Three Dimensional Geometry

Direction Cosines and Direction Ratios of line

- If a directed line L_1 passing through origin makes angles α, β, and γ with the x, y, and z axes respectively, then α, β, and γ are called direction angles and cosines of these angles are called direction cosines of the line.

  If the direction of line L_1 is reversed, then the direction angles will be $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$ and the direction cosines will be $\cos(\pi - \alpha)$, $\cos(\pi - \beta)$, $\cos(\pi - \gamma)$ or $-\cos\alpha$, $-\cos\beta$, $-\cos\gamma$.

  Thus, a line has two sets of direction cosines. In order to have a unique set of direction cosines for a given line in space, we must take the given line as a directed line. These unique direction cosines are denoted by l, m,and n.

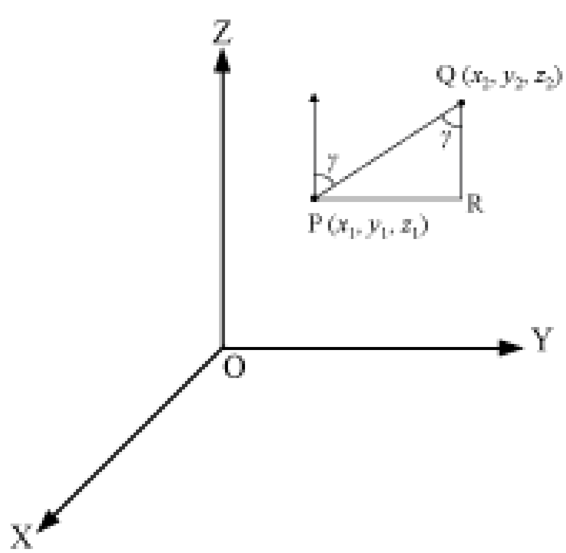
- Two parallel lines have the same direction cosines. Therefore, in order to find the direction cosines of a line which does not pass through origin, we may draw a line parallel to the line through the origin.

- Direction ratios (or number) are proportional to the direction cosines. Thus, for any line, there are infinitely many sets of direction ratios.

  If a, b, c are direction ratios of a line and l, m, n are direction cosines of a line,

  then $$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

  It can be noted that $l^2 + m^2 + n^2 = 1$



- Direction cosines l, m, n of the line passing through the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are given by,

  cos γ=RQPQ=z2−z1PQcos γ=RQPQ=z2-z1PQ

  Similarly, $\cos\alpha = \frac{x_2 - x_1}{PQ}$ and $\cos\beta = \frac{y_2 - y_1}{PQ}$

  $\therefore l = \frac{x_2 - x_1}{PQ},\ m = \frac{y_2 - y_1}{PQ}, n = \frac{z_2 - z_1}{PQ}$

  Where, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Solved examples

Example 1:

Which of the following sets is possible for the direction angles of a line? Assume that all α, β, and γ are measured along the positive x, y, and z axes respectively.

(a) $\alpha = 90º$, $\beta = 60º$, $\gamma = 60º$

(b) $\alpha = 30º$, $\beta = 60º$, $\gamma = 45º$

(c) $\alpha = 90º$, $\beta = 30º$, $\gamma = 30º$

(d) $\alpha = 90º$, $\beta = 45º$, $\gamma = 45º$

Solution:

(a) We know that $l^2 + m^2 + n^2 = 1$

$$l = \cos\alpha = \cos 90° = 0$$
$$m = \cos\beta = \cos 60° = \frac{1}{2}$$
$$n = \cos\gamma = \cos 60° = \frac{1}{2}$$
$$l^2 + m^2 + n^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Hence, this set is not possible.

(b) $l = \cos 30° = \frac{\sqrt{3}}{2}$

$$m = \cos 60° = \frac{1}{2}$$
$$n = \cos 45° = \frac{1}{\sqrt{2}}$$
$$l^2 + m^2 + n^2 = \frac{3}{4} + \frac{1}{4} + \frac{1}{2} = \frac{3}{2}$$

Hence, this set is not possible.

(c) $l = \cos\alpha = \cos 90° = 0$

$$m = \cos\beta = \cos 30° = \frac{\sqrt{3}}{2}$$
$$n = \cos\gamma = \cos 30° = \frac{\sqrt{3}}{2}$$
$$l^2 + m^2 + n^2 = 0 + \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$$

Hence, this set is not possible.

(d) $l = \cos 90° = 0$

$$m = \cos 45° = \frac{1}{\sqrt{2}}$$
$$n = \cos 45° = \frac{1}{\sqrt{2}}$$
$$l^2 + m^2 + n^2 = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

Hence, this set is possible.

Example 2:

If a line has direction ratios as 0, 4, −3, then find the direction cosines of the line.

Solution:

Direction cosines are

$$\frac{0}{\sqrt{0^2+(4)^2+(-3)^2}}, \frac{4}{\sqrt{0^2+(4)^2+(-3)^2}}, \frac{-3}{\sqrt{0^2+(4)^2+(-3)^2}}$$

or $0, \frac{4}{5}, \frac{-3}{5}$

Example 3:

Are the points A (1, 5, −1), B (4, 3, 4), and C (10, −1, 14) collinear or not?

Solution:

Direction ratios of the line joining A and B are 4 − 1, 3 − 5, 4 − (−1) i.e., 3, −2, 5.

Direction rations of line joining B and C are

10 − 4, − 1 − 3, 14 − 4 i.e., 6, −4, 10

Direction ratios of AB and BC are proportional. Hence, AB is parallel to BC. Also, B is common in both the lines.

