CHAPTER 6

PERMUTATION AND COMBINATION

6.1 INTRODUCTION

Permutations and combinations is the art of counting without counting i.e., we study various principles and techniques of counting to obtain the total number of ways an event can occur without counting each and every way individually.

6.2 FUNDAMENTAL PRINCIPLES OF COUNTING

6.2.1 Addition Rule

If an event (operation) E_1 can occur in n_1 ways, E_2 can occur in n_2 ways ,…, and E_k can occur in n_k ways (where $k \geq 1$). And these ways for the above events to occur are pair-wise disjoint, then the number of ways for at least one of the events (E_1, E_2, E_3, …, or E_k) to occurs is $(n_1 + n_2 + n_3 + \ldots + n_k) = \sum_{i=1}^{i=k} n_i$

- An equivalent form of above rule using set-theoretic terminology is given below:
  Let $A_1, A_2, \ldots. A_k$ be any k finite sets, where $k \geq 1$. If the given sets are pairs wise disjoint, i.e., $A_i \cap A_j = \phi$ for $i, j = 1, 2, \ldots, k$, $i \neq j$, then $\left|\bigcup_{i=1}^{k} A_i\right| = |A_1 \cup A_2 \cup \ldots \cup A_k| = \sum_{i=1}^{k} |A_i|$ where $|A_i|$ denotes the number of elements in the set A_i.

6.2.2 Multiplication Rule

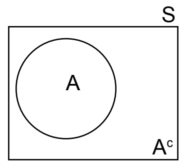
If an event E can be decomposed into ***n*** ordered event $E_1, E_2, \ldots, E_r$ and that there are n_1 ways for the event E_1 to occurs; n_2 ways for the event E_2 to occur, …, n_r ways for the event E_r to occur. Then the total number of ways for the event E to occur is given by: $n(E_1 \text{ and } E_2 \text{ and } \ldots., \text{ and } E_r) = n_1 \times n_2 \times \ldots, \times n_r = \prod_{i=1}^{r} n_i$.

- An equivalent form of (MP), using set-theoretic terminology is stated below
  $\prod_{i=1}^{r} A_i = A_1 \times A_2 \times \ldots \times A_r = \{(a_1, a_2, \ldots., a_n)\} \mid a_i \in A_i, i = 1, 2, \ldots, r\}$ denote the cartesian product of the finite sets $A_1, A_2, .., A_r$. Then $\left|\prod_{i=1}^{r} A_i\right| = |A_1| \times |A_2| \times \ldots \times |A_r| = \prod_{i=1}^{r} A_i$

Notes:

- *And stands for intersection (∩) or multiplication.*
- *Or stands for union (∪) or addition.*
- *Both addition and multiplication rules can be extended to any finite number of mutually exclusive operations.*

6.2.3 Complementation Rule



If A and $\bar{A}$ are two complementary sets and S be universal set, then

$\because \quad n(A) + n(\bar{A}) = n(S) \qquad \Rightarrow \quad n(A) = n(S) - n(\bar{A})$

So, we count $n(\bar{A})$ or $n(A)$ whichever is easier to count, then subtract from n(S) to get the other.

6.2.4 Principles of Inclusion-Exclusion

Let X be a finite set of m elements and $x_1, x_2, x_3, ..., x_r$ be some properties which the elements of X may or may not have if the subset of X having the property x_i (where i = 1, 2, 3,...., r) is X_i and those having both properties x_i and x_j is denoted by $X_i \cap X_j$ and so on.

Then the number of elements of X which have at least one of the properties $x_1, x_2, x_3, ..., x_r$ is given by $n\left(\bigcup_{i=1}^{r} X_i\right) = S_1 - S_2 + S_3 - S_4 + ... + (-1)^{r-1}S_r$ and the number of elements of U which have none of the properties $x_1, x_2, x_3, ...x_r$ is given by

$$n\left(\bigcup_{i=1}^{r} X_i^c\right) = m - S_1 - S_2 + S_3 - S_4 + ... + (-1)^{r-1}S_r; \text{where } S_1 = \sum_{i=1}^{r} n(X_i),\ S_2 = \sum_{1\le i<1}^{r}\sum_{\le r}^{r} n(X_i \cap X_j)$$

e.g., For r = 2, $n(X_1 \cup X_2) = n(X_1) + n(X_2) - n(X_1 \cap X_2)$.

For r = 3, $n(X_1 \cup X_2 \cup X_3) = n(X_1) + n(X_2) + n(X_3) - n(X_1 \cap X_2) - n(X_1 \cap X_3) - n(X_2 \cap X_3) + n(X_1 \cap X_2 \cap X_3)$.

6.2.5 Injection and Bijection Principles

Suppose that a group of n students attend a lecture in a lecture theater which has 100 seats, assuming that no student occupies more than one seat and no two students share a seat, if it is known that every student has a seat, then we must have n ≤ 100. If it known, furthermore, that no seat is vacant, then we are sure that n = 100 without actually counting the number of students.

6.2.5.1 Injection principle (IP)

Let A and B be two finite sets, if there is an injection from A to B, then |A| ≤ |B|.

6.2.5.2 Bijection principle (BP)

Let A and B be two finite sets, if there is a bijection from A to B, then |A| = |B|.

6.3 COMBINATIONS AND PERMUTATIONS

Each of the groups or selections which can be made by taking some or all of a number of things without considering the order in which the objects are taken is called a **combination.** Whereas a selection of objects where the order in which the objects are taken is also taken into account is called as an **arrangement/permutation**.

