Session 3

Slope of Tangent and Normal, Equation of Tangent, Equation of Normal

Slope of Tangent and Normal

Slope of Tangent

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be a point on it.

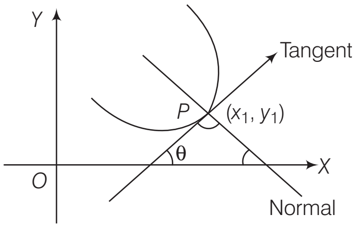


Figure 7.3

Then, $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$ **is the slope of tangent to the curve $y = f(x)$ at a point $P(x_1, y_1)$.**

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_P = \tan\theta = \text{Slope of tangent at } P$$

where, θ is the angle which the tangent at $P(x_1, y_1)$ forms with the positive direction of X-axis as shown in the figure.

Remarks

(i) **Horizontal tangent** If tangent is parallel to X-axis, then

$$\theta = 0° \Rightarrow \tan\theta = 0$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

(ii) **Vertical tangent** If tangent is perpendicular to X-axis or parallel to Y-axis, then

$$\theta = 90° \quad \Rightarrow \quad \tan\theta = \infty \quad \text{or} \quad \cot\theta = 0$$

$$\therefore \quad \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

Slope of Normal

We know that the normal to the curve at $P(x_1, y_1)$ is a line perpendicular to tangent at $P(x_1, y_1)$ and passes through P.

$\therefore$ **Slope of the normal at**

$$P = -\frac{1}{\textbf{Slope of the tangent at } P}$$

$$\Rightarrow \text{Slope of normal at } P(x_1, y_1) = -\frac{1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}$$

$$\text{or} \quad \text{Slope of normal at } P(x_1, y_1) = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

Remarks

(i) **Horizontal normal** If normal is parallel to X-axis, then

$$-\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0 \quad \text{or} \quad \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

(ii) **Vertical normal** If normal is perpendicular to X-axis or parallel to Y-axis, then $-\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = 0$

Example 14 ***Find the slopes of the tangent and normal to the curve $x^3 + 3xy + y^3 = 2$ at (1, 1).***

Sol. Given equation of curve is $x^3 + 3xy + y^3 = 2$.

Differentiating it w.r.t. x, we get

$$3x^2 + 3x\frac{dy}{dx} + 3y + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(3x^2 + 3y)}{(3x + 3y^2)} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{(x^2 + y)}{(x + y^2)}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = -\left(\frac{2}{2}\right) = -1$$

$$\therefore \quad \text{Slope of tangent at } (1,1) = \left(\frac{dy}{dx}\right)_{(1,1)} = -1$$

$$\text{and slope of normal at } (1,1) = -\frac{1}{\left(\dfrac{dy}{dx}\right)_{(1,1)}} = \frac{-1}{-1} = 1$$

Example 15 ***Find the point on the curve $y = x^3 - 3x$ at which tangent is parallel to X-axis.***

Sol. Let the point at which tangent is parallel to X-axis be $P(x_1, y_1)$.

Then, it must lie on curve.

Therefore, we have $y_1 = x_1^3 - 3x_1$ ...(i)

Differentiating $y = x^3 - 3x$ w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 3 \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 - 3$$

Since, the tangent is parallel to X-axis.

$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 0 \Rightarrow 3x_1^2 - 3 = 0$

$\Rightarrow \quad x_1 = \pm 1 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

When $x_1 = 1$, then $y_1 = 1 - 3 = -2$

When $x_1 = -1$, then $y_1 = -1 + 3 = 2$

$\therefore$ Points at which tangent is parallel to X-axis are $(1, -2)$ and $(-1, 2)$.

Example 16 Find the point on the curve $y = x^3 - 2x^2 - x$ at which the tangent line is parallel to the line $y = 3x - 2$.

Sol. Let $P(x_1, y_1)$ be the required point.

Then, we have $\quad y_1 = x_1^3 - 2x_1^2 - x_1 \quad$ ...(i)

Differentiating the curve $y = x^3 - 2x^2 - x$ w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 4x - 1 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 3x_1^2 - 4x_1 - 1$$

Since, tangent at (x_1, y_1) is parallel to the line $y = 3x - 2$.