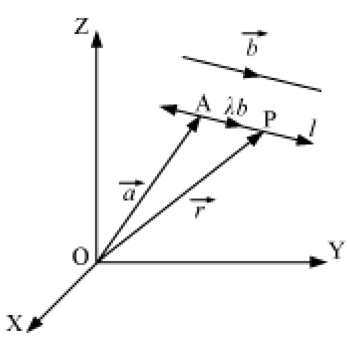
Thus, the points A, B, C lie on the same line and hence, they are collinear.

Equation of a Line

Equation of a Line Passing Through a Point and Parallel to a Given Vector $\vec{b}$

- For a line *l* passing through a point A, having its position vector as $\vec{a}$ and parallel to a given vector $\vec{b}$, position vector $\vec{r}$ of any point P on line *l* is given as:

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

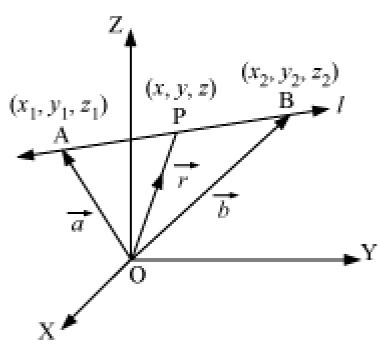


- Similarly, the line l passing through a point A (x_1, y_1, z_1) and parallel to other line whose direction ratios are a, b, and c is given by,
  $\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=\left(x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\right)+\lambda\left(a\hat{i}+b\hat{j}+c\hat{k}\right)$
  The Cartesian equation of line is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

- If the direction cosines of a line are given as l, m, and n, then equation of line is given by $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

Equation of a Line Passing Through Two Given Points

- The equation of a line passing through two points A and B having their position vectors as $\vec{a}$ and $\vec{b}$ respectively is given by, $\vec{r}-\vec{a}=\lambda\left(\vec{b}-\vec{a}\right)$ or $\vec{r}=\vec{a}+\lambda\left(\vec{b}-\vec{a}\right)$



- The equation of a line passing through two points A(x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by,

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\right)=\lambda\left[\left(x_2\hat{i}+y_2\hat{j}+z_2\hat{k}\right)-\left(x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\right)\right]$$

$$\text{or } \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Solved Examples

Example 1:

What are the vector and Cartesian equations of a line passing through point (3, − 1, 2) and parallel to vector $2\hat{i}+3\hat{j}-4\hat{k}$?

Solution:

Here,

$$\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$$
$$\vec{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$$
$$\vec{r} = \vec{a} + \lambda\vec{b}$$
$$\therefore \vec{r} = \left(3\hat{i} - \hat{j} + 2\hat{k}\right) + \lambda\left(2\hat{i} + 3\hat{j} - 4\hat{k}\right)$$

Here, $\vec{r}$ represents the position vector of any point P (x, y, z) on this line.

$$\therefore \left(x\hat{i} + y\hat{j} + z\hat{k}\right) = (3 + 2\lambda)\hat{i} + \hat{j}(-1 + 3\lambda) + (2 - 4\lambda)\hat{k}$$
$$\Rightarrow x = 3 + 2\lambda,\ y = -1 + 3\lambda,\ z = 2 - 4\lambda$$

Thus, the Cartesian equation of the required line is $\dfrac{x-3}{2} = \dfrac{y+1}{3} = \dfrac{z-2}{-4}$

Example 2:

Find the vector equation of a line passing through two points (0, −3, 2) and (4, −1, −5).

Solution:

$$\vec{a} = -3\hat{j} + 2\hat{k}$$
$$\vec{b} = 4\hat{i} - \hat{j} - 5\hat{k}$$
$$\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$$

$$\Rightarrow \vec{r} = \left(-3\hat{j} + 2\hat{k}\right) + \lambda\left(4\hat{i} - \hat{j} - 5\hat{k} + 3\hat{j} - 2\hat{k}\right)$$
$$\Rightarrow \vec{r} = \left(-3\hat{j} + 2\hat{k}\right) + \lambda\left(4\hat{i} + 2\hat{j} - 7\hat{k}\right)$$

Example 3:

If vector equation of a line *l* is $\vec{r} = \left(4\hat{i} - 5\hat{j} + 9\hat{k}\right) + \lambda\left(-2\hat{i} + 3\hat{j} - 4\hat{k}\right)$, then what is the Cartesian equation of line?

Solution:

$$\vec{r} = \left(4\hat{i} - 5\hat{j} + 9\hat{k}\right) + \lambda\left(-2\hat{i} + 3\hat{j} - 4\hat{k}\right)$$

The given equation represents a line passing through a point (4, −5, 9) and parallel to a line whose direction ratios are −2, 3, −4.

Thus, the Cartesian equation of the given line is $\frac{x-4}{-2}=\frac{y+5}{3}=\frac{z-9}{-4}$

Co-planarity of Two Lines

- Two lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, are coplanar, if and only if $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$
- Two lines, $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$,

  are coplanar, if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$

Solved Examples

Example 1:

Show that the lines $\frac{x+4}{5}=\frac{y+3}{-3}=\frac{z-4}{1}$ and $\frac{x+4}{5}=\frac{y+1}{-1}=\frac{z-5}{2}$ are co-planar.