To understand the concept of combination and permutation, let us consider the combinations which can be made by taking the letters from a, b, c, d, two at a time namely,

$$\underbrace{\overbrace{\begin{array}{ccc} ab & ac & ad \\ & bc & bd \\ & & cd \end{array}}^{\text{Combinations (total no.=6)}} \quad \begin{array}{ccc} da & ca & ba \\ bd & cb & \\ dc & & \end{array}}_{\text{Permutations (total number = 12)}}$$

Number of combinations of 'n' distinct objects taken r at a time denoted as ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$.

Note:

From the above illustration, it is simply clear that in combinations we are only concerned with the number of things each selection contains without taking into account the order in which the objects are being selected. (i.e., ab and ba are regarded as same selection). Whereas in permutation the order of objects is taken into account.

6.4 PERMUTATION OF DIFFERENT OBJECTS

Case I: When repetition of objects is not allowed.

Number of permutation of n distinct things taken r at a time $(0 \le r \le n)$ is denoted by ${}^{n}P_{r}$ and it is equivalent to filling up of r vacancies by n different person, clearly first place can be filled in n ways and after which 2nd place can be filled in (n –1) ways and 3rd place can be filled in (n – 2) ways and similarly rth place can be filled in (n – r + 1) ways.

$$\Rightarrow \quad {}^{n}P_{r} = n(n-1)(n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!} = r!\,.\frac{n!}{r!(n-r)!} = {}^{n}C_{r} \times r!$$

Case II: When repetition of objects is allowed.

Number of permutation = $\underbrace{n \times n \times n \times \ldots \times n}_{r \text{ times}}$ = n^{r}, because now each of the vacancies can be filled up in n ways.

Notes:

- *The word indicating permutation are arrangement, standing in a line, seated in a row, problems on digits, word formation, rank of word, number of vectors joining given points and number of greetings sent among a group etc.*
- *The number of permutations of n distinct objects taken all at a time = n!.*
- *The number of all permutations of n different object taken r at a time, when a particular set of k objects is to be always included in each arrangement is* $r!\,{}^{n-k}C_{r-k}$.
- *Number of permutations of n different things, taken all at a time, when r specified things always remain together is* $r!(n-r+1)!$.

- *Number of permutations of n different things, taken all at a time, when r specified things never occur together is $n! - r!(n - r + 1)!$.*
- *The number of permutations of n different things, taken all at a time, when no two of the r particular things come together is ${}^{n-r+1}C_r\,(n - r)!\,r!$.*

6.5 PERMUTATION OF IDENTICAL OBJECTS (TAKING ALL OF THEM AT A TIME)

Number of permutations (N) of 'n' things taken all at a time when 'p' are of one kind, 'q' of a second kind, 'r' of a third kind and so on is given by $N = \dfrac{n!}{p!q!r!}$

Explanation let N be the required number of permutations. From any of these, if the p like things were different we could make p! new permutations. Thus if the p like things were all different, we would have got N(p!) new permutations. Similarly, if the q like things were different, we would get N (q!) new permutations from each of the second set of permutations.

Thus if the p like things and the q like things were all different, we would have got N.p!q! permutations in all. The process is continued untill all the sets of like things are different, and we then get the number of permutations of n things taken all at a time when they are all different (which is n!)

$$\therefore \quad N.p!.q!.r!... = n! \qquad \Rightarrow \quad N = \frac{n!}{p!q!r!}$$

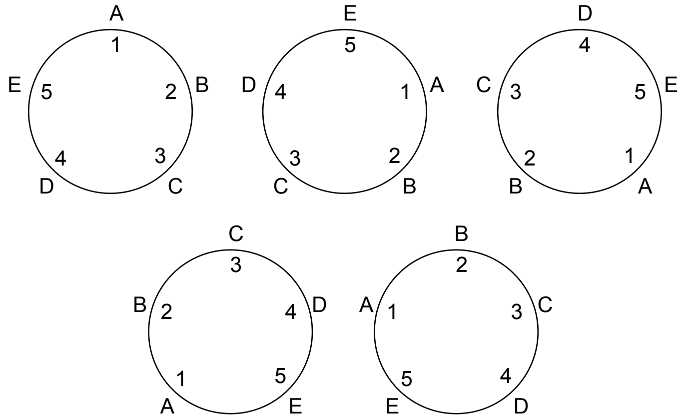
6.6 RANK OF WORDS

When all the letters of a word are arranged in all possible ways to form different words and the words formed are further arranged as per the order of ordinary dictionary, then the position occupied by that word is called as its rank. e.g., rank of the word MAT is 3 because it occupied third position in the alphabetical list (AMT, ATM, MAT, MTA, TAM, TMA) of words formed using letters A, M, T.

Shortcut to Find Rank of a Word	Example Banana	Example Large
1. Write the letters of the word in alphabetical order	AAABNN	A E G L R
2. Pick the letters one-by-one in the order in which they are heard while speaking.	B, A, N, A, N, A	L, A, R, G, E
3. For each of the letters in this order using representation in Step (1) find $x_n = \dfrac{\text{number of letter in left on n}}{p!q!}$ p, q are number of identical letters. Cross the letters as done with it.	$x_1 = \dfrac{3}{3!2!} \to AAA\cancel{B}NN$ $x_2 = 0 \to AANN$ $x_3 = \dfrac{2}{2!2!} \to AAN$ $x_4 = 0$ $x_5 = 1 \to A$	$x_1 = 3 \to AEG\cancel{L}R$ $x_2 = 0 \to \cancel{A}EG\cancel{L}R$ $x_3 = 2 \to EG\cancel{R}$ $x_4 = 1$
Rank = $(x_1)5! + (x_2)4! + (x_3)3! + (x_4)2! + x_5! + 0!$	$\dfrac{5!\times 3}{3!2!} + \dfrac{3!\times 2}{2!2!} + 1\times 1 + 0! = 34$	Rank $3 \times 4! + 0 \times 3! + 2.2! + 1.1! + 0! = 78$

