Session 3

Circles Connected with Triangle

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Circumcircle of a Triangle

The circle which passes through the angular points of a ΔABC is called its Circumcircle. The centre of this circle is the point of intersection of perpendicular bisectors of the sides and called the Circumcentre. Its radius is always denoted by $\boldsymbol{R}$.

Note

1. Circumcentre of an a cute-angled triangle lies inside the triangle.
2. Circumcentre of an obtuse-angled triangle lies outside the triangle.
3. In a right angled triangle the circumcentre is the mid-point of hypotenuse.

Circum-radius

The radius of the circumcircle of a ΔABC is called the circum-radius given by;

(i) $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C}$ (ii) $R = \dfrac{abc}{4\Delta}$

Proof

(i) Here, the perpendicular bisectors of the sides BC, CA and AB intersect at O.

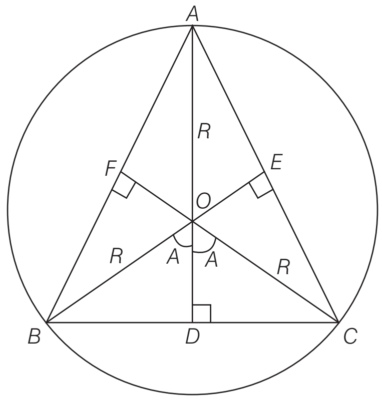
$\therefore$ O is the circumcentre such that,

$$OA = OB = OC = R$$

We have, $\angle BOC = 2\angle A$

$\therefore$ $\angle BOD = \angle COD = \angle A$

In ΔOBD, $\sin A = \frac{BD}{OB} = \frac{a/2}{R}$



$\Rightarrow \quad R = \frac{a}{2\sin A}$

Similarly, $R = \frac{b}{2\sin B}$ and $R = \frac{c}{2\sin C}$

Hence, $\boxed{R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}}$

(ii) As discussed, Area of $\Delta = \frac{1}{2} bc\sin A$

$\Rightarrow \quad \sin A = \frac{2\Delta}{bc}$ ...(i)

Also, $R = \frac{a}{2\sin A}$ ...(ii)

$\therefore$ From Eqs. (i) and (ii);

$$R = \frac{a}{2\left(\frac{2\Delta}{bc}\right)} = \frac{abc}{4\Delta} \Rightarrow R = \frac{abc}{4\Delta}.$$

In-circle or Inscribed Circle of a Triangle

The circle that can be inscribed with in a triangle so as to touch each of its sides is called its inscribed circle or In-circle. The centre of this circle is the points of intersection of bisectors of the angles of the triangle. The radius of the circle is always denoted by 'r' and is equal to the length of perpendicular from its centre to any one of the sides of triangle.

In-radius The radius of the inscribed circle of a triangle is called the in-radius. It is denoted by 'r' and is given by

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$.

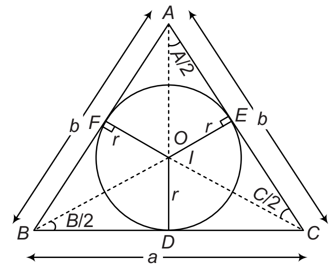
(iii) $r = \frac{a\sin B/2 \cdot \sin C/2}{\cos A/2}$

$r = \frac{b\sin C/2 \cdot \sin A/2}{\cos B/2} \Rightarrow r = \frac{c\sin A/2 . \sin B/2}{\cos C/2}$

(iv) $r = 4R\sin A/2 \sin B/2 \sin C/2$

Proof Let the internal bisectors of the angles of the triangle ABC meet at I. Suppose the circle touches the sides BC, CA and AB at D, E and F, respectively.

Then, ID, IE, IF are perpendicular to these sides and $ID = IE = IF = r$.



(i) We have, area of ΔABC = area of ΔIBC + area of ΔIAB + area of ΔICA

$$\Delta = \frac{1}{2}ar + \frac{1}{2}cr + \frac{1}{2}br$$

$$\Delta = \frac{1}{2}(a+b+c)r = sr \quad \left[\text{as; } s = \frac{a+b+c}{2}\right]$$

$\Rightarrow \quad \Delta = sr$

or $\quad r = \frac{\Delta}{s}$.

