

Ans. (c)

Solution: As the roots are real and distinct
$$(a+1)^2 - 4(a^2 + a - 8) > 0$$
$$3a^2 + 2a - 33 < 0$$
$\Rightarrow \quad (3a + 11)(a - 3) < 0 \Rightarrow -\frac{11}{3} < a < 3 \qquad (1)$



Also, for
$$2^2 - 2(a+1) + a^2 + a - 8 < 0$$
$\Rightarrow \quad a^2 - a - 6 < 0 \Rightarrow (a-3)(a+2) < 0$

$\Rightarrow \quad -2 < a < 3 \qquad (2)$

From (1) and (2), we get $-2 < a < 3$.

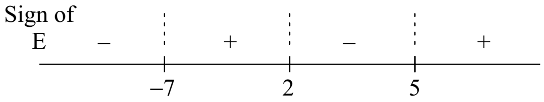
Example 27: The least integral value α of x such that $\dfrac{x-5}{x^2+5x-14} > 0$, satisfies the relation:

(a) $\alpha^2 + 3\alpha - 4 = 0$ (b) $\alpha^2 - 5\alpha + 4 = 0$
(c) $\alpha^2 - 7\alpha + 6 = 0$ (d) $\alpha^2 + 5\alpha - 6 = 0$

Ans. (d)

Solution: $0 < \dfrac{x-5}{x^2+5x-14} = \dfrac{x-5}{(x+7)(x-2)} = \text{E(say)}$

Sign of E in different intervals are shown below



Thus, $\dfrac{x-5}{(x+7)(x-2)} > 0$

$\Rightarrow \quad -7 < x < 2$ or $x > 5$.

Therefore the least integral value α of x is -6. This value of α satisfies the relation $\alpha^2 + 5\alpha - 6 = 0$.

Example 28: Let p and q be two non-zero real numbers and α, β are two numbers such that $\alpha^3 + \beta^3 = -p$, $\alpha\beta = q$, then the quadratic equation whose roots are α^2/β and β^2/α is

(a) $px^2 - qx + p^2 = 0$ (b) $qx^2 + px + q^2 = 0$
(c) $px^2 + qx + p^2 = 0$ (d) $qx^2 - px + q^2 = 0$

Ans. (b)

Solution: $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3+\beta^3}{\alpha\beta} = \dfrac{-p}{q}$

and $\left(\dfrac{\alpha^2}{\beta}\right)\left(\dfrac{\beta^2}{\alpha}\right) = \alpha\beta = q$

Thus, required equation is

$$x^2 - \left(-\frac{p}{q}\right)x + q = 0$$

or $\quad qx^2 + px + q^2 = 0.$

Example 29: The set of values of α for which the quadratic equation

$$(\alpha+2)x^2 - 2\alpha x - \alpha = 0$$

has two roots on the number line symmetrically placed about the point 1 is

(a) $\{-1, 0\}$ (b) $\{0, 2\}$
(c) ϕ (d) $\{0, 1\}$

Ans. (c)

Solution: Let roots of the equation be

$1 - k$ and $1 + k$, where $k > 0$

Then $\quad 2 = (1-k) + (1+k) = \dfrac{2\alpha}{\alpha+2} \Rightarrow 1 = \dfrac{\alpha}{\alpha+2}$

This is not possible for any value of α.

Example 30: Suppose $k \in \mathbf{R}$ and the quadratic equation $x^2 - (k-3)x + k = 0$ has at least one positive roots, then k lies in the set:

(a) $(-\infty, 0) \cup [9, 16]$ (b) $(-\infty, 0) \cup [16, 8)$
(c) $(-\infty, 0) \cup [9, \infty)$ (d) $(-\infty, 0) \cup (1, \infty)$

Ans. (c)

Solution: Both the roots α, β will be non-positive if

$D \geq 0,\ \alpha + \beta \leq 0,\ \alpha\beta \geq 0$

$\Rightarrow \quad (k-3)^2 - 4k \geq 0,\ (k-3) \leq 0,\ k \geq 0$

$\Rightarrow \quad (k-1)(k-9) \geq 0,\ k \leq 3,\ k \geq 0$

$\Rightarrow \quad 0 \leq k \leq 1.$

Thus, quadratic equation will have at least one positive root if $k < 0$ or $k > 1$ and ($k \leq 1$ or $k \geq 9$)

$\Rightarrow \quad k \in (-\infty, 0) \cup [9, \infty)$

Example 31: Two non-integer roots of the equation

$$(x^2+3x)^2 - (x^2+3x) - 6 = 0 \qquad (1)$$

are

(a) $\dfrac{1}{2}(-3+\sqrt{11}), \dfrac{1}{2}(-3-\sqrt{11})$

(b) $\dfrac{1}{2}(-3+\sqrt{7}), \dfrac{1}{2}(-3-\sqrt{7})$

(c) $\dfrac{1}{2}(-3+\sqrt{21}), \dfrac{1}{2}(-3-\sqrt{21})$

(d) none of these

Ans. (c)

Solution: TIP: It is an equation which is reducible to quadratic. Put $x^2 + 3x = y$.

The equation (1) becomes

$y^2 - y - 6 = 0 \quad \Rightarrow \quad (y+2)(y-3) = 0$

$\Rightarrow \quad y = -2, y = 3$

When $\quad y = 2,$

$x^2 + 3x = -2 \quad \Rightarrow \quad x^2 + 3x + 2 = 0$

$\Rightarrow \quad (x+1)(x+2) = 0 \quad \Rightarrow \quad x = -1, -2$

When $\quad y = 3,\ x^2 + 3x = 3 \quad \Rightarrow \quad x^2 + 3x - 3 = 0$

$\Rightarrow \quad x = \dfrac{1}{2}(-3 \pm \sqrt{9+12}) = \dfrac{1}{2}(-3 \pm \sqrt{21})$

Example 32: Two non-integer roots of

$$\left(\frac{3x-1}{2x+3}\right)^4 - 5\left(\frac{3x-1}{2x+3}\right)^2 + 4 = 0 \qquad (1)$$

are

(a) $-5/7, -2/5$ (b) $-2/5, 7/5$
(c) $5/7, 7/5$ (d) $-2/5, 3/5$

Ans. (a)

Solution: TIP: It is an equation which is reducible to quadratic. Put $\left(\dfrac{3x-1}{2x+3}\right)^2 = y$. The equation (1) becomes

$y^2 - 5y + 4 = 0 \Rightarrow (y-1)(y-4) = 0 \Rightarrow y = 1, 4$

When $\quad y = 1, \left(\frac{3x-1}{2x+3}\right)^2 = 1 \Rightarrow \frac{3x-1}{2x+3} = \pm 1$

$\Rightarrow \quad x = 4, -2/5$

When $\quad y = 4, \left(\frac{3x-1}{2x+3}\right)^2 = 4 \Rightarrow \frac{3x-1}{2x+3} = \pm 2$

$\Rightarrow \quad x = -7, -5/7$

Thus, required roots are – 2/5, – 5/7.

Example 33: Sum of the roots of the equation

is $\quad 4^x - 3(2^{x+3}) + 128 = 0 \quad (1)$

(a) 5 (b) 6
(c) 7 (d) 8

Ans. (c)

Solution: Put $2^x = y$. Equation (1) becomes

$y^2 - 3(8y) + 128 = 0 \quad \Rightarrow \quad y^2 - 24y + 128 = 0$

$\Rightarrow \quad (y-8)(y-16) = 0 \quad \Rightarrow \quad y = 16, 8$

$\Rightarrow \quad 2^x = 16, 8 \quad \Rightarrow \quad x = 4, 3$

$\therefore$ Sum of the roots is 7.

Example 34: The only real value of x satisfying the equation is

$$6\sqrt{\frac{x}{x+4}} - 2\sqrt{\frac{x+4}{x}} = 11 \qquad (1)$$

where $x \in \mathbf{R}$

(a) 4/35 (b) – 4/35
(c) 16/3 (d) none of these.

Ans. (d)

Solution: For (1) to be defined either $x < -4$ or $x > 0$.

Put $\sqrt{\frac{x}{x+4}} = y$. Equation (1) becomes

$6y - 2/y = 11 \quad \Rightarrow \quad 6y^2 - 11y - 2 = 0$

$\Rightarrow \quad (6y+1)(y-2) = 0 \quad \Rightarrow \quad y = -1/6, y = 2.$

As $\quad y = \sqrt{\frac{x}{x+4}} \geq 0$, we reject $y = -\frac{1}{6}$.

Thus, $\quad y = 2 \quad \Rightarrow \quad \frac{x}{x+4} = y^2 = 4$

$\Rightarrow \quad x = 4x + 16 \quad \Rightarrow \quad x = -16/3 < -4$

Example 35: The number of real values of x satisfying the equation

$$2\left(x^2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) + 14 = 0 \qquad (1)$$

is

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (c)

Solution: Put $x + \frac{1}{x} = y$, so that $x^2 + \frac{1}{x} = y^2 - 2$. Then equation (1) becomes

$2(y^2 - 2) - 9y + 14 = 0$

$\Rightarrow \quad 2y^2 - 9y + 10 = 0 \quad \Rightarrow \quad (2y-5)(y-2) = 0$

$\Rightarrow \quad y = 5/2, 2$

When $x + \frac{1}{x} = \frac{5}{2}$, we get $x = 2, \frac{1}{2}$

When $x + \frac{1}{x} = 2$, we get $x = 1$.

Example 36: The non-integral roots of

$$x^4 - 3x^3 - 2x^2 + 3x + 1 = 0 \qquad (1)$$

are

(a) $\frac{1}{2}(3+\sqrt{13}), \frac{1}{2}(3-\sqrt{13})$

(b) $\frac{1}{2}(3-\sqrt{13}), \frac{1}{2}(-3-\sqrt{13})$

(c) $\frac{1}{2}(3+\sqrt{17}), \frac{1}{2}(3-\sqrt{17})$

(d) none of these.

Ans. (a)

Solution: Dividing (1) by x^2, we get

$$x^2 - 3x - 2 + \frac{3}{x} + \frac{1}{x^2} = 0$$

or $\quad \left(x^2 + \frac{1}{x^2}\right) - 3\left(x - \frac{1}{x}\right) - 2 = 0 \qquad (2)$

Put $x - 1/x = y$, so that

$$x^2 + \frac{1}{x^2} - 2 = y^2 \text{ or } x^2 + \frac{1}{x^2} = y^2 + 2$$

Equation (2) now becomes

$y^2 + 2 - 3y - 2 = 0 \quad \Rightarrow \quad y^2 - 3y = 0$

$\Rightarrow \quad y(y-3) = 0 \quad \Rightarrow \quad y = 0 \quad$ or $\quad y = 3$

When $y = 0$, we get $x - \frac{1}{x} = 0 \quad \Rightarrow \quad x = \pm 1$

When $y = 3$, we get $x - \frac{1}{x} = 3$

$\Rightarrow \quad x^2 - 3x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}(3 \pm \sqrt{13})$

Example 37: Suppose $a, b, c \in \mathbf{R}$ and $b \neq c$. If α, β are roots of $x^2 + ax + b = 0$ and γ, δ are roots of $x^2 + ax + c = 0$, then value of

$$\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)} \text{ is independent of:}$$

(a) a, b (b) b, c
(c) c, a (d) a, b, c

Ans. (d)

◎ **Solution:** $x^2 + ax + c = (x - \gamma)(x - \delta)$

Thus,

$$\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)} = \frac{\alpha^2 + a\alpha + c}{\beta^2 + a\beta + c}$$

$$= \frac{-b+c}{-b+c} = 1$$

$[\because\ \alpha, \beta$ are roots of $x^2 + ax + b = 0]$

Example 38: Number of real solutions of

$$(x-1)(x+1)(2x+1)(2x-3) = 15 \qquad (1)$$

is

(a) 0 (b) 2
(c) 3 (d) 4

Ans. (b)

◎ **Solution:** We write (1) as

$$(x-1)(2x+1)(x+1)(2x-3) = 15$$

$$\Rightarrow \quad (2x^2 - x - 1)(2x^2 - x - 3) = 15 \qquad (2)$$

Put $2x^2 - x = y$, so that (2) becomes

$$(y-1)(y-3) = 15 \Rightarrow y^2 - 4y + 3 = 15$$

$$\Rightarrow \quad y^2 - 4y - 12 = 0 \Rightarrow (y-6)(y+2) = 0$$

$$\Rightarrow \quad y = 6, -2.$$

When $y = 6$, $2x^2 - x = 6 \Rightarrow 2x^2 - x - 6 = 0$

$$\Rightarrow \quad 2x^2 - 4x + 3x - 6 = 0 \Rightarrow (2x+3)(x-2) = 0$$

$$\Rightarrow \quad x = -3/2, 2$$

When $y = -2$, $2x^2 - x = -2$

or $\quad 2x^2 - x + 2 = 0$

As $D = 1 - 4(2)(2) = -15 < 0$, we get $2x^2 - x + 2 = 0$ does not have real roots.

Example 39: If $p, q \in \mathbf{R}$ and $2 + \sqrt{3}i$ a root of $x^2 + px + q = 0$, then

(a) $p = -2, q = \sqrt{3}$ (b) $p = -4, q = 7$
(c) $p = 3, q = 2$ (d) $p = -4, q = 2$

Ans. (b)

◎ **Solution:** Other root of the equation is $2 - \sqrt{3}i$. Thus

$$-p = \left(2+\sqrt{3}i\right) + \left(2-\sqrt{3}i\right) \qquad (2)$$

and $\quad q = \left(2+\sqrt{3}i\right)\left(2-\sqrt{3}i\right)$

$\Rightarrow \quad p = -4, q = 7$

Example 40: The number of solutions of

$$\sqrt{3x^2 + x + 5} = x - 3, \text{ where } x \in \mathbf{R} \qquad (1)$$

is:

(a) 0 (b) 1
(c) 2 (d) 4

Ans. (a)

◎ **Solution:** Note that we must have $3x^2 + x + 5 \ge 0$ and $x - 3 \ge 0$ or $x \ge 3$. Squaring both sides of (1), we get

$$3x^2 + x + 5 = x^2 - 6x + 9$$

$$\Rightarrow \quad 2x^2 + 7x - 4 = 0 \Rightarrow (2x-1)(x+4) = 0$$

$$\Rightarrow \quad x = 1/2, -4$$

None of these satisfy the inequality $x \ge 3$. Thus, (1) has no solution.

Example 41: The number of solutions of

$$\sqrt{4-x} + \sqrt{x+9} = 5 \qquad (1)$$

is:

(a) 0 (b) 1
(c) 2 (d) 3

Ans. (c)

◎ **Solution:** Note that $4 - x \ge 0$ and $x + 9 \ge 0 \Rightarrow -9 \le x \le 4$.

We can write (1) as $\sqrt{x+9} = 5 - \sqrt{4-x}$

Squaring both the sides we get

$$x + 9 = 25 - 10\sqrt{4-x} + 4 - x$$

$$\Rightarrow \quad 10\sqrt{4-x} = 20 - 2x \Rightarrow 5\sqrt{4-x} = 10 - x$$

Squaring both the sides we get

$$25(4-x) = 100 - 20x + x^2 \Rightarrow x^2 + 5x = 0$$

$$\Rightarrow \quad x(x+5) = 0 \Rightarrow x = 0 \text{ or } x = -5$$

Both $x = 0$ and $x = -5$ satisfy (1).

Example 42: Suppose $a \in \mathbf{R}$ and α, β are roots of $x^2 - 4ax + 5a^2 - 6a = 0$. The maximum possible distance between α and β is

(a) 6 (b) 5
(c) 3 (d) 1

Ans. (a)

◎ **Solution:** $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$= 16a^2 - 4(5a^2 - 6a)$$

$$= 4(6a - a^2)$$

$$= 4[9 - (3-a)^2]$$

As $a \in \mathbf{R}$, $(\alpha - \beta)^2$ is maximum when $a = 3$. Thus, maximum possible distance between α and β is 6.

Example 43: The value of a for which one root of the quadratic equation.

$$(a^2 - 5a + 3)x^2 + (3a-1)x + 2 = 0 \qquad (1)$$

is twice the other, is

(a) $-2/3$ (b) $1/3$
(c) $-1/3$ (d) $2/3$

Ans. (d)

◎ **Solution:** Let α and 2α be the roots of (1), then

$$(a^2 - 5a + 3)\alpha^2 + (3a-1)\alpha + 2 = 0 \qquad (2)$$

and $\quad (a^2 - 5a + 3)(4\alpha^2) + (3a-1)(2\alpha) + 2 = 0 \qquad (3)$

Multiplying (2) by 4 and subtracting it form (3) we get

$$(3a - 1)(2\alpha) + 6 = 0$$

Clearly $a \neq 1/3$. Therefore, $\alpha = -3/(3a - 1)$

Putting this value in (2) we get

$$(a^2 - 5a + 3)(9) - (3a - 1)^2 (3) + 2(3a - 1)^2 = 0$$

$$\Rightarrow \quad 9a^2 - 45a + 27 - (9a^2 - 6a + 1) = 0$$

$$\Rightarrow \quad -39a + 26 = 0$$

$$\Rightarrow \quad a = 2/3.$$

For $a = 2/3$, the equation becomes $x^2 + 9x + 18 = 0$, whose roots are $-3, -6$.

Example 44: Range of function $f(x) = \dfrac{x^2 + x + 2}{x^2 + x + 1}$, $x \in \mathbf{R}$ is

(a) $(1, \infty)$ (b) $(1, 3/2)$
(c) $(1, 7/3]$ (d) $(1, 7/5]$

Ans. (c)

Solution: Let $y = \dfrac{x^2 + x + 2}{x^2 + x + 1} = 1 + \dfrac{1}{x^2 + x + 1}$

$$= 1 + \frac{1}{(x + 1/2)^2 + 3/4} > 1$$

$$\Rightarrow \quad \frac{1}{y - 1} = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$$

Thus, $y > 1$ and $y - 1 \leq 4/3 \Rightarrow y \leq 7/3$

$\therefore \quad 1 < y \leq 7/3$

Example 45: If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are such that min $f(x) >$ max $g(x)$, then relation between b and c, is:

(a) no relation (b) $0 < c < b/2$
(c) $|c| < \dfrac{|b|}{\sqrt{2}}$ (d) $|c| > \sqrt{2}\,|b|$.

Ans. (d)

Solution: $f(x) = (x + b)^2 + 2c^2 - b^2$

$\Rightarrow \quad \min f(x) = 2c^2 - b^2$

Also $\quad g(x) = -x^2 - 2cx + b^2 = b^2 + c^2 - (x + c)^2$

$\Rightarrow \quad \max g(x) = b^2 + c^2$

As min $f(x) >$ max $g(x)$, we get $2c^2 - b^2 > b^2 + c^2$

$\Rightarrow \quad c^2 > 2b^2 \Rightarrow |c| > \sqrt{2}\,|b|$

Example 46: If a, b are the roots of $x^2 + px + 1 = 0$, and c, d are the roots of $x^2 + qx + 1 = 0$, then the value of

$$E = (a - c)(b - c)(a + d)(b + d)$$

is:

(a) $p^2 - q^2$ (b) $q^2 - p^2$
(c) $q^2 + p^2$ (d) none of these

Ans. (b)

Solution: We have

$$x^2 + px + 1 = (x - a)(x - b)$$

Thus,

$$E = (c - a)(c - b)(-d - a)(-d - b)$$

$$= (c^2 + pc + 1)[(-d)^2 - pd + 1]$$

$[\because a + b = -p]$

$$= (c^2 + pc + 1)(d^2 - pd + 1)$$

But $c^2 + qc + 1 = 0$ and $d^2 + qd + 1 = 0$

$$\therefore \quad E = (-qc + pc)(-qd - pd)$$

$$= cd(q - p)(q + p)$$

$$= cd(q^2 - p^2) = q^2 - p^2 \qquad [\because cd = 1]$$

Example 47: If $4^x - 3^{x - 1/2} = 3^{x + 1/2} - 2^{2x - 1}$, then value of x is equal to

(a) 5/2 (b) 2
(c) 3/2 (d) 1

Ans. (c)

Solution: $4^x - 3^{x - 1/2} = 3^{x + 1/2} - 2^{2x - 1}$

$$\Rightarrow \quad 4^x + 2^{2x - 1} = 3^{x + 1/2} + 3^{x - 1/2}$$

$$\Rightarrow \quad 4^x + \frac{4^x}{2} = 3^x\sqrt{3} + \frac{3^x}{\sqrt{3}}$$

$$\Rightarrow \quad \left(\frac{3}{2}\right)4^x = \left(\frac{4}{\sqrt{3}}\right)3^x$$

$$\Rightarrow \quad \left(\frac{4}{3}\right)^x = \frac{2}{3} \times \frac{4}{\sqrt{3}} = \frac{8}{3\sqrt{3}} = \left(\frac{2}{3^{1/2}}\right)^3 = \left(\frac{4}{3}\right)^{3/2}$$

$$\Rightarrow \quad x = 3/2$$

Example 48: If $x^2 + 2ax + 10 - 3a > 0$ for each $x \in \mathbf{R}$, then

(a) $a < -5$ (b) $-5 < a < 2$
(c) $a > 5$ (d) $2 < a < 5$

Ans. (b)

Solution: $x^2 + 2ax + 10 - 3a > 0 \;\forall\; x \in \mathbf{R}$

$\Rightarrow \quad (x + a)^2 - (a^2 + 10 - 3a) > 0 \;\forall\; x \in \mathbf{R}$

$\Rightarrow \quad a^2 + 3a - 10 < 0$

$\Rightarrow \quad (a + 5)(a - 2) < 0$

$\Rightarrow \quad -5 < a < 2$

Example 49: The number of solutions of

$$\sqrt{x + 1} - \sqrt{x - 1} = 1 \qquad (x \in \mathbf{R})$$

is

(a) 1 (b) 2
(c) 4 (d) infinite

Ans. (a)

Solution: For the equation to make sense we must have $x + 1 \geq 0$ and $x - 1 \geq 0 \Rightarrow x \geq -1$, $x \geq 1$ i.e. $x \geq 1$.

We rewrite equation as

$$\sqrt{x + 1} = 1 + \sqrt{x - 1}$$

and square both the sides to obtain

$$x + 1 = 1 + x - 1 + 2\sqrt{x - 1}$$

$$\Rightarrow \quad \frac{1}{2} = \sqrt{x-1} \Rightarrow \frac{1}{4} = x - 1 \text{ or } x = \frac{5}{4}$$

Also, $x = 5/4$ satisfies the given equation.

Example 50: If a, b, c are real and $a \neq b$, then the roots of the equation

$$2(a - b)x^2 - 11(a + b + c)x - 3(a - b) = 0 \qquad (1)$$

are

(a) real and equal (b) real and unequal
(c) purely imaginary (d) none of these

Ans. (b)

Solution: The discriminant D of the quadratic equation (1) is given by

$$D = 121(a + b + c)^2 + 24(a - b)^2$$

As a, b, c are real, $121(a + b + c)^2 \geq 0$

Also, as $a \neq b$, $(a - b)^2 > 0$

Thus, $D > 0$.

Therefore, equation (1) has real and unequal roots.

Example 51: Let $a > 0$, $b > 0$ and $c > 0$. Then both the roots of the equation

$$2ax^2 + 3bx + 5c = 0 \qquad (1)$$

(a) are real and negative
(b) have negative real parts
(c) have positive real parts
(d) none of these

Ans. (b)

Solution: The discriminant D of the equation (1) is given by

$$D = (3b)^2 - 4(2a)(5c) = 9b^2 - 40ac$$

If $D \geq 0$, then both the roots of (1) are real.

Also, since $ac > 0$, $D < 9b^2$.

$$\Rightarrow \quad \sqrt{D} < 3b \quad [\because\ b > 0]$$

Therefore, in this case both the roots $\dfrac{-3b - \sqrt{D}}{4a}$ and $\dfrac{-3b + \sqrt{D}}{4a}$ are negative.

If $D < 0$, both the roots of (1) are imaginary and are given by

$$x = \frac{-3b \pm i\sqrt{40ac - 9b^2}}{2}$$

Both these roots have negative real parts.

Example 52: If a, b, c are real, then both the roots of the equation

$$(x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b) = 0 \qquad (1)$$

are always

(a) positive (b) negative
(c) real (d) none of these

Ans. (c)

Solution: We can write (1) as

$$3x^2 - 2(a + b + c)x + bc + ca + ab = 0 \qquad (2)$$

The discriminant D of (2) is given by

$$\begin{aligned} D &= 4(a + b + c)^2 - 4 \times 3(bc + ca + ab) \\ &= 4[a^2 + b^2 + c^2 + 2bc + 2ca + 2ab - 3bc - 3ca - 3ab] \\ &= 4[a^2 + b^2 + c^2 - bc - ca - ab] \\ &= 2[(b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ca) + (a^2 + b^2 - 2ab)] \\ &= 2[(b - c)^2 + (c - a)^2 + (a - b)^2] \end{aligned}$$

As a, b, c are real, we get $D \geq 0$. Thus, roots of (1) are real.

Example 53: If the roots of the equation $(4a - a^2 - 5)x^2 - (2a - 1)x + 3a = 0$, $a \in \mathbf{R}$, are real and lie on opposite sides of unity, then the a lies in the interval:

(a) $(1, 5)$ (b) $(1, 4)$
(c) $(3, \infty)$ (d) $(-\infty, 4)$

Ans. (b)

Solution: **TIP:** If roots of a quadratic equation $f(x) = 0$ are on the opposite sides of a real number p, then $f(x + p) = 0$ has roots of opposite sings.

Put $\quad y = x - 1$, so that $x = y + 1$

The equation now becomes:

$$\begin{aligned} & (4a - a^2 - 5)(y + 1)^2 - (2a - 1)(y + 1) + 3a = 0 \\ \Rightarrow\ & (4a - a^2 - 5)y^2 + (8a - 2a^2 - 10 - 2a + 1)y \\ & + (4a - a^2 - 5 - 2a + 1 + 3a) = 0 \\ \Rightarrow\ & (4a - a^2 - 5)y^2 + (6a - 2a^2 - 9)y \\ & + (5a - a^2 - 4) = 0 \end{aligned}$$

This equation must have roots of opposite signs. This is possible if and only if

$$\frac{5a - a^2 - 4}{4a - a^2 - 5} < 0 \text{ or } \frac{a^2 - 5a + 4}{a^2 - 4a + 5} < 0$$

As $a^2 - 4a + 5 = (a - 2)^2 + 1 > 0 \ \forall\ a$,
we get $(a - 1)(a - 4) < 0 \Rightarrow 1 < a < 4$.

Example 54: If α, β are roots of $ax^2 + bx + c = 0$, then roots of

$$a^3x^2 + abcx + c^3 = 0 \text{ are}$$

(a) $\alpha\beta, \alpha + \beta$ (b) $\alpha^2\beta, \alpha\beta^2$
(c) $\alpha\beta, \alpha^2\beta^2$ (d) α^3, β^3

Ans. (b)

Solution: $\alpha + \beta = \dfrac{-b}{a}$, $\alpha\beta = \dfrac{c}{a}$.

Write $\quad a^3x^2 + abcx + c^3 = 0 \quad$ as

$$x^2 + \frac{bc}{a^2}x + \left(\frac{c}{a}\right)^3 = 0$$

$$\Rightarrow \quad x^2 - (\alpha + \beta)\,\alpha\beta x + \alpha^3\beta^3 = 0$$

$$\Rightarrow \quad x^2 - (\alpha^2\beta + \alpha\beta^2)x + (\alpha^2\beta)(\alpha\beta^2) = 0$$

$$\Rightarrow \quad (x - \alpha^2\beta)(x - \alpha\beta^2) = 0$$

$$\Rightarrow \quad x = \alpha^2\beta, \alpha\beta^2$$

Example 55: If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $a, b, c \in \mathbf{R}$, then $P(x)\,Q(x) = 0$ has

(a) no real root
(b) exactly two real roots
(c) at least two real roots
(d) none of these

Ans. (c)

Solution: Let $D_1 = b^2 - 4ac$ and $D_2 = d^2 + 4ac$

We have $D_1 + D_2 = b^2 + d^2 \geq 0$

$\Rightarrow$ at least one of $D_1, D_2 \geq 0$

$\Rightarrow$ one of $P(x) = 0$ or $Q(x) = 0$

has real roots. Thus, $P(x)\,Q(x) = 0$

has at least two real roots.

Example 56: If the product of the roots of the equation $x^2 - 5kx + 2e^{4\ln k} - 1 = 0$ is 31, then sum of the root is

(a) -10 (b) 5
(c) -8 (d) 10

Ans. (d)

Solution: We have product of the roots

$$= 2e^{4\ln k} - 1 = 31 \text{ (given)}$$

$\Rightarrow$ $e^{4\ln k} = 16$

$\Rightarrow$ $k^4 = 16 \Rightarrow k^4 - 16 = 0$

$\Rightarrow$ $(k-2)(k+2)(k^2+4) = 0$

$\Rightarrow$ $k = 2, -2$

As $k > 0$, we get $k = 2$.

Sum of the roots $= 5k = 10$

Example 57: Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\dfrac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is :

(a) $\frac{2}{9}(p-q)(2q-p)$ (b) $\frac{2}{9}(q-p)(2p-q)$

(c) $\frac{2}{9}(q-2p)(2q-p)$ (d) $\frac{2}{9}(2p-q)(2q-p)$

Ans. (d)

Solution: $\alpha + \beta = p$, $\alpha\beta = r$

and $\dfrac{\alpha}{2} + 2\beta = q.$

But $\alpha + \beta = p$ and $\alpha + 4\beta = 2q$

$\Rightarrow$ $\beta = \frac{1}{3}(2q-p)$ and $\alpha = \frac{2}{3}(2p-q)$

Thus, $\alpha\beta = r$

$\Rightarrow$ $\frac{2}{9}(2p-q)(2q-p) = r$

Example 58: Sum of all the values of x satisfying the equation

$$\log_{17}\log_{11}\left(\sqrt{x+11} + \sqrt{x}\right) = 0 \qquad (1)$$

is:

(a) 25 (b) 36
(c) 171 (d) 0

Ans. (a)

Solution: Equation (1) is defined if $x \geq 0$.

