3D-COORDINATE GEOMETRY

1. DISTANCE FORMULA :

The distance between two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by $AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

2. SECTION FORMULAE :

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let R (x, y, z) divide PQ in the ratio $m_1 : m_2$. Then R is

$$(x, y, z) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$$

If (m_1/m_2) is positive, R divides PQ internally and if (m_1/m_2) is negative, then externally.

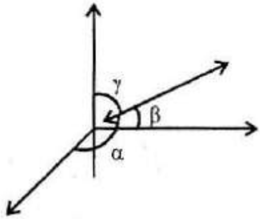
Mid point of PQ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

3. CENTROID OF A TRIANGLE :

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of a triangle ABC. Then its centroid G is given by

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

4. DIRECTION COSINES OF LINE :



If α, β, γ be the angles made by a line with x-axis, y-axis & z-axis respectively then cosα, cosβ & cosγ are called direction cosines of a line, denoted by l, m & n respectively and the relation between ℓ, m, n is given by $\ell^2 + m^2 + n^2 = 1$

D. cosine of x-axis, y-axis & z-axis are respectively
1, 0, 0; 0, 1, 0; 0, 0, 1

5. DIRECTION RATIOS :

Any three numbers a, b, c proportional to direction cosines ℓ, m, n are called direction ratios of the line.

i.e. $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$

It is easy to see that there can be infinitely many sets of direction ratios for a given line.

6. RELATION BETWEEN D.C'S & D.R'S :

$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$$

$$\therefore \quad \frac{\ell^2}{a^2} = \frac{m^2}{b^2} = \frac{n^2}{c^2} = \frac{\ell^2 + m^2 + n^2}{a^2 + b^2 + c^2}$$

$$\therefore \quad \ell = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}}; \quad m = \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}}; \quad n = \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}}$$

7. DIRECTION COSINE OF AXES :

Direction ratios and Direction cosines of the line joining two points :

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points, then d.r.'s of AB are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and the d.c.'s of AB are $\frac{1}{r}(x_2 - x_1)$, $\frac{1}{r}(y_2 - y_1)$, $\frac{1}{r}(z_2 - z_1)$ where $r = \sqrt{[\Sigma(x_2 - x_1)^2]} = |\overrightarrow{AB}|$

8. PROJECTION OF A LINE ON ANOTHER LINE :

Let PQ be a line segment with $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and let L be a straight line whose d.c.'s are l, m, n. Then the length of projection of PQ on the line L is $|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$

9. ANGLE BETWEEN TWO LINES :

Let θ be the angle between the lines with d.c.'s l_1, m_1, n_1 and l_2, m_2, n_2 then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$. If a_1, b_1, c_1 and a_2, b_2, c_2 be D.R.'s of two lines then angle θ between them is given by

$$\cos\theta = \frac{(a_1a_2 + b_1b_2 + c_1c_2)}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

10. PERPENDICULARITY AND PARALLELISM :

Let the two lines have their d.c.'s given by l_1, m_1, n_1 and l_2, m_2, n_2 respectively then they are perpendicular if $\theta = 90°$ i.e. $\cos\theta = 0$, i.e. $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

Also the two lines are parallel if $\theta = 0$ i.e. $\sin\theta = 0$, i.e. $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Note:
If instead of d.c.'s, d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are given, then the lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if $a_1/a_2 = b_1/b_2 = c_1/c_2$.