(ii) Since, the lengths of the tangents to a circle from a given points are equal, therefore

$AE = AF$, $BD = BF$ and $CD = CE$. ...(I)

Now, $2s = a + b + c = BC + CA + AB$

$= (BD + DC) + (CE + EA) + (AF + FB)$

$= (BD + BF) + (AE + AF) + (CD + CE)$

$= 2(BD + AE + CD) = 2(BC + AE) = 2(a + AE)$

$\Rightarrow \quad s = a + AE$

$\Rightarrow \quad AE = (s-a)$

Now, in ΔIAE,

$$\tan\frac{A}{2} = \frac{r}{AE}$$

$\Rightarrow \quad r = AE\tan(A/2) = (s-a)\tan A/2$

$\therefore \quad r = (s-a)\tan A/2$

Similarly, $r = (s-b)\tan B/2$ and $r = (s-c)\tan C/2$

Hence, $r = (s-a)\tan A/2$

$= (s-b)\tan B/2 = (s-c)\tan C/2$

(iii) In ΔIBD and ΔICD, we have,

$$\tan B/2 = \frac{r}{BD} \text{ and } \tan\frac{C}{2} = \frac{r}{CD}$$

$$\therefore \quad BD = \frac{r}{\tan B/2} \text{ and } CD = \frac{r}{\tan C/2}$$

Now, $\quad a = BD + CD$

$$\Rightarrow \quad a = \frac{r}{\tan\left(\frac{B}{2}\right)} + \frac{r}{\tan\left(\frac{C}{2}\right)}$$

$$\Rightarrow \quad a = r\left[\frac{\cos B/2}{\sin B/2} + \frac{\cos C/2}{\sin C/2}\right]$$

$$\Rightarrow \quad a = r\left[\frac{\cos B/2 \cdot \sin C/2 + \sin B/2 \cdot \cos C/2}{\sin B/2 \cdot \sin C/2}\right]$$

$$a = \frac{r\sin(B/2 + C/2)}{\sin B/2 \cdot \sin C/2} \qquad [\because A + B + C = \pi]$$

$$\therefore \quad \sin(B/2 + C/2) = \sin\left(\frac{\pi}{2} - \frac{A}{2}\right) = \cos A/2 .$$

$$\Rightarrow \quad a = \frac{r\cos A/2}{\sin B/2 \cdot \sin C/2}$$

$$\therefore \quad r = \frac{a\sin B/2 \cdot \sin C/2}{\cos A/2}, r = \frac{b\sin A/2 \cdot \sin C/2}{\cos B/2} \text{ and}$$

$$r = \frac{c\sin A/2 \cdot \sin B/2}{\cos C/2}$$

(iv) We have, $r = \dfrac{a\sin B/2 \cdot \sin C/2}{\cos A/2}$ and $R = \dfrac{a}{2\sin A}$

$$\Rightarrow \quad r = \frac{2R\sin A \cdot \sin B/2 \cdot \sin C/2}{\cos A/2}$$

$$\Rightarrow \quad r = \frac{2R \cdot (2\sin A/2 \cdot \cos A/2) \cdot \sin B/2 \cdot \sin C/2}{\cos A/2}$$

$$\Rightarrow \quad r = 4R\sin A/2 \cdot \sin B/2 \cdot \sin C/2$$

Escribed Circles of a Triangle

The circle which touches the sides BC and two sides AB and AC produced of a triangle ABC is called the **Escribed circle opposite to the angle A.**

Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C, respectively. The centres of the escribed circles are called the ex-centres.

The centre of the escribed circles opposite to the angle A is the point of Intersection of external bisector of angles B and C. The internal bisector also passes through the same point. This centre is generally denoted by I_1.