$\therefore$ Slope of the tangent at $P(x_1, y_1)$ = Slope of the line

$$y = 3x - 2 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 3$$

$\Rightarrow \quad 3x_1^2 - 4x_1 - 1 = 3 \Rightarrow 3x_1^2 - 4x_1 - 4 = 0$

$\Rightarrow \quad (x_1 - 2)(3x_1 + 2) = 0 \Rightarrow x_1 = 2, -2/3 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

When $x_1 = 2$, then

$$y_1 = 8 - 8 - 2 \Rightarrow y_1 = -2$$

When $\quad x_1 = -2/3$, then $y_1 = x_1^3 - 2x_1^2 - x_1$

$\Rightarrow \quad y_1 = \frac{-8}{27} - \frac{8}{9} + \frac{2}{3} \Rightarrow y_1 = \frac{-14}{27}$

Thus, the point at which tangent is parallel to $y = 3x - 2$ are $(2, -2)$ and $\left(-\frac{2}{3}, -\frac{14}{27}\right)$.

Example 17 ***In which of the following cases, the function $f(x)$ has a vertical tangent at $x = 0$?***

(i) $f(x) = x^{1/3}$ (ii) $f(x) = \text{sgn } x$

(iii) $f(x) = x^{2/3}$ (iv) $f(x) = \sqrt{|x|}$

(v) $f(x) = \begin{cases} 0, \text{if } x < 0 \\ 1, \text{if } x \ge 0 \end{cases}$

Sol. **Vertical Tangent**

Concept $y = f(x)$ has a vertical tangent at the points $x = x_0$, if

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \infty \text{ or } -\infty \text{ but not both.}$$

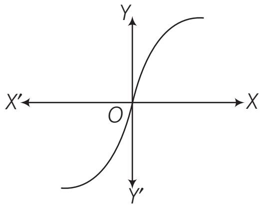
Here, the functions $f(x) = x^{1/3}$ and $f(x) = \text{sgn } x$ both have a vertical tangent at $x = 0$, but $f(x) = x^{2/3}, f(x) = \sqrt{|x|}$ and $f(x) = \begin{cases} 0, \text{ if } x < 0 \\ 1, \text{ if } x \ge 0 \end{cases}$ have no vertical tangent.

Explanation

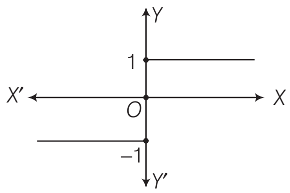
(i) $f(x) = x^{1/3}$

$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{h^{1/3}}{h} = \frac{1}{h^{2/3}} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{(-h)^{1/3}}{-h} = \frac{1}{(-h)^{2/3}} = \frac{1}{h^{2/3}} \to \infty \end{aligned}\right\}$$

$\Rightarrow f(x)$ has a vertical tangent at $x = 0$.



(ii) $f(x) = \text{sgn } x = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$



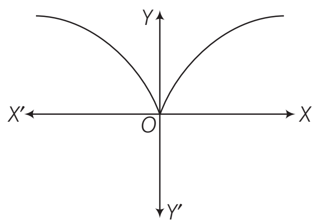
$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{1 - 0}{h} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{-1}{-h} \to \infty \end{aligned}\right\}$$

$\Rightarrow f(x)$ has a vertical tangent at $x = 0$.

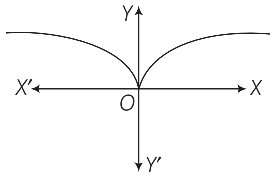
(iii) $f(x) = x^{2/3}$

$$\left.\begin{aligned} f'(0^+) &= \lim_{h \to 0} \frac{h^{2/3}}{h} \to \infty \\ f'(0^-) &= \lim_{h \to 0} \frac{(-h)^{2/3}}{-h} = -\frac{1}{h^{1/3}} \to -\infty \end{aligned}\right\}$$

$\Rightarrow$ No vertical tangent at $x = 0$.



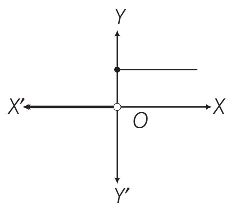
(iv) $f(x)=\sqrt{|x|}=\begin{cases}\sqrt{x}, & \text{if } x\ge 0\\ \sqrt{-x}, & \text{if } x<0\end{cases}$



$$\left.\begin{aligned} f'(0^+) &= \lim_{h\to 0}\frac{\sqrt{h}-0}{h}\to\infty \\ f'(0^-) &= \lim_{h\to 0}\frac{\sqrt{h}}{-h}\to-\infty \end{aligned}\right\}$$

$\Rightarrow$ No vertical tangent at $x=0$.