Solution:

The given lines can be rewritten as $\frac{x-(-4)}{5}=\frac{y-(-3)}{-3}=\frac{z-4}{1}$ and $\frac{x-(-4)}{5}=\frac{y-(-1)}{-1}=\frac{z-5}{2}$

Comparing with $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$, we obtain

$$x_1 = -4,\ y_1 = -3,\ z_1 = 4$$
$$x_2 = -4,\ y_2 = -1,\ z_2 = 5$$
$$a_1 = 5, b_1 = -3, c_1 = 1$$
$$a_2 = 5, b_2 = -1, c_2 = 2$$
$$x_2 - x_1 = -4 - (-4) = 0,\ y_2 - y_1 = -1 - (-3) = 2,\ z_2 - z_1 = 5 - 4 = 1$$
$$\therefore \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 1 \\ 5 & -3 & 1 \\ 5 & -1 & 2 \end{vmatrix}$$
$$= 0(-6+1) + 2(5-10) + 1(-5+15) = -10 + 10 = 0$$

Hence, the given lines are co-planar.

Example 2:

For what value of *p* are the following two lines co-planar?

$$\vec{r} = 2\hat{i} - 2\hat{j} + 5\hat{k} + \lambda\left(\hat{i} + 2\hat{k}\right)$$
$$\vec{r} = 4\hat{i} + 3\hat{j} + 4\hat{k} + \mu\left(\hat{i} + p\hat{j} - 3\hat{k}\right)$$

Solution:

Comparing the given lines with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, we obtain

$$\overrightarrow{a_1} = 2\hat{i} - 2\hat{j} + 5\hat{k},\ \overrightarrow{a_2} = 4\hat{i} + 3\hat{j} + 4\hat{k},\ \overrightarrow{b_1} = \left(\hat{i} + 2\hat{k}\right),\ \overrightarrow{b_2} = \left(\hat{i} + p\hat{j} - 3\hat{k}\right)$$
$$\therefore \overrightarrow{a_2} - \overrightarrow{a_1} = \left(4\hat{i} + 3\hat{j} + 4\hat{k}\right) - \left(2\hat{i} - 2\hat{j} + 5\hat{k}\right) = 2\hat{i} + 5\hat{j} - \hat{k}$$

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 1 & p & -3 \end{vmatrix} = \hat{i}(-2p) + \hat{j}(2+3) + \hat{k}p = -2p\hat{i} + 5\hat{j} + p\hat{k}$$

$$\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right).\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right) = \left(2\hat{i} + 5\hat{j} - \hat{k}\right).\left(-2p\hat{i} + 5\hat{j} + p\hat{k}\right) = -4p + 25 - p = -5p + 25$$

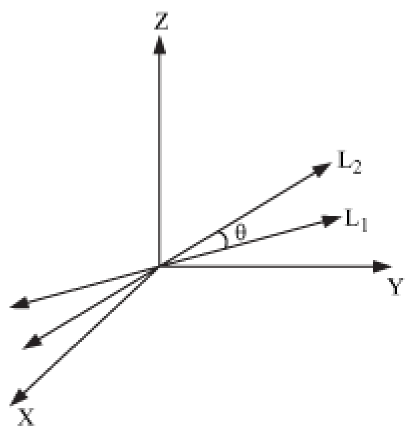
If these two lines are co-planar, then

$$\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)\cdot\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right)=0$$
$$-5p+25=0$$
$$5p=25$$
$$p=5$$

Thus, the required value of p is 5.

Angle between Two Lines

- The acute angle θ between 2 lines with direction ratios as a_1, b_1, c_1 and a_2, b_2, c_2 is given by:



$$\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$
$$\sin\theta=\frac{\sqrt{\left(a_1b_2-a_2b_1\right)^2+\left(b_1c_2-b_2c_1\right)^2+\left(c_1a_2-c_2a_1\right)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

  - It l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of lines L_1 and L_2 respectively, then

$$\cos\theta=\left|l_1l_2+m_1m_2+n_1n_2\right|$$
$$\sin\theta=\sqrt{\left(l_1m_2-l_2m_1\right)^2+\left(m_1n_2-m_2n_1\right)^2+\left(n_1l_2-n_2l_1\right)^2}$$

- For 2 given lines L_1 and L_2 with direction ratios as a_1, b_1, c_1 and a_2, b_2, c_2 respectively, we have
- L_1 and L_2 are perpendicular to each other, if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

- L_1 and L_2 are parallel to each other, if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- The acute angle θ between two lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is given by,

  $$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} \right|$$

Solved Examples

Example 1:

Find the angle between the pair of lines:

$$\vec{r} = 2\hat{i} + 2\hat{j} - 7\hat{k} + \lambda\left(7\hat{i} + 4\hat{j} + 4\hat{k}\right)$$
$$\vec{r} = 4\hat{i} - 3\hat{k} + \mu\left(2\hat{i} + 3\hat{j} - 6\hat{k}\right)$$

Solution:

Here, $\vec{b}_1 = 7\hat{i} + 4\hat{j} + 4\hat{k}$ and $\vec{b}_2 = 2\hat{i} + 3\hat{j} - 6\hat{k}$

$$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} \right| = \left| \frac{\left(7\hat{i} + 4\hat{j} + 4\hat{k}\right)\cdot\left(2\hat{i} + 3\hat{j} - 6\hat{k}\right)}{\sqrt{(7)^2 + (4)^2 + (4)^2}\sqrt{(2)^2 + (3)^2 + (-6)^2}} \right|$$

$$= \left| \frac{14 + 12 - 24}{\sqrt{81}\sqrt{49}} \right| = \frac{2}{63}$$

$$\therefore \theta = \cos^{-1}\left(\frac{2}{63}\right)$$

Thus, the angle between the two given lines is $\cos^{-1}\left(\frac{2}{63}\right)$.