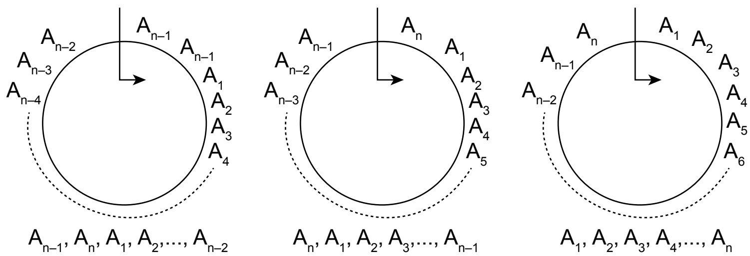
6.7 CIRCULAR PERMUTATION

The arrangement of objects around a circle is called, 'circular permutation'. Two circular permutations are called 'identical' iff one of them can be super imposed on the other by a suitable rotation without overturning and without changing the relative position of object. e.g., following 5 circular permutations are identical



6.7.1 Circular Permutation of n Objects

When 'n' distinct objects (A_1, A_2, A_3,..., A_n) are to be arranged around a circle, then each circular arrangement generates 'n' number of distinct linear arrangements by rotating the objects around the circle by $\left(\frac{360}{n}\right)^0$ at a time **(keeping their relative position fixed)**.



$\Rightarrow$ Each circular array generates 'n' linear permutation.
Let the total number of circular array be x.

$\Rightarrow$ Number of linear arrays = nx $\quad\Rightarrow\quad$ nx = n! $\quad\Rightarrow\quad x = \frac{n!}{n} = n-1!$

Remark:

- *As in circular permutation (unlike linear permutations) there is no initial and terminal position therefore fixing the position of one object around the circle its position acts as a terminal, consequently the remaining (n – 1) positions become as distinct as in linear permutations. Therefore, rest of (n – 1) object can be arranged in these position in (n – 1)! ways.*

Explanation:

In a circular permutation the relative position among the things is important whereas the place of a thing has no significance. Thus, in a circular permutation the first thing can be placed anywhere. This operation can be done only in one way, then relative order begins. Thus the ways for performing remaining parts of the operation can be calculated just like the calculation of linear permutation for an example to place 8 different things round a circle, first we place any one thing at any place, there will be only one numbers of ways = 7!. Thus, required number of circular permutations if 7!.

- *Since each circular arrangement has its unique counter-clockwise arrangement therefore, the number of clockwise array = number of counter-clockwise arrays =* $\frac{(n-1)!}{2}$.
- *In a garland of flowers or a necklace of beads (since the overturning of permutations is possible). It is difficult to distinguish clockwise and anti-clockwise orders of things, so a circular permutation under both these orders (the clockwise and anti-clockwise) is considered to be the same.*

  Therefore, the number of ways of arranging n beads along a circular wire is $\frac{(n-1)!}{2}$.
- *The total number of circular arrangements of n distinct objects taken r at a time is.*

  (i) ${}^nC_r(r-1)! = \frac{{}^nP_r}{r}$, *when clockwise and anticlockwise orders are treated as different.*

  (ii) $\frac{1}{2}\,{}^nC_r(r-1)! = \frac{{}^nP_r}{2r}$, *when the above two orders are treated as same.*

6.8 NUMBER OF NUMBERS AND THEIR SUM

Case I: Number of r digit numbers formed using n digits $\{D_1, D_2, \ldots, D_n\}$ when repetition allowed:

- Number of numbers = n^r
- Sum of all numbers = $\left(\frac{10^r-1}{9}\right).\left(\sum_{k=1}^{r} D_k\right).n^{r-1}$

Proof: When all the numbers formed are arranged vertically for summation. Any digit gets repeated n^{r-1} times in each column keeping a particular digit say D_k fixed at some place out of r, then remaining (r – 1) places can be arranged using n digits in n^{r-1} ways.

Summation of digits in any column = sum of all digits × repetition of digit $\left(\sum_{k=1}^{n} D_k\right)\left(n^{r-1}\right)$.

$\Rightarrow$ Sum of all numbers $\left(\sum_{k=1}^{n} D_k\right)\left(n^{r-1}\right)$ $(1 + 10 + 10^2 + \ldots + 10^{r-1})$

Case II: Number of r digit numbers formed using n digits $\{D_1, D_2, \ldots, D_n\}$ when repetition not allowed

- Number of numbers = $\begin{cases} {}^nP_r; & \text{if } r \le n \\ 0; & \text{if } r > n \end{cases}$.
- Sum of all numbers = $\left(\frac{10^r-1}{9}\right).\left(\sum_{k=1}^{r} D_k\right).{}^{n-1}P_{r-1}$.

Proof: When all the numbers formed are arranged vertically for summation. Any digit gets repeated ${}^{n-1}P_{r-1}$ times in each column keeping a particular digit, say D_k fixed at some place out of r, then remaining (r – 1) places can be arranged using n – 1 digits in ${}^{n-1}P_{r-1}$ ways.

Summation of digits in any column = sum of all digits × repetition of digit = $\left(\sum_{k=1}^{n} D_k\right) . {}^{n-1}P_{r-1}$.

$\Rightarrow$ Sum of all numbers = $\left(\sum_{k=1}^{n} D_k\right)\left({}^{n-1}P_{r-1}\right)(1 + 10 + 10^2 + \ldots + 10^{r-1}) = \left(\frac{10^r - 1}{9}\right).\left(\sum_{k=1}^{r} D_k\right).{}^{n-1}P_{r-1}$.