(v) $f(x)=\begin{cases}0, & \text{if } x<0\\ 1, & \text{if } x\ge 0\end{cases}$



$$\left.\begin{aligned} f'(0^+) &= \lim_{h\to 0}\frac{1-1}{h}=0 \\ f'(0^-) &= \lim_{h\to 0}\frac{0-1}{h}\to-\infty \end{aligned}\right\}$$

$\Rightarrow$ No vertical tangent at $x=0$.

Equation of Tangent

Let $y=f(x)$ be the equation of curve and point (x_1, y_1) be any point on the curve. Let PT be the tangent at point (x_1, y_1).

Since, tangent is a line passing through the point $P(x_1, y_1)$ and having slope $m=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$, therefore by coordinate geometry, the equation of tangent is

$$\boldsymbol{y-y_1=m(x-x_1)\Rightarrow y-y_1=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}\cdot(x-x_1).}$$

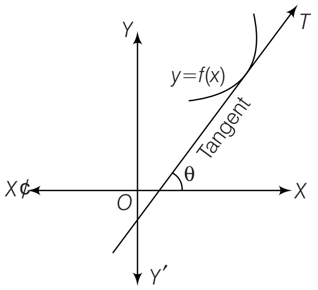


Figure 7.4

Tangent from External Point

If a point $P(a, b)$ does not lie on the curve $y=f(x)$, then the equation of all possible tangents to the curve $y=f(x)$, passing through (a, b), can be found by solving for the point of contact Q.

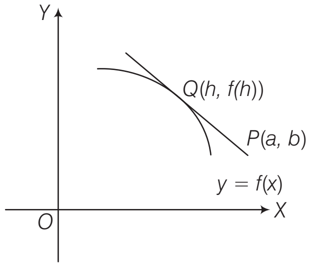


Figure 7.5

Then, $\quad f'(h)=\dfrac{f(h)-b}{h-a}$

and equation of tangent is

$$\boldsymbol{y-b=\frac{f(h)-b}{h-a}\cdot(x-a)}$$

Example 18 Find value of c such that line joining points $(0, 3)$ and $(5, -2)$ becomes tangent to $y=\dfrac{c}{x+1}$.

Sol. Equation of line joining $A(0, 3)$ and $B(5, -2)$ is $x+y=3$.

Solving the line and curve, we get

$$3-x=\frac{c}{x+1}$$

$\Rightarrow \quad x^2-2x+(c-3)=0 \quad$ ...(i)

For tangency, roots of this equation must be equal.

Hence, discriminant of quadratic equation = 0.

$\Rightarrow \quad 4=4(c-3)\Rightarrow c=4$

Hence, required value of c is 4.

Equation of Normal

We know that normal to curve at any point is a straight line passes through that point and is perpendicular to the tangent to the curve at that point.

Since, the slope of tangent at $P(x_1, y_1)=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

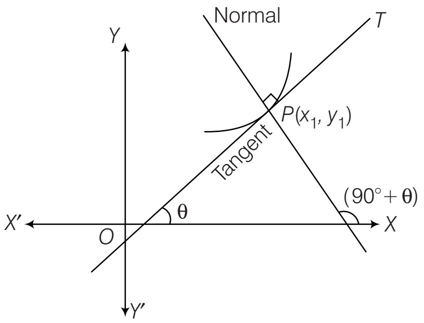


Figure 7.6

Now, as normal is perpendicular to the tangent.

$\therefore$ Slope of normal at $P(x_1, y_1) = \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}$

Hence, from coordinate geometry equation of normal is

$$y - y_1 = -\frac{1}{\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$$

or $$y - y_1 = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}(x - x_1)$$

Example 19 Find the equation of tangent and normal to the curve $2y = 3 - x^2$ at $(1, 1)$.