Example 2:

Find the angle between the pair of lines:

$$\frac{x+4}{2}=\frac{y-5}{7}=\frac{z+4}{3}$$
$$\frac{x-3}{4}=\frac{y+7}{2}=\frac{z-9}{4}$$

Solution:

Directions ratios of first line are 2, 7, and 3 and that of second line are 4, 2, and 4.

$$\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$
$$=\left|\frac{(2)(4)+(7)(2)+(3)(4)}{\sqrt{(2)^2+(7)^2+(3)^2}\sqrt{(4)^2+(2)^2+(4)^2}}\right|$$
$$=\frac{34}{(\sqrt{62})(6)}=\frac{17}{3\sqrt{62}}$$
$$\theta=\cos^{-1}\left(\frac{17}{3\sqrt{62}}\right)$$

Thus, the angle between the two given lines is $\cos^{-1}\left(\frac{17}{3\sqrt{62}}\right)$.

Shortest Distance between two Skew Lines

- The shortest distance d between two skew lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is given by $d=\left|\frac{(\vec{b}_1\times\vec{b}_2).(\vec{a}_1-\vec{a}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$

The shortest distance d between two lines

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is given by,

$$d=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\left(b_1c_2-b_2c_1\right)^2+\left(c_1a_2-c_2a_1\right)^2+\left(a_1b_2-a_2b_1\right)^2}}\right|$$

Solved examples

Example 1:

Find the shortest distance between the two lines whose vector equations are

$\vec{r}=(1-2u)\hat{i}+(u-4)\hat{j}+(5-u)\hat{k}$ and $\vec{r}=(4+3v)\hat{i}+(4v-3)\hat{j}+(12v-8)\hat{k}$,
where u and v are real numbers.

Solution:

$$\vec{r}=(1-2u)\hat{i}+(u-4)\hat{j}+(5-u)\hat{k}=\left(\hat{i}-4\hat{j}+5\hat{k}\right)+u\left(-2\hat{i}+\hat{j}-\hat{k}\right)$$
$$\vec{r}=(4+3v)\hat{i}+(4v-3)\hat{j}+(12v-8)\hat{k}=\left(4\hat{i}-3\hat{j}-8\hat{k}\right)+v\left(3\hat{i}+4\hat{j}+12\hat{k}\right)$$

Comparing these lines with $r=\vec{a}_1+u\vec{b}_1$ and $r=\vec{a}_2+v\vec{b}_2$, we obtain

$$\vec{a}_1 = \hat{i} - 4\hat{j} + 5\hat{k}$$
$$\vec{a}_2 = 4\hat{i} - 3\hat{j} - 8\hat{k}$$
$$\vec{b}_1 = -2\hat{i} + \hat{j} - \hat{k}$$
$$\vec{b}_2 = 3\hat{i} + 4\hat{j} + 12\hat{k}$$
$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ 3 & 4 & 12 \end{vmatrix} = \hat{i}(12+4) + \hat{j}\left(-3+24\right) + \hat{k}\left(-8-3\right) = 16\hat{i} + 21\hat{j} - 11\hat{k}$$
$$\left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(16)^2 + (21)^2 + (-11)^2} = \sqrt{256+441+121} = \sqrt{818}$$
$$\vec{a}_2 - \vec{a}_1 = \left(4\hat{i} - 3\hat{j} - 8\hat{k}\right) - \left(\hat{i} - 4\hat{j} + 5\hat{k}\right) = 3\hat{i} + \hat{j} - 13\hat{k}$$
$$\left(\vec{b}_1 \times \vec{b}_2\right).\left(\vec{a}_2 - \vec{a}_1\right) = \left(16\hat{i} + 21\hat{j} - 11\hat{k}\right).\left(3\hat{i} + \hat{j} - 13\hat{k}\right) = 48 + 21 + 143 = 212$$
$$\therefore d = \left|\frac{\left(\vec{b}_1 \times \vec{b}_2\right).\left(\vec{a}_2 - \vec{a}_1\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|}\right| = \frac{212}{\sqrt{818}} \text{ units}$$

Example 2:

Find the shortest distance between the lines: $\frac{x+3}{4} = \frac{y-2}{5} = \frac{z+1}{2}$ and $\frac{x+1}{2} = \frac{y+3}{4} = \frac{z-2}{1}$

Solution:

The given lines can be rewritten as

$$\frac{x-(-3)}{4} = \frac{y-2}{5} = \frac{z-(-1)}{2}$$
$$\frac{x-(-1)}{2} = \frac{y-(-3)}{4} = \frac{z-2}{1}$$

Comparing with $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$, we obtain

$$\left(x_1, y_1, z_1\right) = \left(-3, 2, -1\right), \ \left(x_2, y_2, z_2\right) = \left(-1, -3, 2\right)$$
$$a_1 = 4, \ b_1 = 5, \ c_1 = 2, \ a_2 = 2, \ b_2 = 4, \ c_2 = 1$$

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} -1-(-3) & -3-2 & 2-(-1) \\ 4 & 5 & 2 \\ 2 & 4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -5 & 3 \\ 4 & 5 & 2 \\ 2 & 4 & 1 \end{vmatrix} = 2(5-8)-5(4-4)+3(16-10) = -6+0+18 = 12$$

$$\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2} = \sqrt{(5-8)^2+(4-4)^2+(16-10)^2} = \sqrt{9+0+36} = 3\sqrt{5}$$

$$\therefore d = \frac{12}{3\sqrt{5}} = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5} \text{ units}$$

Shortest Distance between Two Parallel Lines

- The shortest distance d between two parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is given by $d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

Solved Examples

Examples 1:

Find the distance between the lines:

$$\vec{r} = \hat{i} - 3\hat{j} + 4\hat{k} + \lambda\left(7\hat{i} - 4\hat{j} + 4\hat{k}\right) \text{ and } \vec{r} = -2\hat{i} + 4\hat{j} - 3\hat{k} + \mu\left(7\hat{i} - 4\hat{j} + 4\hat{k}\right)$$