Formulae for r_1, r_2, r_3

In any ΔABC, we have

(i) $r_1 = \dfrac{\Delta}{s-a}, r_2 = \dfrac{\Delta}{s-b}, r_3 = \dfrac{\Delta}{s-c}$

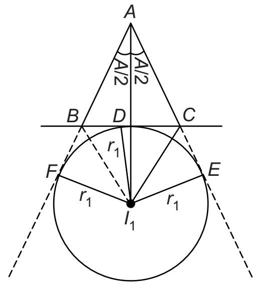
(ii) $r_1 = s\tan A/2, r_2 = s\tan B/2, r_3 = s\tan C/2$

(iii) $r_1 = \dfrac{a\cos B/2 \cdot \cos C/2}{\cos A/2}, r_2 = \dfrac{b\cos A/2 \cdot \cos C/2}{\cos B/2},$

$r_3 = \dfrac{c\cos A/2 \cdot \cos B/2}{\cos C/2}$

(iv) $r_1 = 4R\sin A/2\cos B/2.\cos C/2,$
$r_2 = 4R\cos A/2\sin B/2.\cos C/2,$
$r_3 = 4R\cos A/2\cos B/2.\sin C/2$

Proof (i) Let the ΔABC be as;



We have,

$$I_1D = I_1E = I_1F = r_1$$

Now, area of ΔABC = area of ΔI_1AC + area of ΔI_1AB − area of ΔI_1BC

$$\Rightarrow \quad \Delta = \frac{1}{2}I_1E \cdot AC + \frac{1}{2}I_1F \cdot AB - \frac{1}{2}I_1D \cdot BC$$

$$\Delta = \frac{1}{2}r_1b + \frac{1}{2}r_1c - \frac{1}{2}r_1a$$

$$\Delta = \frac{r_1}{2}(b + c - a)$$

$$\Delta = \frac{r_1}{2}(2s - 2a) \qquad [\text{using } a + b + c = 2s]$$

$$\Rightarrow \quad r_1 = \frac{\Delta}{s-a}$$

Similarly, $\quad r_2 = \dfrac{\Delta}{s-b}$ and $r_3 = \dfrac{\Delta}{s-c}$

(ii) Since, the lengths of tangents to a circle from an external points are equal,

$\therefore \quad AE = AF, BD = BF$ and $CD = CE$

Now, $\quad AE + AF = (AC + CE) + (AB + BF)$

$$= (AC + CD) + (AB + BD)$$

$$= AC + AB + CD + BD$$

$$= AC + AB + BC$$
$$= a + b + c = 2s.$$
$$\Rightarrow \quad 2AF = 2s$$
$$\Rightarrow \quad AE = AF = s$$

In $\Delta I_1 AF$, $\tan A/2 = \dfrac{I_1 F}{AF} = \dfrac{r_1}{AF}$

$$\Rightarrow \quad \tan A/2 = \frac{r_1}{s}$$
$$\Rightarrow \quad r_1 = s \tan A/2$$

Similarly, $r_2 = s \tan B/2$ and $r_3 = s \tan C/2$.

(iii) In $\Delta I_1 BD$, we have

$$\tan\left(\frac{B}{2}\right) = \frac{I_1 D}{BD} = \frac{r_1}{BD}$$
$$\Rightarrow \quad BD = r_1 \tan\frac{B}{2}$$

Similarly, in $\Delta I_1 CD$, we have

$$CD = r_1 \tan\frac{C}{2}$$

Now, $a = BC = BD + CD = r_1 \tan\dfrac{B}{2} + r_1 \tan\dfrac{C}{2}$

$$= r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = r_1 \frac{\cos A/2}{\cos B/2 \cos C/2}$$
$$\Rightarrow \quad r_1 = \frac{a \cos B/2 \cos C/2}{\cos A/2}$$

Similarly,

$$r_2 = \frac{b \cos A/2 \cdot \cos C/2}{\cos B/2} \text{ and } r_3 = \frac{c \cos A/2 \cdot \cos B/2}{\cos C/2}$$

(iv) We have, $r_1 = \dfrac{a \cos B/2 \cdot \cos C/2}{\cos A/2}$ and $R = \dfrac{a}{2 \sin A}$

$$\Rightarrow \quad r_1 = \frac{2R \sin A \cdot \cos B/2 \cdot \cos C/2}{\cos A/2}$$
$$\Rightarrow \quad r_1 = \frac{4R \sin A/2 \cdot \cos A/2 \cdot \cos B/2 \cdot \cos C/2}{\cos A/2}$$
$$\therefore \quad r_1 = 4R \sin A/2 \cdot \cos B/2 \cdot \cos C/2$$