We can rewrite (1) as

$$\log_{11}\left(\sqrt{x+11} + \sqrt{x}\right) = 17^0 = 1$$

$\Rightarrow$ $\sqrt{x+11} + \sqrt{x} = 11^1 = 11$

$\Rightarrow$ $\sqrt{x+11} = 11 - \sqrt{x}$

Squaring both the sides we get

$$x + 11 = 121 - 22\sqrt{x} + x$$

$\Rightarrow$ $22\sqrt{x} = 110 \Rightarrow \sqrt{x} = 5$ or $x = 25$

This clearly satisfies (1).

Thus, sum of all the values satisfying (1) is 25.

Example 59: If $\dfrac{1}{\sqrt{\alpha}}$ and $\dfrac{1}{\sqrt{\beta}}$ are the roots of the equation $ax^2 + bx + 1 = 0$ ($a \neq 0, a, b, \in \mathbf{R}$), then the equation, $x(x + b^3) + (a^3 - 3abx) = 0$ has roots:

(a) $\alpha^{3/2}$ and $\beta^{3/2}$ (b) $\alpha\beta^{-3/2}$ and $\alpha^{1/2}\beta$

(c) $\sqrt{\alpha\beta}$ and $\alpha\beta$ (d) $\alpha^{-3/2}$ and $\beta^{-3/2}$

Ans. (a)

Solution: The equation $x(x + b^3) + (a^3 - 3abx) = 0$ can be written as

$$x^2 + (b^3 - 3ab)x + a^3 = 0$$

We have

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = -\frac{b}{a}, \frac{1}{\sqrt{\alpha}\sqrt{\beta}} = \frac{1}{a}$$

$$\sqrt{\alpha} + \sqrt{\beta} = -b, \sqrt{\alpha}\sqrt{\beta} = a$$

Now

$$-(b^3 - 3ab) = (-b)^3 - 3a(-b)$$

$$= \left(\sqrt{\alpha} + \sqrt{\beta}\right)^3 - 3\left(\sqrt{\alpha} + \sqrt{\beta}\right)\sqrt{\alpha}\sqrt{\beta}$$

$$= \left(\sqrt{\alpha}\right)^3 + \left(\sqrt{\beta}\right)^3 = \alpha^{3/2} + \beta^{3/2}$$

and $a^3 = \alpha^{3/2}\beta^{3/2}$

$\therefore$ The roots of the equation

$$x^2 + (b^2 - 3ab)x + a^3 = 0 \text{ are } \alpha^{3/2}, \beta^{3/2}$$

Example 60: If p, q are roots of $x^2 + px + q = 0$, then

(a) $p = 1$ (b) $p = 1$ or 0
(c) $p = -2$ (d) $p = -2$ or 0

Ans. (b)

Solution: We have $p + q = -p$, $pq = q$

As $pq = q$,

we get $q(p-1) = 0$

$\Rightarrow$ $q = 0$ or $p = 1$

If $q = 0$, we get $p = 0$

If $p = 1$, we get $p + q = -p \Rightarrow q = -2$.

Thus, $p = 1$ or $p = 0$.

Example 61: The equation

$$\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}, \ (x \in \mathbf{R})$$

(a) no solution
(b) one solution
(c) two solutions
(d) more than two solutions

Ans. (a)

Solution: The given equation is valid if $x + 1 \geq 0$, $x - 1 \geq 0$ and $4x - 1 \geq 0$ i.e. if $x \geq 1$.

Squaring both the sides we get

$$x + 1 + x - 1 - 2\sqrt{(x+1)(x-1)} = 4x - 1$$

$$\Rightarrow \quad 1 - 2x = 2\sqrt{(x+1)(x-1)}$$

Squaring again, we get

$$1 - 4x + 4x^2 = 4(x^2 - 1)$$

$$\Rightarrow \quad 4x = 5 \text{ or } x = 5/4.$$

Putting this value of x in the given equation, we get

$$\sqrt{\frac{5}{4}+1} - \sqrt{\frac{5}{4}-1} = \sqrt{4\left(\frac{5}{4}\right)-1}$$

$$\Rightarrow \quad \frac{3}{2} - \frac{1}{2} = 2 \text{ or } 1 = 2$$

which is not true.

Thus, the given equation has no solution.

Example 62: The sum of all the real roots of the equation

$$|x - 2|^2 + |x - 2| - 2 = 0 \qquad (1)$$

is

(a) 7 (b) 4
(c) 1 (d) none of these

Ans. (b)

Solution: Putting $y = |x - 2|$, we can rewrite (1) as

$$y^2 + y - 2 = 0 \text{ or } y^2 + 2y - y - 2 = 0$$

$$\Rightarrow \quad y(y + 2) - (y + 2) = 0 \Rightarrow (y - 1)(y + 2) = 0$$

As $\quad y = |x - 2| \geq 0,\ y + 2 \geq 2$

$$\therefore \quad y - 1 = 0 \Rightarrow y = 1$$

$$\Rightarrow \quad |x - 2| = 1 \Rightarrow x - 2 = \pm 1$$

$$\Rightarrow \quad x = 2 \pm 1 \Rightarrow x = 3 \text{ or } 1$$

$\therefore \quad$ Sum of the roots = 4

Example 63: Let p and q be the roots of $x^2 - 2x + A = 0$ and let r and s be the roots of $x^2 - 18x + B = 0$. If $p < q < r < s$ are in A.P. then ordered pair (A, B) is equal to

(a) $(-3, 77)$ (b) $(77, -3)$
(c) $(-3, -77)$ (d) none of these

Ans. (a)

Solution: We have $p + q = 2$, $pq = A$ (1)

and $\quad r + s = 18$, $rs = B$ (2)

As p, q, r, s are in AP, we take

$$p = a - 3d,\ q = a - d,\ r = a + d,\ s = a + 3d.$$

As $p < q < r < s$, we have $d > 0$

Now, $\quad 2 = p + q = 2a - 4d$

and $\quad 18 = r + s = 2a + 4d$

Solving above equations, we get $a = 5$ and $d = 2$

$\therefore \quad p = -1$, $q = 3$, $r = 7$ and $s = 11$

Thus, $\quad A = pq = -3$ and $B = rs = 77$.

Example 64: In a triangle PQR, $\angle R = \pi/2$. If $\tan(P/2)$ and $\tan(Q/2)$ are the roots of the equation $ax^2 + bx + c = 0$ where $a \neq 0$, then

(a) $a + b = c$ (b) $b + c = a$
(c) $a + c = b$ (d) $b = c$

Ans. (a)

Solution: We have

$$\tan\left(\frac{P}{2}\right) + \tan\left(\frac{Q}{2}\right) = \frac{-b}{a}$$

and $$\tan\left(\frac{P}{2}\right)\tan\left(\frac{Q}{2}\right) = \frac{c}{a}$$

Now, $$P + Q = \frac{\pi}{2} \Rightarrow 1 = \tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{P}{2}+\frac{Q}{2}\right)$$

$$= \frac{\tan(P/2) + \tan(Q/2)}{1 - \tan(P/2)\tan(Q/2)}$$

$$\Rightarrow \quad 1 = \frac{-b/a}{1 - c/a} = \frac{-b}{a - c}$$

$$\Rightarrow \quad a - c = -b \Rightarrow c = a + b$$

Example 65: If α and β $(\alpha < \beta)$, are the roots of the equation $x^2 + bx + c = 0$ where $c < 0 < b$, then

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$

Ans. (b)

Solution: We have $\alpha + \beta = -b$, $\alpha\beta = c$

As $c < 0$, $b > 0$, we get $\alpha < 0 < \beta$

Also, $\quad \beta = -b - \alpha < -\alpha = |\alpha|$

Thus, $\quad \alpha < 0 < \beta < |\alpha|$

Example 66: For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the roots is square of the other, than p is equal to

(a) 1/3 (b) 1
(c) 3 (d) 2/3

Ans. (c)

Solution: Let α, α^2 be the roots of $3x^2 + px + 3 = 0$.

Then $\alpha + \alpha^2 = -p/3$ and $\alpha \cdot \alpha^2 = 1$

As $\quad \alpha^3 = 1$, we get $\alpha = 1$, ω or ω^2.

If $\quad \alpha = 1$, then $p = -3(\alpha + \alpha^2) = -6 < 0$.

Not possible as $p > 0$.

Thus, $\alpha \neq 1$.

We take $\alpha = \omega$ and $\alpha^2 = \omega^2$, so that $\dfrac{-p}{3} = \omega + \omega^2 = -1$

$$\Rightarrow \quad p = 3$$

Example 67: Let α, β be roots of $ax^2 + bx + c = 0$, where $ac \neq 0$. Roots of $cx^2 - bx + a = 0$ are

(a) $1/\alpha, 1/\beta$ (b) $-1/\alpha, -1/\beta$
(c) $1/\sqrt{\alpha}, 1/\sqrt{\beta}$ (d) $-1/\sqrt{\alpha}, -1/\sqrt{\beta}$

Ans. (b)

◎ **Solution:** $\alpha + \beta = -b/a$, $\alpha\beta = c/a$.

Now, $$\frac{b}{c} = -\frac{b/a}{c/a} = -\frac{\alpha+\beta}{\alpha\beta}$$
$$= \left(-\frac{1}{\alpha}\right) + \left(-\frac{1}{\beta}\right)$$

And $$\frac{a}{c} = \frac{1}{\alpha\beta} = \left(-\frac{1}{\alpha}\right)\left(-\frac{1}{\beta}\right)$$

Thus, roots of $cx^2 - bx + a = 0$ are $-1/\alpha, -1/\beta$.

Example 68: If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$ has
(a) both roots in $[a, b]$
(b) both roots in $(-\infty, a)$
(c) both roots in (b, ∞)
(d) one root in $(-\infty, a)$ and other in (b, ∞).

Ans. (d)

◎ **Solution:** Graph of $y = (x - a)(x - b) - 1$ is given in Fig. 3.17.

It is a parabola which open upwards. Also, $y < 0$ for $x = a$ and $x = b$.

$\therefore$ $y = (x - a)(x - b) - 1$ meets the x-axis at two points once in $(-\infty, a)$ and once in (b, ∞). Thus, one root lies in $(-\infty, a)$ and one in (b, ∞).

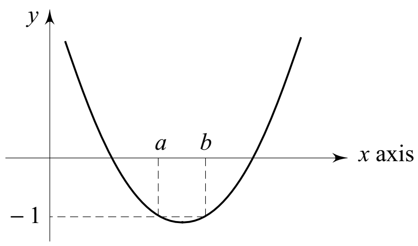


Fig. 3.17

Example 69: Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in G.P. then the integral values of p and q respectively, are
(a) $-2, -32$ (b) $-2, 3$
(c) $-6, 3$ (d) $-6, -32$

Ans. (a)

◎ **Solution:** We have
$$\alpha + \beta = 1,\ \alpha\beta = p,$$
$$\gamma + \delta = 4,\ \gamma\delta = q$$
Let r be the common ratio of the GP $\alpha, \beta, \gamma, \delta$. Then
$$\alpha + \beta = 1 \Rightarrow \alpha + \alpha r = 1 \Rightarrow \alpha(1 + r) = 1$$
$$\gamma + \delta = 4 \Rightarrow \alpha r^2 + \alpha r^3 = 4$$
$\Rightarrow$ $$\alpha r^2(1 + r) = 4$$
Thus, $$\frac{\alpha r^2(1+r)}{\alpha(1+r)} = 4 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$$
When $r = 2$,
we get $\alpha(1 + r) = 1 \Rightarrow \alpha = 1/3$
In this case $p = \alpha\beta = \alpha(\alpha r) = \alpha^2 r$
$$= \frac{1}{9}(2) = \frac{2}{9}$$
which is not an integer.
Thus, $r = -2$. In this case, $\alpha(1 + r) = 1 \Rightarrow \alpha = -1$.
$\therefore$ $p = \alpha^2 r = (-1)^2(-2) = -2$
and $q = \gamma\delta = (\alpha r^2)(\alpha r^3) = \alpha^2 r^5 = -32$
Hence, $p = -2$, $q = -32$.

Example 70: If a, b, and c are not all equal and α and β be the roots of the equation $ax^2 + bx + c = 0$, then value of $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$ is
(a) 0 (b) positive
(c) negative (d) non-negative

Ans. (d)

◎ **Solution:** We have $\alpha + \beta = -b/a$, $\alpha\beta = c/a$
Now, $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$
$$= 1 + \alpha + \beta + \alpha^2 + \beta^2 + \alpha\beta + \alpha^2\beta + \alpha\beta^2 + \alpha^2\beta^2$$
$$= 1 + (\alpha + \beta) + (\alpha + \beta)^2 - 2\alpha\beta + \alpha\beta[1 + \alpha + \beta + \alpha\beta]$$
$$= 1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{2c}{a} + \frac{c}{a} + \frac{c}{a}\left(-\frac{b}{a}\right) + \frac{c^2}{a^2}$$
$$= \frac{1}{a^2}\{a^2 + b^2 + c^2 - bc - ca - ab\}$$
$$= \frac{1}{2a^2}[(b - c)^2 + (c - a)^2 + (a - b)^2] \geq 0$$

Example 71: If a, b, c are in A.P. and if the equations
$$(b - c)x^2 + (c - a)x + (a - b) = 0 \quad (1)$$
and $$2(c + a)x^2 + (b + c)x = 0 \quad (2)$$
have a common root, then
(a) a^2, b^2, c^2 are in A.P.
(b) a^2, c^2, b^2 are in A.P.
(c) c^2, a^2, b^2 are in A.P.
(d) none of these

Ans. (b)

◎ **Solution:** Clearly $x = 1$ is a root of (1). If α is the other root of (1), then
$$\alpha = 1 \cdot \alpha = \frac{a-b}{b-c} \text{ [product of roots]}$$
$$= 1 \text{ [}\because a, b, c \text{ are in A. P.]}$$

Thus, the roots of (1) are 1, 1. Now, (1) and (2) will have a common root if 1 is also a root of (2).

$\Rightarrow \quad 2(c + a) + b + c = 0$

$\Rightarrow \quad 2(2b) + b + c = 0 \Rightarrow c = -5b \quad [\because\ a, b, c \text{ are in AP}]$

Also, $\quad a + c = 2b$

$\Rightarrow \quad a = 2b - c = 2b + 5b = 7b$

$\therefore \quad a^2 = 49b^2,\ c^2 = 25b^2$

This, show that a^2, c^2, b^2 are in A.P.

Example 72: Value of

$$x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots \text{up to } \infty}}} \qquad (1)$$

is

(a) 3 (b) 2
(c) 1 (d) none of these

Ans. (a)

Solution: We can write (1) as

$$x = \sqrt{6 + x} \Rightarrow x^2 = 6 + x$$

$\Rightarrow \quad x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0$

$\Rightarrow \quad x = 3, -2$

As $x > 0$, we take $x = 3$

Example 73: Two complex numbers α and β are such that $\alpha + \beta = 2$ and $\alpha^4 + \beta^4 = 272$, then the quadratic equation whose roots are α and β can be

(a) $x^2 - 2x - 16 = 0$ (b) $x^2 - 2x + 12 = 0$
(c) $x^2 - 2x - 8 = 0$ (d) none of these

Ans. (c)

Solution: $\alpha + \beta = 2 \Rightarrow \alpha^2 + \beta^2 + 2\alpha\beta = 4$

$\Rightarrow \quad \alpha^2 + \beta^2 = 4 - 2\alpha\beta \Rightarrow (\alpha^2 + \beta^2)^2 = 16 - 16\alpha\beta + 4\alpha^2\beta^2$

$\Rightarrow \quad \alpha^4 + \beta^4 + 2\alpha^2\beta^2 = 16 - 16\alpha\beta + 4\alpha^2\beta^2$

$\Rightarrow \quad 272 + 2(\alpha\beta)^2 = 16 - 16\alpha\beta + 4(\alpha\beta)^2$

$\Rightarrow \quad 2(\alpha\beta)^2 - 16(\alpha\beta) - 256 = 0$

$\Rightarrow \quad (\alpha\beta)^2 - 8(\alpha\beta) - 128 = 0$

$\Rightarrow \quad (\alpha\beta - 16)(\alpha\beta + 8) = 0$

$\Rightarrow \quad \alpha\beta = 16 \text{ or } \alpha\beta = -8$

Thus, required equation is either

$$x^2 - 2x + 16 = 0$$

or $\quad x^2 - 2x - 8 = 0$

$\therefore$ Required answer is (c).

Example 74: If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$, then the equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ is

(a) $3x^2 - 25x + 3 = 0$ (b) $x^2 - 5x + 3 = 0$
(c) $x^2 + 5x - 3 = 0$ (d) $3x^2 - 19x + 3 = 0$

Ans. (d)

Solution: α, β are roots of $x^2 = 5x - 3$ or $x^2 - 5x + 3 = 0$. Thus,

$$\alpha + \beta = 5,\ \alpha\beta = 3.$$

Next, $\quad \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = \dfrac{(\alpha + \beta)^2}{\alpha\beta} - 2$

$$= \frac{25}{3} - 2 = \frac{19}{3}$$

and $\quad \left(\dfrac{\alpha}{\beta}\right)\left(\dfrac{\beta}{\alpha}\right) = 1.$

Thus, the quadratic equation whose roots are $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$ is

$$x^2 - \frac{19}{3}x + 1 = 0 \text{ or } 3x^2 - 19x + 3 = 0.$$

Example 75: If $a \neq b$ and difference between the roots of the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ is the same, then

(a) $a + b + 4 = 0$ (b) $a + b - 4 = 0$
(c) $a - b + 4 = 0$ (d) $a - b - 4 = 0$

Ans. (a)

Solution: Let α, β be the roots of $x^2 + ax + b = 0$ and γ, δ be the roots of $x^2 + bx + a = 0$.

We are given

$$|\alpha - \beta| = |\gamma - \delta|$$

$\Rightarrow \quad |\alpha - \beta|^2 = |\gamma - \delta|^2$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = (\gamma + \delta)^2 - 4\gamma\delta$

$\Rightarrow \quad a^2 - 4b = b^2 - 4a$

$\Rightarrow \quad a^2 - b^2 + 4(a - b) = 0$

$\Rightarrow \quad (a - b)(a + b + 4) = 0$

As $\quad a \neq b,\ a + b + 4 = 0.$

Example 76: Product of real roots of the equation

$$t^2x^2 + |x| + 9 = 0.\ t \in \mathbf{R} \text{ is always}$$

(a) positive (b) negative
(c) zero (d) does not exist.

Ans. (d)

Solution: Note that $t^2x^2 + |x| + 9 \geq 0 + 9 > 9\ \forall\ x \in \mathbf{R}$,
Thus, $t^2x^2 + |x| + 9 = 0$ does not have real roots.

Example 77: Let $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$, $x \in \mathbf{R}$. The range of $f(x)$ is

(a) $[0, \infty)$ (b) $[1/3, 3]$
(c) $[3, \infty)$ (d) $[0, 3]$

Ans. (b)

Solution: Let $y = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4} = \dfrac{(x-1)^2 + 3}{(x+1)^2 + 3} > 0$ for each $x \in \mathbf{R}$.

$\Rightarrow \quad (y - 1)x^2 + 2(y + 1)x + 4(y - 1) = 0$

As x is real,

$$4(y + 1)^2 - 16(y - 1)^2 \geq 0$$

$\Rightarrow (y + 1 + 2y - 2)(y + 1 - 2y + 2) \geq 0$

$\Rightarrow 3(y - 1/3)(y - 3) \leq 0$

$\Rightarrow 1/3 \leq y \leq 3.$

Example 78: The values of $a \in \mathbf{R}$ for which one root of the equation $f(x) = x^2 - (a + 1)x + a^2 + a - 8 = 0$ exceeds 2 and the other is less than 2 are given by

(a) $3 < a < 10$ (b) $a > 0$
(c) $-2 < a < 3$ (d) $a \leq -2$

Ans. (c)

TIP: If $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbf{R}$ with $a > 0$ and $f(\alpha) < 0$ for some $\alpha \in \mathbf{R}$ then $D = b^2 - 4ac > 0$, and hence $f(x) = 0$ has real and distinct roots.

Solution: As one of the roots is less than 2 and other is more than 2, $f(2) < 0$.

$\Rightarrow \quad 2^2 - (a + 1)(2) + a^2 + a - 8 < 0$

$\Rightarrow \quad a^2 - a - 6 < 0 \Rightarrow (a + 2)(a - 3) < 0$

$\Rightarrow \quad -2 < a < 3.$

Example 79: If one root of the equation $x^2 + px + 12 = 0$ is 4, while equation $x^2 + px + q = 0$ has equal roots, then the value of q is

(a) 3 (b) 12
(c) $\frac{49}{4}$ (d) 4

Ans. (c)

Solution: $4^2 + 4p + 12 = 0 \quad \Rightarrow \quad p = -7$

Since $x^2 - 7x + q = 0$ has equal roots, we get $49 - 4q = 0$

$\Rightarrow q = \frac{49}{4}$

Example 80: Suppose a is an integer and x_1 and x_2 are positive real roots of $x^2 + (2a - 1)x + a^2 = 0$, then value of $\left|\sqrt{x_1} - \sqrt{x_2}\right|$ is

(a) 1 (b) 2
(c) a (d) $1 - 4a$

Ans.(a)

Solution: Two real roots are positive implies $2a - 1 \leq 0$, that is, $a \leq 1/2$. As a is an integer, we get $a \leq 0$. Now,

$$\left|\sqrt{x_1} - \sqrt{x_2}\right|^2 = \left(\sqrt{x_1} - \sqrt{x_2}\right)^2$$
$$= x_1 + x_2 - 2\sqrt{x_1 x_2}$$
$$= (1 - 2a) - 2\sqrt{a^2} = 1 - 2a - 2|a|$$
$$= 1 - 2(a + |a|) = 1$$

$\Rightarrow \quad \left|\sqrt{x_1} - \sqrt{x_2}\right| = 1$

Example 81: If a, b are two real number satisfying the relations $2a^2 - 3a - 1 = 0$ and $b^2 + 3b - 2 = 0$ and $ab \neq 1$, then value of $\frac{ab + a + 1}{b}$ is

(a) -1 (b) 0
(c) 1 (d) 2

Ans. (c)

Solution: $b^2 + 3b - 2 = 0 \Rightarrow 1 + \frac{3}{b} - \frac{2}{b^2} = 0$

$\Rightarrow \quad \frac{2}{b^2} - \frac{3}{b} - 1 = 0$

Thus, a and $1/b$ are roots of $2x^2 - 3x - 1 = 0$

$\therefore \quad a + \frac{1}{b} = \frac{3}{2}$ and $\frac{a}{b} = \frac{-1}{2}$

Now, $\quad \frac{ab + a + 1}{b} = a + \frac{1}{b} + \frac{a}{b} = \frac{3}{2} - \frac{1}{2} = 1$

Example 82: If α, β are roots of $x^2 - 2x - 1 = 0$, then value of $5\alpha^4 + 12\beta^3$ is

(a) 153 (b) 169
(c) 183 (d) none of these

Ans. (b)

Solution: We have $\alpha^2 = 2\alpha + 1$

$\Rightarrow \quad \alpha^4 = (2\alpha + 1)^2 = 4\alpha^2 + 4\alpha + 1$

$= 4(2\alpha + 1) + 4\alpha + 1 = 12\alpha + 5$, and

$\beta^2 = 2\beta + 1 \Rightarrow \quad \beta^3 = 2\beta^2 + \beta = 2(2\beta + 1) + \beta$
$= 5\beta + 2$

Thus, $\quad 5\alpha^4 + 12\beta^3 = (60\alpha + 25) + (60\beta + 24)$
$= 60(\alpha + \beta) + 49 = 60(2) + 44 = 169$

Example 83: Suppose a, b, c are the lengths of three sides of a ΔABC, $a > b > c$, $2b = a + c$ and b is a positive integer. If $a^2 + b^2 + c^2 = 84$, then value of b is

(a) 7 (b) 6
(c) 5 (d) 4

Ans. (c)

Solution: We have

$$ac = \frac{1}{2}[(a + c)^2 - (a^2 + c^2)] = \frac{1}{2}[4b^2 - (84 - b^2)]$$
$$= 5b^2/2 - 42$$

Thus, a and c are the roots of the equation

$$x^2 - 2bx + (5b^2/2 - 42) = 0$$

As a and c are distinct real numbers the discriminant of the above must be positive, that is, $4b^2 - 4(5b^2/2 - 42) > 0$

$\Rightarrow \quad 6b^2 < 168$ or $b^2 < 28$.

Also, $\quad ac > 0 \Rightarrow 5b^2 > 84$.

$\therefore \quad 84/5 < b^2 < 28.$

As b is a positive integer, we get $b = 5$.

Example 84: Suppose $a, b \in \mathbf{R}$, $a \neq 0$ and $2a + b \neq 0$. A root of the equation

$(a + b)(ax + b)(a - bx) = (a^2x - b^2)(a + bx)$ is

(a) $\frac{a + 2b}{2a + b}$ (b) $\frac{2a + b}{a + 2b}$

(c) $-\dfrac{a-2b}{a+2b}$ (d) $-\dfrac{a+2b}{2a+b}$

Ans. (d)

◎ **Solution:** We can write the quation as

$[a^2b + (a + b)\, ab]x^2 + [(a^3 - b^3) - (a + b)\,(a^2 - b^2)]x$
$\qquad - ab^2 - ab(a + b) = 0$

$\Rightarrow (2a + b)x^2 - (a - b)x - (a + 2b) = 0$

Since the sum of coefficients is 0 one of the roots is 1 and the other root is $-\dfrac{a+2b}{2a+b}$.

Example 85: Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ as its roots is

(a) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(b) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(c) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(d) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

Ans. (b)

◎ **Solution:** $q = \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)$
$= -p^3 + 3\alpha\beta p$

$\Rightarrow \qquad \alpha\beta = \dfrac{p^3+q}{3p}$

We have

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$$
$$= \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} = \frac{(\alpha+\beta)^2}{\alpha\beta}-2$$
$$= \frac{p^2}{(p^3+q)/3p}-2$$
$$= \frac{3p^3-2p^3-2q}{p^3+q} = \frac{p^3-2q}{p^3+q}$$

and $\qquad \dfrac{\alpha}{\beta}\cdot\dfrac{\beta}{\alpha} = 1$

Thus, required quadratic equation is

$$x^2 - \frac{(p^3-2q)}{p^3+q}\,x + 1 = 0$$

or $\qquad (p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

Example 86: Suppose $a, b \in \mathbf{R}$. If the roots of $x^2 - ax + b = 0$ are real and distinct and differ by at most 1, then $4b$ lies in the interval

(a) $(a^2 - 1, a^2)$ (b) $[a^2 - 1, a^2)$
(c) $(a^2 - 1, \infty)$ (d) (a^2, ∞)

Ans. (b)

◎ **Solution:** As roots of $x^2 - ax + b = 0$ are real and distinct, $a^2 - 4b > 0 \Rightarrow 4b < a^2$.

If α, β are roots of $x^2 - ax + b = 0$, then

$|\alpha - \beta| \le 1 \Rightarrow (\alpha - \beta)^2 \le 1$

$\Rightarrow \qquad (\alpha + \beta)^2 - 4\alpha\beta \le 1$

$\Rightarrow \qquad a^2 - 4b \le 1 \Rightarrow a^2 - 1 \le 4b.$

Thus, $a^2 - 1 \le 4b < a^2$.

Example 87: If a is the minimum root of the equation $x^2 - 3|x| - 2 = 0$, then value of $-1/a$ is

(a) $\left(\sqrt{17}-3\right)/4$ (b) $\left(\sqrt{17}+3\right)/4$
(c) 2 (d) -3

Ans. (a)

◎ **Solution:** $x^2 - 3|x| - 2 = 0$ can be written as $|x|^2 - 3|x| - 2 = 0$

$\Rightarrow \qquad |x| = \dfrac{1}{2}\left(3 \pm \sqrt{17}\right)$

As $\qquad |x| \ge 0$, we get $|x| = \dfrac{1}{2}\left(3+\sqrt{17}\right)$

$\Rightarrow \qquad x = \pm\dfrac{1}{2}\left(3+\sqrt{17}\right)$

$\therefore \qquad a = -\dfrac{1}{2}\left(3+\sqrt{17}\right) \Rightarrow -\dfrac{1}{a} = \dfrac{2}{\sqrt{17}+3} = \dfrac{\sqrt{17}-3}{4}$

Example 88: Let $f(x) = \dfrac{x^2-2x+3}{x^2-2x-8}$, $x \in \mathbf{R} - \{-2, 4\}$

The range of f is

(a) $\left(\dfrac{-2}{9}, 1\right]$ (b) $\mathbf{R} - \left(\dfrac{-2}{9}, 1\right)$
(c) $\left(-\infty, \dfrac{-2}{9}\right]$ (d) $\mathbf{R} - \left(\dfrac{-2}{9}, 1\right]$

Ans. (d)

◎ **Solution:** Let $y = \dfrac{x^2-2x+3}{x^2-2x-8}$, $x \in \mathbf{R} - \{-2, 4\}$

Note that $y \neq 1$.