6.8.1 Divisor of Composite Number

A natural number $x = p^{\alpha} q^{\beta} r^{\gamma}$ is called divisor of $N = p^a . q^b . r^c$ iff N is completely divisible by x. For Example, when all the prime factors of x are present in N which is possible only if $0 \le \alpha \le a$; $0 \le \beta \le b$ and $0 \le \gamma \le c$, where $\alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}$.

- **Set of all divisors of *N* is given as:** $\{x : x = p^{\alpha} q^{\beta} r^{\gamma}$; where $0 \le \alpha \le a; 0 \le \beta \le b; 0 \le \gamma \le c\}$.
- **Number of divisor:** number of divisors = $n\{(\alpha, \beta, \gamma) : 0 \le \alpha \le a; 0 \le \beta \le b; 0 \le \gamma \le c\}$
  $= n\{\alpha : 0 \le \alpha \le a\} \times n\{\beta : 0 \le \beta \le b\} \times n\{\gamma : 0 \le \gamma \le c\} = (a + 1)(b + 1)(c + 1)$.
- Number of divisors are given by number of distinct terms in the product
  $= (1+ p + p^2 + \ldots. + p^a)(1+ q + q^2 + \ldots. + q^b)(1+ r + r^2 + \ldots. + r^c)$
  $= (a + 1)(b + 1)(c + 1)$.... (which includes 1 and the N it self)

6.8.2 Sum of Divisor

Since each individual divisor is given as terms of the expansion, $(p^0 + p^1 + p^2 + \ldots. + p^a)(1+ q + q^2 + \ldots. + q^b)(1+ r + r^2 + \ldots. + r^c)$ therefore the sum of all divisors is $= 1 + p + q + r + p^2 + q^2 + r^2 + pq + pr + \ldots\ldots + p^a . q^b . r^c$

$$= \left(\frac{p^{a+1} - 1}{p - 1}\right)\left(\frac{q^{b+1} - 1}{q - 1}\right)\left(\frac{r^{c+1} - 1}{r - 1}\right) \quad \ldots.(i)$$

Notes:

- *Improper/Proper divisors of $N = p^a . q^b . r^c$: When $\alpha = \beta = \gamma = 0$*
  $\Rightarrow$ *$x = 1$ which is divisor of every integer and $\alpha = a$, $\beta = b$ and $\gamma = c$, then x becomes number N itself. These two are called 'improper divisor.'*
  $\Rightarrow$ *The number of proper divisors of $N = (a + 1).(b + 1).(c + 1) - 2$.*
- *If $p = 2$, then number of even divisors $= a(b + 1)(c + 1)$, number of odd divisors $= (b + 1)(c + 1)$.*

6.8.3 Number/Sum of Divisors Divisible by a Given Number

If $x = p^{\alpha} . q^{\beta} . r^{\gamma}$ is divisor of $N = p^a . q^b . r^c$ and completely divisible by $y = p^{\alpha_1}.q^{\beta_1}.r^{\gamma_1}$.

- **Set of all divisors of N is given as:** $\{ x : x = p^{\alpha} q^{\beta} r^{\gamma}$; where $\alpha_1 \le \alpha \le a; \beta_1 \le \beta \le b; \gamma_1 \le \gamma \le c\}$.

$\Rightarrow$ Number of divisors = $n\{(\alpha, \beta, \gamma)\}$: $\{\alpha_1 \le \alpha \le a; \beta_1 \le \beta \le b; \gamma_1 \le \gamma \le c\} = (a - \alpha_1 + 1).(b - \beta_1 + 1)(c - \gamma_1 + 1)$.

6.8.4 Factorizing a Number into Two Integer Factors

If x and y be two factors of the Natural Number $N = p^a . q^b . r^c$: $N = x.y$

$\Rightarrow$ x and y are divisors of N.

Case I: If number N is not a perfect square:

Number of two factor products $= \dfrac{\text{(number of total divisors)}}{2}$

Case II: If number N is a perfect square:

Number of two factor products $= \dfrac{\text{(number of total divisors)} + 1}{2}$

Case III: Number of integer solution of equation $x.y = p^a . q^b . r^c . s^d = 2 \times$ total number of divisor

Since number of natural number solution of the equation:

$x.y = p^a . q^b . r^c . s^d$ = Number of divisors = $(a + 1)(b + 1)(c + 1)(d + 1)$

$\Rightarrow$ Number of integer solution of the equation = $2(a + 1)(b + 1)(c + 1)(d + 1)$

6.9 COMBINATION

Combination of n objects taken r at a time is denoted as ${}^nC_r$ and defined as ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

6.9.1 Properties of Combinations

1. The number of combination of n different things taken r at a time is denoted by ${}^nC_r$ or C(n, r) or $\binom{n}{r}$ and it is empirically calculated as: ${}^nC_r = \dfrac{n!}{r!(n-r)!}$; $(0 \le r \le n)$; where $n \in N$ and $r \in W$ {whole numbers} = 0 (if r > n).
2. ${}^nC_r$ is always an integer.
   The following important conclusions can be made out of the above statement:
   (a) Product of r consecutive integers is always divisible by r!
   $\because \quad {}^nC_r = \dfrac{n(n-1)(n-2)(n-3).....(n-r+1)}{r!} \in I$
   Clearly, the numerator is completely divisible by r!
   (b) $0! = 1; \left\{ {}^nC_0 = {}^nC_n = \dfrac{n!}{n!\,0!} = 1 \right\}$ and ${}^nC_1 = n$
   (c) $k! = \infty$ if $k < 0$ (Think why?)
3. ${}^nC_r = {}^nC_{n-r}$ this is simply selection of r things means rejection of n – r at the same time.
4. ${}^nC_x + {}^nC_y \Rightarrow x = y$ or $x + y = n$
5. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ $(1 < r < n)$ this is also known as Pascal Rule.