Sol. The equation of given curve is,

$$2y = 3 - x^2 \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$2\left(\frac{dy}{dx}\right) = -2x \Rightarrow \frac{dy}{dx} = -x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = -1 \quad \text{...(ii)}$$

Now, the equation of tangent at (1, 1) is,

$$\Rightarrow \frac{y-1}{x-1} = \left(\frac{dy}{dx}\right)_{(1,1)} \Rightarrow \frac{y-1}{x-1} = -1$$

$$\Rightarrow y - 1 = -x + 1$$

$$\Rightarrow y + x = 2 \quad \text{[required equation of tangent]}$$

and the equation of the normal at (1, 1) is,

$$\frac{y-1}{x-1} = -\frac{1}{(dy/dx)_{(1,1)}} = 1$$

$$\therefore y - x = 0 \quad \text{[required equation of normal]}$$

Example 20 ***Find the equation of tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.***

Sol. The equation of given curve is,

$$y^2 = 4ax \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get $2y\dfrac{dy}{dx} = 4a$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(at^2, 2at)} = \frac{4a}{4at} = \frac{1}{t} \quad \text{...(ii)}$$

Now, the equation of tangent at $(at^2, 2at)$ is

$$\frac{y - 2at}{x - at^2} = \left(\frac{dy}{dx}\right)_{(at^2, 2at)} = \frac{1}{t} \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow (y - 2at)t = x - at^2$$

$$\Rightarrow yt - 2at^2 = x - at^2 \Rightarrow yt = x + at^2$$

[required equation of tangent]

and the equation of normal at $(at^2, 2at)$ is,

$$\frac{y - 2at}{x - at^2} = -\frac{1}{\left(\dfrac{dy}{dx}\right)_{(at^2, 2at)}} = -t \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow y - 2at = -xt + at^3$$

$$\Rightarrow y + xt = 2at + at^3 \quad \text{[required equation of normal]}$$

Example 21 ***Find the point on the curve $y - e^{xy} + x = 0$ at which we have vertical tangent.***

Sol. The equation of given curve is,

$$y - e^{xy} + x = 0 \quad \text{...(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} - e^{xy}\left\{1 \cdot y + x \cdot \frac{dy}{dx}\right\} + 1 = 0$$

$$\Rightarrow \frac{dy}{dx}(1 - xe^{xy}) = -1 + y \cdot e^{xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1 + y \cdot e^{xy}}{1 - xe^{xy}} \quad \text{...(ii)}$$

Let at point (x_1, y_1) on the curve, we have a vertical tangent (i.e. a tangent parallel to Y-axis). Then,

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \infty \quad \text{or} \quad \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

$$\Rightarrow \frac{1 - x_1 e^{x_1 y_1}}{-1 + y_1 e^{x_1 y_1}} = 0 \Rightarrow 1 - x_1 e^{x_1 y_1} = 0$$

$$\Rightarrow x_1 e^{x_1 y_1} = 1,$$

which is possible only if $x_1 = 1$ and $y_1 = 0$.

Thus, the required point is (1, 0).

Remarks

For standard curves students are advised to use direct method of finding equation of tangent.

For the curve of the form

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, replace

x^2 by xx_1; $2x$ by $x + x_1$;

y^2 by yy_1; $2y$ by $y + y_1$

and xy by $\dfrac{xy_1 + x_1 y}{2}$

Then, equation of tangent is,

$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$

e.g.

(i) Find equation of tangent to the curve $2y = x^2 + 3$ at (x_1, y_1).

Sol. On replacing $2y$ by $y + y_1$ and x^2 by xx_1, we get

$(y + y_1) = xx_1 + 3$, which is the required equation of tangent.

(ii) Find equation of tangent to the curve $y^2 = 4ax$ at $(at^2, 2at)$.

Sol. Clearly, the equation of curve $y^2 = 4ax$ is a standard equation of curve, therefore on replacing y^2 by yy_1 and $2x$ by $x + x_1$, we can get the equation of tangent. Thus, the required equation is given by

$$yy_1 = 2a(x + x_1)$$

where $(x_1, y_1) = (at^2, 2at)$

Hence, the required equation of tangent is

$$y(2at) = 2a(x + at^2) \Rightarrow yt = x + at^2$$

(iii) Find the equation of tangent, at point $P(x_1, y_1)$, to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Sol. Clearly, the equation of curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a standard equation of curve, therefore equation of the tangent can be obtained by replacing x^2 *by* xx_1 and y^2 *by* yy_1.

i.e. $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

Some Important Points Regarding Tangent and Normal

1. If a curve passes through the origin, then the equation of the tangent at the origin can be directly written by equating the lowest degree terms appearing in the equation of the curve to zero.