Similarly, $r_2 = 4R \cos A/2 \cdot \sin B/2 \cdot \cos C/2$

$$r_3 = 4R \cos A/2 \cdot \cos B/2 \cdot \sin C/2$$

Example 12. Show that $\dfrac{b-c}{r_1} + \dfrac{c-a}{r_2} + \dfrac{a-b}{r_3} = 0.$

Sol. LHS $\dfrac{(b-c)}{r_1} + \dfrac{(c-a)}{r_2} + \dfrac{(a-b)}{r_3}$

$$\Rightarrow \quad (b-c)\cdot\left(\frac{s-a}{\Delta}\right) + (c-a)\cdot\left(\frac{s-b}{\Delta}\right) + (a-b)\cdot\left(\frac{s-c}{\Delta}\right)$$
$$\Rightarrow \quad \frac{(s-a)(b-c) + (s-b)(c-a) + (s-c)(a-b)}{\Delta}$$
$$= \frac{s(b-c+c-a+a-b) - [ab - ac + bc - ba + ac - bc]}{\Delta}$$
$$= \frac{0}{\Delta} = 0 = \text{RHS}$$

Thus, $\dfrac{b-c}{r_1} + \dfrac{c-a}{r_2} + \dfrac{a-b}{r_3} = 0$

Example 13. *If $r_1 = r_2 + r_3 + r$, then prove that the triangle is right angled.*

Sol. We have, $r_1 - r = r_2 + r_3$

$$\Rightarrow \quad \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$
$$\Rightarrow \quad \frac{s-s+a}{s(s-a)} = \frac{s-c+s-b}{(s-b)(s-c)}$$
$$\Rightarrow \quad \frac{a}{s(s-a)} = \frac{2s-(b+c)}{(s-b)(s-c)} \quad [\text{as, } 2s = a+b+c]$$
$$\Rightarrow \quad \frac{a}{s(s-a)} = \frac{a}{(s-b)(s-c)}$$
$$\Rightarrow \quad s^2 - (b+c)s + bc = s^2 - as$$
$$\Rightarrow \quad s(-a+b+c) = bc$$
$$\Rightarrow \quad \frac{(b+c-a)(a+b+c)}{2} = bc$$
$$\Rightarrow \quad (b+c)^2 - (a)^2 = 2bc$$
$$\Rightarrow \quad b^2 + c^2 + 2bc - a^2 = 2bc$$
$$\Rightarrow \quad b^2 + c^2 = a^2$$
$$\therefore \quad \angle A = 90^\circ$$

Example 14. Prove that $r \cot\dfrac{B}{2} \cdot \cot\dfrac{C}{2} = r_1$.

Sol. LHS $r \cot B/2 \cdot \cot C/2$

$$\Rightarrow \quad 4R \sin A/2 \cdot \sin B/2 \cdot \sin C/2 \cdot \frac{\cos B/2}{\sin B/2} \cdot \frac{\cos C/2}{\sin C/2}$$

[as, $r = 4R \sin A/2 \cdot \sin B/2 \cdot \sin C/2$]

$$\Rightarrow \quad 4R \cdot \sin A/2 \cdot \cos B/2 \cdot \cos C/2$$

$\Rightarrow \quad r_1 = \text{RHS}$ [as, $r_1 = 4R \sin A/2 \cdot \cos B/2 \cdot \cos C/2$]

$\therefore \; r \cot B/2 \cdot \cot C/2 = r_1$

Example 15. In a right angled triangle, prove that $r + 2R = s$.

Sol. In a right angled triangle, the circum centre lies on the hypotenuse.

$$\Rightarrow \quad R = \frac{a}{2} \quad \text{...(i) } [\because \angle A = 90^\circ]$$

Also, $r = (s-a)\tan A/2 = (s-a)\tan 45^\circ$

$$r = (s-a) \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get $r = s - 2R$

$$\Rightarrow \quad r + 2R = s.$$

Example 16. The ex-radii r_1, r_2, r_3 of a ΔABC are in HP, show that its sides a, b, c are in AP.