Now,

$(y - 1)x^2 - 2x(y - 1) - (8y + 3) = 0$

As x is real

$4(y - 1)^2 + 4(y - 1)\,(8y + 3) \ge 0$

$\Rightarrow \qquad (y - 1)\,[y - 1 + 8y + 3] \ge 0.$

$\Rightarrow \qquad (y - 1)\,(y + 2/9) \ge 0$

$\Rightarrow \qquad y \le -2/9$ or $y > 1$. [$\because$ $y \neq 1$]

Thus, $\qquad y \in \mathbf{R} - (-2/9, 1]$.

Example 89: Suppose a, b, c are three non-zero real numbers. The equation

$$x^2 + (a + b + c)x + (a^2 + b^2 + c^2) = 0 \qquad (1)$$

has

(a) two negative real roots,
(b) two positive real roots,
(c) two real roots with opposite signs,
(d) no real roots

Ans. (d)

◎ **Solution:** Discriminant D of (1) is given by

$$D = (a + b + c)^2 - 4(a^2 + b^2 + c^2)$$
$$= -\{(b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ca) + (a^2 + b^2 - 2ab) + (a^2 + b^2 + c^2)\}$$
$$= -[(b - c)^2 + (c - a)^2 + (a - b)^2 + (a^2 + b^2 + c^2)] < 0$$

Thus, (1) cannot have real roots.

Example 90: Suppose $a, b \in \mathbf{R}$. If the equation

$$x^2 - (2a + b)x + (2a^2 + b^2 - b + 1/2) = 0 \qquad (1)$$

has two real roots, then

(a) $a = \frac{1}{2}, b = -1$ (b) $a = -\frac{1}{2}, b = 1$
(c) $a = 2, b = 1$ (d) $a = -2, b = -1$

Ans. (a)

◎ **Solution:** As (1) has real roots,

$$(2a + b)^2 - 4(2a^2 + b^2 - b + 1/2) \geq 0$$
$$\Rightarrow \quad 4a^2 + 3b^2 - 4ab - 4b + 2 \leq 0$$
$$\Rightarrow \quad (2a - b)^2 + 2(b - 1)^2 \leq 0 \Rightarrow b = 1, a = \frac{1}{2}$$

Example 91: The equation $e^{\sin x} - e^{-\sin x} = 4$ has:

(a) no real roots
(b) exactly one real root
(c) exactly four real roots
(d) infinite number of real roots.

Ans. (a)

◎ **Solution:** Put $e^{\sin x} = y$. Note that $1/e \leq y \leq e$.

Also, the given equation can be written as

$$y - 1/y = 4 \quad \text{or} \quad y^2 - 4y - 1 = 0$$
$$\Rightarrow \quad y = 2 \pm \sqrt{5}.$$

As $1/e \leq y \leq e$, none of the two values of y is possible.

Example 92: If $x^2 - 3x + 2$ is a factor of $x^4 - ax^2 + b$ then the equation whose roots are a and b is

(a) $x^2 - 9x - 20 = 0$ (b) $x^2 - 9x + 20 = 0$
(c) $x^2 + 9x - 20 = 0$ (c) $x^2 + 9x + 20 = 0$.

Ans. (b)

◎ **Solution:** As $x^2 - 3x + 2 = (x - 1)(x - 2)$ is a factor of $x^4 - ax^2 + b$, $x = 1$ and $x = 2$ are zeros of $x^4 - ax^2 + b$, therefore,

$$1 - a + b = 0, \quad 16 - 4a + b = 0$$
$$\Rightarrow \quad a = 5, b = 4.$$

Thus, equation whose roots a and b is $x^2 - 9x + 20 = 0$.

Example 93: Let for $a, a_1 \neq 0, a \neq a_1$

$$f(x) = ax^2 + bx + \text{c}, \quad g(x) = a_1x^2 + b_1x + c_1$$

and $p(x) = f(x) - g(x)$. If $p(x) = 0$ only for $x = -1$

and $p(-2) = 2$, then the value of $p(2)$ is

(a) 3 (b) 9
(c) 6 (d) 18

Ans. (d)

◎ **Solution:** As $a \neq a_1$, $p(x) = f(x) - g(x)$ is a quadratic polynomial.

As $p(x) = 0$ only for $x = -1$, we get

$p(x) = k(x + 1)^2$ where $k = a - a_1$

As $p(-2) = 2$, we get $2 = k(-2 + 1)^2 = k$.

Thus, $p(2) = 2(2 + 1)^2 = 18$

Example 94: If a, b, c are non-zero rational numbers such that $a + b + c = 0$, then the roots of the equation

$$(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$$

are

(a) both rational
(b) both irrational
(c) both purely imaginary
(d) one rational and one irrational

Ans. (a)

◎ **Solution:** As $a + b + c = 0$, we can write the equation

$$2ax^2 + 2bx + 2c = 0$$

or

$$ax^2 + bx + c = 0.$$

Note that $x = 1$ satisfies this equation and its other root is c/a.

Example 95: If α and β are the roots of the equation $ax^2 + bx + c = 0$, then roots of

$$ax^2 - bx(x - 1) + c(x - 1)^2 = 0 \qquad (1)$$

are

(a) $\frac{1}{\alpha}, \frac{1}{\beta}$ (b) $\frac{1}{\alpha}, \beta$
(c) $\frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta}$ (d) $\alpha, \frac{1}{\beta}$

Ans. (c)

◎ **Solution:** We can write (1) as

$$a\left(-\frac{x}{x-1}\right)^2 + b\left(\frac{-x}{x-1}\right) + c = 0$$
$$\Rightarrow \quad \frac{-x}{x-1} = \alpha, \beta \quad \Rightarrow \quad x = \frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta}$$

Example 96: If $3p^2 = 5p + 2$ and $3q^2 = 5q + 2$ then the equation whose roots $3p - 2q$ and $3q - 2p$ is

(a) $x^2 - 5x + 100 = 0$ (b) $3x^2 - 5x - 100 = 0$
(c) $3x^2 + 5x + 100 = 0$ (d) $5x^2 - x + 7 = 0$

Ans. (b)

◎ **Solution:** Note that p, q are roots of

$$3x^2 = 5x + 2 \quad \text{or} \quad 3x^2 - 5x - 2 = 0$$

$$p + q = 5/3, \; pq = -2/3.$$

Let $\alpha = 3p - 2q, \; \beta = 3q - 2p,$

then $\alpha + \beta = p + q = 5/3$

and $\alpha\beta = (3p - 2q)(3q - 2p)$

$$= -6(p^2 + q^2) + 13pq$$

$$= -6(p + q)^2 + 25pq = -100/3.$$

Thus, required quadratic equation is

$$x^2 - (5/3)x - 100/3 = 0$$

$$\Rightarrow \quad 3x^2 - 5x - 100 = 0$$

Example 97: The number of distinct real roots of the equation

$$(x + 3)^4 + (x + 5)^4 = 16 \qquad (1)$$

is

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (b)

◎ **Solution:** Put $x + 4 = t$, so that (1) becomes

$$(t - 1)^4 + (t + 1)^4 = 16$$

$$\Rightarrow \quad 2(t^4 + 6t^2 + 1) = 16$$

$$\Rightarrow \quad t^4 + 6t^2 - 7 = 0$$

$$\Rightarrow \quad t^2 = 1, -7 \quad \Rightarrow \quad t = \pm 1, \sqrt{7}i$$

Thus, the equation (1) has two real roots.

Example 98: If $a, b, c, \in \mathbf{R}$ and the equation $x^2 + (a + b)x + c = 0$ has no real roots, then $c(a + b + c)$ more than

(a) 2 (b) -2
(c) 0 (d) none of these.

Ans. (c)

◎ **Solution:** We have $f(x) = x^2 + (a + b)x + c > 0 \; \forall \; x \in \mathbf{R}$

Thus, $f(0) > 0, f(1) > 0$

$\Rightarrow c(a + b + c) > 0$

Example 99: The value of

$$y = 6 + \log_{3/2}\left[\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\ldots.,}}}\right]$$

is

(a) 2 (b) 3
(c) 4 (d) 8

Ans. (c)

◎ **Solution:** Let $x =$

$$\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\ldots.,}}}$$

then $x = \dfrac{1}{3\sqrt{2}}\sqrt{4 - x}$

$$\Rightarrow \quad 18x^2 + x - 4 = 0$$

$$\Rightarrow \quad (9x - 4)(2x + 1) = 0$$

$$\Rightarrow \quad x = 4/9, -1/2$$

As $x > 0$, we get $x = 4/9$.

Thus, $y = 6 + \log_{3/2}(4/9)$

$$= 6 + \log_{3/2}(3/2)^{-2} = 4$$

Example 100: The sum of the roots of the equation

$$x + 1 - 2\log_2(2^x + 3) + 2\log_4(10 - 2^{-x}) = 0 \qquad (1)$$

is

(a) $\log_2 11$ (b) $\log_2 12$
(c) $\log_2 13$ (d) $\log_2 14$

Ans. (a)

◎ **Solution:** Put $2^x = t$, then (1) can be written as

$$1 + \log_2 t - 2\log_2(t + 3) + \frac{2}{2}\log_2\left(10 - \frac{1}{t}\right) = 0$$

$$\Rightarrow \quad 1 + \log_2 t - \log_2(t + 3)^2 + \log_2(10t - 1) - \log_2 t = 0$$

$$\Rightarrow \quad \log_2(10t - 1)/(t + 3)^2 = -1$$

$$\Rightarrow \quad 10t - 1 = \frac{1}{2}(t + 3)^2$$

$$\Rightarrow \quad t^2 - 14t + 11 = 0 \qquad (2)$$

If $t_1 = 2^{x_1}, t_2 = 2^{x_2}$ are roots of (2),

then, $t_1 t_2 = 2^{x_1} . 2^{x_2} = 11$

$$\Rightarrow \quad x_1 + x_2 = \log_2 11$$

Assertion-Reason Type Questions

Example 101: Suppose $a, b, c, p \in \mathbf{R}$ $a \neq 0, c \neq 0$ and $b^2 - 4ac > 0$.

Statement-1: If the Roots of $f(x) = ax^2 + bx + c = 0$ are symmetrically placed on the real line about the point p, then $p = b/2a$ and $a(ap^2 + bp + c) < 0$

Statement-2: If the roots of $ax^2 + bx + c = 0$ are equal in magnitude but opposite in signs, then $b = 0$ and $ac < 0$.

Ans. (a)

◎ **Solution:** If $\alpha, -\alpha$ are roots of $ax^2 + bx + c = 0$,

then $0 = \alpha + (-\alpha) = -b/a \Rightarrow b = 0$

Also, $\alpha(-\alpha) = c/a$.

As $c \neq 0$, $-\alpha \neq 0$ and therefore $c/a = -\alpha^2 < 0 \Rightarrow ac < 0$.

Thus, Statement-2 is true.

If $p - \alpha$ and $p + \alpha$ are roots of $f(x) = ax^2 + bx + c = 0$, then $\alpha, -\alpha$ are roots of $f(x + p) = a(x + p)^2 + b(x + p) + c = 0$, that is, $\alpha, -\alpha$ are roots of

$$ax^2 + (2ap + b)x + ap^2 + bp + c = 0.$$

By Statement-2: $2ap + b = 0 \Rightarrow p = -b/2a$ and $a(ap^2 + bp + c) < 0$.

Example 102: Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$, $b^2 - 4ac > 0$.
Statement-1: If α, β are roots of $f(x) = ax^2 + bx + c = 0$, such that $\alpha < -1$ and $\beta > 1$, then $a + |b| + c < 0$.
Statement-2: If α, β are roots of $f(x) = ax^2 + bx + c = 0$ such that $\alpha < -1$, $\beta > 1$, then $ac < 0$.
Ans. (d)

Solution: If $\alpha < -1$, $\beta > 1$, then $\alpha < 0$, $\beta > 0$

$$\Rightarrow \quad \alpha\beta < 0 \Leftrightarrow \frac{c}{a} < 0 \Leftrightarrow ac < 0.$$

Thus, Statement-2 is true.

As $ax^2 + bx + c = 0$ and $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ have the same roots, roots of

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \text{ are } \alpha, \beta.$$

$$\therefore \quad g(x) = x^2 + \frac{b}{a}x + \frac{c}{a} = (x - \alpha)(x - \beta).$$

Note that $g(x) < 0$ for $\alpha < x <$ and $\alpha < -1 < 1 < \beta$

$$\therefore \quad g(-1) < 0, g(1) < 0$$

$$\Rightarrow \quad 1 - \frac{b}{a} + \frac{c}{a} < 0, 1 + \frac{b}{a} + \frac{c}{a} < 0$$

$$\Rightarrow \quad 1 + \left|\frac{b}{a}\right| + \frac{c}{a} < 0.$$

We cannot conclude $a + |b| + c < 0$ unless $a > 0$.

Example 103: Suppose $a, b, c \in \mathbf{R}$

Statement-1: If $a > 0$, $b^2 - 4ac < 0$, and $f(x) = \sqrt{a + bx + cx^2}$ then domain of f is $\mathbf{R}$
Statement-2: $ax^2 + bx + c > 0\ \forall\, x \in \mathbf{R}$ if $a > 0$ and $b^2 - 4ac > 0$.
Ans. (a)

Solution: Statement-2 is true. See Theory.
As $ax^2 + bx + c > 0\ \forall\, x \in \mathbf{R}$, we get $c = a(0)^2 + b(0) + c > 0$.
Thus, $cx^2 + bx + a > 0\ \forall\, x \in \mathbf{R}$.
Therefore, domain of f is **R.**
Hence, Statement-1 is true and
Statement-2 is a correct explanation for it.

Example 104: Suppose $a, b, c, d \in \mathbf{R}$.
Statement-1: If $a < b < c < d$, then the quadratic equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ has real and distinct roots.
Statement-2: Let $f(x) = ax^2 + bx + c$:
If $f(\alpha) f(\beta) < 0$ for some $\alpha, \beta \in \mathbf{R}$, then $f(x) = 0$ has real and distinct roots.
Ans. (a)

Solution: As $f(x) = ax^2 + bx + c$ is a continuous function and $f(\alpha) f(\beta) < 0$, there is at least one $\gamma \in \mathbf{R}$ lying between α and β such that $f(\gamma) = 0$. As $a, b, c \in \mathbf{R}$ and one root of $f(x) = 0$ is real, other must be real. Also, roots are distinct. Therefore, Statement-2 is true.
Now, let $f(x) = (x - a)(x - c) + 2(x - b)(x - d)$,
then $f(a) = 2(a - b)(a - d) > 0$,

$$f(b) = (b - a)(b - c) > 0$$

Thus, $f(x) = 0$ has a real root lying in the interval (a, b). As coefficients are real and one of the roots is real, the quadratic equation must have real and distinct roots.

Example 105: Suppose $a, b, c\ \alpha \in \mathbf{R}$
Statement-1: The quadratic equation
$(x - \sin\alpha)(x - \cos\alpha) + \sqrt{3} = 0$ has imaginary roots.
Statement-2: If $a > 0$ and $b^2 - 4ac < 0$, then the graph of $y = ax^2 + bx + c$ lies above the x-axis.
Ans. (a)

Solution: For truth of Statement-2, see Theory.
We can write
$(x - \sin\alpha)(x - \cos\alpha) - \sqrt{3} = 0$
as $\quad f(x) = 0$
where $\quad f(x) = x^2 - (\sin\alpha + \cos\alpha) + \sin\alpha\cos\alpha + \sqrt{3}$
coefficient of $x^2 = 1 > 0$, and

$$\begin{aligned} b^2 - 4ac &= (\sin\alpha + \cos\alpha)^2 - 4(\sin\alpha\cos\alpha + \sqrt{3}) \\ &= (\sin\alpha - \cos\alpha)^2 - 4\sqrt{3} \\ &\leq (\sqrt{2})^2 - 4\sqrt{3} < 0 \end{aligned}$$

as maximum possible value of $\sin\alpha - \cos\alpha$ is $\sqrt{2}$.
Thus graph of $y = f(x)$ lies above the x-axis and hence $f(x) = 0$ has imaginary roots.

Example 106: Suppose $a, b, c \in \mathbf{R}$ and $a \neq 0$.
Statement-1 If $ac > 0$, $ab < 0$, $b^2 - 4ac > 0$ the equation $ax^4 + bx^2 + c = 0$ has four distinct roots.
Statement-2 If $ac > 0$, $ab < 0$, then the equation $ax^2 + bx + c = 0$ has distinct positive roots.
Ans. (c)

Solution: Statement-2 is false as $x^2 - x + 1 = 0$ satisfies the condition $ac > 0$ and $ab < 0$ but $x^2 - x + 1 = 0$ does not have positive roots.
Statement-1 is true. Put $x^2 = y$, so that $ay^2 + by + c = 0$.
As $b^2 - 4ac > 0$. This equation has two distinct roots.

As $-\frac{b}{a} = -\frac{ab}{a^2} > 0$ and $\frac{c}{a} = \frac{ac}{a^2} > 0$ both the roots of $ay^2 + by + c = 0$ are positive. If the roots are $\alpha, \beta > 0$, then $x^2 = \alpha, \beta$
$\Rightarrow x = \pm\sqrt{\alpha}, \pm\sqrt{\beta}$.

Thus, $ax^4 + bx^2 + c = 0$ has four distinct roots.

Example 107: Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$, and let $f(x) = ax^2 + bx + c$.

Statement-1: If $f(x) = 0$ has imaginary roots and $a + c > 0$, then $f(x) > 0\ \forall\ x \in \mathbf{R}$.

Statement-2: If $f(x) > 0\ \forall\ x \in \mathbf{R}$, then $g(x) = f(x) + f'(x) + f''(x) > 0\ \forall\ x \in \mathbf{R}$.

Ans. (b)

Solution: $f(x) > 0\ \forall\ x \in \mathbf{R}$,

$\Leftrightarrow \quad a > 0,\ b^2 - 4ac < 0.$

We have
$$g(x) = ax^2 + bx + c + (2ax + b) + 2a$$
$$= ax^2 + (b + 2a)x + c + b + 2a$$

Note that
$$(b + 2a)^2 - 4a\,(c + b + 2a)$$
$$= (b^2 - 4ac) - 4a^2 < 0$$

and $\quad a > 0$, thus $g(x) > 0\ \forall\ x \in \mathbf{R}$

$\therefore$ Statement-2 is true.

As $f(x) = 0$ has imaginary roots, $f(x) > 0\ \forall \in \mathbf{R}$ or $f(x) < 0\ \forall\ x \in \mathbf{R}$

We have $f(1) + f(-1) = 2(a + c) > 0$,

therefore $f(x) > 0\ \forall\ x \in \mathbf{R}$.

Thus, statement-1 is true but statement-2 is not a correct explanation for it.

Example 108: Suppose $a, b, c \in \mathbf{R}$ are three distinct real numbers.

Statement-1:
$$a\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\frac{(x-c)(x-a)}{(b-c)(b-a)} + c\frac{(x-a)(x-b)}{(c-a)(c-b)} = x\ \forall\ x \in \mathbf{R}.$$

Statement-2: If $Ax^2 + Bx + C = 0$ for $x = a, b, c$, then $A = B = C = 0$.

Ans. (a)

Solution: If three distinct values of x satisfy a quadratic equation, then it is identically equal to zero. Thus, Statement-2 is true.

Let
$$f(x) = a\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\frac{(x-c)(x-a)}{(b-c)(b-a)} + c\frac{(x-a)(x-b)}{(c-a)(c-b)} - x.$$

Then $f(x) = 0$ is a quadratic and $f(a) = f(b) = f(c) = 0$.

Thus, $f(x) \equiv 0$.

Therefore, Statement-1 is true and statement-2 is a correct reason for it.

Example 109: Suppose $a, b, c \in \mathbf{I}$, $a \neq 0$ and $f(x) = ax^2 + bx + c$.

Statement-1: If $f(x) = 0$ has no rational roots, then $\left|f\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^2}\ \forall\ p, q \in \mathbf{I}$, $q \neq 0$ and

Statement-2: If $\alpha + \sqrt{\beta}$, $\alpha, \beta \in \mathbf{Q}$ and $\beta \neq 0$ is a zero of $f(x)$, then its other zero must be $\alpha - \sqrt{\beta}$.

Ans. (b)

Solution: Statement-2 is true. (See Theory)

Let $\quad x = p/q$, $p, q \in \mathbf{I}$ and $q \neq 0$.

Then
$$f\left(\frac{p}{q}\right) = \frac{1}{q^2}\,(ap^2 + bpq + cq^2)$$

As $\quad f(x) = 0$ has no rational roots, $f(p/q) \neq 0$

$\Rightarrow ap^2 + bpq + cq^2 \neq 0$

As $ap^2 + bpq + cq^2$ is an integer, we get $|ap^2 + bpq + cq^2| \geq 1$.

Thus, $q^2 \left|f\left(\frac{b}{q}\right)\right| \geq 1 \Rightarrow \left|f\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^2}$.

Example 110: Suppose $a, b, c \in \mathbf{R}$ and $a < b < c$.

Let $f : \mathbf{R} \to \mathbf{R}$ be defined by
$$f(x) = \begin{cases} \dfrac{(x-a)(x-c)}{x-b} & \text{if } x \neq b \\ 0 & \text{if } x = b \end{cases}$$

Statement-1: Range of f is $\mathbf{R}$.

Statemetnt-2: For each $y \in \mathbf{R}$, $y = f(x)$ has exactly two distinct solutions.

Ans. (c)

Solution: Statement-2 is false as $f(a) = f(b) = f(c) = 0$

$\Rightarrow \quad f(x) = 0$ has three distinct solutions.

Note that Statement-1, put $t = x - b$, so that
$$f(x) = \frac{(t+b-a)(t+b-c)}{t}$$
$$= t - (a + c - 2b) + \frac{(a-b)(c-b)}{t}$$
$$= x - (a + c - b) + \frac{(a-b)(c-b)}{x-b}$$

note that f is continuous on $(-\infty, b)$. Also, $\lim_{x\to-\infty} f(x) = -\infty$

and $\lim_{x \to b-} f(x) = \infty$

Thus, range of f is $(-\infty, \infty) = \mathbf{R}$.

$\therefore$ Statement-1 is true.

LEVEL 2

Straight Objective Type Questions

Example 111: Suppose $a, b, c, d \in \mathbf{R}$ and $2ac = b + d$. Consider the quadratic equations $x^2 + 2ax + b = 0$ and $x^2 + 2cx + d = 0$. Then

(a) none of these equations have real roots
(b) both the equations have real roots
(c) exactly one of the equation has real roots
(d) at least one of the equation has real roots

Ans. (d)

Solution: Let D_1 be the discriminant of $x^2 + 2ax + b = 0$ and D_2 be the discriminant of $x^2 + 2cx + d = 0$

Then

$$D_1 + D_2 = 4(a^2 - b) + 4(c^2 - d)$$
$$= 4(a^2 + c^2) - 8ac$$
$$= 4(a - c)^2 \geq 0$$

$\Rightarrow$ at least one of D_1, D_2 is non-negative. Thus, at least one of the equations has real roots.

Example 112: If the roots of the equation

$$\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c} \quad (1)$$

are equal in magnitude but opposite in sign, then their product is

(a) $\frac{1}{2}(a^2 + b^2)$ (b) $-\frac{1}{2}(a^2 + b^2)$
(c) $\frac{1}{2}ab$ (d) $-\frac{1}{2}ab$

Ans. (b)

Solution: The equation (1) can be written as

$$c(x + b + x + a) = (x + a)(x + b)$$

or $$x^2 + (a + b - 2c)x + ab - ac - bc = 0 \quad (2)$$

Let α and $-\alpha$ be the roots of (2) then

$$0 = \alpha + (-\alpha) = a + b - 2c \Rightarrow c = \frac{1}{2}(a + b)$$

Also, $$2\alpha(-\alpha) = 2ab - 2(a + b)c = 2ab - (a + b)^2$$
$$= -(a^2 + b^2)$$

$\Rightarrow$ $$\alpha(-\alpha) = -\frac{1}{2}(a^2 + b^2)$$

Example 113: If c, d are roots of $x^2 - 10ax - 11b = 0$ and a, b are root of $x^2 - 10cx - 11d = 0$, then value of $a + b + c + d$ is

(a) 1210 (b) -1
(c) 2530 (d) -11

Ans. (a)

Solution: $c + d = 10a$ (1)

$a + b = 10c$ (2)

Subtracting (1) from (2) we get

$$(a - c) + (b - d) = 10(c - a)$$

$\Rightarrow$ $b - d = 11(c - a)$ (3)

As c is a root of $x^2 - 10ax - 11b = 0$,
we get

$$c^2 - 10ac - 11b = 0 \quad (3)$$

Similarly, $$a^2 - 10ac - 11d = 0 \quad (4)$$

Subtracting (4) from (3), we get

$$c^2 - a^2 = 11(b - d)$$

$\Rightarrow$ $(c - a)(c + a) = (11)11(c - a)$

$\Rightarrow$ $c + a = 121$

$\therefore$ $a + b + c + d = 10(a + c)$ [from (1) and (2)]

$= 10(121) = 1210$

Example 114: If α, β are the roots of the equation $ax^2 + bx + c = 0$, then the value of $\alpha^3 + \beta^3$ is

(a) $\frac{3abc + b^3}{a^3}$ (b) $\frac{a^3 + b^3}{3abc}$
(c) $\frac{3abc - b^3}{a^3}$ (d) $\frac{-(3abc + b^3)}{a^3}$

Ans. (c)

Solution: We have

$$\alpha + \beta = \frac{-b}{a}, \ \alpha\beta = \frac{c}{a}$$

Now, $$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$= \left(\frac{-b}{a}\right)^3 - \frac{3c}{a}\left(\frac{-b}{a}\right) = \frac{3abc - b^3}{a^3}$$

Example 115: Suppose $a \in \mathbf{I}$ and the equation $(x - a)(x - 5) = 3$ has integral roots, then the set of values which a can take is:

(a) ϕ (b) $\{-11, -13\}$
(c) $\{3, 7\}$ (d) $\{-3, -7\}$

Ans. (c)

Solution: Let $m \in \mathbf{I}$ be a roots of $(x - a)(x - 5) = 3$, then

$$(m - a)(m - 5) = 3$$

As $m - a$ and $m - 5$ are integers, $m - 5 = \pm 1$ or $m - 5 = \pm 3$
$\Rightarrow m = 2, 4, 6, 8.$
Thus, $a = 3, 7.$

Example 116: If the ratio of the roots of the equation $x^2 + bx + c = 0$ is the same as that of the ratio of the roots of $x^2 + qx + r = 0$, then

(a) $br^2 = qc^2$ (b) $cq^2 = rb^2$
(c) $q^2c^2 = b^2r^2$ (d) $bq = rc$

Ans. (b)

Solution: Let α, β be the roots of the equation $x^2 + bx + c = 0$ and γ, δ be the roots of the equation $x^2 + qx + r = 0$. We are given

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \frac{\alpha-\beta}{\alpha+\beta} = \frac{\gamma-\delta}{\gamma+\delta}$$

$$\Rightarrow \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 = \left(\frac{\gamma-\delta}{\gamma+\delta}\right)^2$$

$$\Rightarrow \frac{(\alpha+\beta)^2 - 4\alpha\beta}{(\alpha+\beta)^2} = \frac{(\gamma+\delta)^2 - 4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \frac{\alpha\beta}{(\alpha+\beta)^2} = \frac{\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \frac{c}{(-b)^2} = \frac{r}{(-q)^2} \Rightarrow cq^2 = rb^2$$

Example 117: If a and b are the non-zero distinct roots of $x^2 + ax + b = 0$, then the least value of $x^2 + ax + b$ is

(a) 2/3 (b) 9/4
(c) – 9/4 (d) 1

Ans. (c)

Solution: We have $a + b = -a$, $ab = b$

As $b \neq 0$, we get $a = 1$

$\therefore$ $1 + b = -1 \Rightarrow b = -2$

Thus, $x^2 + ax + b = x^2 + x - 2$

$$= \left(x + \frac{1}{2}\right)^2 - \frac{9}{4} \geq -\frac{9}{4}$$

$\therefore$ least value of $x^2 + ax + b$ is $-9/4$ which is attained at $x = -1/2$.

Example 118: If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has

(a) at least one root in [0, 1]
(b) one root in [2, 3] and other is [– 2, – 1]
(c) imaginary roots
(d) none of these

Ans. (a)

Solution: Let $f(x) = ax^3 + bx^2 + cx$, $x \in [0, 1]$. Since f is a polynomial function, f is differentiable on the whole real line and in particular on [0, 1].

Also, $f(0) = 0$ and $f(1) = a + b + c = 0$.

Thus, all the conditions in the hypothesis of the Rolle's theorem are satisfied. By the Rolle's theorem there exists at least one $\alpha \in (0, 1)$ such that

$$f'(\alpha) = 0$$

But $f'(x) = 3ax^2 + 2bx + c$

Hence, $3ax^2 + 2bx + c = 0$ has a root in [0, 1].

Example 119: If $a < b < c < d$, then the equation

$3(x - a)(x - c) + 5(x - b)(x - d) = 0$

has

(a) real and distinct roots
(b) real and equal roots
(c) purely imaginary roots
(d) none of these

Ans. (a)

Solution: Let $f(x) = 3(x - a)(x - c) + 5(x - b)(x - d)$

Since f is a polynomial, f is continuous on **R**. Also, since $a < b < c < d$,

$$f(a) = 5(a - b)(a - d) > 0$$
$$f(b) = 3(b - a)(b - c) < 0$$
$$f(c) = 5(c - b)(c - d) < 0$$
$$f(d) = 3(d - a)(d - c) > 0$$

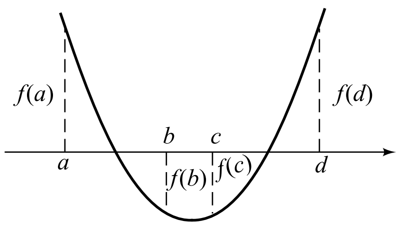


Fig. 3.18

As f is continuous on ***R***, $y = f(x)$ crosses x-axis at least once between a and b and once between c and d. See Fig. 3.18.