${}^{n+1}C_r$	=	${}^nC_r$	+	${}^nC_{r-1}$
$\downarrow$		$\downarrow$		$\downarrow$
Choosing any r objects out of (n+1) given objects	≡	Choosing r objects excluding one particular out of (n+1) objects	or	Choosing r objects always excluding one particular out of (n+1) objects

6. $r.{}^nC_r = n.{}^{n-1}C_{r-1}$ $\Rightarrow$ $${}^nC_r = \frac{n}{r}({}^{n-1}C_{r-1}) = \frac{n}{r}\left(\frac{n-1}{r-1}\right){}^{n-2}C_{r-2} = ...$$

Thus we can work out as:

Choosing r MP's from n citizens (${}^nC_r$ ways) Choosing 1 PM from r Choosen MP's (r ways) $r \times {}^rC_r$ ways MP : Member of Parliament	$\equiv$	Choosing 1 PM from n citizens (n ways) and Choosing remaining (r–1) MP's from remaining (n – 1) citizens (${}^{n-1}C_{r-1}$) ways MP : Prime Minister

7. $${}^nC_r = {}^nC_{r+1}.\left(\frac{r+1}{n+1}\right) = \frac{(r+1(r+2)}{(n+1)(n+2)}\,{}^{n+2}C_{r+2}$$

8. ${}^nC_r\,{}^rC_s = {}^nC_s\,.\,{}^{n-s}C_{r-s}$; $(n \geq r \geq s)$**.** This we can work out as

Choosing r MP's (${}^nC_r$ ways) and Choosing s ministers out of r MP's (${}^rC_s$ ways) ${}^nC_r \times {}^rC_s$	$\equiv$	Choosing s ministers (${}^rC_s$ ways) and Choosing remaining (r – s) MP's out of remaining (n – s) citizens ${}^nC_s \times {}^{n-s}C_{r-s}$

9. $$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

10. ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$ this is selection of any number of objects out of given n objects. For each object we have only two possibilities selection or rejection which is 2^n.

11. ${}^nC_0 + {}^nC_2 + {}^nC_4 + ...= {}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$.

12. ${}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + ... + {}^mC_m = {}^{n+1}C_{m+1}$.

6.9.2 Restricted Combinations

The number of combinations of n different things taking r at a time

(a) When p particular things are always to be excluded = ${}^{n-p}C_r$.

(b) When p particular things are always to be included = ${}^{n-p}C_{r-p}$.

(c) When p particular things are always included and q particular things are always excluded = ${}^{n-p-q}C_{r-z}$.

6.9.3 Combination of Objects Taking any Number of Them at a Time

- Number of selections of objects when any number of them can be selected is given as ${}^nC_0 + {}^nC_1 +....... + {}^nC_n = 2^n$
  Where ${}^nC_r$ corresponds to the case when r objects are selected out of n different objects. In above case r varies from 0 to n. The right hand side value 2^n can be explained as number of ways of dealing with all n objects each in exactly two ways either selected or rejected.
- Number of selection of objects (at least one) out of n different objects $\sum_{r=0}^{n} {}^nC_r = {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$.
- Number of selection of atleast two object out of n = ${}^{2n-n}C_0 - {}^nC_1$.

6.9.4 Combination when Some Objects are Identical (Taking any Number of Them at a Time)

1. **Combination when some objects are identical.** The total number of ways in which it is possible to make a selection taking some or all out of (p + q + r) things, where p are alike of the first kind, q are alike of the second kind and r alike of the third kind and s are different = $(p + 1)(q + 1)(r + 1)2^s$ ways

   Explanation: Out of p alike things, we may select none or one or two or three, ... or all p. Hence they may be disposed off in (p + 1) ways. Similarly, q alike things may be disposed of in (q + 1) ways, similarly for r. And s different things may be disposed of in 2^s ways. (This includes the case in which all of them are rejected).

   - Number of ways (if at least one object to be selected) = $(p + 1)(q + 1)(r + 1)2^s - 1$.
   - Number of ways (if at least one from s different object to be selected) = $(p + 1)(q + 1)(r + 1)(2^s - 1)$.
   - Number of ways (if at least one object of each identical type lot is to be selected) = $(p . q . r)2^s$.

6.9.5 Combination when Some Objects are Identical (Taking specific number of them at a time)

Case 1: If a group has n things in which p are identical, then the number of ways of selecting r things from a group is $\sum_{k=0}^{r} {}^{n-p}C_k$ or $\sum_{k=r-p}^{r} {}^{n-p}C_k$ according as $r \le p$ or $r > p$.

Explanation: It can be obtained by assuming the selection of k distinct object and rest r − k objects identical and taking the values of variable k from 0 to r (or p) whichever is less.

For an instance when no object is selected from identical objects (k = 0), then the number of selection = ${}^{n-p}C_r$.

And when one object is selected from identical objects (k = 2), then the number of selection = ${}^{n-p}C_{r-1}$

Similarly, for k = 3 the number of selection = ${}^{n-p}C_{r-2}$ and so on.

Notes:

(i) The number of ways of selecting r objects out of n identical objects is 1.

(ii) The number of ways of selecting any number of objects out of n identical objects is n + 1.

Case 2: If there are p_1 objects of one kind, p_2 objects of second kind, ..., p_n objects of nth kind then the number of ways of choosing r objects out of these $(p_1 + p_2 + ... + p_n)$ objects

= coefficients of x^r in $(1 + x + ... + x^{p_1})(1 + x + ... + x^{p_2})...(1 + x^2 + ... + x^{p_n})$

If one object of each kind is to be included in such a collection, then the number of ways of choosing r objects.