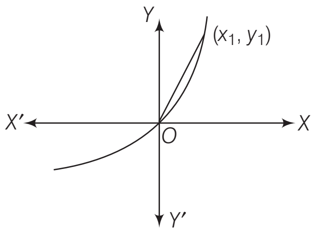


Figure 7.7

Proof Let the equation of the curve be

$$a_1 x + b_1 y + a_2 x^2 + b_2 xy + c_2 y^2 = 0 \quad \text{...(i)}$$

Tangent $y - 0 = \lim_{\substack{x_1 \to 0 \\ y_1 \to 0}} \frac{y_1}{x_1}(x - 0)$

Now, Eq. (i) becomes

$$a_1 + b_1 \frac{y_1}{x_1} + a_2 x_1 + b_2 \frac{y_1}{x_1} \cdot x_1 + c_2 \frac{y_1}{x_1} \cdot y_1 = 0 \quad \text{...(ii)}$$

at $x_1 \to 0$ and $y_1 \to 0, \frac{y_1}{x_1} \to m$

From Eq. (ii), $a_1 + b_1 m = 0$ $\left[\because m = -\frac{a_1}{b_1}\right]$

Hence, tangent is $y = -\frac{a_1}{b_1} x$

$\Rightarrow$ $a_1 x + b_1 y = 0$

e.g.

(i) Equation of tangent at origin, to the curve $x^2 + y^2 + 2gx + 2fy = 0$ is $gx + fy = 0$.

(ii) Equation of tangent at origin to the curve $x^3 + y^3 - 3x^2y + 3xy^2 + x^2 - y^2 = 0$ is $x^2 - y^2 = 0$.

(iii) Equation of tangent at origin, to the curve $x^3 + y^3 - 3xy = 0$ is $xy = 0$.

2. If the curve is $x^4 + y^4 = x^2 + y^2$, then the equation of the tangent would be $x^2 + y^2 = 0$ which would indicate that the origin is an isolated point on the graph.

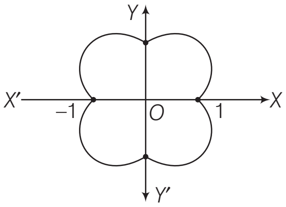


Figure 7.8

3. Same line could be the tangent as well as normal to a given curve at a given point.

e.g. In $x^3 + y^3 - 3xy = 0$ [folium of descartes]

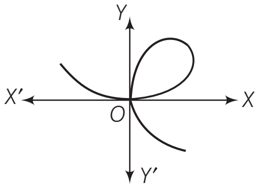


Figure 7.9

The line pair $xy = 0$ is both the tangent as well as normal at $x = 0$.

4. Some common parametric coordinates on a curve

(i) For $x^{2/3} + y^{2/3} = a^{2/3}$, take parametric coordinate $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

(ii) For $\sqrt{x} + \sqrt{y} = \sqrt{a}$, take $x = a\cos^4\theta$ and $y = a\sin^4\theta$.

(iii) For $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$, take $x = a(\cos\theta)^{2/n}$ and $y = b(\sin\theta)^{2/n}$.

(iv) For $c^2(x^2 + y^2) = x^2y^2$, take $x = c\sec\theta$ and $y = c\operatorname{cosec}\theta$.

(v) For $y^2 = x^3$, take $x = t^2$ and $y = t^3$.

Example 22 ***Find the sum of the intercepts on the axes of coordinates by any tangent to the curve $\sqrt{x} + \sqrt{y} = 2$.***

Sol. Here, equation of curve is $\sqrt{x} + \sqrt{y} = 2$.

Whose parametric coordinates are given by,

$$\sqrt{x} = 2\cos^2\theta$$

and $\quad \sqrt{y} = 2\sin^2\theta$

i.e. $\quad x = 4\cos^4\theta$

and $\quad y = 4\sin^4\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{4\times 4\sin^3\theta\cdot\cos\theta}{4\times 4\cos^3\theta(-\sin\theta)} = -\tan^2\theta$$

Now, equation of tangent is $\dfrac{y-4\sin^4\theta}{x-4\cos^4\theta} = -\tan^2\theta$

$$\therefore \quad x\text{-intercept, } \left|4\cos^4\theta + \frac{4\sin^4\theta}{\tan^2\theta}\right| = 4\cos^2\theta$$

$$\text{and } y\text{-intercept, } \left|4\sin^4\theta + \frac{4\sin^4\theta}{\tan^2\theta}\right| = 4\sin^2\theta$$