Thus, f has two distinct real roots, one lying between a and b and one lying between c and d.

Example 120: For real x, the function $\dfrac{(x-a)(x-c)}{x-b}$ will assume all real values provided

(a) $a < b < c$ (b) $b < c < a$
(c) $c < a < b$ (d) none of these

Ans. (a)

Solution: Let $y = \dfrac{(x-a)(x-c)}{x-b} = \dfrac{x^2 - (a+c)x + ac}{x-b}$

$$\Rightarrow y(x - b) = x^2 - (a + c)x + ac$$

$$\Rightarrow x^2 - (a + c + y)x + ac + by = 0 \quad (1)$$

Since x is real, the discriminant of (1)

$$(a + c + y)^2 - 4(ac + by) \geq 0$$

$$\Rightarrow y^2 + 2(a + c)y + (a + c)^2 - 4ac - 4by \geq 0$$

$$\Rightarrow y^2 + 2(a + c - 2b)y + (a - c)^2 \geq 0 \quad (2)$$

Since y takes all real values, (2) is possible if and only if

$$4(a + c - 2b)^2 - 4(a - c)^2 < 0$$

[$\because$ coeff. of $y^2 = 1 > 0$]

$$\Leftrightarrow (a + c - 2b + a - c)(a + c - 2b - a + c) < 0$$

$\Leftrightarrow \quad (2a - 2b)(2c - 2b) < 0$

$\Leftrightarrow \quad (a - b)(c - b) < 0$

$\Leftrightarrow \quad b$ lies between a and c. Thus, one of the possibilities $a < b < c$.

Example 121: Let $a, b, c \in \boldsymbol{R}$ and $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies

(a) $\gamma = \dfrac{1}{2}(\alpha + \beta)$ (b) $\gamma = \alpha + \dfrac{1}{2}\beta$

(c) $\gamma = \alpha + \beta$ (d) $\alpha < \gamma < \beta$

Ans. (d)

Solution: By the hypothesis,

$$a^2\alpha^2 + b\alpha + c = 0 \qquad (1)$$

and

$$a^2\beta^2 - b\beta - c = 0 \qquad (2)$$

where $0 < \alpha < \beta$, and $a \neq 0$.

Let $f(x) = a^2x^2 + 2bx + 2c$

Since f is a polynomial function, f is continuous on $\boldsymbol{R}$ and hence on $[\alpha, \beta]$. Also,

$$\begin{aligned} f(\alpha) &= a^2\alpha^2 + 2b\alpha + 2c \\ &= 2(a^2\alpha^2 + b\alpha + c) - a^2\alpha^2 \\ &= 2(0) - a^2\alpha^2 = -a^2\alpha^2 < 0 \qquad \text{[using (1)]} \end{aligned}$$

and

$$\begin{aligned} f(\beta) &= a^2\beta^2 + 2b\beta + 2c \\ &= 3a^2\beta^2 - 2(a^2\beta^2 - b\beta - c) \\ &= 3a^2\beta^2 - 2(0) = 3a^2\beta^2 > 0 \quad \text{[using (2)]} \end{aligned}$$

Since f is continuous on $[\alpha, \beta]$ and $f(\alpha) < 0$, $f(\beta) > 0$, f must vanish at least once in (α, β).

Example 122: Suppose $p, q, r, s \in \boldsymbol{R}$ and α, β be the roots of $x^2 + px + q = 0$ and α^4, β^4 be the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always

(a) two imaginary roots
(b) two positive roots
(c) two negative roots
(d) one positive and one negative root

Ans. (d)

Solution: We have

$$\alpha + \beta = -p,\ \alpha\beta = q,\ \alpha^4 + \beta^4 = r,\ \alpha^4\beta^4 = s$$

Discriminant D of the equation

$$x^2 - 4qx + 2q^2 - r = 0 \qquad (1)$$

is given by $D = 16q^2 - 4(2q^2 - r) = 8q^2 + 4r = 4[2q^2 + r]$

$$= 4[2\alpha^2\beta^2 + \alpha^4 + \beta^4] = 4(\alpha^2 + \beta^2)^2 \geq 0$$

Thus, the equation (1) has real roots.

The roots of (1) are given by

$$x = \frac{4q \pm \sqrt{D}}{2} = 2\alpha\beta \pm (\alpha^2 + \beta^2)$$
$$= (\alpha + \beta)^2, -(\alpha - \beta)^2$$

Hence, (1) has one positive and one negative root.

Example 123: The equation

$$x^{(3/4)(\log_2 x)^2 + \log_2 x - 5/4} = \sqrt{2} \qquad (1)$$

has

(a) exactly two real roots
(b) no real root
(c) one irrational root
(d) none of these

Ans. (c)

Solution: Taking log of the sides in (1), we get

$$\left[\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right]\log_2 x = \log_2(\sqrt{2})$$

$\Rightarrow \quad \left(\dfrac{3}{4}y^2 + y - \dfrac{5}{4}\right)y = \dfrac{1}{2}\log_2 2$ where $y = \log_2 x$

$\Rightarrow \quad 3y^3 + 4y^2 - 5y - 2 = 0$

$\Rightarrow \quad 3y^3 - 3y^2 + 7y^2 - 7y + 2y - 2 = 0$

$\Rightarrow \quad (y - 1)(3y^2 + 7y + 2) = 0$

$\Rightarrow \quad (y - 1)[3y^2 + 6y + y + 2] = 0$

$\Rightarrow \quad (y - 1)(y + 2)(3y + 1) = 0$

$\Rightarrow \quad y = 1, -2, -1/3$

$\Rightarrow \quad \log_2 x = 1, -2, -1/3 \Rightarrow x = 2, \dfrac{1}{4}, 2^{-1/3}$

Thus, (1) has one irrational root viz. $2^{-1/3}$.

Example 124: Let $f(x)$ be a quadratic expression which is positive for all x. If $g(x) = f(x) + f'(x) + f''(x)$ then for all real x,

(a) $g(x) < 0$ (b) $g(x) > 0$

(c) $g(x) = 0$ (d) $g(x) \geq 0$

Ans. (b)

Solution: Let $f(x) = ax^2 + bx + c$

As $f(x) > 0\ \forall\ x \in \boldsymbol{R}$, we must have $a > 0$ and $b^2 - 4ac < 0$.

Also, $f'(x) = 2ax + b$ and $f''(x) = 2a$

Thus, $g(x) = ax^2 + (2a + b)x + 2a + b + c$.

Since $a > 0$ and discriminant

$$\begin{aligned} &(2a + b)^2 - 4a(2a + b + c) \\ &= 4a^2 + 4ab + b^2 - 8a^2 - 4ab - 4ac \\ &= -4a^2 + (b^2 - 4ac) < 0 \qquad [\because b^2 - 4ac < 0] \end{aligned}$$

we get $g(x) > 0\ \forall\ x \in \mathbf{R}$

Example 125: If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then the quadratic equation whose roots are α^3, β^3 is

(a) $a^3y^2 + (b^3 - 3abc)y + c^3 = 0$
(b) $a^3y^2 + (3abc - b^3)y - c^3 = 0$
(c) $a^2y^2 + 2aby + c^2 = 0$
(d) none of these

Ans. (a)

Solution: Put $y = \alpha^3 \Rightarrow \alpha = y^{1/3}$.

As α is a root of $ax^2 + bx + c = 0$, we get

$$ay^{2/3} + by^{1/3} + c = 0 \Rightarrow y^{1/3}(ay^{1/3} + b) = -c \qquad (1)$$

Cubing both the side, we get

$$y(ay^{1/3} + b)^3 = -c^3$$

$\Rightarrow$ $y[a^3y + b^3 + 3aby^{1/3}(ay^{1/3} + b)] = -c^3$

$\Rightarrow$ $y[a^3y + b^3 + 3ab(-c)] = -c^3$ [using (1)]

$\Rightarrow$ $a^3y^2 + (b^3 - 3abc)y + c^3 = 0$

Example 126: If $a, b, c \in \mathbf{R}$ and the equations $ax^2 + bx + c = 0$ and $x^3 + 3x^2 + 3x + 2 = 0$ have two common roots, then

(a) $a = b = -c$ (b) $a = -b = c$
(c) $a = b = c$ (d) none of these

Ans. (c)

Solution: $x^3 + 3x^2 + 3x + 2 = 0$

$\Rightarrow$ $(x + 1)^3 = -1$

$\Rightarrow$ $x + 1 = -1, -\omega, -\omega^2$

$\Rightarrow$ $x = -2, -1 - \omega, -1 - \omega^2$

As $a, b, c \in \mathbf{R}$, the roots of $ax^2 + bx + c = 0$ are both real or both imaginary.

$\therefore$ roots of $ax^2 + bx + c = 0$ must be $-1 - \omega, -1 - \omega^2$.

Thus, $-1 - \omega - 1 - \omega^2 = -\dfrac{b}{a}$

and $(-1 - \omega)(-1 - \omega^2) = \dfrac{c}{a}$

$\Rightarrow$ $1 = \dfrac{b}{a}$ and $1 = \dfrac{c}{a}$

$\Rightarrow$ $a = b = c$

Example 127: Let a, b, c be non-zero real number such that

$$\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)\,dx = \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)\,dx$$

Then the quadratic equation $ax^2 + bx + c = 0$ has

(a) no root in (0, 2)
(b) at least one root in (1, 2)
(c) a double root (0, 2)
(d) none of these

Ans. (b)

Solution: Let $f(x) = (1 + \cos^8 x)(ax^2 + bx + c)$

We are given

$$\int_0^1 f(x)\,dx = \int_0^2 f(x)\,dx$$

$\Rightarrow$ $\int_0^1 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$\Rightarrow$ $\int_1^2 f(x)\,dx = 0$

If $f(x) > 0\ (f(x) < 0)\ \forall\ x \in [1, 2]$,

then $\int_1^2 f(x)\,dx > 0\ \left(\int_1^2 f(x)\,dx < 0\right)$

But $\int_1^2 f(x)\,dx = 0$

$\therefore$ $f(x)$ is partly positive and partly negative on [1, 2].

$\Rightarrow$ there exist $\alpha, \beta \in [1, 2]$ such that

$$f(\alpha) > 0 \text{ and } f(\beta) < 0.$$

As f is continuous on [1, 2] there exists γ lying between α and β (and hence between 1 and 2) such that $f(\gamma) = 0$

$\Rightarrow$ $(1 + \cos^8 \gamma)(a\gamma^2 + b\gamma + c) = 0$

$\Rightarrow$ $a\gamma^2 + b\gamma + c = 0$ $[\because 1 + \cos^8 \gamma \ge 1]$

Thus, $ax^2 + bx + c = 0$ has at least one root in [1, 2].

Example 128: The number of real solutions of the equation

$$27^{1/x} + 12^{1/x} = 2(8^{1/x}) \quad (1)$$

is

(a) 0 (b) 1
(c) infinite (d) none of these

Ans. (a)

Solution: We can write (1) as

$$\left(\frac{27}{8}\right)^{1/x} + \left(\frac{12}{8}\right)^{1/x} = 2$$

or $\left(\dfrac{3}{2}\right)^{3/x} + \left(\dfrac{3}{2}\right)^{1/x} = 2$ (1)

As $3/2 > 1$, $(3/2)^t > 1$ if $t > 0$

and $(3/2)^t < 1$ if $t < 0$.

Thus, $\left(\dfrac{3}{2}\right)^{3/x} + \left(\dfrac{3}{2}\right)^{1/x} < 2$ if $x < 0$

and $\left(\dfrac{3}{2}\right)^{3/x} + \left(\dfrac{3}{2}\right)^{1/x} > 2$ if $x > 0$.

Therefore, (1) is possible if $1/x = 0$.

But this is not true for any real value of x.

Example 129: If $0 < a < b < c < d$, then the quadratic equation

$$ax^2 + \{1 - a(b + c)\}x + abc - d = 0 \quad (1)$$

has

(a) real and distinct roots out of which one lies between c and d.
(b) real and distinct roots out of which one lies between a and b
(c) real and distinct roots out of which one lies between b and c
(d) non-real roots

Ans. (a)

Solution: We can rewrite (1) as

$$ax^2 - a(b + c)x + abc + x - d = 0$$

or $a(x - b)(x - c) + x - d = 0$

Let $f(x) = a(x - b)(x - c) + x - d$.

As $a > 0$, $y = f(x)$ represents a parabola which open upwards. See Fig. 3.19.

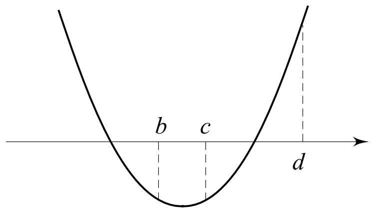


Fig. 3.19

Also, $f(b) = b - d < 0$

$f(c) = c - d < 0$, and $f(d) = a(d - b)(d - c) > 0$

Thus, $f(x) = 0$ has a root between $-\infty$ and b and a root between c and d.

Example 130: If $ax^2 + 2bx - 3c = 0$ has no real roots and $c < \frac{4}{3}(a + b)$, then range of c is

(a) (0, b) (b) $(-1, b)$
(c) $(-\infty, -b^2/3a)$ (d) $(-\infty, -b/12a)$

Ans. (c)

Solution: As $f(x) = ax^2 + 2bx - 3c = 0$ has no real roots, $f(x) > 0\ \forall\ x \in \mathbf{R}$ or $f(x) < 0\ \forall\ x \in \mathbf{R}$

Since $4a + 4b - 3c > 0, f(2) > 0$.

$\therefore\quad f(x) = ax^2 + 2bx - 3c > 0$

$\Rightarrow\quad f(0) = -3c > 0 \Rightarrow c < 0$

Also, $a > 0$ and $b^2 + 3ca < 0$

$\Rightarrow\quad c < -b^2/3a \Rightarrow c \in (-\infty, -b^2/3a)$

Example 131: Suppose $a, b, c \in \mathbf{R}$ and $a \neq b$. If $ax + \frac{b}{x} + c$, $x \in \mathbf{R} - \{0\}$, assume all real values, then a and b satisfy the relation

(a) $ab \leq 0$ (b) $ab \geq 0$
(c) $ab \geq 1$ (d) $ab \leq 1$

Ans. (a)

Solution: Let $y = ax + \frac{b}{x} + c$, then

$$ax^2 + (c - y)x + b = 0.$$

As x is real, $(c - y)^2 - 4ab \geq 0$

This is true for each $y \in \mathbf{R}$ if $ab \leq 0$

EXERCISE

Concept-based Straight Objective Type Questions

1. The number of real solutions of $x^2 - 2x + 2 + |x - 1| = 0$ is:
(a) 0 (b) 1
(c) 2 (d) infinite

2. If roots of $7x^2 - 11x + k = 0$, $k \neq 0$ are reciprocal of each other, then k is equal to
(a) –1 (b) 7/11
(c) 7 (d) 11/7

3. If $a + b \neq 0$ and the roots of $x^2 - px + q = 0$ differ by –1, then $p^2 - 4q$ equals:
(a) –1 (b) 0
(c) 1 (d) 2

4. If the equations $x^2 - ax + b = 0$ and $x^2 + bx - a = 0$ have a common root then
(a) $a = b$ (b) $a + b = -1$
(c) $a - b = -1$ (d) $a - b = 1$

5. If both the roots of $x^2 + x + a = 0$ exceed a, then a belongs to:
(a) $(-\infty, -1)$ (b) $(-\infty, -2)$
(c) (0, 1) (d) $(1, \infty)$

6. The number of real solutions of $x^2 - 4|x| - 2 = 0$ is
(a) 1 (b) 2
(c) 3 (d) 4

7. Suppose α and β are roots of the equation $x^2 + px + \frac{3}{4}p = 0$. If $|\alpha - \beta| = \sqrt{10}$, then p belongs to the set
(a) $\{-2, 5\}$ (b) $\{-3, 2\}$
(c) $\{2, -5\}$ (d) $\{3, -5\}$

8. The number of solutions of $\sqrt{5 + x} + \sqrt{x} = 2$ is
(a) 0 (b) 1
(c) 2 (d) infinite

9. Suppose $a \in \mathbf{R}$. The set of values of a for which the quadratic equation $x^2 - 2(a + 1)x + a^2 - 4a + 3 = 0$ has two negative roots is
(a) $(-\infty, -1)$ (b) (1, 3)
(c) $(-\infty, 1) \cup (3, \infty)$ (d) ϕ

10. Suppose $\alpha > 0$, $\beta > 0$ and $\alpha + \beta = \pi/4$. If $\tan\alpha$, $\tan\beta$ are roots of $x^2 - ax + b = 0$, then

(a) $a + b = 1$ (b) $a + b = 1, 0 < b < 1$
(c) $a = b$ (d) $0 < a + b < 2$

11. If $b > a$, and $c > 0$ then the equation $(x - a)(x - b) - c = 0$ has:
(a) both roots in $(-\infty, a)$
(b) both roots in $(a, b]$
(c) one root in $(-\infty, a)$ and other root in (b, ∞)
(d) one root in $(-\infty, a)$ and other root in $[a, b]$

12. If the quadratic equation $x^2 + 2(k + 1)x + 9k - 5 = 0$ has exactly one positive root, then k lies in the set
(a) $[5/9, \infty)$ (b) $(-\infty, 1) \cup (6, \infty)$
(c) $(-\infty, 5/9]$ (d) $[1, 6]$

13. If $k \in \mathbf{R}$ lies between the roots of $ax^2 + 2bx + c = 0$, then
(a) $ak^2 + 2bk + c < 0$
(b) $a^2k^2 + 2abk + ac < 0$
(c) $a^2k^2 + 2abk + ac > 0$
(d) $ak^2 + 2bk + c > 0$

14. If both the roots of the quadratic equation $x^2 - 4ax + 2a^2 - 3a + 5$ are less than 2, then a lies in the set
(a) $(9/2, \infty)$
(b) $(-\infty, 9/2)$
(c) $(-1, \infty)$
(d) $(2, \infty)$

15. Fig. 3.20 Shows graph of $y = ax^2 + bx + c$. Then which one of the following is not true.

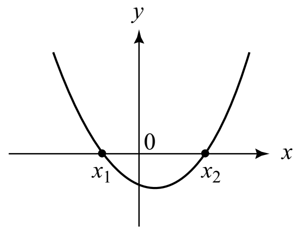


Fig. 3.20

(a) $a > 0$
(b) $c < 0$
(c) $b^2 - 4ac > 0$
(d) $b > 0$

16. Greatest value of the expression $\dfrac{8}{9x^2 - 6x + 5}$ is
(a) 2 (b) 5
(c) $8\frac{1}{3}$ (d) 9.2

17. If $a, b \in \{1, 2, 3, 4\}$ the number of quadratic equation of the form $ax^2 + bx + c = 0$ which have non-real complex roots is:
(a) 27 (b) 35
(c) 52 (d) 56

18. The number of real roots of the equation $(x - 1)^2 + (x - 2)^2 + (x - 3)^2 = 0$ is
(a) 0 (b) 2
(c) 3 (d) infinite

19. Suppose $0 < a < b < c$. If the roots α, β of $ax^2 + bx + c = 0$ are imaginary, then
(a) $|\alpha| = \sqrt{\dfrac{c}{a}}$ (b) $|\beta| = \sqrt{\dfrac{a}{c}}$
(c) $\alpha + \beta = 0$ (d) $\alpha - \beta = -b/2a$

20. If $a > 0$ and both the roots of $ax^2 + bx + c = 0$ are more than 1, then
(a) $a + b + c > 0$ (b) $a + b + 4c = 0$
(c) $a + b + c < 0$ (d) $a + 4b + c = 0$

LEVEL 1

Straight Objective Type Questions

21. In Fig. 3.21 graph of $y = ax^2 + 2bx + c$ is given. Which one of the following is not true?

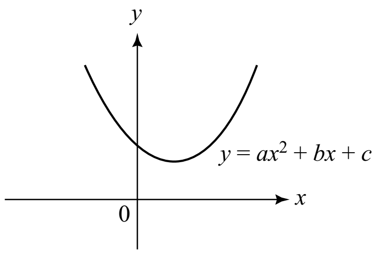


Fig. 3.21

(a) $a > 0$ (b) $b > 0$
(c) $c > 0$ (d) $b^2 < ac$

22. If $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0$, $b \neq c$ have a common root, then $\dfrac{a + b + c}{a}$ is equal to
(a) -2 (b) -1
(c) $3/4$ (d) $-1/4$

23. Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$ and $4a - 6b + 9c < 0$. If $ax^2 + bx + c = 0$ does not have real roots, then $\dfrac{b + c}{a}$ is less than
(a) 0 (b) 1
(c) -1 (d) -2

24. Suppose α, β are roots of $4x^2 - 16x + c = 0$, where $c \in \mathbf{N}$, the set of natural numbers. The number of values of c for which $1 < \alpha < 2 < \beta < 3$ is
(a) 2 (b) 3
(c) 4 (d) 9

25. Suppose $a, b \in \mathbf{R}$, $ab \neq 0$. If all the three quadratic equations $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common negative root, then (a, b) is equal to
(a) $(-7, -8)$ (b) $(-8, -7)$
(c) $(7, 8)$ (d) $(8, 7)$

26. Suppose $a, b \in \mathbf{R}$. If the equation $\dfrac{a}{x} = \dfrac{1}{x-b} + \dfrac{1}{x+b}$ is not satisfied by any real value of x, then
(a) $0 < a < b < 2$ (b) $0 < a < 2$
(c) $0 < b < 2$ (d) $0 < b < a < 2$

27. Suppose $a \in \mathbf{R}$, $a \neq -1/2$. Let α, β be roots of $(2a + 1)x^2 - ax + a - 2 = 0$ If $\alpha < 1 < \beta$, then
(a) $\frac{1}{7}\left(6 - 2\sqrt{23}\right) < a < 1$ (b) $\frac{1}{7}\left(6 - 2\sqrt{23}\right) < a < \frac{1}{2}$
(c) $\frac{1}{2} < a < \frac{1}{7}\left(6 + 2\sqrt{23}\right)$ (d) none of these

28. Suppose α, β are roots of $8x^2 - 10x + 3 = 0$, then $\sum_{n=0}^{\infty}\left(\alpha^n + \beta^n\right)$ is
(a) 7/4
(b) 3/7
(c) 6
(d) 7

29. Suppose $a, b, c \in \mathbf{R}$ $a \neq 0$. If $a + |b| + 2c = 0$, then roots of $ax^2 + bx + c = 0$ are
(a) real and distinct
(b) real and equal
(c) purely imaginary
(d) non-real complex numbers

30. Suppose $0 < b < c$ and $f(x) = \dfrac{x^2 - bc}{2x - (b + c)}$, $x \in \mathbf{R}$, then $f(x)$ cannot lie in
(a) (b, c) (b) $(-\infty, b)$
(c) (c, ∞) (d) $(0, b) \cup (b, c)$

31. If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, then value of $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ is
(a) $(r - p)^2 - (q - s)^2$
(b) $(r - p)^2 + (q - s)^2$
(c) $(r - p)^2 + (q - s)^2 - 2rp(r - p)(q - s)$
(d) none of these

32. The number of real solutions of $x^2 + 5|x| + 4 = 0$ is
(a) 4 (b) 2
(c) 1 (d) 0

33. The number of real solutions of $x^2 - 3|x| + 2 = 0$ is
(a) 4 (b) 2
(c) 1 (d) 0

34. If $2 + \sqrt{5}\, i$ is a root of $x^2 - px + q = 0$ where p and q are real, then the ordered pair (p, q) is equal to
(a) $(4, 9)$ (b) $(9, 4)$
(c) $(3, 3)$ (d) $(2, 3)$

35. If the quadratic equation $2x^2 + ax + b = 0$ and $2x^2 + bx + a = 0$, $(a \neq b)$ have a common root, the value of $a + b$ is
(a) -3 (b) -2
(c) -1 (d) 0

36. If a, b, c, d and p are distinct real numbers such that
$(a^2 + b^2 + c^2)p^2 + 2(ab + bc + cd)p +$
$(b^2 + c^2 + d^2) \leq 0,$
then a, b, c, d
(a) are in A.P. (b) are in G.P.
(c) are in H.P. (d) none of these

37. The number of real roots of the equation $\sin(e^x) = 5^x + 5^{-x}$ is
(a) 0 (b) 1
(c) 2 (d) infinitely many

38. The value of a for which the equations $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$ have a common root is
(a) 2 (b) -2
(c) 0 (d) none of these

39. The roots of the equation $|x^2 - x - 6| = x + 2$ are
(a) $-2, 1, 4$ (b) $0, 2, 4$
(c) $0, 1, 4$ (d) $-2, 2, 4$

40. The number of real roots of the equation
$x^2 - 3|x| + 2 = 0$
(a) 4 (b) 1
(c) 2 (d) infinite

41. The least value of $n \in \mathbf{N}$ for which $(n - 4)x^2 + 8x + n + 2 > 0$ $\forall\, x \in \mathbf{R}$, is
(a) 11 (b) 10
(c) 8 (d) 7

42. Let $f(x)$ be a quadratic expression such that $f(x) < 0$ $\forall\, x \in \boldsymbol{R}$. If $g(x) = f(x) + f'(x) + f''(x)$ then for $x \in \boldsymbol{R}$.
(a) $g(x) < 0$ (b) $g(x) \leq 0$
(c) $g(x) > 0$ (d) $g(x) \geq 0$

43. If $x + 1$ is a factor of $x^4 + (p - 3)x^3 - (3p - 5)x^2 + (2p + 9)x + 12$, then value of p is

(a) -2 (b) 2
(c) 1 (d) -1

44. If both the roots of $x^2 - (p - 4)x + 2e^{2\log p} - 4 = 0$ are negative, then p belongs to

(a) $(-\sqrt{2}, 4)$ (b) $(\sqrt{2}, 4)$
(c) $(4, \infty)$ (d) none of these

45. Let f and g be two real valued functions and $S = \{x|f(x) = 0\}$ and $T = \{x|g(x) = 0\}$, then $S \cap T$ represent the set of roots of

(a) $f(x)\, g(x) = 0$ (b) $f(x)^2 + g(x)^2 = 0$
(c) $f(x) + g(x) = 0$ (d) $\frac{f(x)}{g(x)} = 0$

46. If domain of $f(x) = \sqrt{x^2 + bx + 4}$ is **R**, then maximum possible integral value of b is

(a) 2 (b) 3
(c) 4 (d) 5

47. If $p \in (-1, 1)$, then roots of the quadratic equation $(p - 1)x^2 + px + \sqrt{1 - p^2} = 0$ are

(a) purely imaginary
(b) non-real complex numbers
(c) real and equal
(d) real and distinct.