Solution. r_1, r_2, r_3 are in HP.

$$\Rightarrow \quad \frac{2}{r_2} = \frac{1}{r_1} + \frac{1}{r_3}$$

$$\Rightarrow \quad \frac{2(s-b)}{\Delta} = \frac{(s-a)}{\Delta} + \frac{(s-c)}{\Delta}$$

$$\Rightarrow \quad 2s - 2b = 2s - (a+c)$$

$$\Rightarrow \quad 2b = a + c$$

Hence, a, b, c are in AP.

Example 17. If A, B, C are the angles of a triangle, then prove that

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Sol. $\cos A + \cos B + \cos C$

$$= 2\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right) + \cos C$$

$$= 2\sin\frac{C}{2}\cdot\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right)\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$\left[\because \frac{C}{2} = 90^\circ - \left(\frac{A+B}{2}\right)\right]$$

$$= 1 + 2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\cdot\sin\frac{B}{2}$$

$$= 1 + 4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2} = 1 + \frac{r}{R}$$

$$[\text{as, } r = 4R\sin A/2\cdot\sin B/2\cdot\sin C/2]$$

$$\Rightarrow \quad \cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

Example 18. Find the ratio of the circum-radius and the inradius of ΔABC, whose sides are in the ratio $4:5:6$.

Sol. Here, $a = 4k,\ b = 5k,\ c = 6k$

$$\therefore \quad s = \frac{15k}{2} \quad \text{...(i)}$$

$$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{\frac{15k}{2}\left(\frac{15k}{2} - 4k\right)\left(\frac{15k}{2} - 5k\right)\left(\frac{15k}{2} - 6k\right)}$$

$$= \frac{15\sqrt{7}}{4}k^2 \quad \text{...(ii)}$$

and $$R = \frac{abc}{4\Delta} = \frac{4k\cdot 5k\cdot 6k}{4\cdot\frac{15}{4}\sqrt{7}k^2} \quad \text{[using Eq. (ii)]}$$

$$\therefore \quad R = \frac{8}{\sqrt{7}}k \quad \text{...(iii)}$$

and $$r = \frac{\Delta}{s} = \frac{15\sqrt{7}}{4}k^2\cdot\frac{2}{15k} \quad \text{[using Eqs. (i) and (ii)]}$$

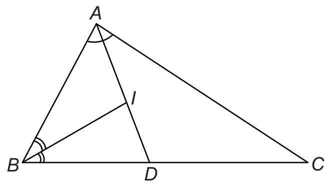
$$r = \frac{\sqrt{7}}{2}k \quad \text{...(iv)}$$

$$\therefore \quad \frac{R}{r} = \frac{8k/\sqrt{7}}{\sqrt{7}k/2} = \frac{16}{7} \quad \text{[using Eqs. (iii) and (iv)]}$$

$$\Rightarrow \quad R : r = 16 : 7$$

Example 19. Find the ratio of $IA : IB : IC$, where I is the incentre of ΔABC.

Sol. Here, $BD : DC = c : b$



But $BD + DC = a$;

$$\therefore \quad BD = \frac{c}{b+c}\cdot a \quad \text{...(i)}$$

In ΔABD, $$\frac{BD}{\sin A/2} = \frac{AD}{\sin B}$$

$$\therefore \quad AD = \frac{ac}{b+c}\cdot\frac{\sin B}{\sin A/2} = \frac{2\Delta}{b+c}\cdot\operatorname{cosec} A/2 \quad \text{...(ii)}$$

Also, $$\frac{AI}{ID} = \frac{AB}{BD} = \frac{c}{\frac{ac}{b+c}} = \frac{b+c}{a} \quad \text{[using Eq. (i)]}$$

or $$\frac{ID}{AI} = \frac{a}{b+c}$$

On adding '1', we get

$$\frac{ID}{AI} + 1 = \frac{a}{b+c} + 1 \Rightarrow \frac{ID + AI}{AI} = \frac{a+b+c}{b+c}$$

$$\Rightarrow \quad AI = \frac{b+c}{a+b+c}\cdot AD$$

$$\therefore \quad AI = \frac{b+c}{a+b+c}\cdot\frac{2\Delta}{b+c}\cdot\operatorname{cosec} A/2 = \frac{\Delta}{s}\cdot\operatorname{cosec} A/2$$