Solution:

Comparing the given lines with $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$, we obtain

$$\vec{a}_1 = \hat{i} - 3\hat{j} + 4\hat{k},\ \vec{a}_2 = -2\hat{i} + 4\hat{j} - 3\hat{k},\ \vec{b} = 7\hat{i} - 4\hat{j} + 4\hat{k}$$

$$(\vec{a}_2 - \vec{a}_1) = \left(-2\hat{i} + 4\hat{j} - 3\hat{k}\right) - \left(\hat{i} - 3\hat{j} + 4\hat{k}\right) = \left(-3\hat{i} + 7\hat{j} - 7\hat{k}\right)$$

$$\vec{b} \times (\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -4 & 4 \\ -3 & 7 & -7 \end{vmatrix} = \hat{i}(28-28) + \hat{j}(-12+49) + \hat{k}(49-12) = 37\hat{j} + 37\hat{k}$$

$$\left|\vec{b}\right| = \sqrt{(7)^2 + (-4)^2 + (4)^2} = \sqrt{49+16+16} = 9$$

$$\left|\frac{\vec{b}\times(\vec{a}_2 - \vec{a}_1)}{\left|\vec{b}\right|}\right| = \frac{\left|37\hat{j} + 37\hat{k}\right|}{9} = \frac{37\sqrt{2}}{9} \text{ units}$$

Thus, the distance between the given lines is $\frac{37\sqrt{2}}{9}$ units.

Example 2:

Find the shortest distance between the lines:

$\vec{r} = (4-6s)\hat{i} + (3+2s)\hat{j} + 3(1+s)\hat{k}$ and and $\vec{r} = -3(1+2t)\hat{i} + (1+2t)\hat{j} + (4+3t)\hat{k}$, where *s* and *t* are real numbers

Solution:

The given lines can be written as

$$\vec{r} = (4-6s)\hat{i} + (3+2s)\hat{j} + 3(1+s)\hat{k} = \left(4\hat{i} + 3\hat{j} + 3\hat{k}\right) + s\left(-6\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$\vec{r} = -3(1+2t)\hat{i} + (1+2t)\hat{j} + (4+3t)\hat{k} = \left(-3\hat{i} + \hat{j} + 4\hat{k}\right) + t\left(-6\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

Comparing with $\vec{r} = \vec{a}_1 + s\vec{b}$ and $\vec{r} = \vec{a}_2 + t\vec{b}$, we obtain

$$\vec{a}_1 = 4\hat{i} + 3\hat{j} + 3\hat{k},\ \vec{a}_2 = -3\hat{i} + \hat{j} + 4\hat{k},\ \vec{b} = -6\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = \left(-3\hat{i} + \hat{j} + 4\hat{k}\right) - \left(4\hat{i} + 3\hat{j} + 3\hat{k}\right) = -7\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{b}\times(\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 2 & 3 \\ -7 & -2 & 1 \end{vmatrix} = \hat{i}(2+6) + \hat{j}(-21+6) + \hat{k}(12+14) = 8\hat{i} - 15\hat{j} + 26\hat{k}$$

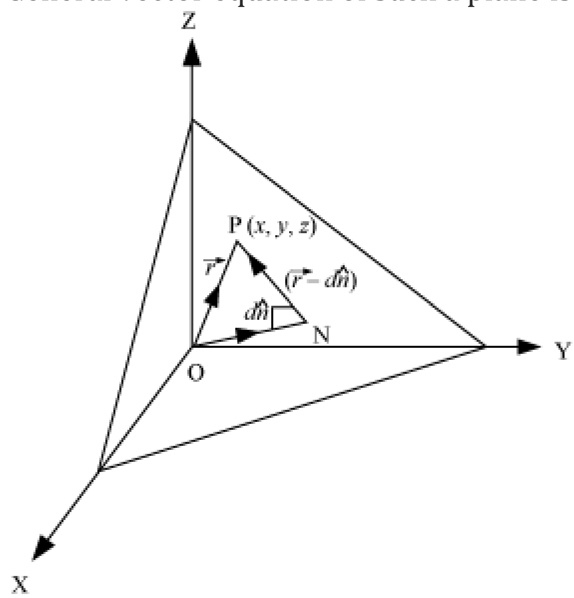
$$\left|\vec{b}\right| = \sqrt{(-6)^2 + (2)^2 + (3)^2} = 7$$

$$\left|\frac{\vec{b}\times(\vec{a}_2 - \vec{a}_1)}{\left|\vec{b}\right|}\right| = \frac{\left|8\hat{i} - 15\hat{j} + 26\hat{k}\right|}{7} = \frac{\sqrt{64+225+676}}{7} = \frac{\sqrt{965}}{7} \text{ units}$$

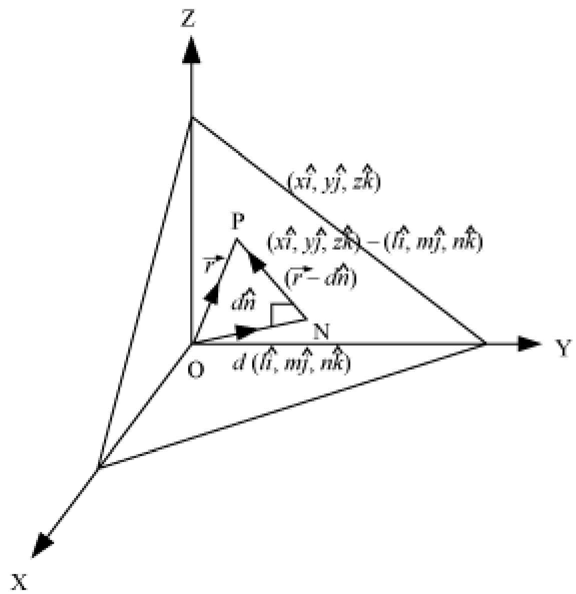
Thus, the distance between the given lines is $\frac{\sqrt{965}}{7}$ units.