= coefficients of x^r in the product $(x + ... + x^{p_1})(x + ... + x^{p_2})...(x^2 + ... + x^{p_n})$

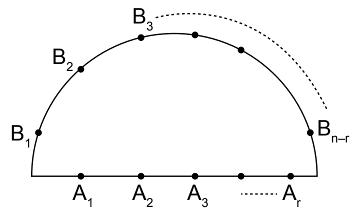
This problem can also be stated as

Let there be n distinct objects x_1, ... x_n; x_1 can be used at the most p_1 times, x_2 at the most p_2 times, ..., x_n at the most p_n times, then the number of ways to have r things.

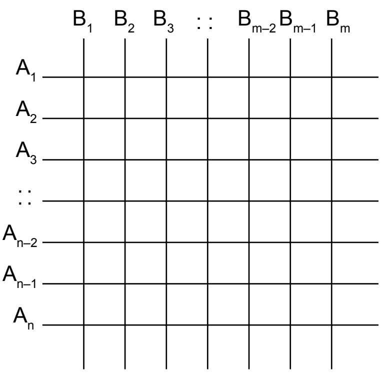
Renarks:

- *Given n distinct points in a plane, no three of which are collinear, then the number of line segments they determine is ${}^nC_2$.*

- *The number of diagonals in n-polygon (n sides closed polygon) is* $^nC_2 - n$.
  If in which m points are collinear ($m \geq 3$), *then the number of line segments is* $(^nC_2 - {}^mC_2) + 1$.
- *Given n distinct points in a plane, no three of which are collinear then the number of triangles formed =* $^nC_3$. *If in which m points are collinear* ($m \geq 3$), *then the number of triangles is* $^nC_3 - {}^mC_3$.
- *Given n distinct points of which no three points are collinear:*
  (i) *Number of straight lines =* $^nC_2$
  (ii) *Number of triangles =* $^nC_3$.
  (iii) *Number of quadrilaterals =* $^nC_4$.
  (iv) *Number of pentagon =* $^nC_5$.
- *Given n points in a plane out of which r of them are collinear. Except these r points no other three points are collinear. Then number of different geometric figures constructed by joining these points are expressed as below.*



  (i) *number of line segments (L.S.) =* $^nC_2$
  (ii) *number of directed line segments vectors (D.L.S.) =* $^nP_2$
  (iii) *number of lines formed*
  $= {}^nC_2 - {}^rC_2 + 1$ *or* ${}^{n-r}C_2 + (n-r)r + 1$.
  (iv) *number of triangles formed*
  $= {}^nC_3 - {}^rC_3$ *or* ${}^{n-r}C_3 + ({}^{n-r}C_2)r + (n-r)\,{}^rC_2$.
  (v) *number of quadrilateral*
  $= {}^nC_4 - ({}^rC_4 + (n-r)\,{}^rC_3))$ *or* ${}^{n-r}C_4 + {}^{n-r}C_3 \cdot {}^rC_1 + {}^{n-r}C_2 \cdot {}^rC_2$.
  (vi) *number of rectangles/squares formed put of m horizontal lines and n vertical lines such that distance between conjugative line both set of parallel lines is unity.*
- *Given* $A_1, A_2, A_3, \ldots, A_n$ *are horizontal lines* $B_1, B_2, B_3, \ldots B_m$ *are vertical lines as shown in figure:*



  (i) *Number of rectangles = number of ways of choosing two lines from each set =* $n\left(\overline{A_iA_j} \text{ and } \overline{B_kB_l}\right)$
  $= {}^nC_2 \times {}^mC_2$.
  (ii) *Number of square of size* $k \times k$ *= number of ways of choosing two lines* $\overline{A_iA_{j+i}}$ *horizontal line =* $\underbrace{n\left(\overline{A_iA_{j+k}}\right)}_{1 \le i \le n-k} \times \underbrace{n\left(\overline{B_jB_{j+k}}\right)}_{1 \le j \le m-k}$ $= (n-k)(m-k)$.
  (iii) *Total number of squares =* $\sum_{k=1}^{k=r}(n-k)\ (m-k)$ *where* $r = \min\{m-1, n-1\}$.

6.10 DISTRIBUTION

6.10.1 Distribution Among Unequal Groups

To find the number of ways in which m + n things can be divide into two groups containing m and n things respectively. This is clearly equivalent to finding the number of combinations of m + n things taking m at a time, for every time we select a group of m things we leave a group of n things behind.

Thus, the required number = $\dfrac{(m+n)!}{m!n!}$.

6.10.2 To Find the Number of Ways in Which the m + n + p Things Can be Divided into Three Groups Containing m, n, p Things Separately

First divide the m + n + p things into two groups containing m and n + p things respectively; the number of ways in which this can be done is ${}^{m+n+p}C_m = \frac{(m+n+p)!}{m!(n+p)!}$. And the number of ways in which the group of n + p things can be divided into two groups containing n and p things respectively is ${}^{n+p}C_n = \frac{(n+p)!}{n!p!}$. Hence the number of ways in which the subdivision into three groups containing m, n, p things can be made follows:

6.10.3 Distribution Among Equal Groups

When name of groups is not specified: If 2m objects are to be distributed among two equal groups containing m objects each. Then it can be done in $\frac{(m+m)!}{m!\,m!2!} = \frac{2m!}{(m!)^2\,2!}$ because each division it is possible to arrange the groups into 2! ways without obtaining new distribution.