48. If a, b, c are positive real numbers, then the number of positive real roots of the equation $ax^2 + bx + c = 0$ is

(a) 0 (b) 1
(c) 2 (d) infinite

49. If the roots of the equation $x^2 + p^2 = 8x + 6p$ are real, then p belongs to the interval

(a) $[2, 8]$ (b) $[-8, 2]$
(c) $[-2, 8]$ (d) $[-8, -2]$

50. If sum of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ is -3, then the product of the roots is

(a) 1 (b) 4
(c) 2 (d) -2

51. If $3 - 4i$ is a root of $x^2 - px + q = 0$ where $p, q \in \boldsymbol{R}$, then value $\frac{2p - q}{p + q}$ is

(a) $-12/31$ (b) $-13/31$
(c) $-15/31$ (d) none of these

52. If $x = 1 + i$ is a root of $x^3 - ix + 1 - i = 0$, then the quadratic equation whose roots are the remaining two roots of $x^3 - ix + 1 - i = 0$ is

(a) $x^2 + (1 + i)x + 1 + i = 0$
(b) $x^2 + (1 + i)x + i = 0$
(c) $x^2 + 2(1 + i)x - 2 = 0$
(d) none of these

53. If α and β be the roots of the equation $x^2 + px - \frac{1}{2p^2} = 0$, where $p \in \boldsymbol{R}$. Then the minimum possible value of $\alpha^2 + \beta^2$ is

(a) 2 (b) $2\sqrt{2}$
(c) $2 + \sqrt{2}$ (d) none of these

54. The equation $\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1$ has

(a) no solution
(b) exactly one solution
(c) exactly two solutions
(d) more than two solution

55. The equation $|x - x^2 - 1| = |2x - 3 - x^2|$ has

(a) no solution
(b) exactly one solution
(c) exactly two solutions
(d) more than two solutions

56. If $\sin \alpha$, $\cos \alpha$ are the roots of the equation $ax^2 + bx + c = 0$, $(a \neq 0)$, then

(a) $a^2 - b^2 + 2ac = 0$
(b) $a^2 + b^2 - 2ac = 0$
(c) $(a - c)^2 = b^2 + c^2$
(d) none of these

57. If $x \in \boldsymbol{R}$, and $k = \frac{x^2 - x + 1}{x^2 + x + 1}$, then

(a) $1/3 \leq k \leq 3$ (b) $k \geq 5$
(c) $k \leq 0$ (d) none of these

58. If the equation $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have a common root and $b \neq c$, then their other roots will satisfy the equation

(a) $x^2 + (b + c)x + bc = 0$ (b) $x^2 - ax + bc = 0$
(c) $x^2 + ax + bc = 0$ (d) none of these

59. If the inequality $\frac{mx^2 + 3x + 4}{x^2 + 2x + 2} < 5$

is satisfied for all $x \in \boldsymbol{R}$, then
(a) $m < 5$ (b) $m > 5$
(c) $m < 71/24$ (d) $m > 71/24$

60. If a, b, c are distinct real numbers, then the quadratic expression

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)}$$

is identically equal to

(a) 1 (b) x
(c) x^2 (d) none of these

61. If $ax^2 + bx + c$, $a, b, c \in \boldsymbol{R}$, $a \neq 0$ has no real zero and $a - b + c < 0$, then value of ac is
(a) positive (b) zero
(c) negative (d) non-negative

62. Suppose $p \in \mathbf{R}$. Let α, β be roots of $x^2 - 2x - (p^2 - 1) = 0$ and γ, δ $(\gamma < \delta)$ be roots of $x^2 - 2(p + 1)x + p(p - 1) = 0$. If $\alpha, \beta \in (\gamma, \delta)$, then p lies in the interval
(a) $(-1/4, 1)$ (b) $(-1, 1)$
(c) $(0, \infty)$ (d) $(\sqrt{2}, \infty)$

63. If α, β are the roots of the equation $ax^2 + 2bx + c = 0$ and $\alpha + h$, $\beta + h$ are the roots of the equation $Ax^2 + 2Bx + C = 0$, then
(a) $\dfrac{b^2 - ac}{B^2 - AC} = \dfrac{a}{A}$ (b) $b^2 - ac = B^2 - AC$
(c) $h = \dfrac{bA - aB}{2Aa}$ (d) $h = \dfrac{Ac + aC}{Aa + Bb}$

64. The quadratic equation $x^2 + 7x = 14(q^2 + 1)$, where q is an integer has
(a) real and distinct roots
(b) integral roots
(c) imaginary roots
(d) none of these

65. Let $a, b, c \in \boldsymbol{R}$ and $a > 0$. If the quadratic equation $ax^2 + bx + c = 0$ has two real roots α and β such that $\alpha < -1$ and $\beta > 1$, then value of $\dfrac{c}{a} + \left|\dfrac{b}{a}\right|$ is
(a) less than 2 (b) less than 1
(c) less than 0 (d) less than -1

66. Let $a, b, c \in \boldsymbol{R}$ and $a \neq 0$ be such that $(a + c)^2 < b^2$, then the quadratic equation $ax^2 + bx + c = 0$ has
(a) imaginary roots
(b) real roots
(c) two real roots lying between $(-1, 1)$
(d) none of these

67. The integral values of a for which the quadratic equation
$(x - a)(x - 10) + 1 = 0$ has integeral roots are
(a) $-1, 3$ (b) $2, 3$
(c) $12, 8$ (d) $-8, -12$

68. The number of real solution of $4^{x + 1.5} + 9^{x + 0.5} = (10)(6^x)$ is
(a) zero (b) one
(c) two (d) infinite

69. The number of real solution of $2^{\sin^2 x} + 5(2^{\cos^2 x}) = 7$ is
(a) zero (b) 1
(c) finitely many (d) infinitely many

70. The number of values of k for which the equation $x^2 - 2x + k = 0$ has two distinct roots lying in the interval $(0, 1)$ is
(a) 0 (b) 1
(c) 2 (d) infinitely many

71. If roots of the quadratic equation $ax^2 + bx + c = 0$ are real and are of the form $\dfrac{\alpha}{\alpha - 1}, \dfrac{\alpha + 1}{\alpha}$, then value of $(a + b + c)^2$ is
(a) $4ac - b^2$ (b) $b^2 - 4ac$
(c) $c^2 + a^2 - 2b^2$ (d) none of these

72. Let x be an integer and $x^2 + x + 1$ is divisible by 3. When x is divided by 3, it leaves remainder
(a) 0 (b) 1
(c) 2 (d) any of (a), (b) and (c)

73. If α, β are the roots of the equation $x^2 + ax + b = 0$, then maximum value of $-x^2 + ax + b + \dfrac{1}{4}(\alpha - \beta)^2$ is
(a) $\dfrac{1}{4}(a^2 - 4b)$ (b) $\dfrac{1}{4}(b^2 - 4a)$
(c) $\dfrac{a^2}{2}$ (d) none of these

74. If both the roots of the equation $x^2 + bx + c = 0$ lie in the interval $(0, 1)$, then
(a) $b = -1, c = 2$ (b) $b > -2, c < 1$
(c) $b = -5, c < 2$ (d) none of these

75. Let $a, b, c \in \mathbf{R}$ be such that $a + b + c < 0$, $a - b + c < 0$ and $c > 0$. If α and β are roots of the equation $ax^2 + bx + c = 0$, then value of $[\alpha] + [\beta]$ is
(a) 2 (b) 1
(c) -1 (d) 0

76. If roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ lie in the interval $(-2, 4)$, then
(a) $-1 < m < 3$ (b) $1 < m < 5$
(c) $1 < m < 3$ (d) $-1 < m < 5$

77. The value of $\sqrt{8 + 2\sqrt{8 + 2\sqrt{8 + 2\sqrt{8 + \cdots}}}}$ is
(a) 10 (b) 6
(c) 8 (d) none of these

78. The number of solutions of the equation
$$\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}\,x + 4 \text{ is}$$

(a) 1 (b) 2
(c) 0 (d) infinite

79. The number of solutions of $|x + 2| = 2(3 - x)$ is

(a) 1 (b) 2
(c) 3 (d) 0

80. If the equation $ax^2 + bx + c = 0$, $a > 0$ has two distinct roots α and β such that $\alpha < -2$ and $\beta > 3$, then

(a) $c > 0$ (b) $c = 0$
(c) $c < 0$ (d) $c = a - b$

81. Two non-integer roots of

$$(x^2 - 5x)^2 - 7(x^2 - 5x) + 6 = 0$$

are

(a) $\frac{1}{2}(5 + \sqrt{29}), \frac{1}{2}(5 - \sqrt{29})$

(b) $\frac{1}{2}(-5 + \sqrt{29}), \frac{1}{2}(-5 + \sqrt{29})$

(c) $\frac{1}{2}(-5 + \sqrt{14}), \frac{1}{2}(-5 - \sqrt{41})$

(d) none of these

82. The number of real roots of

$$\left(\frac{x-1}{x+1}\right)^4 - 13\left(\frac{x-1}{x+1}\right)^2 + 36 = 0, x \neq -1$$

is

(a) 0 (b) 2
(c) 3 (d) 4

83. The number of negative roots of

$$9^{x+2} - 6(3^{x+1}) + 1 = 0$$

is

(a) 0 (b) 1
(c) 2 (d) 4

84. The number of rational roots of

$$81\left(\frac{2x-5}{3x+1}\right)^4 - 45\left(\frac{2x-5}{3x+1}\right)^2 + 4 = 0,\ x \neq 1/3$$

is

(a) 1 (b) 2
(c) 3 (d) 4

85. The number irrational roots of

$$(x^2 + 3x + 2)^2 - 8(x^2 + 3x) - 4 = 0$$

is

(a) 0 (b) 2
(c) 3 (d) 4

86. The number of roots of the equation

$$\sqrt{\frac{x}{x-3}} + \sqrt{\frac{x-3}{x}} = \frac{5}{2}, x \neq 0, x \neq 3$$

is

(a) 0 (b) 2
(c) 3 (d) 4

87. The number of irrational roots of the equation

$$4\left(x - \frac{1}{x}\right)^2 + 8\left(x + \frac{1}{x}\right) = 29$$

is

(a) 0 (b) 2
(c) 4 (d) infinite

88. Irrational roots of the equation

$$2x^4 + 9x^3 + 8x^2 + 9x + 2 = 0$$

are

(a) $-2 - \sqrt{3}, 2 + \sqrt{3}$ (b) $2 - \sqrt{3}, 2 + \sqrt{3}$

(c) $-2 + \sqrt{3}, -2 - \sqrt{3}$ (d) none of these

89. Sum of the roots of the equation

$$4\left(x - \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right) + 1 = 0$$

is

(a) 5 (b) 1
(c) – 5/2 (d) – 1

90. The number of irrational roots of the equation

$$(x - 1)(x - 2)(3x - 2)(3x + 1) = 21$$

is

(a) 0 (b) 2
(c) 3 (d) 4

91. Product of roots of the equation

$$x - \sqrt{3x - 6} = 2$$

is

(a) 2 (b) 5
(c) 7 (d) 10

92. The number of real roots of

$$2\sqrt{2x + 1} = 2x - 1$$

is

(a) 1 (b) 2
(c) 3 (d) 4

93. Product of roots of the equation

$$\sqrt{13 - x^2} = x + 5$$

is

(a) – 6 (b) 7
(c) 6 (d) – 7

94. The number of roots of the equation

$$\sqrt{x^2 - 4} - (x - 2) = \sqrt{x^2 - 5x + 6}$$ is

(a) 0 (b) 1
(c) 2 (d) 3

95. The product of the roots of the equation

$\sqrt{x^2-4x+3}+\sqrt{x^2-7x+12} = 3\sqrt{x-3}$ is

(a) 15 (b) – 15
(c) 20 (d) – 20

96. Suppose a and b satisfy the equations $18a^2 + 77a + 2 = 0$ and $2b^2 + 77b + 18 = 0$ then value of $\frac{ab+a+1}{b}$ is
(a) – 25 (b) – 25/6
(c) 6/25 (d) – 1/25

97. Suppose α, β are roots of $x^2 - 7x + 8 = 0$, with $\alpha > \beta$, then value of $\frac{16}{\alpha}+3\beta^2-19\beta$ is
(a) – 10 (b) 10
(c) – 23 (d) 17

98. If α is a root of $x^4 + x^2 - 1 = 0$, the value of $(\alpha^6 + 2\alpha^4)^{2012}$ is
(a) 0 (b) – 1
(c) 1 (d) none of these

99. Sum and product of all the roots of the equation $(x^2 - x - 1)(x^2 - x - 2)(x^2 - x - 3) \ldots (x^2 - x - 2012) = 0$ is
(a) 2012, 2012! (b) – 2012, 2012!
(c) – 2012, – 2012! (d) 2012, – 2012!

100. Suppose three distinct non-zero real numbers satisfy $a^2(a + k) = b^2(b + k) = c^2(c + k)$, where k is some real number, then value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is
(a) 0 (b) k
(c) $-k$ (d) $2k$

Assertion-Reason Type Questions

101. Suppose $a, b, c, \alpha, \beta \in \mathbf{R}$ $a \neq 0$ and $0 < \alpha < \beta$. Let $f(x) = a^2x^2 + bx + c$, $g(x) = a^2x^2 - bx - c$

Statement-1: If α is a root of $f(x) = 0$ and β is a root of $g(x) = 0$, then there exists $\gamma \in (\alpha, \beta)$ such that γ is a root of $a^2x^2 + 2bx + 2c = 0$.

Statement-2: If a function $h : [\alpha, \beta] \to \mathbf{R}$ is continuous on $[\alpha, \beta]$ and $h(\alpha)\, h(\beta) < 0$, then there exists $\gamma \in (\alpha, \beta)$ such that $h(\gamma) = 0$.

102. Suppose $p \in \mathbf{R}$ and $f(x) = x^2 + 2px + p$

Statement-1: If $f(x) < 0\ \forall\ x \in [1, 2]$ then $p \in (-\infty, -4/5)$.

Statement-2: a lies between the roots of $f(x) = 0$, if and only if $f(a) < 0$.

103. Suppose $a \in \mathbf{R}$.

Statement-1: The equation $\sqrt{a(2^x-2)+1} = 1 - 2^x$ has a real solution for all $a \in (0, 1]$.

Statement-2: $ax^2 + bx + c = 0$, $a \neq 0$ has two positive roots if $ab > 0$ and $ac > 0$.

104 Suppose $a, b, c \in \mathbf{R}$ and $f(x) = 2(a-x)\left(x+\sqrt{x^2+b^2}\right)$ $x \in \mathbf{R}$.

Statement-1: Maximum value of $f(x)$ is $a^2 + b^2$.

Statement-2: If $a < 0$, the maximum value of $f(x) = ax^2 + bx + c$ is $(b^2 - 4ac)/4a$.

105. Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$.

Statement-1: If $ax^2 + bx + c = x$ does not have real roots, then $a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c = x$ has two real and two imaginary roots.

Statement-2: If $\alpha + i\beta$, where $\alpha, \beta \in \mathbf{R}$, $\beta \neq 0$ is a root of $ax^2 + bx + c = x$, then its other root must be $\alpha - i\beta$

106. Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$ and $a - b + c < 0$.

Statement-1: If $ax^2 + bx + c = 0$ has imaginary roots, and

$$\left|a\left(x^2+\frac{y}{a}\right)+(b+1)x+c\right| = |ax^2+bx+c|+|x+y|$$

then (x, y) lies in the half plane $\{(x, y) | x + y \leq 0\}$

Statement-2: $|a + b| = |a| + |b| \Leftrightarrow ab \geq 0$

107. $a, b, c \in \mathbf{R}$ and $c < \min\{a, b\}$.

Let $f(x) = \frac{(x+a)(x+b)}{x+c}$, $x > -c$.

Statement-1: Minimum value of f is $\left(\sqrt{a-c}+\sqrt{b-c}\right)^2$

Statement-2: If $a < 0$, then $ax^2 + bx + c$ has no minimum value.

108. Suppose $a, b, c, p, q, r \in \mathbf{R}$, $a, p \neq 0$.

Let $f(x) = ax^2 + bx + c$ and $g(x) = px^2 + qx + r$

Statement-1: If $f(x) = g(x)$ for three distinct real values of x then $a = p$, $b = q$ and $c = r$.

Statement-2: If $a \neq p$, then $f(x) - g(x) = 0$ has two real roots.

109. Suppose $p \in \mathbf{R}$.

Statement-1: If $\sin^4 x + p\sin^2 + 1 = 0$ has a solution, then $p \in (-\infty, -2]$

Statement-2: $y + \dfrac{1}{y} \geq 2 \;\; \forall \; y > 0$.

110. Suppose $a, b, c \in \mathbf{R}$ $a \neq 0$ and $c > 0$.

Let $f(x) = ax^2 + bx + c$.

Statement-1: If $f(x) = 0$, does not have real roots, then

$a > \max \left\{\dfrac{b^2}{4c}, b - c\right\}$

Statement-2: If $b^2 - 4ac < 0$ then $f(x) > 0 \;\forall\; x \in \mathbf{R}$.

LEVEL 2

Straight Objective Type Questions

111. If $\alpha \in (0, \pi/2)$, then the expression

$$\sqrt{x^2 + x} + \frac{\tan^2 \alpha}{\sqrt{x^2 + x}}$$

is always greater than or equal to

(a) $2 \tan \alpha$ (b) 2
(c) 1 (d) $\sec^2 \alpha$

112. If $a, b \in \mathbf{R}$, and the equation

$$x^2 + (a - b)x - a - b + 1 = 0$$

has real roots for all $b \in \mathbf{R}$, then a lies in the interval

(a) $(1, \infty)$ (b) $(0, \infty)$
(c) $(-\infty, 1)$ (d) $(-1, 1)$

113. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) none of these.

114. Suppose $a, b, c \in \mathbf{R}$ and the equation $x^2 + (a + b)x + c = 0$ has no real roots, then which one of the following is **not** true.

(a) $c + c\,(a + b + c) > 0$
(b) $c - c\,(a + b - c) > 0$
(c) $(a + b)^2 - 4c < 0$
(d) $c\,(a + b + c) < 0$

115. If $[x]$ denotes the greatest integer $\leq x$, then number of solutions of the equation $x^2 - 2 - 2[x] = 0$ is

(a) 4 (b) 2
(c) 3 (d) none of these

116. If $[x]$ denotes the greatest integer $\leq x$, and a, b are two odd integers, then number of solutions of $[x]^2 + a[x] + b = 0$ is

(a) 1 (b) 0
(c) 2 (d) infinite

117. Let α, β, γ be distinct real numbers lying in $(0, \pi/2)$, then the equation

$$\frac{1}{x - \sin\alpha} + \frac{1}{x - \sin\beta} + \frac{1}{x - \sin\gamma} = 0, \text{ has}$$

(a) two distinct real roots
(b) two equal roots
(c) two imaginary roots
(d) one real and one imaginary root.

118. If roots of the equation $x^2 + ax + b = 0$ are α, β, then the roots of $x^2 + (2\alpha + a)x + \alpha^2 + a\alpha + b = 0$ are

(a) $1, \beta - \alpha$ (b) $0, \alpha - \beta$
(c) $0, \beta - \alpha$ (d) $0, 1$.

119. Let a, b, c be the sides of a triangle with $a \neq c$ and $\lambda \in \boldsymbol{R}$. If the roots of $x^2 + 2(a + b + c)x + 3\lambda\,(ab + bc + ca) = 0$ are real, then λ lies in

(a) $(-\infty, 4/3)$ (b) $(5/3, \infty)$
(c) $(1/3, 5/3)$ (c) $(4/3, 5{,}3)$

120. If $\tan\theta$ and $\cot\theta$ are roots of $x^2 + 2ax + b = 0$, then least value of $|a|$ is

(a) $\dfrac{1}{2}$ (b) 1
(c) 2 (d) cannot be found.

121. Let α, β, γ be the roots of $x^3 + x^2 - 5x - 1 = 0$, then value of $[\alpha] + [\beta] + [\gamma]$, where $[x]$ denotes the greatest integer $\leq x$, is

(a) 1 (b) 2
(c) -2 (d) -3.

[**Hint:** Show that $f(-3) < 0, f(-2) > 0, f(-1) > 0, f(0) < 0, f(1) < 0, f(2) > 0$]

122. If tan A and tan B are roots of the quadratic equation $x^2 - px + q = 0$, then the value of $\sin^2 (A + B)$ is

(a) $\dfrac{p^2}{p^2 + q^2}$ (b) $\dfrac{p^2}{(q + p)^2}$

(c) $\dfrac{p^2}{(1 - q)^2 + p^2}$ (d) $1 - \dfrac{p^2}{(1 - q)^2}$

123. The equation

$$\sqrt{x+3+4\sqrt{x-1}}+\sqrt{x+8+6\sqrt{x-1}}=1 \text{ has}$$

(a) no solution
(b) only one solution
(c) only two solutions
(d) infinite number of solutions

124. The number of irrational solutions of the equation

$$\sqrt{x^2+\sqrt{x^2+11}}+\sqrt{x^2-\sqrt{x^2+11}}=4 \text{ is}$$

(a) 0 (b) 2
(c) 4 (d) infinite

125. Let a, b, c, p, q be five different non-zero real numbers and x, y, z be three numbers satisfying the system of equations

$$\frac{x}{a}+\frac{y}{a-p}+\frac{z}{a-q}=1$$

$$\frac{x}{b}+\frac{y}{b-p}+\frac{z}{b-q}=1$$

and $$\frac{x}{c}+\frac{y}{c-p}+\frac{z}{c-q}=1$$

then x equals

(a) $\frac{abc}{pq}$ (b) $\frac{pq}{abc}$
(c) $\frac{abc}{p+q}$ (d) none of these.

126. Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{R}$. Suppose $|f(x)| \le 1\ \forall\ x \in [0, 1]$, then $|a|$ cannot exceed

(a) 5 (b) 6
(c) 7 (d) 8

127. If $a(p + q)^2 + 2bpq + c = 0$ and
$a(p + r)^2 + 2bpr + c = 0$,
then $|q - r|$ equals

(a) $\frac{2}{|a|}\sqrt{(2a+b)bp^2-ac}$ (b) $\frac{2}{|a|}\sqrt{p^2-4ac}$
(c) $p^2+\frac{c}{a}$ (d) none of these.

128. Let $a, b, c \in \mathbf{R}$ be such that $b^2 \ge 4ac$. All the four roots of the equation $ax^4 + bx^2 + c = 0$ will be real if

(a) $a > 0, b < 0, c > 0$ or $a < 0, b > 0, c < 0$
(b) $a > 0, b > 0, c > 0$ or $a < 0, b < 0, c < 0$
(c) $a > 0, b < 0, c > 0$ or $a > 0, b > 0, c < 0$
(d) none of these.

129. If the equations
$x^2 + mx + 1 = 0$ and
$(b - c)x^2 + (c - a)x + (a - b) = 0$ have a common root, then

(a) $m = -2$ (b) $m = -1$
(c) $m = 0$ (d) $m = 1$

130. If $x, y, z \in \mathbf{R}$, $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$, then the maximum possible value of z is

(a) 1 (b) 2
(c) $\frac{3}{2}$ (d) $\frac{4}{3}$

Previous Years' AIEEE/JEE Main Questions

1. If $\alpha \ne \beta$ and $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$, then the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is

(a) $3x^2 - 25x + 3 = 0$ (b) $x^2 - 5x + 3 = 0$
(c) $x^2 + 5x - 3 = 0$ (d) $3x^2 - 19x + 3 = 0$
[2002]

2. If $a \ne b$ and differences between the roots of the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ is the same, then

(a) $a + b + 4 = 0$ (b) $a + b - 4 = 0$
(c) $a - b + 4 = 0$ (d) $a - b - 4 = 0$
[2002]

3. If $a, b, c \in \mathbf{R}$ and $2a + 3b + 6c = 0$, then the equation $ax^2 + bx + c = 0$ has

(a) at least one root in [0, 1]
(b) at least one root in [2, 3]
(c) at least one root in [– 1, 0]
(d) at least one root in $(-\infty, 1)$ **[2002]**

4. Product of real roots of the equation $t^2x^2 + |x| + 9 = 0$, $t \in \mathbf{R}$ is always

(a) positive (b) negative
(c) zero (d) does not exit **[2002]**

5. The value of a for which one root of the quadratic equation

$$(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$$

is twice as large as the other is

(a) $-\frac{2}{3}$ (b) $\frac{1}{3}$
(c) $-\frac{1}{3}$ (d) $\frac{2}{3}$ **[2003]**

6. If sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then $\frac{a}{c}$, $\frac{b}{a}$ and $\frac{c}{b}$ are in
(a) G.P. (b) H.P.
(c) A.G.P. (d) A.P. **[2003]**
7. The number of real solution of the equation $x^2 - 3|x| + 2 = 0$ is
(a) 4 (b) 1
(c) 3 (d) 2 **[2004]**
8. Let two numbers have arithmetic mean 9 and geometric mean 4. Then these numbers are roots of the quadratic equation
(a) $x^2 + 18x - 16 = 0$ (b) $x^2 - 18x + 16 = 0$
(c) $x^2 + 18x + 16 = 0$ (d) $x^2 - 18x - 16 = 0$ **[2004]**
9. If $1 - p$ is a root of quadratic equation $x^2 + px + 1 - p = 0$, then its roots are
(a) $0, -1$ (b) $-1, 1$
(c) $0, 1$ (d) $-1, 2$ **[2004]**
10. If one root of the equation $x^2 + px + 12 = 0$ is 4, while equation $x^2 + px + q = 0$ has equal roots, then the value of q is
(a) 3 (b) 12
(c) $\frac{49}{4}$ (d) 4 **[2004]**
11. If $2a + 3b + 6c = 0$, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval
(a) (2, 3) (b) (1, 2)
(c) (0, 1) (d) (1, 3) **[2004]**
12. In a triangle PQR, $\angle R = \frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of $ax^2 + bx + c = 0$, $a \neq 0$, then
(a) $b = c$ (b) $b = a + c$
(c) $a = b + c$ (d) $c = a + b$ **[2005]**
13. The value of a for which sum of the squares of the roots of the equation of the roots of the equation
$$x^2 - (a - 2)x - a - 1 = 0$$
assume least value is
(a) 3 (b) 2
(c) 1 (d) 0 **[2005]**
14. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers then $b^2 - 4c$ equals
(a) 2 (b) 1
(c) -2 (d) 3 **[2005]**
15. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k lies in the interval
(a) $(-\infty, 4)$ (b) [4, 5]
(c) (5, 6) (d) $(6, \infty)$ **[2005]**
16. If the equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x = 0$, $a_1 \neq 0$, $n \geq 2$ has a positive root $x = \alpha$, then the equation
$n\, a_n x^{n-1} + (n - 1)\, a_{n-1} x^{n-2} + \ldots + a_1 = 0$
has a positive root, which is
(a) greater than or equal to α
(b) equal to α
(c) greater than α
(d) smaller than α **[2005]**
17. If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15° respectively, then the value of $2 + q - p$ is
(a) 1 (b) 2
(c) 3 (d) 0 **[2006]**
18. All the values of m for which both the roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4, lie in the interval
(a) $1 < m < 4$ (b) $-2 < m < 0$
(c) $m > 3$ (d) $-1 < m < 3$ **[2006]**
19. If x is real, the maximum value of $\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is
(a) $\frac{17}{7}$ (b) $\frac{1}{4}$
(c) 41 (d) 1 **[2006]**
20. If the difference between the roots of $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then set of possible values of a lie in the interval
(a) $(-3, 3)$ (b) $(-3, \infty)$
(c) $(3, \infty)$ (d) $(-\infty, -3)$ **[2007]**
21. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one common root. The other roots of first and second equation are integers in the ratio 4 : 3. Then common root is
(a) 1 (b) 4
(c) 3 (d) 2 **[2008]**
22. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is
(a) greater then $-4ab$ (b) less then $-4ab$
(c) greater than $4ab$ (d) less than $4ab$ **[2009]**
23. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009}$ equals
(a) 1 (b) 2
(c) -2 (d) -1 **[2010]**
24. Let for $a, a_1 \neq 0$, $a \neq a_1$
$f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and
$p(x) = f(x) - g(x)$.
If $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then value of $p(2)$ is:
(a) 3 (b) 9
(c) 6 (d) 18 **[2011]**

25. The equation $e^{\sin x} - e^{-\sin x} = 4$ has
   (a) no real roots
   (b) exactly one real root
   (c) exactly four real roots
   (d) infinite number of real roots. **[2012]**

26. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in \mathbf{R}$ have a common root, then $a: b: c$ is
   (a) $3 : 2 : 1$ (b) $1 : 3 : 2$
   (c) $3 : 1 : 2$ (d) $1 : 2 : 3$ **[2013]**

27. If α and β are the roots of the equation $x^2 + px + \frac{3}{4}p = 0$, such that $|\alpha - \beta| = \sqrt{10}$, then p belongs to the set
   (a) $\{2, -5\}$ (b) $\{-3, 2\}$
   (c) $\{-2, 5\}$ (d) $\{-3, 5\}$ **[2013, online]**

28. If p and q are non-zero real number such that $\alpha^3 + \beta^3 = -p$ and $\alpha\beta = q$, then a quadratic equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$ is
   (a) $px^2 - qx + p^2 = 0$
   (b) $qx^2 + px + q^2 = 0$
   (c) $px^2 + qx + p^2 = 0$
   (d) $qx^2 - px + q^2 = 0$ **[2013, online]**

29. The values of a for which one root of the equation $x^2 - (a + 1)x + a^2 + a - 8 = 0$ exceeds 2 and the other is less than 2 are given by
   (a) $3 < a < 10$ (b) $a > 10$
   (c) $-2 < a < 3$ (d) $a \le -2$ **[2013, online]**

30. The least integral value α of x such that $\frac{x-5}{x^2+5x-14} > 0$, satisfies
   (a) $\alpha^2 + 3\alpha - 4 = 0$ (b) $\alpha^2 - 5\alpha + 4 = 0$
   (c) $\alpha^2 - 7\alpha + 6 = 0$ (d) $\alpha^2 + 5\alpha - 6 = 0$ **[2013, online]**

31. Let α and β be the roots of the equation $px^2 + qx + r = 0$, $p \ne 0$. If p, q, r are in A.P. and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then value of $|\alpha - \beta|$ is
   (a) $\frac{1}{9}\sqrt{61}$ (b) $\frac{2}{9}\sqrt{17}$
   (c) $\frac{1}{9}\sqrt{34}$ (d) $\frac{2}{9}\sqrt{13}$ **[2014]**

32. If equations $ax^2 + bx + c = 0$, $(a, b, c \in \mathbf{R}, a \ne 0)$ and $2x^2 + 3x + 4 = 0$ have a common root, then $a : b : c$ equals:
   (a) $1 : 2 : 3$ (b) $2 : 3 : 4$
   (c) $4 : 3 : 2$ (d) $3 : 2 : 1$ **[2014, online]**

33. The sum of the roots of the equation $x^2 + |2x - 3| + 4 = 0$, is:
   (a) 2 (b) -2
   (c) $\sqrt{2}$ (d) $-\sqrt{2}$ **[2014, online]**

34. If α and β are the roots of $x^2 - 4\sqrt{2}\,kx + 2e^{4\ln k} - 1 = 0$ for some k, and $\alpha^2 + \beta^2 = 66$, then $\alpha^3 + \beta^3$ is equal to:
   (a) $248\sqrt{2}$ (b) $280\sqrt{2}$
   (c) $-32\sqrt{2}$ (d) $-280\sqrt{2}$ **[2014, online]**

35. The equation $\sqrt{3x^2 + x + 5} = x - 3$, where x is real, has:
   (a) has no solution
   (b) exactly one solution
   (c) exactly two solutions
   (d) exactly four solutions **[2014, online]**

36. Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$, for $n \ge 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to:
   (a) 6 (b) -6
   (c) 3 (d) -3 **[2015]**

37. If $2 + 3i$ is one of the roots of the equation $2x^3 - 9x^2 + kx - 13 = 0$, $k \in \mathbf{R}$, then the real root of the equation:
   (a) does not exist.
   (b) exists and is equal to $\frac{1}{2}$
   (c) exists and is equal to $-\frac{1}{2}$
   (d) exists and is equal to 1 **[2015, online]**

38. If the two roots of the equation, $(a - 1)(x^4 + x^2 + 1) + (a + 1)(x^2 + x + 1)^2 = 0$ are real and distinct, then the set of all values of a is:
   (a) $\left(-\frac{1}{2}, 0\right)$ (b) $(-\infty, -2) \cup (2, \infty)$
   (c) $\left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$ (d) $\left(0, \frac{1}{2}\right)$ **[2015, online]**