11. EQUATION OF A STRAIGHT LINE IN SYMMETRICAL FORM :

(a) One point form : Let $A(x_1, y_1, z_1)$ be a given point on the straight line and l, m, n the d.c's of the line, then its equation is

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{(say)}$$

It should be noted that $P(x_1 + lr, y_1 + mr, z_1 + nr)$ is a general point on this line at a distance r from the point $A(x_1, y_1, z_1)$ i.e. $AP = r$. One should note that for $AP = r$; l, m, n must be d.c.'s not d.r.'s. If a, b, c are direction ratios of the line, then equation of the line is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$ but here $AP \neq r$

(b) Equation of the line through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

12. FOOT, LENGTH AND EQUATION OF PERPENDICULAR FROM A POINT TO A LINE :

Let equation of the line be

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{(say)} \qquad \text{.......... (i)}$$

and $A(\alpha, \beta, \gamma)$ be the point. Any point on the line (i) is

$P(\ell r + x_1, mr + y_1, nr + z_1)$ .... (ii)

If it is the foot of the perpendicular, from A on the line, then AP is $\perp$ to the line, so $\ell(\ell r + x_1 - \alpha) + m(mr + y_1 - \beta) + n(nr + z_1 - \gamma) = 0$

i.e. $r = (\alpha - x_1)\ell + (\beta - y_1)m + (\gamma - z_1)n$

since $\ell^2 + m^2 + n^2 = 1$

Putting this value of r in (ii), we get the foot of perpendicular from point A to the line.

Length : Since foot of perpendicular P is known, length of perpendicular,

$$AP = \sqrt{[(\ell r + x_1 - \alpha)^2 + (mr + y_1 - \beta)^2 + (nr + z_1 - \gamma)^2]}$$

Equation of perpendicular is given by

$$\frac{x - \alpha}{\ell r + x_1 - \alpha} = \frac{y - \beta}{mr + y_1 - \beta} = \frac{z - \gamma}{nr + z_1 - \gamma}$$

13. EQUATIONS OF A PLANE :

The equation of every plane is of the first degree i.e. of the form $ax + by + cz + d = 0$, in which a, b, c are constants, where $a^2 + b^2 + c^2 \neq 0$ (i.e. a, b, c $\neq 0$ simultaneously).

(a) Vector form of equation of plane :

If $\vec{a}$ be the position vector of a point on the plane and $\vec{n}$ be a vector normal to the plane then it's vectorial equation is given by $(\vec{r} - \vec{a}).\vec{n} = 0 \Rightarrow \vec{r}.\vec{n} = d$ where $d = \vec{a}.\vec{n}$ = constant

(b) Plane Parallel to the Coordinate Planes :

(i) Equation of y-z plane is $x = 0$.

(ii) Equation of z-x plane is $y = 0$.

(iii) Equation of x-y plane is $z = 0$.

(iv) Equation of the plane parallel to x-y plane at a distance c is $z = c$. Similarly, planes parallel to y-z plane and z-x plane are respectively $x = c$ and $y = c$.

(c) Equations of Planes Parallel to the Axes :

If $a = 0$, the plane is parallel to x-axis i.e. equation of the plane parallel to x-axis is $by + cz + d = 0$.

Similarly, equations of planes parallel to y-axis and parallel to z-axis are $ax + cz + d = 0$ and $ax + by + d = 0$ respectively.

(d) Equation of a Plane in Intercept Form :

Equation of the plane which cuts off intercepts a, b, c from the axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(e) Equation of a Plane in Normal Form :

If the length of the perpendicular distance of the plane from the origin is p and direction cosines of this perpendicular are (l, m, n), then the equation of the plane is $lx + my + nz = p$.

(f) Vectorial form of Normal equation of plane :

If $\hat{n}$ is a unit vector normal to the plane from the origin to the plane and d be the perpendicular distance of plane from origin then its vector equation is $\vec{r}.\hat{n} = d$.

(g) Equation of a Plane through three points :

The equation of the plane through three non-collinear points

$(x_1, y_1, z_1), (x_2, y_2, z_2)$ (x_3, y_3, z_3) is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

14. ANGLE BETWEEN TWO PLANES :

Consider two planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$.

Angle between these planes is the angle between their normals.

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}$$

$\therefore$ Planes are perpendicular if $aa' + bb' + cc' = 0$ and they are parallel if $a/a' = b/b' = c/c'$.