Explanation: Then we divide the total number of arrangements obtained normally by k! where k is | number of groups among which the objects are distributed. If we put n = p = m we obtain $\frac{3m!}{m!\,m!\,m!}$; but since this include 3! times the actual number of divisions because of the arrangement of groups among them selves, therefore the number of different ways in which subdivision into three equal groups can be made is $= \frac{3m!}{m!m!\,m!\,3!}$.

6.10.4 When Name of Groups Specified

If the name of groups among which the objects are distributed are specified (e.g., distributing books to students, dividing soldiers into regiment, distributing students into sections etc.) If we put n = p = m , we obtain $\frac{3m!}{m!\,m!\,m!}$.

- The number of ways of dividing pq objects among p groups of same size, each group containing q objects = $\frac{(pq)!}{(q!)^p.p!}$.
- The number of ways of distributing pq objects among n people, each person getting q objects = $\frac{(pq)!}{(q!)^p}$.

6.11 MULTINOMIAL THEOREM

The expansion of $[x_1 + x_2 + x_3 + ... + x_n]^r$; where n and r are integers ($0 < r \leq n$) is a homogenous expression in $x_1, x_2, x_3, ... x_n$ and given as: $[x_1+ x_2 + x_3 + ... + x_n]^r = \sum\left(\frac{r!}{\lambda_1!\,\lambda_2!\,\lambda_3!......\lambda_n!}\right) x_1^{\lambda_1}\,x_2^{\lambda_2}\,x_3^{\lambda_3}\,....x_n^{\lambda_n}$;

(where n and r are integers $0 < r \le n$ and λ_1, λ_2, ..., λ_n are non-negative integers). Such that $\lambda_1 + \lambda_2 + ...+ \lambda_n = r$ (valid only if $x_1, x_2, x_3, ..., x_n$ are independent of each other) coefficient of $x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}....$ = total number of arrangements of r objects out of which λ_1 number of x_1's are identical λ_2 number of x_2's are identical and so on $= \dfrac{(\lambda_1+\lambda_2+\lambda_3+......\lambda_n)!}{\lambda_1!\lambda_2!\lambda_3!......\lambda_n!} = \dfrac{(r)!}{\lambda_1!\lambda_2!\lambda_3!......\lambda_n!}$

6.11.1 Number of Distinct Terms

Since $(x_1 + x_2 + x_3 + ... + x_n)^r$ is multiplication of $(x_1 + x_2 + x_3 + ... + x_n)^r$ times and will be a homogeneous expansion of r^{th} degree in $x_1, x_2, ...x_n$ .So in each term sum of powers of variables must be r.

So number of distinct terms will be total number of non-negative integral solution of equation is $\lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n = r$ = Number of ways of distributing r identical objects among n persons = number of ways of distributing r balls among n people

= number of arrangements of r balls and n – 1 identical separators $= \dfrac{(n-1+r)!}{(n-1)!r!} = {}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$.

6.12 DEARRANGEMENTS AND DISTRIBUTION IN PARCELS

Any change in the order of the things in a group is called a derangement. If n things are arranged in a row, the number of ways in which they can be dearranged so that none of them occupies its original position is $n!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-......+(-1)^n\frac{1}{n!}\right\}$

- If r objects go to wrong places out of n thing, then (n – r) objects go to their original place. If $\Delta_n \to$ number of group, and if all objects go to the wrong places and $\Delta_r \to$ number of ways if r objects go to wrong places out of n, then (n – r) objects go to correct places. Then $\Delta_n = {}^nC_{n-r}\,\Delta_t$; where $\Delta_r = r!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-......+(-1)^r\frac{1}{r!}\right\}$
- Derangement of a given n-permutations $\underbrace{P_1P_2P_3...P_{n-1}P_n}_{n-\text{permutation}}$ is an arrangement in which at least one object does not occupy its assigned position. ⇒ Total number of dearrangements = n! – 1.
- Let A_i denotes set of arrays when i^{th} objects occupies i^{th} place $n(A_i) = (n-1)!$

  ⇒ $n(A_1 \cup A_j) = (n-2)!$

  ⇒ Number of arrays in which atleast one object occupies its correct place $= n(A_1 \cap A_2 \cap A_3 .... \cap A_n) = \Sigma n(A_i) - \Sigma n(A_i \cup A_j) + \Sigma n(A_i \cup A_j \cup A_k) - .... + (-1)^{n-1}\, n(A_1 \cup A_2 \cup A_3 .... \cup A_n)$

  $= {}^nC_1\,(n-1)! - {}^nC_2(n-2)! + {}^nC_3\,(n-3)! - ..... + (-1)^{n-1}\,.\,{}^nC_nO!$

  $= \dfrac{n!}{1!}-\dfrac{n!}{2!}+\dfrac{n!}{3!}-....+\dfrac{(-1)^n n!}{n!} = n!\left(\dfrac{1}{1!}-\dfrac{1}{2!}+\dfrac{1}{3!}-....+\dfrac{(-1)^{n-1}}{n!}\right)$

  the total number of dearrangement in which no object occupies its correct place = n! – n $(A_1 \cap A_2 \cap A_3 .... \cap A_n)$

  $= n!-n!\left(\dfrac{1}{1!}-\dfrac{1}{2!}+\dfrac{1}{3!}....+\dfrac{(-1)^{n-1}}{n!}\right) = n!\left(1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+\dfrac{1}{4!}...+\left(\dfrac{(-1)^n}{n!}\right)\right) = n!\left(\dfrac{1}{2!}-\dfrac{1}{3!}+\dfrac{1}{4!}....+\dfrac{(-1)^n}{n!}\right)$