Similarly, $$BI = \frac{\Delta}{s}\cdot\operatorname{cosec} B/2$$

$$CI = \frac{\Delta}{s}\operatorname{cosec} C/2$$

$$\Rightarrow \quad IA : IB : IC = \frac{\Delta}{s}\cdot\operatorname{cosec} A/2 : \frac{\Delta}{s}\operatorname{cosec} B/2 : \frac{\Delta}{s}\operatorname{cosec} C/2$$

$$\therefore \quad IA : IB : IC = \operatorname{cosec} A/2 : \operatorname{cosec} B/2 : \operatorname{cosec} C/2$$

Note

Student are advised to remember the above result i.e.
$IA = r \operatorname{cosec} A/2$, $IB = r \operatorname{cosec} B/2$, $IC = r \operatorname{cosec} C/2$.

Example 20. If the sides of a triangle are in GP and the largest angle is twice the smallest angle, then find the relation for r.

Sol. Let the sides of Δ be $a, b = ar, c = ar^2$, where $r > 1$

Here, $c = 2A$ (given)

So, $B = \pi - A - C = \pi - 3A$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{1}{\sin A} = \frac{r}{\sin B} = \frac{r^2}{\sin C}$$

$$\Rightarrow \quad \frac{1}{\sin A} = \frac{r}{\sin 3A} = \frac{r^2}{\sin 2A}$$

$$\therefore \quad r^2 = 2\cos A \text{ and } r = \frac{\sin 3A}{\sin A} = 3 - 4\sin^2 A$$

$$r = 4\cos^2 A - 1$$

$$\therefore \quad r = r^4 - 1$$

Thus, the required relation is $r^4 - r - 1 = 0$.

Example 21. The equation $ax^2 + bx + c = 0$, where a, b, c are the sides of a ΔABC, and the equation $x^2 + \sqrt{2}x + 1 = 0$ have a common root. Find measure for $\angle C$.

Sol. Clearly, the roots of $x^2 + \sqrt{2}x + 1 = 0$ are non-real complex.

So, the one root common implies both roots are common.

So, $$\frac{a}{1} = \frac{b}{\sqrt{2}} = \frac{c}{1} = k$$

$$\therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{k^2 + 2k^2 - k^2}{2k.\sqrt{2}k} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \angle C = 45°$$

Example 22. If in a ΔABC, the value of $\cot A, \cot B, \cot C$ are in AP show a^2, b^2, c^2 are in A.P.

Sol. Here; $\cot A$, $\cot B$, $\cot C$ are in AP.

$$\Rightarrow \quad \frac{b^2 + c^2 - a^2}{2bc \cdot \frac{a}{2R}}, \frac{c^2 + a^2 - b^2}{2ac \cdot \frac{b}{2R}}, \frac{a^2 + b^2 - c^2}{2ab \cdot \frac{c}{2R}} \text{ are in AP}$$

$$\Rightarrow \quad b^2 + c^2 - a^2, c^2 + a^2 - b^2, a^2 + b^2 - c^2 \text{ are in AP}$$

$$\left[\text{multiplying by } \frac{abc}{R}\right]$$

$$\Rightarrow \quad -2a^2, -2b^2, -2c^2 \text{ are in AP}$$

[subtracting $a^2 + b^2 + c^2$ from each]

$$\Rightarrow \quad a^2, b^2, c^2 \text{ are in AP.}$$

Aliter $2\cot B = \cot A + \cot C$

$$\Rightarrow \quad \frac{2(a^2 + c^2 - b^2)}{2ac \cdot kb} = \frac{b^2 + c^2 - a^2}{2bc \cdot ka} + \frac{a^2 + b^2 - c^2}{2ab \cdot kc}$$

[using sine and cosine law]

$$\Rightarrow \quad 2(a^2 + c^2 - b^2) = b^2 + c^2 - a^2 + a^2 + b^2 - c^2$$

$$\Rightarrow \quad 2(a^2 + c^2 - b^2) = 2b^2$$

$$\Rightarrow \quad a^2 + c^2 - b^2 = b^2 \quad \text{or} \quad a^2 + c^2 = 2b^2$$

i.e. a^2, b^2, c^2 are in AP.