39. The sum of all real values of x satisfying the equation $(x^2 - 5x + 5)^{x^2+4x-60} = 1$ is
   (a) 3 (b) -4
   (c) 6 (d) 5 **[2016]**

40. If the equations $x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$ have a common root different from -1, then $|b|$ is equal to:
   (a) 2 (b) 3
   (c) $\sqrt{3}$ (d) $\sqrt{2}$ **[2016, online]**

41. If x is a solution of the equation, $\sqrt{2x+1} - \sqrt{2x-1} = 1$, $\left(x \ge \frac{1}{2}\right)$, then $\sqrt{4x^2 - 1}$ is equal to:

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$

(c) $2\sqrt{2}$ (d) 2 **[2016, online]**

42. Let x, y, z be positive real numbers such that $x + y + z = 12$ and $x^3y^4z^5 = (0.1)(600)^3$. Then $x^3 + y^3 + z^3$ is equal to
(a) 342 (b) 216
(c) 258 (d) 270 **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. If the roots of the quadratic equation $x^2 + 2px + q = 0$ are tan 30° and tan 15° respectively, then q is
(a) $1 + p$ (b) $1 - p$
(c) $1 - 2p$ (d) $1 + 2p$ **[2006]**

2. The set of values of α for which the quadratic equation $(\alpha + 2)\,x^2 - 2\alpha x - \alpha = 0$ has two roots on the number line symmetrically placed about the point 1 is
(a) $\{-1, 0\}$ (b) $\{0, 2\}$
(c) ϕ (d) $\{0, 1\}$ **[2007]**

3. The number of solutions of the equation $x^2 - 4|x| - 2 = 0$ is:
(a) 1 (b) 2
(c) 3 (d) 4 **[2008]**

4. The quadratic equation whose roots are a/b and b/a, $a \neq b \neq 0$, where $a^2 = 5a - 3$, and $b^2 = 5b - 3$, is:
(a) $3x^2 - 19x + 3 = 0$
(b) $3x^2 + 19x - 3 = 0$
(c) $3x^2 + 19x + 3 = 0$
(d) $3x^2 - 19x - 3 = 0$ **[2009]**

5. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are α, β, then the roots of the quadratic equation $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$, are
(a) $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$ (b) $\frac{\alpha}{\alpha-1}, \frac{\beta}{\beta-1}$
(c) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (d) $1 - \alpha, 1 - \beta$ **[2010]**

6. If $x^2 - 3x + 2$ is a factor of $x^4 - ax^2 + b = 0$ then the equation whose roots are a and b is
(a) $x^2 + 9x + 20 = 0$ (b) $x^2 - 9x - 20 = 0$
(c) $x^2 - 9x + 20 = 0$ (d) $x^2 + 9x - 20 = 0$ **[2011]**

7. Let a, b, $c \in \mathbf{R}$, $a > 0$ and the function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = ax^2 + bx + c$.
Statement-1: $b^2 < 4ac \Rightarrow f(x) > 0$ for every value of x.
Statement-2: f is strictly decreasing in the interval $(-\infty, -b/2a)$ and strictly increasing in the interval $(-b/2a, \infty)$. **[2012]**

8. If the quadratic equation $3x^2 + 2(a^2 + 1)x + a^2 - 3a + 2 = 0$ posseses roots of opposite signs, then a lies in the interval:
(a) $(-\infty, -1)$ (b) $(-1, 1)$
(c) $(1, 2)$ (d) $(2, 3)$ **[2013]**

9. If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude and opposite in sign, then product of roots is:
(a) $p^2 + q^2$ (b) $\frac{1}{2}(p^2 + q^2)$
(c) $-\frac{1}{2}(p^2 + q^2)$ (d) $-\frac{1}{2}(p^2 - q^2)$ **[2014]**

10. The values of k for which each root of the equation, $x^2 - 6kx + 2 - 2k + 9k^2 = 0$ is greater than 3, always satisfy the inequality:
(a) $7 - 9y > 0$ (b) $11 - 9y < 0$
(c) $29 - 11y > 0$ (d) $29 - 11y < 0$ **[2015]**

11. The number of integral values of m for which the equation, $(1 + m^2)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$, has no real root, is:
(a) 2 (b) 3
(c) infinitely many (d) 1 **[2016]**

Answers

Concept-based

1. (a)	**2.** (c)	**3.** (c)	**4.** (d)
5. (b)	**6.** (b)	**7.** (a)	**8.** (a)
9. (d)	**10.** (b)	**11.** (c)	**12.** (c)
13. (b)	**14.** (b)	**15.** (d)	**16.** (a)
17. (c)	**18.** (a)	**19.** (a)	**20.** (a)

Level 1

21. (b)	**22.** (c)	**23.** (c)	**24.** (b)
25. (a)	**26.** (b)	**27.** (a)	**28.** (c)
29. (a)	**30.** (a)	**31.** (d)	**32.** (d)
33. (a)	**34.** (a)	**35.** (b)	**36.** (b)
37. (a)	**38.** (b)	**39.** (d)	**40.** (a)

41. (d)	**42.** (a)	**43.** (b)	**44.** (d)
45. (b)	**46.** (c)	**47.** (d)	**48.** (a)
49. (c)	**50.** (b)	**51.** (b)	**52.** (b)
53. (a)	**54.** (d)	**55.** (b)	**56.** (a)
57. (a)	**58.** (c)	**59.** (c)	**60.** (a)
61. (a)	**62.** (a)	**63.** (c)	**64.** (a)
65. (d)	**66.** (b)	**67.** (c)	**68.** (c)
69. (d)	**70.** (a)	**71.** (b)	**72.** (b)
73. (c)	**74.** (d)	**75.** (c)	**76.** (a)
77. (d)	**78.** (a)	**79.** (a)	**80.** (c)
81. (a)	**82.** (d)	**83.** (b)	**84.** (d)
85. (d)	**86.** (b)	**87.** (b)	**88.** (c)
89. (b)	**90.** (b)	**91.** (d)	**92.** (a)
93. (c)	**94.** (d)	**95.** (a)	**96.** (b)
97. (a)	**98.** (c)	**99.** (a)	**100.** (a)
101. (a)	**102.** (a)	**103.** (c)	**104.** (c)
105. (d)	**106.** (a)	**107.** (b)	**108.** (b)
109. (a)	**110.** (a)		

Level 2

111. (a)	**112.** (a)	**113.** (b)	**114.** (d)
115. (d)	**116.** (b)	**117.** (a)	**118.** (c)
119. (a)	**120.** (b)	**121.** (d)	**122.** (c)
123. (a)	**124.** (b)	**125.** (a)	**126.** (d)
127. (a)	**128.** (a)	**129.** (a)	**130.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (d)	**2.** (a)	**3.** (a)	**4.** (d)
5. (d)	**6.** (b)	**7.** (a)	**8.** (b)
9. (a)	**10.** (c)	**11.** (c)	**12.** (d)
13. (c)	**14.** (b)	**15.** (a)	**16.** (d)
17. (c)	**18.** (d)	**19.** (c)	**20.** (a)
21. (d)	**22.** (a)	**23.** (a)	**24.** (d)
25. (a)	**26.** (d)	**27.** (c)	**28.** (b)
29. (c)	**30.** (d)	**31.** (d)	**32.** (b)
33. (c)	**34.** (b)	**35.** (a)	**36.** (c)
37. (b)	**38.** (c)	**39.** (a)	**40.** (c)
41. (a)	**42.** (b)		

Previous Years' B-Architecture Entrance Examination Questions

1. (d)	**2.** (c)	**3.** (b)	**4.** (a)
5. (c)	**6.** (c)	**7.** (a)	**8.** (c)
9. (c)	**10.** (b)	**11.** (c)	

Hints and Solutions

Concept-based

1. $(x-1)^2 + 1 + |x-1| \geq 1 \ \forall \ x \in \mathbf{R}$
2. $1 = \alpha\left(\dfrac{1}{\alpha}\right) = \dfrac{k}{7}$
3. $\alpha - \beta = -1 = \Rightarrow (\alpha+\beta)^2 - 4\alpha\beta = 1$ or $p^2 - 4q = 1$
4. If α is a common root, then $\alpha^2 - a\alpha + b = 0$, $\alpha^2 + b\alpha - a = 0$

   Subtracting, we get $-(a+b)\ \alpha + b + a = 0$

   $\Rightarrow \alpha = 1$. Thus, $1 - a + b = 0 \Rightarrow a - b = 1$
5. $x^2 + x + a = 0 \Rightarrow \left(x + \dfrac{1}{2}\right)^2 = \dfrac{1}{4} - a$

   We must have $\dfrac{1}{4} - a \geq 0 \Rightarrow a \leq \dfrac{1}{4}$.

   Roots of the equation are $-\dfrac{1}{2} \pm \sqrt{\dfrac{1}{4} - a}$

   These will exceed a, if $-\dfrac{1}{2} - \sqrt{\dfrac{1}{4} - a} > a$

   $\Rightarrow -\sqrt{\dfrac{1}{4} - a} > a + \dfrac{1}{2}$

   This is possible if $a + 1/2 < 0$ and $\sqrt{\dfrac{1}{4} - a} < \left(a + \dfrac{1}{2}\right)$

   $\Rightarrow \dfrac{1}{4} - a < a^2 + a + \dfrac{1}{4}$

   $\Rightarrow a^2 + 2a > 0 \Rightarrow a < -2$ as $a < 0$.
6. $(|x| - 2)^2 = 6 \Rightarrow |x| = 2 \pm \sqrt{6}$.

   As $|x| \geq 0$, $|x| = 2 + \sqrt{6} \Rightarrow x = \pm(2 + \sqrt{6})$
7. $10 = (\sqrt{10})^2 = |\alpha - \beta|^2 = (\alpha+\beta)^2 - 4\alpha\beta = p^2 - 3p$

   $\Rightarrow p^2 - 3p - 10 = 0 \Rightarrow (p-5)(p+2) = 0$

   $\Rightarrow p = -2, 5$
8. For the equation to be defined, $x \geq 0$, $5 + x \geq 0$. For $x \geq 0$, $x + 5 \geq 5 \Rightarrow \sqrt{x} + \sqrt{x+5} \geq \sqrt{5} > 2$
9. $D = 4(a+1)^2 - 4(a^2 - 4a + 3) \geq 0$,

   $2(a+1) < 0$ and $a^2 - 4a + 3 > 0$

   First two imply $a \geq 1/3$, $a < -1$. Not possible.
10. $\tan\alpha + \tan\beta = -a$, $\tan\alpha\tan\beta = b$.

    $1 = \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{a}{1-b}$

    Also, $0 < \alpha < \pi/4$, $0 < \beta < \pi/4$

$\Rightarrow 0 < \tan \alpha < 1, 0 < \tan \beta < 1$

$\Rightarrow 0 < \tan \alpha \tan \beta < 1 \Rightarrow 0 < b < 1.$

11. Let $f(x) = (x - a)(x - b) - c$
Use coefficient of $x^2 > 0$,
$f(a) = - c = f(b) < 0.$

12. $D \geq 0$ and product of roots ≤ 0.
$\Rightarrow (k + 1)^2 - (9k - 5) \geq 0$ and $9k - 5 \leq 0$
$\Rightarrow k^2 - 7k + 6 \geq 0$ and $\leq 5/9$
$\Rightarrow (k - 1)(k - 6) \geq 0$ and $k \leq 5/9$
Thus, $k \leq 5/9$

13. Use : a and $a(k - \alpha)(k - \beta)$ have the opposite signs, where α, β are roots of
$ax^2 + bx + c = 0.$

14. Write the equation as
$(x - 2a)^2 = 4a^2 - (2a^2 - 3a + 5)$
$= (2a + 5)(a - 1)$
$\Rightarrow x = 2a \pm \sqrt{(2a+5)(a-1)}$
We must have $(2a + 5)(a - 1) \geq 0$
and $2a + \sqrt{(2a+5)(a-1)} < 2$
$\Rightarrow a \leq - 5/2$ or $a \geq 1$
and $\sqrt{(2a+5)(a-1)} < -2(a - 1)$
Clearly, $a \geq 1$ is not possible
For $a \leq - 5/2$, we must have
$\sqrt{-(2a+5)(1-a)} < 2(1 - a)$
$\Rightarrow -(2a + 5) < 4(1 - a) \Rightarrow a < 9/2$

15. As parabola open upwards, $a > 0$.
Also, for $x = 0$ $ax^2 + bx + c = c < 0$.
Since the equation has distinct roots, $b^2 - 4ac > 0$. However, we cannot be sure of sign of b.
If can be positive, negative or zero. For instance, consider $x^2 + 2x - 3 = 0$, $x^2 - 2x - 3 = 0$ and $x^2 - 3 = 0$.

16. $9x^2 - 6x + 5 = (3x - 1)^2 + 4$ attains least value when $x = 1/3$.

17. $ax^2 + bx + c = 0$ has real roots if
$b^2 - 4ac \geq 0$, i.e. if $b^2 \geq 4ac$.
Minimum possible value of b is 2.
For $b = 2$, $(a, c) = (1, 1)$
For $b = 3$, $ac \leq 9/4 \Rightarrow (a, c) = (1, 1), (1, 2)$ or $(2, 1)$
For $b = 4$, $ac \leq 4 \Rightarrow (a, c) = (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 2)$
Thus, there $4^3 - 12 = 52$ such quadratic equations.

18. LHS $> 0\ \forall\ x \in \mathbf{R}$.

19. As the roots are imaginary $b^2 - 4ac < 0$ and
$$\alpha, \beta = \frac{1}{2a}\left(-b \pm i\sqrt{4ac - b^2}\right)$$
$$|\alpha| = \frac{1}{2a}\sqrt{b^2 + 4ac - b^2} = \sqrt{\frac{c}{a}} = |\beta|$$

20. Suppose $\alpha, \beta > 1, \alpha < \beta$ be roots of $ax^2 + bx + c = 0$, then
$ax^2 + bx + c > 0$ if $x < \alpha$ or $x > \beta$
< 0 if $\alpha < x < \beta$
Thus, $a + b + c > 0$.

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21. $a > 0, c > 0, 4b^2 - 4ac < 0$. However nothing can be said about b. For instance, consider parabolas $y = x^2 - x + 1$, $y = x^2 + x + 1$ or $y = x^2 + 1$

22. If α is a common root, then $a\alpha^2 + 2b\alpha + c = 0$, $a\alpha^2 + 2c\alpha + \beta = 0$
Subtracting, we get $2(b - c)\alpha + c - b = 0$
$\Rightarrow \alpha = \frac{1}{2}$. Thus $\frac{1}{4}a + b + c = 0$
$$\Rightarrow \frac{a+b+c}{a} = \frac{3}{4}$$

23. Let $f(x) = ax^2 + bx + c$. As $f(x) = 0$, does not have real roots, $f(x) > 0\ \forall\ x \in \mathbf{R}$ or $f(x) < 0\ \forall\ x \in \mathbf{R}$. But
$f\left(-\frac{2}{3}\right) = \frac{1}{9}(4a - 6b + 9c) < 0$, therefore
$f(x) < 0\ x\ \forall\ x \in \mathbf{R}. \Rightarrow f(1) < 0$
$\Rightarrow a + b + c < 0.$
$$\Rightarrow \frac{b+c}{a} < -1$$

24. We have $16^2 - 16c > 0 \Rightarrow c < 16$
If $f(x) = 4x^2 - 16x + c$, then $1 < \alpha < 2 < \beta < 3$ implies $f(1) > 0, f(2) < 0, f(3) > 0$.
$\Rightarrow c > 12, c < 16, c > 12$
Thus, $c = 13, 14, 15$.

25. If α is a common root of the three equations, then
$\alpha^2 + a\alpha + 12 = 0$, $\alpha^2 + b\alpha + 15 = 0$
and $\alpha^2 + (a + b)\alpha + 36 = 0$
$\Rightarrow b\alpha = - 24, a\alpha = - 21$
Thus, $\alpha^2 - 21 + 12 = 0 \Rightarrow \alpha = \pm 3$
As $\alpha > 0$, we get $\alpha = 3$.
Therefore, $a = - 7, b = - 8$.

26. $\frac{a}{x} = \frac{2x}{x^2 - b^2} \Rightarrow a(x^2 - b^2) = 2x^2$
$\Rightarrow (a - 2)x^2 - ab^2 = 0$
This equation will have no solution if $a - 2$ and a have opposite signs, that is , $a(a - 2) < 0$
$\Rightarrow 0 < a < 2.$

27. As the roots are real and distinct, $\frac{a^2}{(2a+1)^2} - \frac{4(a-2)}{2a+1} > 0$
$\Rightarrow a^2 - 4(a - 2)(2a + 1) > 0$
$$\Rightarrow \frac{6 - 2\sqrt{23}}{7} < a < \frac{6 + 2\sqrt{23}}{7}$$

Write the equation as

$$x^2 - \frac{a}{2a+1}x + \frac{a-2}{2a+1} = 0$$

Note that its roots are α and β.

As $\alpha < 1 < \beta$, we must have

$$1^2 - \frac{a}{2a+1} + \frac{a-2}{2a+1} < 0$$

$\Rightarrow 2a + 1 - a + a - 2 < 0 \Rightarrow a < 1/2$

Thus, $\frac{1}{7}\left(6 - 2\sqrt{23}\right) < a < \frac{1}{2}$

28. $8x^2 - 10x + 3 = 0 \Rightarrow (2x - 1)(4x - 3) = 0$

$\Rightarrow \alpha, \beta = 1/2. 3/4$

$$\therefore \sum_{n=0}^{\infty}\left(\alpha^n + \beta^n\right) = \frac{1}{1-\alpha} + \frac{1}{1-\beta} = 6$$

29. $b^2 - 4ac = (-a - 2c)^2 - 4ac$

$= a^2 + 4c^2 > 0$

30. Let $y = \frac{x^2 - bc}{2x - (b+c)}$, then

$x^2 - bc = 2xy - (b + c)y$

$\Rightarrow x^2 - 2xy + (b + c)y + - bc = 0$

As x is real,

$y^2 - (b + c)y + bc \ge 0$

$\Rightarrow (y - b)(y - c) \ge 0$

$\Rightarrow y$ cannot lie in (b, c).

31. $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$

$= (-p\alpha - q + r\alpha + s)(-p\beta - q + r\beta + s)$

$= [(r - p)\alpha - (q - s)][(r - p)\beta - (q - s)]$

$$= (r-p)^2\left[\frac{q-s}{r-p} - \alpha\right]\left[\frac{q-s}{r-p} - \beta\right]$$

$$= (r-p)^2\left[\left(\frac{q-s}{r-p}\right)^2 + p\left(\frac{q-s}{r-p}\right) + q\right]$$

$= (q - s)^2 + p(q - s)(r - p) + q(r - p)^2$

32. Use $x^2 \ge 0$, $|x| \ge 0$

33. $x^2 - 3|x| + 2 = 0 \Rightarrow |x| = 1, 2 \Rightarrow x = \pm 1, \pm 2$

34. As $p, q \in \mathbf{R}$, other root of the equation is $2 - i\sqrt{5}$. Thus,

$p = 4$, $q = \left(2 + \sqrt{5}i\right)\left(2 - \sqrt{5}i\right) = 9$

35. Subtract one equation from the other to obtain $x = 1$.

36. Write the expression as

$(ap + b)^2 + (bp + c)^2 + (cp + d)^2 \le 0$.

$\Rightarrow ap + b = 0$, $bp + c = 0$, $cp + d = 0$

$$\Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = -p.$$

37. Use $5^x + 5^{-x} \ge 2 \ \forall\ x \in \mathbf{R}$.

38. Multiply the first equation by x and subtract from the second to obtain $x = 1$.

39. $x + 2 \ge 0 \Rightarrow x \ge -2$

Now, $|(x + 2)(x - 3)| = x + 2$

$|x + 2|\,|x - 3| = x + 2$

$\Rightarrow x + 2 = 0$ or $|x - 3| = 1$

$\Rightarrow x = -2$, $x = 3 \pm 1$ *i.e.*, $x = -2, 2, 4$

40. $|x|^2 - 3|x| + 2 = 0 \Rightarrow |x| = 1, 2 \Rightarrow x = \pm 1, \pm 2$

41. Use $n - 4 > 0$ and $D < 0$

42. Let $f(x) = ax^2 + bx + c$

$f(x) < 0 \ \forall\ x \in \mathbf{R} \Leftrightarrow a < 0,\ b^2 - 4ac < 0.$

Now, $g(x) = ax^2 + bx + c + (2ax + b) + 2a$

$= ax^2 + (2a + b)x + (2a + b + c)$

Here $a < 0$ and $(2a + b)^2 - 4a(2a + b + c)$

$= (b^2 - 4ac) - 4a^2 < 0.$

43. Use the fact that expression becomes zero when $x = -1$.

44. Note that $p > 0$, and

$p - 4 < 0$, $2p^2 - 4 > 0$,

and $(p - 4)^2 - 4(2p^2 - 4) \ge 0$

$\Rightarrow p < 4$, $p > \sqrt{2}$ and $7p^2 + 8p - 32 \le 0$.

$$\Rightarrow \sqrt{2} < p < \frac{4}{7}\left(\sqrt{15} - 1\right)$$

45. $f(x)^2 + g(x)^2 = 0 \Leftrightarrow f(x) = 0$ and $g(x) = 0$

46. Use $D < 0$

47. Show that $D > 0$

48. $ax^2 + bx + c > 0 \ \forall\ x > 0$.

49. $x^2 - 8x + p^2 - 6p = 0$ has real roots

$\Rightarrow 64 - 4(p^2 - 6p) \ge 0$

$\Rightarrow p^2 - 6p - 16 \le 0 \Rightarrow (p - 8)(p + 2) \le 0$

$\Rightarrow p \in [-2, 8]$

50. $-\frac{2a+3}{a+1} = -3 \Rightarrow a = 0.$

$\therefore$ Product of roots $= \frac{3a+4}{a+1} = 4$

51. Use that other root of the equation is $3 + 4i$ to obtain $p = 6$, $q = 25$

52. Let other roots be β, γ then

$(1 + i) + \beta + \gamma = 0$ and $(1 + i)\beta\gamma = -(1 - i)$

$\Rightarrow \beta + \gamma = -(1 + i)$, $\beta\gamma = i$

53. $\alpha + \beta = -p$, $\alpha\beta = -\frac{1}{2p^2}$.

$$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 + \frac{1}{p^2}$$

$$\Rightarrow \alpha^2 + \beta^2 = \left(p - \frac{1}{p}\right)^2 + 2 \ge 2.$$

54. Put $\sqrt{x-1} = t$ so that $x = t^2 + 1$

$$\therefore \sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1$$

$\Rightarrow \sqrt{t^2-4t+4}+\sqrt{t^2-6t+9}=1$

$\Rightarrow |t-2| + |t-3| = 1 \Rightarrow 2 \le t \le 3$

$\Rightarrow 2 \le \sqrt{x-1} \le 3$

$\Rightarrow 4 \le x-1 \le 9 \quad \Rightarrow \quad 5 \le x \le 10.$

55. $|x - x^2 - 1| = |2x - 3 - x^2| \Rightarrow \quad |x^2 - x + 1| = |x^2 - 2x + 3|$

$\Rightarrow \left|\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right| = |(x-1)^2 + 2|$

$\Rightarrow x^2 - x + 1 = x^2 - 2x + 3 \quad \Rightarrow \quad x = 2$

56. $\sin\alpha + \cos\alpha = -\frac{b}{a}$, $\sin\alpha\cos\alpha = \frac{c}{a}$

Now $1 = \sin^2\alpha + \cos^2\alpha$
$= (\sin\alpha + \cos\alpha)^2 - 2\sin\alpha\cos\alpha$

$\Rightarrow \quad 1 = \frac{b^2}{a^2}-\frac{2c}{a} \quad \Rightarrow \quad a^2 = b^2 - 2ac$

$\Rightarrow \quad a^2 - b^2 + 2ac = 0.$

57. $k = \frac{x^2-x+1}{x^2+x+1}$

$\Rightarrow \quad (k-1)x^2 + (k+1)x + (k-1) = 0$
As x is real $(k+1)^2 - 4(k-1)^2 \ge 0$
$\Rightarrow (k + 1 + 2k - 2)(k + 1 - 2k + 2) \ge 0$
$\Rightarrow (3k - 1)(3 - k) \ge 0 \Rightarrow 1/3 \le k \le 3.$

58. $\alpha^2 + b\alpha + ca = 0$ and $\alpha^2 + c\alpha + ab = 0$
$\Rightarrow (b - c)\alpha + a(c - b) = 0 \quad \Rightarrow \quad \alpha = a.$
Now, show that $a + b + c = 0$ and use the fact that other roots are b and c.

59. $mx^2 + 3x + 4 < 5(x^2 + 2x + 2)$
$\Rightarrow (m-5)x^2 - 7x - 6 < 0 \ \forall\, x \in \mathbf{R}$
$\Rightarrow m - 5 < 0$ and $(-7)^2 + 24(m-5) < 0$
$\Rightarrow m < 5$ and $24m - 71 < 0 \Rightarrow m < 71/24$

60. Use the fact that expression becomes equal to 1 for $x = a, b, c$.

61. As $ax^2 + bx + c$ has no real zeros.
$ax^2 + bx + c > 0 \ \forall\, x \in \mathbf{R}$ or $ax^2 + bx + c < 0 \ \forall\, x \in \mathbf{R}$.
Since $a(-1)^2 + b(-1) + c < 0$, we get $ax^2 + bx + c < 0 \ \forall\, x \in \mathbf{R}$
Thus, $a < 0$ and $c < 0 \Rightarrow ac > 0.$

62. $x^2 - 2x - p^2 + 1 = 0$
$\Rightarrow (x-1)^2 = p^2$
$\Rightarrow x = 1 \pm p$
Let $\alpha = 1 - p$, $\beta = 1 + p$
Let $f(x) = x^2 - 2(p+1)x + p(p-1)$
As $f(x) = 0$ has distinet real roots,
$(p+1)^2 - p(p-1) > 0$

$\Rightarrow 3p + 1 > 0 \Rightarrow p > -\frac{1}{3}$

Also, $f(1-p) < 0$,
and $f(1+p) < 0$

$\Rightarrow (1-p)^2 - 2(1+p)(1-p) + p(p-1) < 0$
and $(1+p)^2 - 2(1+p)^2 + p(p-1) < 0$

$\Rightarrow -1/4 < p < 1$ and $p > -\frac{1}{3}$

Thus, $p \in \left(-\frac{1}{4}, 1\right)$

63. $\alpha + \beta = -\frac{b}{a}$, $\alpha + \beta + 2h = -\frac{B}{A}$

64. $D = 49 + 56(q^2 + 1) > 0.$

65. Note that α, β are roots of

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

As $\alpha < -1$, $\beta > 1$, we get

$1 - \frac{b}{a} + \frac{c}{a} < 0$

and $1 + \frac{b}{a} + \frac{c}{a} < 0$

$\Rightarrow \quad 1+\left|\frac{b}{a}\right|+\frac{c}{a}<0$

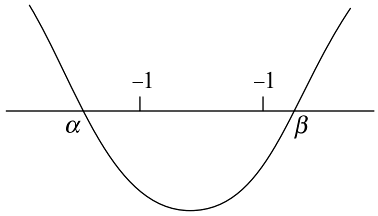


Fig. 3.22

66. Let $f(x) = ax^2 + bx + c$.
Now, $(a + c)^2 < b^2$
$\Leftrightarrow \quad (a - b + c)(a + b + c) < 0$
$\Rightarrow f(-1)\, f(1) < 0.$
Thus, $ax^2 + bx + c = 0$ has a real root between -1 and 1.
The other root also must be real.

67. If x is an integer and
$(x - a)(x - 10) = -1$
$\Rightarrow x - 10 = 1$ and $x - a = -1$ or $x - 10 = -1$ and $x - a = 1$
$\Rightarrow x = 11$ and $a = x + 1 = 12$ or $x = 9$ and $a = x - 1 = 8$

68. $4^{1.5}\left(\frac{2}{3}\right)^x + 9^{0.5}\left(\frac{3}{2}\right)^x = 10$

Put $\left(\frac{3}{2}\right)^x = t$, so that

$3t + 8/t = 10 \quad \Rightarrow \quad 3t^2 - 10t + 8 = 0$

$\Rightarrow (3t - 4)(t - 2) = 0 \quad \Rightarrow \quad t = \frac{4}{3}, 2$

$\Rightarrow \left(\frac{3}{2}\right)^x = \frac{4}{3}, 2$

$\Rightarrow x = \dfrac{\log(4/3)}{\log(3/2)}, \dfrac{\log 2}{\log(3/2)}$

69. Put $2^{\sin^2 x} = t$, so that equation becomes

$t + \dfrac{5\times 2}{t} = 7 \Rightarrow t^2 - 7t + 10 = 0 \quad \Rightarrow \quad t = 2, 5$

But $1 \le t \le 2$. Thus, $\sin^2 x = 1$

70. Let the two distinct roots lying between 0 and 1 be α, β such that $\alpha < \beta$.
As $f(x) = x^2 - 2x + k$ is a differentiable and $f(\alpha) = f(\beta) = 0$, therefore by the Roll's theorem there exists $\gamma \in (\alpha, \beta) \subset (0, 1)$ such that $f'(\gamma) = 0$
$\Rightarrow \quad 2\gamma - 2 = 0 \quad \Rightarrow \quad \gamma = 1$. Not possible.