Hence, the sum of intercepts made on the axes of coordinates is,

$$4\cos^2\theta + 4\sin^2\theta = 4$$

Example 23 The tangent, represented by the graph of the function $y = f(x)$, at the point with abscissa $x = 1$ form an angle of $\pi/6$, at the point $x = 2$ form an angle of $\pi/3$ and at the point $x = 3$ form an angle of $\pi/4$. Then, find the value of,

$$\int_1^3 f'(x)\, f''(x)\, dx + \int_2^3 f''(x)\, dx.$$

Sol. Given, at $x = 1$, $\dfrac{dy}{dx} = \tan\pi/6 = 1/\sqrt{3}$

or at $\quad x = 1, \quad f'(1) = \tan\pi/6 = 1/\sqrt{3}$

Also, at $\quad x = 2, \quad f'(2) = \tan\pi/3 = \sqrt{3}$

and at $\quad x = 3, \quad f'(3) = \tan\pi/4 = 1$

Then, $\quad \displaystyle\int_1^3 f'(x)\, f''(x)\, dx + \int_2^3 f''(x)\, dx$

$$= \int_{f'(1)}^{f'(3)} t\, dt + (f'(x))_2^3$$

[putting $f'(x) = t \Rightarrow f''(x)\, dx = dt$]

$$= \frac{1}{2}(t^2)_{f'(1)}^{f'(3)} + \{f'(3) - f'(2)\}$$

$$= \frac{1}{2}\{(f'(3))^2 - (f'(1))^2\} + \{f'(3) - f'(2)\}$$

$$= \frac{1}{2}\left\{(1)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right\} + \{1 - \sqrt{3}\} = \frac{1}{2}\left(1 - \frac{1}{3}\right) + (1 - \sqrt{3})$$

$$= \frac{4}{3} - \sqrt{3} = \frac{4 - 3\sqrt{3}}{3}$$

Example 24 Find the equation for family of curves for which the length of normal is equal to the radius vector.

Sol. Let $P(x, y)$ be the point on the curve.

$$OP = \text{radius vector} = \sqrt{x^2 + y^2}$$

$$PN = \text{length of normal}$$

Now, $\quad \tan\phi = -\dfrac{1}{\left(\dfrac{dy}{dx}\right)} \quad$ and $\quad PN = \dfrac{y}{\sin\phi}$

It is given $\quad OP = PN$

$$\Rightarrow \quad \sqrt{x^2 + y^2} = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

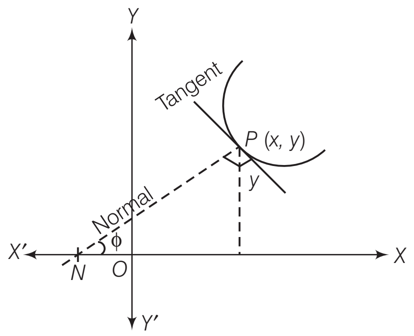


Figure 7.10

$$\Rightarrow \quad x^2 + y^2 = y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow \quad x^2 = y^2\left(\frac{dy}{dx}\right)^2 \quad \Rightarrow \quad \frac{dy}{dx} = \pm\frac{x}{y}$$

or $\quad y\, dy = \pm\, x\, dx$

Integrating both the sides, we get

$y^2 = \pm\, x^2 + C$, which is the required family of curves.

Example 25 ***Find the condition that the line*** $x\cos\alpha + y\sin\alpha = p$ ***may touch the curve*** $\left(\dfrac{x}{a}\right)^m + \left(\dfrac{y}{b}\right)^m = 1$.

Sol. Given equation of curve is $\left(\dfrac{x}{a}\right)^m + \left(\dfrac{y}{b}\right)^m = 1$

Differentiating the equation of curve w.r.t. x, we get

$$m\left(\frac{x}{a}\right)^{m-1}\cdot\frac{1}{a} + m\left(\frac{y}{b}\right)^{m-1}\cdot\frac{1}{b}\cdot\frac{dy}{dx} = 0$$

On simplifying, we get $\dfrac{dy}{dx} = \dfrac{-b^m x^{m-1}}{a^m y^{m-1}}$

Now, at any point $P(x_1, y_1)$ on the curve,

$$\text{slope of tangent} = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \frac{-b^m x_1^{m-1}}{a^m y_1^{m-1}}$$