Exercise for Session 3

1. The side of a triangle are 22 cm, 28 cm and 36 cm. So, find the area of the circumscribed circle.
2. If the lengths of the side of a triangle are 3, 4 and 5 units, then find the circum radius R.
3. In an equilateral triangle of side $2\sqrt{3}$ cm. The find circum-radius.
4. If $8R^2 = a^2 + b^2 + c^2$, then prove that the Δ is right angled.
5. In a ΔABC, show that $2R^2 \sin A \sin B \sin C = \Delta$.
6. In a ΔABC, show that $\dfrac{a\cos A + b\cos B + c\cos C}{a + b + c} = \dfrac{r}{R}$
7. If the sides of a triangles are 3 : 7 : 8, then find ratio $R : r$.
8. In an equilateral triangle show that the in-radius and the circum-radius are connected by $r = \dfrac{R}{2}$.
9. In any ΔABC, find $\sin A + \sin B + \sin C$.

10. In any ΔABC, show that $\cos A + \cos B + \cos C = \left(1 + \frac{r}{R}\right)$.

11. If the side be a, b and c, then show that $\frac{r_1 + r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}$.

12. Show that $r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2$

13. Show that $(r_1 + r_2)(r_2 + r_3)(r_3 + r_1) = 4Rs^2$

14. If $r_1 = r_2 + r_3 + r$, then show that Δ is right angled.

15. In an equilateral triangle, show that the in-radius, circumradius and one of the ex-radii are in the ratio 1 : 2 : 3.

16. Show that $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{16R}{r^2(\Sigma a)^2}$

17. If r_1, r_2, r_3 in a triangle be in HP, then show that the sides are in AP.

18. In a ΔABC, show that $r_1\, r_2\, r_3 = \Delta^2$.

19. If l_1, l_2, l_3 are respectively the perpendicular from the vertices of a triangle on the opposite side, then show that $l_1\, l_2\, l_3 = \frac{a^2 b^2 c^2}{8R}$.

20. If the angle of a triangle are in the ratio 1 : 2 : 3, then show that the sides opposite to the respective angle are in the ratio $1 : \sqrt{3} : 2$.

21. Show that, $4\,Rr \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} = S$

22. If $(a - b)(s - c) = (b - c)(s - a)$, then show that r_1, r_2, r_3 are in HP.

23. To show that $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} = \frac{\Sigma a^2}{S^2}$

24. Show that $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$

25. Show that $\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\left(\frac{1}{r_2} + \frac{1}{r_3}\right)\left(\frac{1}{r_3} + \frac{1}{r_1}\right) = \frac{64R^3}{a^2 b^2 c^2}$

26. If the sides be a, b and c, then find the value of $(r + r_1)\tan\frac{B - C}{2} + (r + r_2)\tan\frac{C - A}{2} + (r + r_3)\tan\frac{A - B}{2}$

27. If the sides be a, b, c, then find value of $\frac{b - c}{r_1} + \frac{c - a}{r_2} + \frac{a - b}{r_3}$.

28. If the sides be a, b, c, then find $(r_1 - r)(r_2 + r_3)$.

29. If a, b, c are in AP, then show that r_1, r_2, r_3 are in HP.

30. Show that $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = 2R - r$

31. Show that $r_1 + r_2 = c\cot\left(\frac{C}{2}\right)$

32. Show that $Rr(\sin A + \sin B + \sin C) = \Delta$

33. Show that $16R^2 r\, r_1\, r_2\, r_3 = a^2 b^2 c^2$

34. If $\frac{r}{r_1} = \frac{r_2}{r_3}$, then show that $c = 90°$.

Answers

Exercise for Session 3

1. 1018.81 sq. cm **2.** 2.5 **3.** 2 cm **7.** 7:2 **9.** $\frac{S}{R}$

26. (1) **27.** 0 **28.** a^2b^2