71. $\dfrac{\alpha}{\alpha-1} + \dfrac{\alpha+1}{\alpha} = -\dfrac{b}{a}$, and $\dfrac{\alpha}{\alpha-1}\cdot\dfrac{\alpha+1}{\alpha} = \dfrac{c}{a}$

Now, $(a + b + c)^2 = a^2\left(1 + \dfrac{b}{a} + \dfrac{c}{a}\right)^2$

$$= a^2\left[1 - \left\{\frac{\alpha}{\alpha-1} + \frac{\alpha+1}{\alpha}\right\} + \frac{\alpha+1}{\alpha-1}\right]^2$$

$$= a^2\left(\frac{\alpha}{\alpha-1} - \frac{\alpha+1}{\alpha}\right)^2$$

$$= a^2\left[\left(\frac{\alpha}{\alpha-1} + \frac{\alpha+1}{\alpha}\right)^2 - 4\frac{\alpha}{\alpha-1}.\frac{\alpha+1}{\alpha}\right]$$

$$= a^2\left[\frac{b^2}{a^2} - \frac{4c}{a}\right] = b^2 - 4ac$$

72. Let $x = 3m + r$, $0 \le r \le 2$.
$x^2 + x + 1 = (9m^2 + 6mr + 3m) + (r^2 + r + 1)$
Note that $x^2 + x + 1$ will be divisible by 3 if and only if $r = 1$.

73. $f(x) = \dfrac{1}{4}(\alpha - \beta)^2 - x^2 - (\alpha + \beta)x + \alpha\beta$

$$= \frac{(\alpha-\beta)^2 + 4\alpha\beta}{4} - \left\{x - \frac{\alpha+\beta}{2}\right\}^2 + \frac{(\alpha+\beta)^2}{4}$$

$$= \frac{1}{2}(\alpha+\beta)^2 - \left\{x + \frac{a}{2}\right\}^2 \le \frac{a^2}{2}$$

Thus, maximum value $f(x)$ is $\dfrac{a^2}{2}$ which is attained when $x = -\dfrac{a}{2}$

74. Let α, β be roots of $f(x) = x^2 + bx + c$, then $0 < \alpha, \beta < 1 \Rightarrow 0 < \alpha + \beta < 2$ and $\alpha\beta < 1$
$\Rightarrow \quad -b < 2$ and $c < 1$
$\Rightarrow \quad b > -2$ and $c < 1$
However, these conditions are not sufficient to guarantee that the roots are real. For instance, $x^2 + 0x + 0.5 = 0$ does not have real roots.

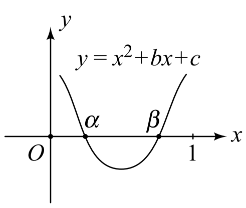


Fig. 3.23

75. Let $f(x) = ax^2 + bx + c$.
Note that $f(0) = c > 0$, $f(-1) = a - b + c < 0$ and $f(1) = a + b + c < 0$
Thus, $y = ax^2 + bx + c$ represents a parabola that open downwards. See Fig.3.21
Thus $[\alpha] = -1$ and $[\beta] = 0$
$\Rightarrow [\alpha] + [\beta] = -1$

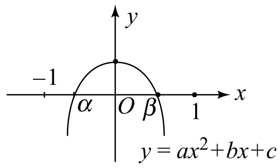


Fig. 3.24

76. $x^2 - 2mx + m^2 - 1 = 0 \Rightarrow (x - m)^2 = 1$
$\Rightarrow x - m = \pm 1 \quad \Rightarrow \quad x = m \pm 1$
Therefore, $-2 < m - 1 < m + 1 < 4$
$\Rightarrow \quad -1 < m < 3$.

77. $y = \sqrt{8 + 2\sqrt{8 + 2\sqrt{8 + 2\sqrt{8 + \cdots}}}}$

$\Rightarrow \quad y^2 = 8 + 2y \quad \Rightarrow \quad y^2 - 2y - 8 = 0$
$\Rightarrow \quad y = 4, -2$. As $y > 0$, $y = 4$

78. $\sin\left(\dfrac{\pi x}{2\sqrt{3}}\right) = \left(x - \sqrt{3}\right)^2 + 1 \ge 1$

But $\sin\left(\dfrac{\pi x}{2\sqrt{3}}\right) \le 1$. Thus, $\sin\left(\dfrac{\pi x}{2\sqrt{3}}\right) = 1$

Therefore, $x = \sqrt{3}$

79. $2(3 - x) = |x + 2| \ge 0 \quad \Rightarrow \quad x \le 3$
If $x \le -2$, the equation becomes
$2(3 - x) = -x - 2 \quad \Rightarrow \quad x = 8$. Not possible
If $-2 < x \le 3$, the equation becomes
$2(3 - x) = x + 2 \quad \Rightarrow \quad 3x = 4 \quad \Rightarrow \quad x = 4/3$.

80. Use product of roots < 0

81. Put $x^2 - 5x = y$;
so that $y^2 - 7y + 6 = 0 \quad \Rightarrow \quad y = 1, 6$
$\therefore\ x^2 - 5x = 1$ or $x^2 - 5x = 6$
$\Rightarrow x^2 - 5x - 1 = 0$ or $x^2 - 5x - 6 = 0$
But $x^2 - 5x - 6 = 0$ has integral roots and roots of $x^2 - 5x - 1 = 0$ are

$x = \dfrac{1}{2}\left(5 \pm \sqrt{29}\right)$

82. Put $\left(\dfrac{x-1}{x+1}\right)^2 = y$ to obtain

$y^2 - 13y + 36 = 0 \quad \Rightarrow \quad y = 4, 9$

$\therefore\ \dfrac{x-1}{x+1} = \pm 2, \pm 3$

Thus, the given equation has four real roots.

83. $9^{x+2} - 6(3^{x+1}) + 1 = 0$
Put $3^{x+1} = t$ so that $t^2 - 6t + 1 = 0$

$\Rightarrow t = \dfrac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$

$\Rightarrow x + 1 = \log_3 (3 + 2\sqrt{2}) > 1$
or $x + 1 = \log_3 (3 - 2\sqrt{2}) < 0$

84. Put $\left(\frac{2x-5}{3x+1}\right)^2 = t$, so that equation becomes
$81t^2 - 45t + 4 = 0$
$\Rightarrow 81t^2 - 36t - 9t + 4 = 0$
$\Rightarrow (9t - 1)(9t - 4) = 0$
$\Rightarrow t = \frac{1}{9}, \frac{4}{9} \Rightarrow \frac{2x-5}{3x+1} = \pm\frac{1}{3}, \pm\frac{2}{3}$
Thus, the given equation has four rational roots.

85. Put $x^2 + 3x + 2 = t$ to obtain $t^2 - 8(t-2) - 4 = 0$
$\Rightarrow t^2 - 8t + 12 = 0 \Rightarrow (t-2)(t-6) = 0$
$\Rightarrow t = 2, 6$
But both $x^2 + 3x + 2 = 2$
and $x^2 + 3x + 2 = 6$ have integral roots.

86. Put $\sqrt{\frac{x}{x-3}} = t$ to obtain $t + \frac{1}{t} = \frac{5}{2}$
$\Rightarrow \quad t = 2, \frac{1}{2}$
$\therefore \sqrt{\frac{x}{x-3}} = 2, \frac{1}{2} \quad \Rightarrow \quad \frac{x}{x-3} = 4, \frac{1}{4}$

87. Put $x + 1/x = t$ to obtain
$4\{t^2 - 4\} + 8t = 29$ or $4t^2 + 8t - 45 = 0$
$\Rightarrow t^2 + 2t = \frac{45}{4} \Rightarrow (t+1)^2 = \frac{45}{4} + 1 = \left(\frac{7}{2}\right)^2$
$\Rightarrow t = -1 \pm \frac{7}{2} = -\frac{9}{2}, \frac{5}{2}$
Thus, $\quad x + \frac{1}{x} = -\frac{9}{2}, \frac{5}{2}$
$\Rightarrow \quad x^2 + \frac{9}{2}x + 1 = 0, \; x^2 - \frac{5}{2}x + 1 = 0$
The first equation has irrational roots and the second equation has rational roots.

88. Divide the given equation by x^2 to obtain
$$2\left(x^2 + \frac{1}{x^2}\right) + 9\left(x + \frac{1}{x}\right) + 8 = 0$$
Put $x + \frac{1}{x} = t$ to get $2(t^2 - 2) + 9t + 8 = 0$
$\Rightarrow t = -\frac{1}{2}, -4$
$\therefore \; x + \frac{1}{x}, = -\frac{1}{2} \; - 4$
$\Rightarrow \quad \frac{x^2+1}{x} = -\frac{1}{2}, \; \frac{x^2+1}{x} = -4$
$\Rightarrow 2x^2 + x + 2 = 0, \quad x^2 + 4x + 1 = 0$
The first equation has non-real complex roots and the second equation has irrational roots. *viz.* $-2 \pm \sqrt{3}$.

89. Put $x - \frac{1}{x} = t$ to obtain $4t^2 - 4t + 1 = 0$
$\Rightarrow \quad t = \frac{1}{2}, \frac{1}{2}$
Now, $x - \frac{1}{x} = \frac{1}{2} \quad \Rightarrow \quad \frac{x^2-1}{x} = \frac{1}{2}$
$\Rightarrow \quad 2x^2 - x - 2 = 0$
$\therefore$ Sum of roots of the given equation $= \frac{1}{2} + \frac{1}{2} = 1$.

90. $(3x-2)(x-1)(3x+1)(x-2) = 21$
$\Rightarrow (3x^2 - 5x + 2)(3x^2 - 5x - 2) = 21$
$\Rightarrow (3x^2 - 5x)^2 - 4 = 21 \quad \Rightarrow \quad 3x^2 - 5x = \pm 5$
$\Rightarrow 3x^2 - 5x - 5 = 0$ or $3x^2 - 5x + 5 = 0$
The first equation has irrational roots and the second has imaginary roots.

91. Note that $x \geq 2$ and
$\Rightarrow x - 2 = \sqrt{3x-6} \quad \Rightarrow \quad x^2 - 4x + 4 = 3x - 6$
$\Rightarrow x^2 - 7x + 10 = 0 \quad \Rightarrow \quad x = 2, 5$
Both of these satisfy the given equation.

92. Note that $x \geq -\frac{1}{2}$. Put $\sqrt{2x+1} = t$, so that
$2t = t^2 - 1 - 1$ or $t^2 - 2t + 1 = 3$
$\Rightarrow t = 1 \pm \sqrt{3}$.
But $t \geq 0$, therefore $t = 1 + \sqrt{3}$
$\Rightarrow 2x + 1 = 1 + 2 + 2\sqrt{3} \quad \Rightarrow \quad x = \frac{1}{2}(3 + 2\sqrt{3})$

93. Note that $-\sqrt{13} \leq x \leq \sqrt{13}$ and
$13 - x^2 = x^2 + 10x + 25$
$\Rightarrow x^2 + 5x + 6 = 0 \Rightarrow x = -2, -3$.

94. $\sqrt{x^2-4} - (x-2) = \sqrt{x^2 - 5x + 6} \qquad (1)$
is defined for $x = 2$ or $x \leq -2$ or $x \geq 3$. Write (1) as
$\sqrt{x-2}\left[\sqrt{x+2} - \sqrt{x-2}\right] = \sqrt{x-2}\sqrt{x-3}$
$\Rightarrow x = 2$ or $\sqrt{x+2} - \sqrt{x-2} = \sqrt{x-3}$
$\Rightarrow x + 2 + x - 2 - 2\sqrt{x^2-4} = x - 3$
$\Rightarrow x + 3 = 2\sqrt{x^2-4} \Rightarrow x^2 + 6x + 9 = 4(x^2 - 4)$
$\Rightarrow 3x^2 - 6x - 25 = 0$
$\Rightarrow \quad (x-1)^2 = \frac{28}{3} \Rightarrow x = 1 \pm \sqrt{\frac{28}{3}}$
Note that $1 - \sqrt{\frac{28}{3}} < -2$ and $1 + \sqrt{\frac{28}{3}} > 3$

95. $\sqrt{x^2 - 4x + 3} + \sqrt{x^2 - 7x + 12} = 3\sqrt{x-3} \qquad (1)$
is defined for $x = 3$ or $x \geq 4$.

Write (1) as

$\sqrt{x-3}\left[\sqrt{x-1}+\sqrt{x-4}\right] = 3\sqrt{x-3}$

$\Rightarrow x = 3$ or $\sqrt{x-1}+\sqrt{x-4} = 3$

$\Rightarrow x = 3$ or $\sqrt{x-1} = 3-\sqrt{x-4}$

$\Rightarrow x = 3$ or $x - 1 = 9-6\sqrt{x-4}+x-4$

$\Rightarrow x = 3$ or $6\sqrt{x-4}=6 \Rightarrow x = 3$ or $x = 5$.

Thus, product of roots = 15.

96. Note that $2b^2 + 77b + 18 = 0$

$\Rightarrow \quad 2 + 77/b + 18(1/b)^2 = 0$

Thus, a and $1/b$ are roots of the equation

$18x^2 + 77x + 2 = 0$

$\therefore \quad a + 1/b = -77/18$ and $\frac{a}{b}=\frac{2}{18}=\frac{1}{9}$

Now, $\frac{ab+a+1}{b} = a+\frac{1}{b}+\frac{a}{b}=-\frac{77}{18}+\frac{1}{9}=-\frac{25}{6}$

97. $\beta^2 = 7\beta - 8$.

Thus, $\frac{16}{\alpha}+3\beta^2-19\beta=\frac{16}{\alpha}+3(7\beta-8)-19\beta$

$= \frac{16}{\alpha}+2\beta-24$

Also, $\alpha = \frac{7+\sqrt{17}}{2}$, $\beta = \frac{7-\sqrt{17}}{2}$

$\therefore \quad \frac{1}{\alpha} = \frac{2}{7+\sqrt{17}}=\frac{2\left(7-\sqrt{17}\right)}{49-17}=\frac{7-\sqrt{17}}{16}$

Thus $\frac{16}{\alpha}+3\beta^2-19\beta$

$= \left(7-\sqrt{17}\right)+\left(7-\sqrt{17}\right)-24 = -10$

98. As α is a root of $x^4 + x^2 - 1 = 0$, we get

$\alpha^4 + \alpha^2 - 1 = 0 \Rightarrow \alpha^6 + \alpha^4 - \alpha^2 = 0$

$\Rightarrow \quad \alpha^6 + \alpha^4 - (1 - \alpha^4) = 0 \Rightarrow \alpha^6 + 2\alpha^2 = 1$

$\therefore \quad (\alpha^6 + 2\alpha^2)^{2012} = 1$

99. Use the fact that sum and product of the roots of $x^2 - x - k = 0$ are 1 and $-k$ respectively.

100. Suppose $a^2(a + k) = b^2(b + k) = c^2(c + k) = t$, then a, b, c are roots of $x^3 + kx^2 - t = 0$ thus, $bc + ca + ab = 0$ and $abc = t$

$\Rightarrow \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{bc+ca+ab}{abc}=0$

101 For truth of Statement-2, see theory

$f(\alpha) = a^2\alpha^2 + b\alpha + c = 0$ and

$g(\beta) = a^2\beta^2 - b\beta - c = 0$

We have $a^2\alpha^2 + 2b\alpha + 2c$

$= a^2\alpha^2 - 2a^2\alpha^2 < 0$

and $a^2\beta^2 + 2b\beta + 2c$

$= a^2\beta^2 + 2a^2\beta^2 > 0$

As $a^2x^2 + 2bx + 2c$ is continuous on $[\alpha, \beta]$, by statement-2 there exists γ such that $a^2\gamma^2 + 2b\gamma + 2c = 0$

102. As $y = f(x)$ open upwards, a lies between the roots of $f(x) = 0$, if and only if $f(a) < 0$. Therefore, statement-2 is true.

If $f(x) < 0$ for $1 \le x \le 2$, then 1 and 2 lie between roots of $f(x) = 0$. Thus, we must have

$4p^2 - 4p > 0 \Rightarrow p(p - 1) > 0$

$\Rightarrow p < 0$ or $p > 1$.

Also, $f(1) = 1 + 2p + p < 0 \Rightarrow p < -1/3$ and

$f(2) = 4 + 4p + p < 0 \Rightarrow p < -4/5$.

$\therefore p \in (-\infty, -4/5)$

103. Statement-2 is false as $x^2 + 5x + 4 = 0$ meets both the conditions but does not have positive roots.

As LHS ≥ 0, $1 - 2^x \ge 0 \Rightarrow 2^x \le 1 \Rightarrow x \le 0$.

Squaring, we get

$a(2^x - 2) + 1 = 1 - 2(2^x) + 2^{2x}$

$\Rightarrow y^2 - (a + 2)y + 2a = 0$

$\Rightarrow (y - a)(y - 2) = 0 \Rightarrow y = 2, a$.

$y = 2^x = 2 \Rightarrow x = 1$. Not possible.

Thus, $y = a$. This implies

$0 < 2^x = a \le 1$

Therefore $0 < a \le 1$, and $x = \log_2 a$

104. Put $t = x + \sqrt{x^2+b^2}$

$\Rightarrow \frac{1}{t}=\frac{1}{\sqrt{x^2+b^2}+x}=\frac{\sqrt{x^2+b^2}-x}{b^2}$

$\therefore t-\frac{b^2}{t} = 2x$

Thus,

$f(x) = (2a - t + \frac{b^2}{t})t = b^2 + 2at - t^2$

$= a^2 + b^2 - (t - a)^2$

Therefore, maximum value of $f(x)$ is $a^2 + b^2$ which is attained when $t = a$ or $x = \frac{1}{2}\left(a-\frac{b^2}{a}\right)$

Statement-2 is false as maximum value of $f(x) = ax^2 + bx + c$, $a < 0$ is $(4ac - b^2)/4a$.

105. For truth of statement-2 see theory

Let $P(x) = ax^2 + bx + c$.

As $P(x) = x$ does not have real roots, $P(x) > x \ \forall\ x \in$ **R** or $P(x) < x \ \forall\ x \in$ **R**.

Suppose $P(x) < x \ \forall \ x \in \mathbf{R}$.

$\Rightarrow P(P(x)) < P(x) < x \ \not\forall \ x \in \mathbf{R}$

$\Rightarrow a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c - x = 0$

cannot have real roots.

106. Statement-2 is true as $|a + b| = |a| + |b|$

$\Leftrightarrow a, b$ are both non-negative or both non-positive.

As $ax^2 + bx + c = 0$ has imaginary roots, $ax^2 + bx + c > 0 \ \forall \ x \in \mathbf{R}$ or $ax^2 + bx + c < 0 \ \forall \ x \in \mathbf{R}$.

Since $a(-1)^2 + b(-1) + c = a - b + c < 0$,

we get $ax^2 + bx + c < 0 \ \forall \ x \in \mathbf{R}$.

Now, $|ax^2 + bx + c| + |x + y|$

$$= \left|a\left(x^2 + \frac{y}{a}\right) + (b+1)\,x + c\right|$$

$= |(ax^2 + bx + c) + (x + y)|$

$\Leftrightarrow (ax^2 + bx + c)\,(x + y) \geq 0$

$\Leftrightarrow x + y \leq 0$ as $ax^2 + bx + c < 0 \ \forall \ x \in \mathbf{R}$.

107. Statement-2 is true. See Theory

Put $x + c = t$, so that

$$f(x) = \frac{[t + (a-c)]\,[t + (b-c)]}{t}$$

$$= t + \frac{(a-c)\,(b-c)}{t} + [(a-c) + (b-c)]$$

$$= \left[\sqrt{t} - \frac{\sqrt{(a-c)\,(b-c)}}{\sqrt{t}}\right]^2 + \left[\sqrt{a-c} + \sqrt{b-c}\right]^2$$

Thus, minimum value of $f(x)$ is $\left(\sqrt{a-c} + \sqrt{b-c}\right)^2$

and it is attained when $t = \sqrt{(a-c)\,(b-c)}$ or

$x = -c + \sqrt{(a-c)\,(b-c)} > -c$.

108. If $a \neq p$, $f(x) - g(x) = 0$ is a quadratic equation whose coefficient of $x^2 \neq 0$.

$\therefore f(x) - g(x) = 0$ has two real roots

Thus, statement-2 is true

Let $h(x) = f(x) - g(x)$

$\qquad = (a - b)\,x^2 + (b - q)\,x + (c - r)$

If $h(x) = f(x) - g(x) = 0$

for three distinct real values of x, then $a - p = 0$, $b - q = 0$, $c - r = 0$

$\Rightarrow a = p, b = q, c = r$.

$\therefore$ Statement-1 is also true.

However, statement-2 is not a correct explanation for Statement-1.

109. As A.M. $\geq$ G.M, we get

$$\frac{1}{2}\left(y + \frac{1}{y}\right) \geq \sqrt{y\left(\frac{1}{y}\right)} = 1$$

$$\Rightarrow y + \frac{1}{y} \geq 2.$$

Thus, statement-2 is true.

Since, $\sin^2 x = 0$,

does not satisty $\sin^4 x + p \sin^2 x + 1 = 0$, we get

$0 < \sin^2 x \leq 1$, for all x satisfying $\sin^4 x + p \sin^2 x + 1 = 0$

For such a value of $\sin x$,

$$-p = \sin^2 x + \frac{1}{\sin^2 x} \geq 2$$

$\Rightarrow p \leq -2$ or $p \in (-\infty, -2]$

Let $f(t) = t^2 + pt + 1$

As $f(0) = 1 > 0$ and $f(1) = 1 - p + 1 \leq 0$,

we get there is at least one value of $t \in (0, 1]$ such that $f(t) = 0$.

110. As $b^2 - 4ac < 0$, $f(x) > 0 \ \forall \ x \in \mathbf{R}$ or $f(x) < 0 \ \forall \ x \in \mathbf{R}$.

As $f(0) = c > 0$, $f(x) > 0 \ \forall \ x \in \mathbf{R}$.

Thus, statement-2 is true.

As $f(x) = 0$ does not have real roots, $b^2 - 4ac < 0$. By the statement-2, $f(x) > 0 \ \forall \ x \in \mathbf{R}$.

$\Rightarrow f(-1) = a - b + c > 0 \Rightarrow a > b - c$.

Also $b^2 - 4ac < 0$

$\Rightarrow a > b^2/4c$

Hence, $a > \max \left\{\dfrac{b^2}{4c}, b - c\right\}$.

Level 2

111. As A.M $\geq$ G.M, we get

$$\sqrt{x^2 + x} + \frac{\tan^2 \alpha}{\sqrt{x^2 + x}} \geq 2\left[\sqrt{x^2 + x}\,\frac{\tan^2 \alpha}{\sqrt{x^2 + x}}\right]^{1/2}$$

$= 2 \tan \alpha$

112. $(a - b)^2 + 4(a + b - 1) \geq 0 \ \forall \ b \in \mathbf{R}$.

$\Rightarrow b^2 - 2b(a - 2) + a^2 + 4a - 4 \geq 0 \ \forall \ b \in \mathbf{R}$.

$\Rightarrow 4(a - 2)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow -8a + 8 < 0 \Rightarrow a > 1$

$\therefore a \in (1, \infty)$

113. $\alpha + \beta = -b < 0$ and $\alpha\beta = c < 0$.

As $\alpha\beta < 0$, and $\alpha < \beta$,

$\alpha < 0 < \beta < -\alpha = |\alpha|$

114. As $x^2 + (a + b)\,x + c = 0$ has no real roots,

and the coefficient of $x^2 = 1 > 0$,

$f(x) = x^2 + (a + b)\,x + c > 0 \ \forall \ x \in \mathbf{R}$.

$\Rightarrow (a + b)^2 - 4c < 0$.

Also $f(0) = c > 0$, $f(1) = a + b + c + 1 > 0$

$\Rightarrow c\,(a + b + c) + c > 0$

Next, $f(0)\,f(-1) > 0$

$\Rightarrow c\,(1 - a - b + c) > 0$

$\Rightarrow c - c(a + b - c) > 0$

115. As $x - 1 < [x] \leq x$, we get $-x \leq -[x] < -x + 1$

$= x^2 - 2x - 2 \leq x^2 - 2[x] - 2 < x^2 - 2x$

$= (x - 1)^2 - 3 \leq x^2 - 2[x] - 2 < (x - 1)^2 - 1$

Note that $(x-1)^2 - 3 \le 0$ for $-\sqrt{3}+1 \le x \le \sqrt{3}+1$
and $(x-1)^2 - 1 > 0$ for $x < 0$ or $x > 2$,
The roots of $x^2 - 2[x] - 2 = 0$
lie in $\left(-\sqrt{3}+1, 0\right)$ or $(2, 1+\sqrt{3})$

If $x \in \left(-\sqrt{3}+1, 0\right)$, $[x] = -1$,

and $x^2 - 2[x] - 2 = 0 \Rightarrow x^2 = 0$ or $x = 0$.
This is not possible.
Suppose $x \in (2, 1+\sqrt{3})$, then $[x] = 2$,
$x^2 - 2[x] - 2 = 0 \Rightarrow x^2 = 6$ or $x = \sqrt{6}$.
Note that $x = \sqrt{6}$ satisfies $x^2 - 2[x] - 2 = 0$

116. If $[x]$ is odd, then
$[x]^2 + a[x] + b$ is odd and if $[x]$ is even, then $[x]^2 + a[x] + b$ is also odd
Thus, $[x]^2 + a[x] + b$ can never take value 0.

117. Let $a = \sin\alpha$, $b = \sin\beta$, $c = \sin\gamma$.
Note that a, b, c are distinct and $0 < a, b, c < 1$.
We can write the given equation as
$f(x) = (x-b)(x-c) + (x-c)(x-a)$
$\qquad + (x-a)(x-b) = 0$.
Assume that $a < b < c$.
Note that $f(a) = (a-b)(a-c) > 0$
$\qquad f(b) = (b-c)(b-a) < 0$
and $\qquad f(c) = (c-a)(c-b) > 0$.
Thus, $f(x) = 0$ has a root in (a, b) and a root in (b, c).

118. Let $f(x) = x^2 + (2\alpha + a)x + \alpha^2 + a\alpha + b$.
$f(0) = \alpha^2 + a\alpha + b = 0$
Let other root of $f(x) = 0$ be r, then
$0 + r = -(2\alpha + a) = -2\alpha + \alpha + \beta = \beta - \alpha$.
Thus, two roots are 0 and $\beta - \alpha$.

119. As roots are real
$(a+b+c)^2 - 3\lambda(ab+bc+ca) \ge 0$
$= 3\lambda \le \dfrac{(a+b+c)^2}{ab+bc+ca}$
Since, a, b, c are sides of triangle.
$b + c - a > 0$, $c + a - b > 0$, $a + b - c > 0$
$\Rightarrow 2(ab+bc+ca) > a^2 + b^2 + c^2$
$\Rightarrow (a+b+c)^2 < 4(ab+bc+ca)$
Thus, $3\lambda < 4 \Rightarrow \lambda \in (-\infty, 4/3)$.

120. $\tan\theta + \cot\theta = 2a$ and $\tan\theta\cot\theta = b$
Now, $2a = \tan\theta + \cot\theta \ge 2$ if $\tan\theta > 0$
and $2a = \tan\theta + \cot\theta \le -2$ if $\tan\theta < 0$.
$\Rightarrow 2|a| \ge 2 \Rightarrow |a| \ge 1$
Thus, least value of $|a|$ is 1.

121. Let $f(x) = x^3 + x^2 - 5x - 1$, then
$f(-3) < 0, f(-2) > 0, f(-1) > 0, f(0) < 0,$
$f(1) < 0, f(2) > 0$.
Thus, a root of $f(x) = 0$ lies in each of the three intervals:
$(-3, -2), (-1, 0), (1, 2)$.
$\therefore\ [\alpha] + [\beta] + [\gamma] = -3 + (-1) + 1 = -3$

122. $\tan A \tan B = p$, $\tan A \tan B = q$.
As, $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \dfrac{p}{1-q}$, we get

$\sin^2(A+B) = \dfrac{\tan^2(A+B)}{1+\tan^2(A+B)}$

$= \dfrac{p^2/(1-q)^2}{1+p^2/(1-q)^2} = \dfrac{p^2}{(1-q)^2+p^2}$

123. Put $\sqrt{x-1} = t \Rightarrow x - 1 = t^2$, so that equation becomes;
$\sqrt{t^2+1+3+4t} + \sqrt{t^2+1+8+6t} = 1$
$\Rightarrow |t+2| + |t+3| = 1$
Not possible as $t \ge 0$.
Thus, the equation has no solution.