Equation of a Plane in Normal Form

- If a plane is given whose perpendicular distance from origin is *d*, and $\hat{n}$ is the unit normal vector along the direction of the normal to the plane from the origin, then:
- General vector equation of such a plane is given by $\vec{r} \cdot \hat{n} = d$



- Equation of the plane when direction cosines of the unit normal vector $\hat{n}$ are given as *l*, *m*, and *n* is $lx + my + nz = d$



- Equation of the plane when the direction ratios of the unit normal vector $\hat{n}$ are given as *a*, *b*, and *c* instead of direction cosines is:

  Vector form− $\vec{r}\cdot\left(a\hat{i}+b\hat{j}+c\hat{k}\right)=d$; Cartesian form− $ax + by + cz = d$

- The normal drawn from origin to plane $lx + my + nz = d$ will intersect the plane at (ld, md, nd).

- The normal drawn from origin to plane $ax + bx + cz = d$ will intersect the plane at $\left(\frac{ad}{a^2+b^2+c^2}, \frac{bd}{a^2+b^2+c^2}, \frac{cd}{a^2+b^2+c^2}\right)$.

Solved Examples

Example 1:

What is the equation of the plane which is at the distance of $\frac{4}{7}$ units from the origin and its normal vector from the origin is $2\hat{i}-6\hat{j}+3\hat{k}$?

Solution:

Normal vector, $\vec{n}=2\hat{i}-6\hat{j}+3\hat{k}$

Thus, unit vector along normal vector $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} - 6\hat{j} + 3\hat{k}}{\sqrt{(2)^2 + (-6)^2 + (3)^2}}$

$$= \frac{2}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{3}{7}\hat{k}$$

Hence, the equation of plane is

$$\vec{r} \cdot \left(\frac{2}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{3}{7}\hat{k}\right) = \frac{4}{7}$$

Example 2:

Find the distance of plane $3x - 4y + 22 - 9 = 0$ from the origin.

Solution:

$3x - 4y + 2z = 9$

Here, it can be noticed that

$(3)^2 + (-4)^2 + (2)^2 = 29 \neq 1$

Hence, equation is in the form of $ax + by + cz = d$

Dividing both sides by $\sqrt{a^2 + b^2 + c^2}$, we obtain

$$\frac{ax}{\sqrt{a^2 + b^2 + c^2}} + \frac{by}{\sqrt{a^2 + b^2 + c^2}} + \frac{cz}{\sqrt{a^2 + b^2 + c^2}} = \frac{d}{\sqrt{a^2 + b^2 + c^2}}$$

Distance of plane from origin = $\frac{d}{\sqrt{a^2 + b^2 + c^2}}$

$$= \frac{9}{\sqrt{(3)^2 + (-4)^2 + (2)^2}}$$

$$= \frac{9}{\sqrt{29}} \text{ units}$$

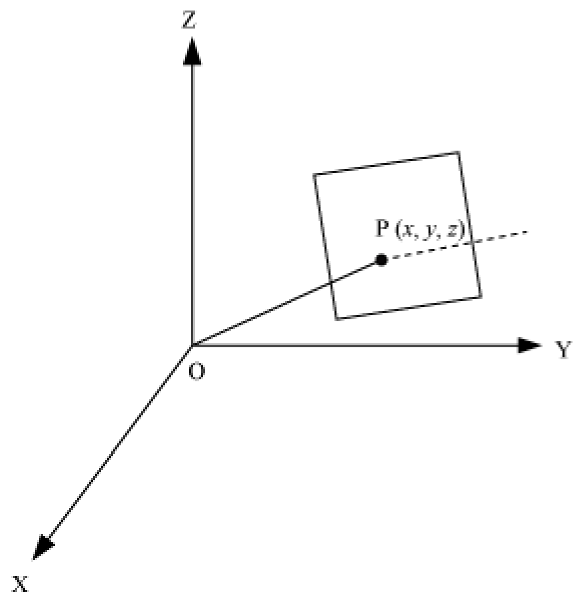
Example 3:

At what points does the perpendicular drawn from the origin to plane $3x + 5y - z - \sqrt{35} = 0$ intersect the plane?

Solution:

Let the perpendicular intersect the given plane at $P\ (x_1, y_1, z_1)$.

Now, it can be noticed that x_1, y_1, z_1 are direction ratios of $\overrightarrow{OP}$.



$$3x+5y-z=\sqrt{35}=0$$
$$3x+5y-z=\sqrt{35}$$

In normal form, the equation can be rewritten as

$$\frac{3}{\sqrt{35}}x+\frac{5}{\sqrt{35}}y-\frac{z}{\sqrt{35}}=1$$

Direction cosines of $\overrightarrow{OP}$ are $\frac{3}{\sqrt{35}},\frac{5}{\sqrt{35}},\frac{-1}{\sqrt{35}}$.