$\therefore$ Equation of tangent at P is, $y - y_1 = \dfrac{-b^m x_1^{m-1}}{a^m y_1^{m-1}}(x - x_1)$

$$\Rightarrow \quad \frac{yy_1^{m-1}}{b^m} - \frac{y_1^m}{b^m} = -\frac{xx_1^{m-1}}{a^m} + \frac{x_1^m}{a^m}$$

$$\text{i.e. } \frac{x}{a}\left(\frac{x_1}{a}\right)^{m-1} + \frac{y}{b}\left(\frac{y_1}{b}\right)^{m-1} = \left(\frac{x_1}{a}\right)^m + \left(\frac{y_1}{b}\right)^m = 1$$

Since, P lies on the curve, therefore the equation of tangent at $P(x_1, y_1)$ on the curve is,

$$\frac{x}{a}\left(\frac{x_1}{a}\right)^{m-1}+\frac{y}{b}\left(\frac{y_1}{b}\right)^{m-1}=1 \quad \text{...(i)}$$

Also, we have $x\cos\alpha + y\sin\alpha = p$ ...(ii)

If Eq. (ii) is the equation of tangent, then coefficients of Eqs. (i) and (ii) must be proportional for point (x_1, y_1).

i.e. $$\frac{\cos\alpha}{\frac{1}{a}\left(\frac{x_1}{a}\right)^{m-1}}=\frac{\sin\alpha}{\frac{1}{b}\left(\frac{y_1}{b}\right)^{m-1}}=\frac{p}{1}$$

This gives $\frac{x_1}{a}=\left(\frac{a\cos\alpha}{p}\right)^{\frac{1}{m-1}}, \frac{y_1}{b}=\left(\frac{b\sin\alpha}{p}\right)^{\frac{1}{m-1}}$

Since, point $P(x_1, y_1)$ lies on the curve.

Therefore, we have $\left(\frac{x_1}{a}\right)^m+\left(\frac{y_1}{b}\right)^m=1$

i.e. $$\left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}}+\left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}}=1$$

i.e. $$(a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}=p^{\frac{m}{m-1}},$$

which is the required condition.

Example 26 If tangent and normal to the curve $y = 2\sin x + \sin 2x$ are drawn at $P\left(x=\frac{\pi}{3}\right)$, then area of the quadrilateral formed by the tangent, the normal at P and the coordinate axes is

(a) $\frac{\pi}{3}$ (b) 3π

(c) $\frac{\pi\sqrt{3}}{2}$ (d) None of these

Sol. Here, $\frac{dy}{dx}=0$ at $\left(x=\frac{\pi}{3}, y=\frac{3\sqrt{3}}{2}\right)$

$\Rightarrow$ Tangent at $x=\frac{\pi}{3}$ is parallel to X-axis.

$\Rightarrow$ Equation of tangent is, $y=\frac{3\sqrt{3}}{2}$

Also, equation of normal is, $x=\frac{\pi}{3}$

Now, area of quadrilateral $=\frac{\pi}{3}\cdot\frac{3\sqrt{3}}{2}=\frac{\pi\sqrt{3}}{2}$ sq unit

Hence, (c) is the correct answer.

Example 27 ***The maximum value of the sum of the intercepts made by any tangent to the curve $(a\sin^2\theta, 2a\sin\theta)$ with the axes is***

(a) $2a$ (b) $a/4$ (c) $a/2$ (d) a

Sol. The curve is $(a\sin^2\theta, 2a\sin\theta) \Rightarrow x = a\sin^2\theta, y = 2a\sin\theta$

Now, the equation of any tangent to the curve is,

$$\frac{y-2a\sin\theta}{x-a\sin^2\theta}=\frac{1}{\sin\theta} \Rightarrow y\sin\theta = x + a\sin^2\theta$$

$$\Rightarrow \frac{x}{-a\sin^2\theta}+\frac{y}{a\sin\theta}=1$$

$\Rightarrow$ Sum of intercepts $= a(\sin^2\theta + \sin\theta)$

$$= a\left\{\left(\sin\theta+\frac{1}{2}\right)^2-\frac{1}{4}\right\}$$

Which is maximum, when $\sin\theta = 1$

i.e. (Sum of intercept)$_{\max} = 2a$

Hence, (a) is the correct answer.