124. Put $\sqrt{x^2+11} = t \Rightarrow x^2 + 11 = t^2$, so that equation becomes

$\sqrt{t^2+t-11} + \sqrt{t^2-t-11} = 4$.
Since $(t^2+t-11) - (t^2-t-11) \equiv 2t$,
we get
$\sqrt{t^2+t-11} - \sqrt{t^2-t-11} = \dfrac{t}{2}$

$\therefore\ 2\sqrt{t^2+t-11} = 4 + \dfrac{t}{2}$

$\Rightarrow 4(t^2+t-11) = 16 + \dfrac{1}{4}t^2 + 4t \Rightarrow t^2 = 16$

$\Rightarrow x^2 + 11 = 16 \Rightarrow x = \pm\sqrt{5}$

125. Note that a, b, c are roots of
$\dfrac{x}{t} + \dfrac{y}{t-p} + \dfrac{z}{t-q} = 1$
$\Leftrightarrow\ t^3 - (x+y+z+p+q)t^2$
$+[(p+q)x + qy + pz]t - pqx = 0$.
$\therefore$ product of roots $= pqx$

$\Rightarrow abc = pqx \Rightarrow x = \dfrac{abc}{pq}$

126. Putting $x = 0, 1, 1/2$, we get
$-1 \le c \le 1$, $-1 \le a + b + c \le 1$,
$-1 \le \dfrac{1}{4}a + \dfrac{1}{2}b + c \le 1$,

$\Rightarrow -4 \le -a - 2b - 4c \le 4$
As $-4 \le 4a + 4b + 4c \le 4$, we get
$-8 \le 3a + 2b \le 8$.
Also, $-8 \le a + 2b \le 8$
$\Rightarrow -16 \le 2a \le 16 \Rightarrow |a| \le 8$

127. $p + q$ and $p + r$ are roots of
$at^2 + 2bp\,(t - p) + c = 0$
or $at^2 + 2bpt + c - 2bp^2 = 0$.
We have
$|q - r|^2 = |(p + q) - (p + r)|^2$
$= [(p + q) + (p + r)]^2 - 4(p + q)(p + r)$
$= \left(\dfrac{-2bp}{a}\right)^2 - \dfrac{4\left(c - 2bp^2\right)}{a}$
$= \dfrac{4}{a^2}\left[b^2p^2 - ac + 2abp^2\right]$
$= \dfrac{4}{|a|^2}\,[(2a + b)\,bp^2 - ac]$
$\Rightarrow |q - r|$
$= \dfrac{2}{|a|}\sqrt{(2a+b)bp^2 - ac}$

128. Put $x^2 = y$. The given equation will have four real roots if
$f(y) = ay^2 + by + c = 0$
has two non-negative roots.
This is possible if
$-\dfrac{b}{a} \ge 0$, $af(0) \ge 0$, $b^2 - 4ac \ge 0$
$\Rightarrow ab \le 0$, $ac \ge 0$.
This condition is met if $a > 0$, $b < 0$, $c > 0$ or $a < 0$, $b > 0$, $c < 0$

129. As $x = 1$ satisfies
$(b - c)\,x^2 + (c - a)\,x + (a - b) = 0$,
$x = 1$ must satisfy $x^2 + mx + 1 = 0$
$\Rightarrow 1 + m + 1 = 0 \Rightarrow m = -2$.

130. Eliminating x from $x + y + z = 4$,
and $x^2 + y^2 + z^2 = 6$, we get
$y^2 + z^2 + (4 - y - z)^2 = 6$
$\Rightarrow 2y^2 - 2(4 - z)y + (4 - z)^2 + z^2 - 6 = 0$
As y is real, we get
$(4 - z)^2 - 2[(4 - z)^2 + z^2 - 6] \ge 0$
$\Rightarrow 3z^2 - 8z + 4 \le 0$
$\Rightarrow (3z - 2)\,(z - 2) \le 0$
$\Rightarrow 2/3 \le z \le 2$.
Thus, maximum value of z is 2.

Previous Years' AIEEE/JEE Main Questions

1. α, β are root of $x^2 = 5x - 3$ or $x^2 - 5x + 3 = 0$
$\Rightarrow \alpha + \beta = 5,\ \alpha\beta = 3.$
Now, $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$
$= \dfrac{25 - 6}{3} = \dfrac{19}{3}$
and $\left(\dfrac{\alpha}{\beta}\right)\left(\dfrac{\beta}{\alpha}\right) = 1$
Thus, equation whose roots are α/β and β/α is
$x^2 - (19/3)x + 1 = 0$
or $3x^2 - 19x + 3 = 0$

2. $|\alpha - \beta| = |\alpha_1 - \beta_1|$
$\Rightarrow (\alpha - \beta)^2 = (\alpha_1 - \beta_1)^2$
$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = (\alpha_1 + \beta_1)^2 - 4\alpha_1\beta_1$
$\Rightarrow a^2 - 4b = b^2 - 4a$
$\Rightarrow (a - b)(a + b + 4) = 0 \Rightarrow a + b + 4 = 0$
$[\because a \ne b]$

3. Let $f(x) = \dfrac{1}{3}ax^3 + \dfrac{1}{2}bx^2 + cx.$
As f is a polynomial f is continuous on [0, 1] and f is differentiable on (0, 1). Also
$$f(0) = 0 \text{ and } f(1) = \frac{1}{6}(2a + 3b + 6c) = 0$$
By the Rolle's theorem there exist some $\alpha \in (0, 1)$ such that $f'(\alpha) = 0$.
Thus, $ax^2 + bx + c = 0$ has at least one root in [0, 1].

4. For each $x \in \mathbf{R}$, $t \in \mathbf{R}$
$t^2x^2 + |x| + 9 \ge 9 > 0$
Thus, $t^2x^2 + |x| + 9 = 0$ does not have real roots.

5. Let α and 2α be the roots of the given equation, then
$(a^2 - 5a + 3)\alpha^2 + (3a - 1)\alpha + 2 = 0 \qquad (1)$
$(a^2 - 5a + 3)(4\alpha^2) + (3a - 1)(2\alpha) + 2 = 0 \qquad (2)$
Multiplying (1) by 4 and subtracting it form (2) we get
$(3a - 1)\,(2\alpha) + 6 = 0$

Clearly $\alpha \neq 1/3$. Therefore, $\alpha = -3/(3a-1)$

Putting this value in (1) we get

$$(a^2 - 5a + 3)(9) - (3a-1)^2(3) + 2(3a-1)^2 = 0$$

$$\Rightarrow \quad 9a^2 - 45a + 27 - (9a^2 - 6a + 1) = 0$$

$$\Rightarrow \quad -39a + 26 = 0$$

$$\Rightarrow \quad a = 2/3.$$

For $a = 2/3$, the equation becomes $x^2 + 9x + 18 = 0$, whose roots are –3, –6.

6. Let α, β be the roots of $ax^2 + bx + c = 0$

We are given

$$\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

$$= \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$\Rightarrow \quad \frac{-b}{a} = \frac{(-b/a)^2 - 2(c/a)}{(c/a)^2}$$

$$\Rightarrow \quad -\frac{b}{a}\cdot\frac{c^2}{a^2} = \frac{b^2}{a^2} - \frac{2c}{a}$$

$$\Rightarrow \quad 2a^2c = ab^2 + bc^2$$

Divide by abc to obtain

$$2\frac{a}{b} = \frac{c}{a} + \frac{b}{c}$$

$\Rightarrow$ $\frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in A.P.

$\Rightarrow$ $\frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in H.P.

7. $$x^2 - 3|x| + 2 = 0$$

$$\Rightarrow \quad |x|^2 - 3|x| + 2 = 0$$

$$\Rightarrow \quad (|x| - 1)(|x| - 2) = 0$$

$$\Rightarrow \quad |x| = 1, 2$$

$$\Rightarrow \quad x = \pm 1, \pm 2$$

8. Let two numbers be α and β. We are given

$$\frac{a+\beta}{2} = 9, \ \sqrt{a\beta} = 4$$

$\Rightarrow$ $\alpha + \beta = 18$ and $\alpha\beta = 16$

$\therefore$ Required equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

or $$x^2 - 18x + 16 = 0$$

9. Let other root be α. Then

$\alpha + (1-p) = -p$ and $\alpha(1-p) = 1-p$

$\Rightarrow$ $\alpha = -1$ and $-(1-p) = 1-p$

$\Rightarrow$ $\alpha = 1$ and $1 - p = 0$

Thus, equation is $x^2 + x = 0$

$\Rightarrow$ $x = 0, -1$

10. Let α be the other root of $x^2 + px + 12 = 0$

$\therefore$ $(\alpha)(4) = 12$ and $\alpha + 4 = p$

$\Rightarrow$ $\alpha = 3$ and $3 + 4 = p$

Thus, $p = 7$,

As $x^2 + px + q = 0$ has equal roots, we must have $p^2 - 4q = 0$

$$\Rightarrow \quad q = \frac{1}{4}p^2 = \frac{49}{4}$$

11. See solution to Question No. 3

12. We have

$$\tan\left(\frac{P}{2}\right) + \tan\left(\frac{Q}{2}\right) = \frac{-b}{a} \quad \text{and} \quad \tan\left(\frac{P}{2}\right)\tan\left(\frac{Q}{2}\right) = \frac{c}{a}$$

Also $P + Q = \pi - R = \pi/2$

$$\Rightarrow \quad \frac{P}{2} + \frac{Q}{2} = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan\left(\frac{P}{2} + \frac{Q}{2}\right) = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \frac{\tan(P/2) + \tan(Q/2)}{1 - \tan(P/2)\tan(Q/2)} = 1$$

$$\Rightarrow \quad \frac{-bla}{1 - cla} = 1 \Rightarrow \frac{-b}{a-c} = 1$$

$\Rightarrow$ $-b = a - c$ or $c = a + b$

13. Let α, β be roots of $x^2 - (a-2)x - a - 1 = 0$

Then $\alpha + \beta = a - 2$, $\alpha\beta = -(a+1)$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= (a-2)^2 + 2(a+1)$$

$$= a^2 - 4a + 4 + 2a + 2$$

$$= a^2 - 2a + 1 + 3$$

$$= (a-1)^2 + 3$$

Thus, $\alpha^2 + \beta^2$ is least when $a = 1$

14. We know that if α and β are roots of $x^2 - bx + c = 0$, then $(\alpha - \beta)^2$ = discriminant = $b^2 - 4c$.

In the present case $\beta = \alpha + 1$.

$\therefore \quad 1 = b^2 - 4c$

15. We can write the given equation as

$$(x-k)^2 = 5-k.$$

Thus, for roots to be real we must have $5 - k \geq 0$ or $k \leq 5$.

Also, for $k \leq 5$,

$$x = k \pm \sqrt{5-k}$$

For both the roots to be less than 5, we must have that larger of the two roots is less than 5. Therefore.

$$\Rightarrow \quad k + \sqrt{5-k} < \quad \Rightarrow 5\sqrt{5-k} < 5-k \qquad (1)$$

This does not hold for $k = 5$. For $k < 5$, we can write (1) as

$1 < \sqrt{5-k}$ or $5 - k > 1$ or $k < 4$

$\therefore \quad k \in (-\infty, 4)$.

16. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x$. Then, $f(0) = 0$ and $f(\alpha) = 0$.

By the Rolle's theorem $\exists\ \beta \in (0, \alpha)$ such that $f'(\beta) = 0$.

Thus, $\quad na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + a_1 = 0$

has a positive root β which is smaller than α.

17. Let $\alpha = \tan 30°$, $\beta = \tan 15°$, then $\alpha = \dfrac{1}{\sqrt{3}}$ and $\beta = 2 - \sqrt{3}$

$$\therefore \quad -p = \frac{1}{\sqrt{3}} + (2-\sqrt{3})$$

$$\text{and} \quad q = \frac{1}{\sqrt{3}}(2-\sqrt{3}) = \frac{2}{\sqrt{3}} - 1$$

$$\text{Thus,} \quad q - p = \frac{3}{\sqrt{3}} + 2 - \sqrt{3} - 1 = 1$$

$\therefore \quad 2 + q - p = 3$

18. $x^2 - 2mx + m^2 - 1 = 0$

$\Rightarrow \quad (x-m)^2 = 1 \quad \Rightarrow x = m \pm 1$

As both the roots lie between –2 and 4, we get

$-2 < m \pm 1 < 4$

$\Rightarrow \quad -1 < m < 3$.

19. Let $\quad y = \dfrac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$

$\Rightarrow \quad 3(y-1)x^2 + 9(y-1)x + 7y - 17 = 0$

As x is real

$81(y-1)^2 - 12(y-1)(7y-17) \geq 0$

$\Rightarrow \quad 3(y-1)[27(y-1) - 4(7y-17)] \geq 0$

$\Rightarrow \quad 3(y-1)(41-y) \geq 0$

$\Rightarrow \quad (y-1)(y-41) \leq 0$

$\Rightarrow \quad 1 \leq y \leq 41$

20. $\alpha + \beta = -a,\ \alpha\beta = 1$

$$|\alpha - \beta| < \sqrt{5}$$

$\Rightarrow \quad |\alpha - \beta|^2 < 5$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta < 5$

$\Rightarrow \quad a^2 - 4 < 5$

$\Rightarrow \quad a^2 < 9$

$\Rightarrow \quad -3 < a < 3$

$\Rightarrow \quad a \in (-3, 3)$

21. Let roots of $x^2 - 6x + a = 0$ be α, $4m$ and that of $x^2 - cx + 6 = 0$ be α, $3m$ where m is an integer We have,

$\alpha + 4m = 6,\ \alpha(4m) = a$

and $\quad \alpha + 3m = c,\quad \alpha(3m) = 6$

$\Rightarrow \quad \alpha m = 2 \Rightarrow \quad a = 8$

Also, $\alpha + \dfrac{8}{\alpha} = 6 \quad \Rightarrow \quad \alpha^2 - 6\alpha + 8 = 0$

$\Rightarrow \quad (\alpha - 2)(\alpha - 4) = 0 \quad \Rightarrow \alpha = 2, 4.$

As m is an integer, $\alpha \neq 4$ ($\because\ \alpha m = 2$)

$\therefore \quad \alpha = 2$

22. As roots of $bx^2 + cx + a = 0$ are imaginary, $c^2 - 4ab < 0$. Now,

$3b^2x^2 + 6bcx + 2c^2$

$$= 3\left[b^2x^2 + 2bcx + \frac{2}{3}c^2\right]$$

$$= 3\left[(bx+c)^2 - \frac{1}{3}c^2\right]$$

$= 3(bx+c)^2 - c^2 \geq -c^2 > -4ab$

$\therefore\ 3b^2x^2 + 6bcx + 2c^2 > -4ab\ \forall\ x \in \mathbf{R}$.

23. $x^2 - x + 1 = 0$

$\Rightarrow \quad x = -\omega, -\omega^2$ where $\omega = -\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$

Let $\quad \alpha = -\omega$ and $\beta = -\omega^2$

We have $\alpha^{2009} + \beta^{2009}$

$= (-\omega)^{2009} + (-\omega^2)^{2009}$

$= (-1)\,[\omega^{2007}\,\omega^2 + (\omega^2)^{2007}\,\omega^4]$

$= (-1)\,(\omega^2 + \omega) = (-1)\,(-1) = 1$

24. As $a \neq a_1$, $p(x) = f(x) - g(x)$ is a quadratic polynomial. As $p(x) = 0$ only for $x = -1$, we get $p(x)$ must be of the form $p(x) = k(x+1)^2$ where $k = a - a_1$.

As $\quad p(-2) = 2$, we get

$$2 = k(-2+1)^2 \Rightarrow k = 2.$$

Thus, $\quad p(x) = 2(x+1)^2$

$\therefore \quad p(2) = 2(2+1)^2 = 18$

25. We can write the given equation as

$y - 1/y - 4 = 0 \quad$ where $y = e^{\sin x}$

$\Rightarrow y^2 - 4y - 1 = 0$

$\Rightarrow \; y = \dfrac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$

As $y > 0$, $y = 2+\sqrt{5}$

$\Rightarrow \sin x = \ln(2+\sqrt{5}) > \ln(4) > 1$

This is not possible.

Thus, equation has no real roots.

26. $x^2 + 2x + 3 = 0$

$\Rightarrow \quad (x+1)^2 = 2i^2$

$\Rightarrow \quad x = -1 \pm \sqrt{2}i$

As $a, b, c \in \mathbf{R}$, if $-1 \pm \sqrt{2}i$ (or $-1-\sqrt{2}i$) is a root of $ax^2 + bx + c = 0$, the other root must be $-1-\sqrt{2}i$ (or $-1 \pm \sqrt{2}i$)

Thus, $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$ have both roots in common, therefore

$$\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$$

$\Rightarrow \quad a : b : c = 1 : 2 : 3$

27. $(\sqrt{10})^2 = |\alpha - \beta|^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow 10 = p^2 - 3p \quad \Rightarrow p^2 - 3p - 10 = 0$

$\Rightarrow (p-5)\,(p+2) = 0 \quad p = -2, 5$

Thus, p lies in the set $\{-2, 5\}$

28. $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha} = \dfrac{a^3 + \beta^3}{\alpha\beta} = \dfrac{-p}{q}$

and $\left(\dfrac{\alpha^2}{\beta}\right)\left(\dfrac{\beta^2}{\alpha}\right) = \alpha\beta = q$

Thus, required equation is

$$x^2 - \left(-\frac{p}{q}\right)x + q = 0$$

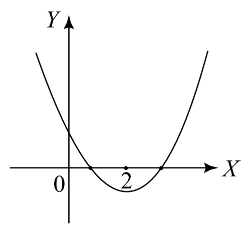
or $\qquad qx^2 + px + q^2 = 0$

29. As the roots are real and distinct

$(a+1)^2 - 4(a^2 + a - 8) > 0$

$3a^2 + 2a - 33 < 0$

$\Rightarrow \quad (3a+11)(a-3) < 0 \Rightarrow -\dfrac{11}{3} < a < 3 \qquad (1)$



Also, for

$2^2 - 2(a+1) + a^2 + a - 8 < 0$

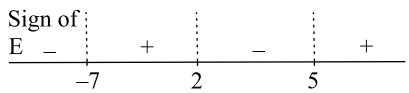
$\Rightarrow \quad a^2 - a - 6 < 0 \Rightarrow (a-3)\,(a+2) < 0$

$\Rightarrow \quad -2 < a < 3 \qquad (2)$

From (1) and (2), we get $-2 < a < 3$

30. $0 < \dfrac{x-5}{x^2+5x-14} = \dfrac{x-5}{(x+7)(x-2)} = \text{E(say)}$

Sign of E in different intervals are shown below



Thus, $\dfrac{x-5}{(x+7)(x-2)} > 0$

$\Rightarrow \quad -7 < x < 2$ or $x > 5$.

Therefore the least integral value α of x is -6. This value of α satisfies the relation $\alpha^2 + 5\alpha - 6 = 0$.

31. $4 = \dfrac{\alpha+\beta}{\alpha\beta} = \dfrac{-q/p}{r/p} = \dfrac{-q}{r}$

$\Rightarrow \quad q + 4r = 0$

Also, $\quad 2q = p + r \qquad [\because\ p, q, r$ are in A.P.]

Thus, $\quad p = -9r,\ q = -4r$

$$|\alpha - \beta|^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$= \left(\frac{-q}{p}\right)^2 - \frac{4r}{p}$$

$$= \frac{16}{81} - 4\left(-\frac{1}{9}\right) = \frac{52}{81}$$

$\Rightarrow \quad |\alpha - \beta| = \dfrac{2}{9}\sqrt{13}$

32. As roots of $2x^2 + 3x + 4 = 0$ are non-real complex numbers, the given quadratic equations have both roots in common. [See Solution Question No. 26.]

Thus, $\dfrac{a}{2} = \dfrac{b}{3} = \dfrac{c}{4} \quad \Rightarrow a : b : c = 2 : 3 : 4$

33. **Case 1:** When $x < 3/2$

In this case the equation becomes

$x^2 - (2x - 3) - 4 = 0$

$\Rightarrow \quad x^2 - 2x + 1 = 2$

$\Rightarrow \quad (x - 1)^2 = 2 \quad \Rightarrow \quad x - 1 = \pm\sqrt{2}$

$\Rightarrow \quad x = 1 \pm \sqrt{2}$

As $x < 3/2$, we take $x = 1 - \sqrt{2}$

Case 2: When $x \ge 3/2$

In this case the equation becomes

$x^2 + (2x - 3) - 4 = 0$

$\Rightarrow \quad (x + 1)^2 = 8 \Rightarrow x + 1 \pm 2\sqrt{2}$

As $x \ge 3/2$, $x = -1 + 2\sqrt{2}$

sum of roots $= \left(1 - \sqrt{2}\right) + \left(-1 + 2\sqrt{2}\right) = \sqrt{2}$

34. $\alpha + \beta = 4\sqrt{2}\,k$, $\alpha\beta = 2e^{4\ln k} - 1 = 2k^4 - 1$

Now,

$66 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 32k^2 - 2(2k^4 - 1)$

$\Rightarrow 33 = 16k^2 - 2k^4 + 1$

$\Rightarrow 2k^4 - 16k^2 + 32 = 0$

$\Rightarrow k^4 - 8k^2 + 16 = 0$

$\Rightarrow (k^2 - 4)^2 = 0 \Rightarrow k = 2 \qquad [\because\ k > 0]$

We have

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)$$

$$= (\alpha + \beta)\,[(\alpha + \beta)^2 - 3\alpha\beta]$$

$$= \left(4\sqrt{2}\right)(2)\left[66 - \left\{2(2^4) - 1\right\}\right]$$

$$= 280\sqrt{2}$$

35. Note that we must have $3x^2 + x + 5 \ge 0$ and $x - 3 \ge 0$ or $x \ge 3$. Squaring both sides of (1), we get

$$3x^2 + x + 5 = x^2 - 6x + 9$$

$\Rightarrow \quad 2x^2 + 7x - 4 = 0$

$\Rightarrow \quad (2x - 1)(x + 4) = 0$

$\Rightarrow \quad x = 1/2, -4$

None of these satisfy the inequality $x \ge 3$. Thus, (1) has no solution.

36. $\alpha + \beta = 6$, $\alpha\beta = -2$.

$a_{10} - 2a_8 = \alpha^{10} + \beta^{10} + \alpha\beta\,(\alpha^8 + \beta^8)$

$= (\alpha + \beta)\,(\alpha^9 + \beta^9) = 6a_9$

$\Rightarrow \dfrac{a_{10} - 2a_8}{2a_9} = 3$

37. As the coefficients of

$2x^3 - 9x^2 - kx - 13 = 0$

are all real, non-real complex roots occur in conjugate pair, so let the roots of (1) be $2 + 3i$, $2 - 3i$ and α, where $\alpha \in \mathbf{R}$. Now,

$(2 + 3i) + (2 - 3i) + \alpha = 9/2 \Rightarrow \alpha = 1/2$

$\therefore$ real root of the equation is 1/2.

[Verification: $(2 + 3i)\,(2 - 3i)\alpha = 13/2 \Rightarrow \alpha = 1/2$]

38. Write

$$x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2$$

$$= (x^2 + 1)^2 - x^2$$

$$= (x^2 + x + 1)\,(x^2 - x + 1)$$

The given equation, now can be written as

$(x^2 + x + 1)\,[(a - 1)(x^2 - x + 1) + (a + 1)\,(x^2 + x + 1)] = 0$

$\Rightarrow (x^2 + x + 1)\,[2ax^2 + 2x + 2a] = 0$

As x is real, we get

$ax^2 + x + a = 0$.

This equation will have real and distinct roots if

$a \neq 0$ and $1 - 4a^2 > 0 \Rightarrow a \neq 0,\ a^2 < 1/4$

$\Rightarrow a \in (-1/2, 0) \cup (0, 1/2)$

39. If $x^2 - 5x + 5 \neq 1$, then $x^2 + 4x - 60 = 0$

$\Rightarrow (x + 10)(x - 6) = 0 \Rightarrow x = -10, 6$

When $x = -10$ and 6, $x^2 - 5x + 5 \neq 1$

If $x^2 - 5x + 5 = 1$, $(x^2 - 5x + 5)^{x^2+4x-60} = 1$

In this case, $x^2 - 5x + 4 = 0$

$\Rightarrow (x - 1)(x - 4) = 0 \Rightarrow x = 1, 4$

If $x^2 - 5x + 5 = -1$, then $x^2 + 4x - 60$ must be even integer.

But $x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

For $x = 2$, $x^2 + 4x - 60 = -48$ is even.

For $x = 3$, $x^2 + 4x - 60 = -39$ is odd.

Thus, sum of desired values is $-10 + 6 + 1 + 4 + 2 = 3$.

40. Let α be common roots of

$x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$.

Then $\alpha^2 + b\alpha - 1 = 0$ and $\alpha^2 + \alpha + b = 0$

$\Rightarrow (\alpha^2 + b\alpha - 1) - (\alpha^2 + \alpha + b) = 0$

$\Rightarrow (b - 1)\alpha - (b + 1) = 0$

$\Rightarrow \alpha = \dfrac{b+1}{b-1}$ if $b \neq 1$

As $\alpha \neq -1$, $b \neq 0$

Also, $\left(\dfrac{b+1}{b-1}\right)^2 + b\left(\dfrac{b+1}{b-1}\right) - 1 = 0$

$\Rightarrow (b + 1)^2 + b(b + 1)(b - 1) - (b - 1)^2 = 0$

$\Rightarrow 4b + b(b + 1)(b - 1) = 0$

As $b \neq 0$, $4 + b^2 - 1 = 0 \Rightarrow b^2 = 3$

$\Rightarrow |b| = \sqrt{3}$

41. We are given

$\sqrt{2x+1} - \sqrt{2x-1} = 1$ (1)

Also, $(2x + 1) - (2x - 1) = 2$ (2)

Dividing (2) by (1),

$\sqrt{2x+1} + \sqrt{2x-1} = 2$ (3)

From (1) and (3), we get

$\sqrt{2x+1} = \dfrac{3}{2}$, $\sqrt{2x-1} = \dfrac{1}{2}$

$\therefore \sqrt{4x^2 - 1} = \dfrac{3}{4}$

42. $x + y + z = 12$, $x^3y^4z^5 = (0.1)(600)^3$

We have

$x^3y^4z^5 = (0.1)(600)(600)^2$

$= (60)(600)^2$

$= 3^3 2^8 5^5$

$\therefore$ we may take

$x = 3, y = 4, z = 5$

Thus, $x^3 + y^3 + z^3 = 3^3 + 4^3 + 5^3$

$= 27 + 64 + 125 = 216$

Previous Years' B-Architecture Entrance Examination Questions

1. $\tan 30^\circ + \tan 15^\circ = -2p$

$\tan 30^\circ \tan 15^\circ = q$

$1 = \tan 45^\circ = \dfrac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ}$

$\Rightarrow \quad 1 = \dfrac{-2p}{1-q} \quad \Rightarrow \quad 1 - q = -2p$

$\Rightarrow \quad q = 1 + 2p$.

2. Let roots of the equation be $1 - k$ and $1 + k$, where $k > 0$.

Then $2 = (1 - k) + (1 + k) = \dfrac{2\alpha}{\alpha+2}$

$\Rightarrow \quad 1 = \dfrac{\alpha}{\alpha+2}$

This is not possible for any value of α.

3. $|x|^2 - 4|x| - 2 = 0$

$\Rightarrow (|x| - 2)^2 = 6 \quad \Rightarrow \quad |x| - 2 = \pm\sqrt{6}$

$\Rightarrow |x| = 2 \pm \sqrt{6}$

As $|x| > 0$, $|x| = 2 + \sqrt{6}$

$\Rightarrow x = \pm(2 + \sqrt{6})$

$\therefore$ there are two values of x.

4. See Solution to Question 1 in Previous Years' AIEEE/JEE Questions.

5. $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$

$\Rightarrow \quad a\left(-\dfrac{x}{x-1}\right)^2 + b\left(-\dfrac{x}{x-1}\right) + c = 0$

$\Rightarrow \quad -\dfrac{x}{x-1} = \alpha, \beta$

$\Rightarrow \quad x = \dfrac{\alpha}{\alpha+1}, \dfrac{\beta}{\beta+1}$

6. $x^4 - ax^2 + b = (x^2 - 3x + 2)\ p\ (x)$

$$= (x - 1)(x - 2)\ p(x)$$

where $p(x)$ is a polynomial of degree 2.

$\therefore\ 1 - a + b = 0$ and $16 - 4a + b = 0$

$\Rightarrow\ a = 5,\ b = 4$

Thus, equation whose roots are a and b is

$x^2 - 9x + 20 = 0$

7. $f'(x) = 2ax + b = a\left(x + \dfrac{b}{2a}\right)$

As $a > 0$,

$$f'(x) < 0 \text{ for } x < -\,b/2a$$

and $\quad f'(x) > 0$ for $x > -\,b/2a$

Thus, f is strictly decreasing in the interval $(-\infty, -\,b/2a)$ and strictly increasing in the interval $(-\,b/2a, \infty)$.

Also, Min $f(x) = f\left(\dfrac{-b}{2a}\right) = \dfrac{4ac - b^2}{4a} > 0$

if $b^2 < 4ac$

As Min $f(x) > 0$, $f(x) > 0\ \forall\ x \in \mathbf{R}$

$\therefore$ Both the statements are true and statement- 2 is a correct explanation for statement-1.

8. As the roots are of opposite signs, the product of roots must be negative, that is,

$$\frac{a^2 - 3a + 2}{3} < 0 \Rightarrow (a-1)(a-2) < 0$$

$\Rightarrow\ a \in (1, 2)$

9. The given equation can be written as

$r(2x + p + q) = (x + p)\ (x + q)$

or $\quad x^2 + (p + q - 2r) + pq - r(p + q) = 0$

Let roots of this equation be α , $-\alpha$.

Then

$$0 = \alpha + (-\alpha) = -\ (p + q - 2r) \Rightarrow 2r = p + q$$

and $\quad \alpha(-\alpha) = pq - r(p + q)$

$$= pq - \frac{1}{2}(p+q)^2 = -\frac{1}{2}(p^2 + q^2)$$

10. Write the equation as

$$(x - 3k)^2 = 2(k - 1)$$

$\Rightarrow \quad x = 3k \pm \sqrt{2}\sqrt{k-1}$

Both the roots will be greater than 3 if smaller root is greater than 3, that is, if

$3k - \sqrt{2}\sqrt{k-1} > 3$

$\Rightarrow\ 3(k-1) > \sqrt{2}\sqrt{k-1}$

$\Rightarrow\ k > 1$ and $9(k - 1) > 2 \Rightarrow\ k > 11/9$

$\Rightarrow\ 11 - 9k < 0.$

11. $(1 + 3m)^2 - (1 + m^2)(1 + 8m) < 0$

$\Rightarrow 1 + 6m + 9m^2 - (1 + m^2 + 8m + 8m^3) < 0$

$\Rightarrow -2m + 8m^2 - 8m^3 < 0$

$\Rightarrow 2m(4m^2 - 4m + 1) > 0$

$\Rightarrow m(2m - 1)^2 > 0$

which is true for each $m \in \mathbf{N}$.