Direction ratios and cosines are always proportional.

$$\therefore \frac{x_1}{\frac{3}{\sqrt{35}}}=\frac{y_1}{\frac{5}{\sqrt{35}}}=\frac{z_1}{\frac{-1}{\sqrt{35}}}=k$$
$$x_1=\frac{3}{\sqrt{35}}k, y_1=\frac{5}{\sqrt{35}}k, z_1=\frac{k}{\sqrt{35}}$$

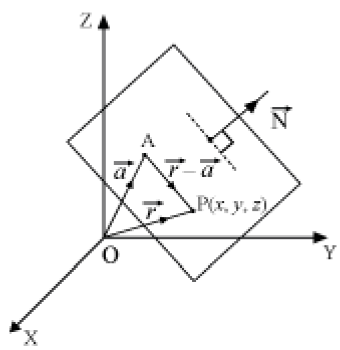
Point $P\ (x_1, y_1, z_1)$ lies on plane. Hence, it must satisfy the equation of plane.

$$\Rightarrow\left(\frac{3}{\sqrt{35}}\right)\left(\frac{3k}{\sqrt{35}}\right)+\left(\frac{5}{\sqrt{35}}\right)\left(\frac{5k}{\sqrt{35}}\right)-\left(\frac{1}{\sqrt{35}}\right)\left(\frac{-1k}{\sqrt{35}}\right)=1$$
$$\Rightarrow\frac{9k}{35}+\frac{25k}{35}+\frac{k}{35}=1$$
$$\Rightarrow k=1$$

Hence, the plane is intersected by the perpendicular from origin at $\left(\frac{3}{\sqrt{35}},\frac{5}{\sqrt{35}},\frac{-1}{\sqrt{35}}\right)$.

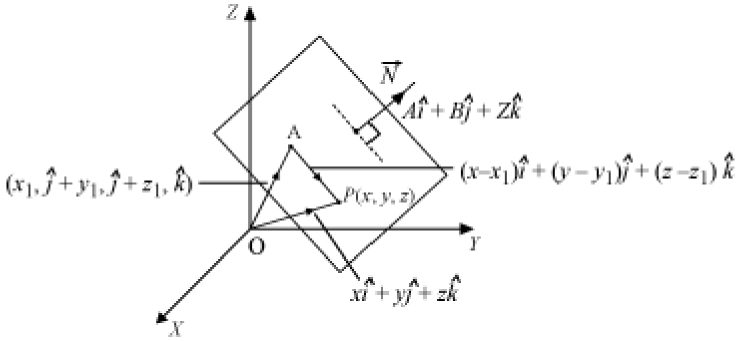
Equation of Plane Perpendicular to Given Vector and Passing Through Given Point

- If a plane passes through a point A with position vector $\vec{a}$ and is perpendicular to the vector $\vec{\mathrm{N}}$, then the vector equation of the plane is given by, $(\vec{r}-\vec{a})\cdot\vec{\mathrm{N}}=0$



- If a plane passes through a point A (x_1, y_1, z_1) and is perpendicular to the vector $\vec{\mathrm{N}}$ whose direction ratios are *A*, *B*, *C*, then the equation of the plane in Cartesian form is given by,

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$



Solved Examples

Example 1:

Find the vector equation of the plane which passes through the point (6, −2, −3) and perpendicular to the line having its direction ratios as 4, 5, and −2.

Solution:

Position vector of the point (6, −2, −3) is $6\hat{i} - 2\hat{j} - 3\hat{k}$.

Normal vector $\vec{N}$ perpendicular to the plane is $\vec{N} = 4\hat{i} + 5\hat{j} - 2\hat{k}$

Therefore, vector equation of the plane is given by,

$(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

Or, $\left[\vec{r} - \left(6\hat{i} - 2\hat{j} - 3\hat{k}\right)\right] \cdot \left[4\hat{i} + 5\hat{j} - 2\hat{k}\right] = 0$

Example 2:

Find the Cartesian equation of the plane passing through the point (3, −7, 6) and perpendicular to the line having its direction ratios 4, −1, and −9.

Solution:

Position vector of the point is given by,

$\vec{a} = 3\hat{i} - 7\hat{j} + 6\hat{k}$

Normal vector $\vec{N}$ perpendicular to the given plane = $4\hat{i} - \hat{j} - 9\hat{k}$

Therefore, Cartesian equation of the plane is given by,

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

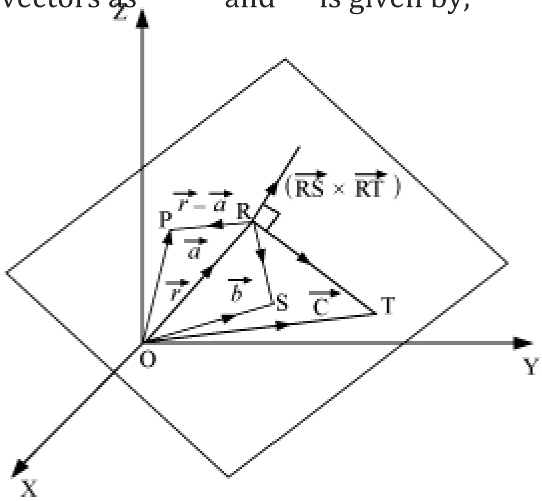
$\therefore 4(x - 3) + (-1)(y + 7) + (-9)(z - 6) = 0$

$4x - 12 - y - 7 - 9z + 54 = 0$

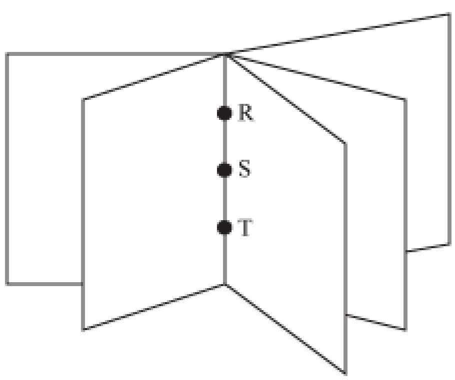
$4x - y - 9z + 35 = 0$

Equation of a Plane Passing Through Three Non-Collinear Points

- The equation of a plane passing through three non-collinear points, which have their position vectors as $\vec{a}, \vec{b},$ and $\vec{c}$ is given by, $(\vec{r}-\vec{a})\cdot\left[\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)\right]=0$



- If the three points are collinear, then infinitely many planes can pass through these points.