- Number of dearrangement in which exactly r objects occupy their assigned places

$$= \underbrace{{}^{n}C_r}_{\substack{\text{choo singr objects}\\ \text{and placing them}\\ \text{at their correct places}}} \underbrace{\times(n-r)!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots+\frac{(-1)^{n-r}}{(n-r)!}\right)}_{\substack{\text{Arrangingg n–r objects so that none}\\ \text{oft hem occupies their assigned positions}}}$$

6.13 DISTRIBUTION IN PARCELS

6.13.1 Distribution in Parcels When Empty Parcels are Allowed

The number of ways in which n different objects can be distributed in r different groups (here distributed means order of objects inside a group is not important) under the condition that empty groups are allowed = r^n. Take any one of the objects which can be put in any one of the groups in r ways. Similarly, all the objects can be put in any one of groups in r number of ways. So number of ways = r . r . r......n times = r^n

= coefficient of x^n in $n!\,(e^x)^r = \sum_{k=0}^{r-1}(-1)^k.\,{}^{r}C_k(r-k)^n$.

6.13.2 When at Least One Parcel is Empty

Number of distribution when at least one parcel is empty:

$= n(A_1 \cup A_2 \cup A_3 \ldots.. \cup A_r)$ {Ai is the set of distribution when i^{th} parcel is empty}

$n(A_i) = (r - n)^n$ and $n(A_i \cap A_j) = (r - 2)^n$

$= \Sigma n(A_i) - \Sigma n(A_i \cap A_j) + \Sigma n(A_i \cap A_j \cap A_k) + \ldots. + (-1)^{r-1} n(A_1 \cap A_2 \cap \ldots. \cap A_r)$

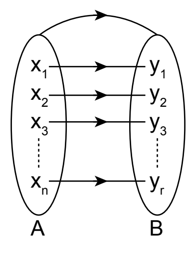
$= {}^{n}C_1(r-1)^n - {}^{r}C_2(r-2)^n + {}^{r}C_3(r-3)^n + \ldots\ldots. + (-1)^{r-1}\,{}^{r}C_{r-1} = \sum_{k=1}^{r-1}(-1)^{k-1}\,{}^{r}C_k(r-k)^n$

The number of ways in which n different objects can be arranged in r different groups:

$= n! \times {}^{n+r-1}C_{r-1}$ if empty groups are allowed = $n! \times {}^{n-1}C_{r-1}$ if empty groups are not allowed

The number of ways in which n different things can be distributed into r different places, blank roots being admissible is r^n.

Remarks



Given two sets $A = \{a_1, a_2, \ldots., a_n\}$ and $B = \{b_1, b_2, b_3, \ldots, b_r\}$, then following holds good.

(i) $n(A \times B) = n(A)\,.\,n(B) = n \times r$.

(ii) Number of relation $R : A \to B$ = number of subsets of $A \times B = 2^{n.r}$.

(iii) Number of functions $f : A \to B$ = number of ways of distributing n elements (objects) of A in to elements (boxes) of B = r^n.

(iv) Number of injective functions $f : A \to B$ = number of permutations of n elements of A (objects) over r elements of B (places) $= \begin{cases} {}^{r}P_n & \text{if } r \geq n \\ 0 & \text{if } r < n \end{cases}$.

(v) Number of into (non surjective) functions $f : A \to B$ = number of ways of distributing n elements (objects) of A into elements (boxes) of B such that atleast one box is empty $= \sum_{k=1}^{r-1}(-1)^{k-1}.\,{}^{r}C_k(r-k)^n$.

(vi) Number of on-to (surjective) functions $f : A \to B$ = number of ways of distributing n elements (objects) of A in to elements (boxes) of B, such that no box is empty $= \sum_{k=0}^{r-1}(-1)^k.\,{}^{r}C_k(r-k)^n$.

6.14 EXPONENT OF A PRIME IN N!

Exponent of prime p in n! is denoted by E_p (n!); where n is natural number, so the last integer amongest 1, 2,.....,(n – 1)n which is divisible by p is [n/p] p when [n] ≤ x.

$$\Rightarrow \quad E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots + \left[\frac{n}{p}\right]^s$$, where s is the largest number, such that $p^s \le n < p^{s+1}$

6.14.1 Exponent of Prime 'P' in n!

Exponent of prime number 'p' in n! is defined as power of p when n! is factorized into prime factor using unique factorization theorem and it is denoted as E_p (n!).

Theorem: The largest natural number divisible by p is less than or equal to 'n' is given as $\left[\frac{n}{p}\right]p$.

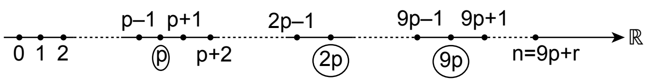
Proof: Division algorithm as n ≤ p, thus there exist:

Two natural number q and r, such that n = p.q + r

where $0 \le r \le p \qquad \Rightarrow \quad \frac{n}{p} = q + \frac{r}{p}$; where $0 \le \frac{r}{p} < 1$

q is called integer part of number n/p, denoted as $\left[\frac{n}{p}\right]$ and $\frac{r}{p}$ is known as fractional part of number n/p denoted as $\left\{\frac{n}{p}\right\}$. Observe the situation on ℝ number lies.

Conclusion! i.e., $\left[\frac{n}{p}\right]$ is the quotient in the division of n by p.



Theorem: The number of natural numbers divisible by p less than or equal to 'n' is equal to $\left[\frac{n}{p}\right]$

⇒ The number of natural numbers divisible by p^2 less than or equal to 'n' is equal to $\left[\frac{n}{p^2}\right]$

⇒ The number of natural numbers divisible by p^3 less than or equal to 'n' is equal to $\left[\frac{n}{p^3}\right]$.

∴ Exponent of prime p in n! $E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots$.