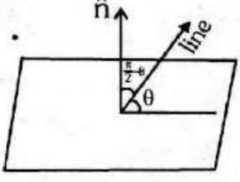
Planes parallel to a given Plane :

Equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + d' = 0$. d' is to be found by other given condition.

15. ANGLE BETWEEN A LINE AND A PLANE :

Let equations of the line and plane be $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and $ax + by + cz + d = 0$ respectively and θ be the angle which line makes with the plane. Then $(\pi/2 - \theta)$ is the angle between the line and the normal to the plane.

So $\sin\theta = \frac{a\ell + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(\ell^2 + m^2 + n^2)}}$



Line is parallel to plane if $\theta = 0$

i.e. if $al + bm + cn = 0$.

Line is ⊥ to the plane if line is parallel to the normal of the plane

i.e. if $\frac{a}{\ell} = \frac{b}{m} = \frac{c}{n}$.

16. CONDITION IN ORDER THAT THE LINE MAY LIE ON THE GIVEN PLANE :

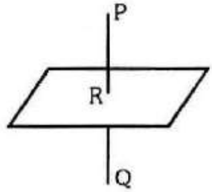
The line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ will lie on the plane $Ax + By + Cz + D = 0$

if **(a)** $A\ell + Bm + Cn = 0$ and **(b)** $Ax_1 + By_1 + Cz_1 + D = 0$

17. POSITION OF TWO POINTS W.R.T. A PLANE :

Two points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ are on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ & $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs.

18. IMAGE OF A POINT IN THE PLANE :



Let the image of a point $P(x_1, y_1, z_1)$ in a plane $ax + by + cz + d = 0$ is $Q(x_2, y_2, z_2)$ and foot of perpendicular of point P on plane is $R(x_3, y_3, z_3)$, then

(a) $$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = \frac{z_3 - z_1}{c} = -\left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right)$$

(b) $$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = -2\left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right)$$

19. CONDITION FOR COPLANARITY OF TWO LINES :

Let the two lines be

$$\frac{x - \alpha_1}{\ell_1} = \frac{y - \beta_1}{m_1} = \frac{z - \gamma_1}{n_1} \quad \text{.......... (i)}$$

and $$\frac{x - \alpha_2}{\ell_2} = \frac{y - \beta_2}{m_2} = \frac{z - \gamma_2}{n_2} \quad \text{.......... (ii)}$$

These lines will coplanar if $\begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 - \beta_1 & \gamma_2 - \gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

the plane containing the two lines is $\begin{vmatrix} x - \alpha_1 & y - \beta_1 & z - \gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

20. PERPENDICULAR DISTANCE OF A POINT FROM THE PLANE :

Perpendicular distance p, of the point $A(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is given by

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{(a^2 + b^2 + c^2)}}$$

Distance between two parallel planes $ax + by + cz + d_1 = 0$

& $ax + by + cz + d_2 = 0$ is - $\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

21. A PLANE THROUGH THE LINE OF INTERSECTION OF TWO GIVEN PLANES :

Consider two planes

$u \equiv ax + by + cz + d = 0$ and $v \equiv a'x + b'y + c'z + d' = 0$.

The equation $u + \lambda v = 0$, λ a real parameter, represents the plane passing through the line of intersection of given planes and if planes are parallel, this represents a plane parallel to them.

22. BISECTORS OF ANGLES BETWEEN TWO PLANES :

Let the equations of the two planes be $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$.

Then equations of bisectors of angles between them are given by

$$\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}$$

(a) Equation of bisector of the angle containing origin : First make both constant terms positive. Then +ve sign give the bisector of the angle which contains the origin.

(b) Bisector of acute/obtuse angle : First making both constant terms positive,

$aa_1 + bb_1 + cc_1 > 0 \Rightarrow$ origin lies in obtuse angle

$aa_1 + bb_1 + cc_1 < 0 \Rightarrow$ origin lies in acute angle