Example 28 ***If $g(x)$ is a curve which is obtained by the reflection of $f(x)=\frac{e^x-e^{-x}}{2}$ by the line $y = x$, then***

(a) $g(x)$ has more than one tangent parallel to X-axis

(b) $g(x)$ has more than one tangent parallel to Y-axis

(c) $y = -x$ is a tangent to $g(x)$ at (0, 0)

(d) $g(x)$ has no extremum

Sol. As $g(x)$ is a curve, obtained by the reflection of $f(x)=\frac{e^x-e^{-x}}{2}$ on $y=x$.

so $g(x)$ is inverse of $f(x)$

$$\therefore \quad g(x)=\log\left(x+\sqrt{1+x^2}\right)=f^{-1}(x)$$

$$\Rightarrow \quad g'(x)=\frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$=\frac{1}{\sqrt{1+x^2}}\neq 0, \forall\, x\in R$$

$\Rightarrow g(x)$ has no tangent parallel to X-axis. Also, $g'(x)$ is always defined, $\forall\, x\in R$.

$\Rightarrow g(x)$ has no tangent parallel to Y-axis.

Since $g'(x) > 0\ \forall x \in R$,
therefore $g(x)$ doesn't have any extremum.

Hence, (d) is the correct answer.

Exercise for Session 3

1. If the line $ax + by + c = 0$ is normal to the $xy + 5 = 0$, then a and b have

(a) same sign
(b) opposite sign
(c) cannot be discussed
(d) None of these

2. The equation of tangent drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2) is given by

(a) $y - 2(1 \pm \sqrt{2}) = \pm 2\sqrt{3}\,(x - 2)$
(b) $y - 2(1 \pm \sqrt{3}) = \pm 2\sqrt{2}\,(x - 2)$
(c) $y - 2\,(1 \pm \sqrt{3}) = \pm 2\sqrt{3}\,(x - 2)$
(d) None of these

3. The equation of the tangents to the curve $(1 + x^2)y = 1$ at the points of its intersection with the curve $(x + 1)y = 1$, is given by

(a) $x + y = 1, y = 1$
(b) $x + 2y = 2, y = 1$
(c) $x - y = 1, y = 1$
(d) None of these

4. The tangent lines for the curve $y = \int_0^x 2\,|t|\,dt$ which are parallel to the bisector of the first coordinate angle, is given by

(a) $y = x + \frac{3}{4}, y = x - \frac{1}{4}$
(b) $y = -x + \frac{1}{4}, y = -x + \frac{3}{4}$
(c) $x + y = 2, x - y = 1$
(d) None of these

5. The equation of normal to $x + y = x^y$, where it intersects X-axis, is given by

(a) $x + y = 1$
(b) $x - y - 1 = 0$
(c) $x - y + 1 = 0$
(d) None of these

6. The equation of normal at any point θ to the curve
$x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$ is always at a distance of

(a) $2a$ unit from origin
(b) a unit from origin
(c) $\frac{1}{2}a$ unit from origin
(d) None of these

7. If the tangent at (x_0, y_0) to the curve $x^3 + y^3 = a^3$ meets the curve again at (x_1, y_1), then $\frac{x_1}{x_0} + \frac{y_1}{y_0}$ is equal to

(a) a
(b) $2a$
(c) 1
(d) None of these

8. The area bounded by the axes of reference and the normal to $y = \log_e x$ at $(1, 0)$, is

(a) 1 sq unit
(b) 2 sq units
(c) $\frac{1}{2}$ sq unit
(d) None of these

9. If $\frac{x}{a} + \frac{y}{b} = 2$ touches the curve $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 2$ at the point (α, β), then

(a) $\alpha = a^2, \beta = b^2$
(b) $\alpha = a, \beta = b$
(c) $\alpha = -2a, \beta = 4b$
(d) $\alpha = 3a, \beta = -2b$

10. The equation of tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$ that are parallel to the line $x + 2y = 0$, is

(a) $x + 2y = \frac{\pi}{2}$ and $x + 2y = -\frac{3\pi}{2}$
(b) $x + 2y = \frac{\pi}{2}$ and $x + 2y = \frac{3\pi}{2}$
(c) $x + 2y = 0$ and $x + 2y = \pi$
(d) None of these

Answers

Exercise for Session 3

1. (a)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (b)
6. (b)	**7.** (d)	**8.** (c)	**9.** (b)	**10.** (a)