- The equation of a plane in Cartesian form passing through points R (x_1, y_1, z_1), S (x_2, y_2, z_2), and T (x_3, y_3, z_3) is given by,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

Solved Examples

Example 1:

Find the vector equation of the plane passing through the points A (3, −1, 2), B (4, −2, 3), and C (−6, 8, 3).

Solution:

Position vector of point $A = \overrightarrow{OA} = 3\hat{i} - \hat{j} + 2\hat{k}$

Position vector of point $B = \overrightarrow{OB} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

Position vector of point $C = \overrightarrow{OC} = -6\hat{i} + 8\hat{j} + 3\hat{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(4\hat{i} - 2\hat{j} + 3\hat{k}\right) - \left(3\hat{i} - \hat{j} + 2\hat{k}\right) = \hat{i} - \hat{j} + \hat{k}$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(-6\hat{i} + 8\hat{j} + 3\hat{k}\right) - \left(3\hat{i} - \hat{j} + 2\hat{k}\right) = -9\hat{i} + 9\hat{j} + \hat{k}$$

The vector equation of plane passing through points A, B, and C is given by,

$$\left(\vec{r} - \overrightarrow{OA}\right) \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = 0$$
$$\left[\vec{r} - \left(3\hat{i} - \hat{j} + 2\hat{k}\right)\right] \cdot \left[\left(\hat{i} - \hat{j} + \hat{k}\right) \times \left(-9\hat{i} + 9\hat{j} + \hat{k}\right)\right] = 0$$

Example 2:

A plane passing through three points A (1, 3, −2), B (2, *p*, 4), and C (−3, 5, *q*) is given as *ax* − 31*y* − 18*z* + *b* = 0. What are the values of *a*, *b*, *p*, and *q*?

Solution:

The equation of plane passing through points A (1, 3, −2), B (2, *p*, 4), and C (−3, 5, *q*) is given by,

$$\begin{vmatrix} x-1 & y-3 & z-(-2) \\ 2-1 & p-3 & 4-(-2) \\ -3-1 & 5-3 & q-(-2) \end{vmatrix} = 0$$
$$\begin{vmatrix} x-1 & y-3 & z+2 \\ 1 & p-3 & 6 \\ -4 & 2 & q+2 \end{vmatrix} = 0$$
$$(x-1)\left[\left(pq-3q+2p-6-12\right)\right] + (y-3)(-24-q-2) + (z+2)(2+4p-12) = 0$$
$$(x-1)(pq-3q+2p-18) + (y-3)(-26-q) + (z+2)(4p-10) = 0$$
$$(pq-3q+2p-18)x + (-26-q)y + z(4p-10) - pq + 3q - 2p + 18 + 78 + 3q + 8p - 20 = 0$$

Comparing the above equation with the given equation, *ax* − 31*y* − 18*z* + *b* = 0, we obtain

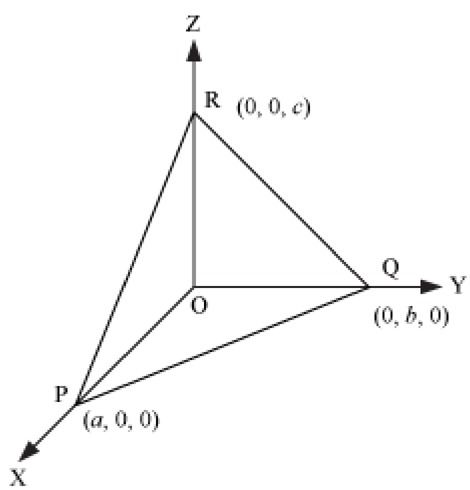
$$-26-q=-31 \Rightarrow q=5$$
$$4p-10=-18 \Rightarrow p=-\frac{8}{4}=-2$$
$$a=(pq-3q+2p-18)=\left[(-2)(5)-3(5)+2(-2)-18\right]=-47$$

$$b=-pq+3q-2p+18+78+3q+8p-20$$
$$b=-(-2)5+3(5)-2(-2)+96+3(5)+8(-2)-20$$
$$b=10+15+4+96+15-36$$
$$b=104$$

Thus, the required values are $a = -47$, $p = -2$, $q = 5$, and $b = 104$

Intercept form of the Equation of Plane

- If a plane $Ax + By + Cz = D$ ($D \neq 0$) makes intercepts a, b, c on x, y, and z axes respectively, then the equation of the plane in intercept form is given by, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$



Solved Examples

Example 1:

How far is the point (0, −4, 0) from the point of intersection of the plane $12x + 4y + 3z = 24$ and the y-axis?

Solution:

The equation of the given plane $12x + 4y + 3z = 24$ can be written in the form

$$\frac{12x}{24}+\frac{4y}{24}+\frac{3z}{24}=1$$

$$\frac{x}{2}+\frac{y}{6}+\frac{z}{8}=1, \text{ which is of the form } \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Thus, the plane intersects the x, y, and z axes at (2, 0, 0), (0, 6, 0), and (0, 0, 8) respectively.

Distance between (0, −4, 0) and (0, 6, 0) = 10 units

Hence, the point (0, −4, 0) is 10 units far from the point of intersection of the given plane and the y-axis.

Example 2:

If the intercepts made by a plane on the x, y, and z axes are in a ratio of 3:15:5 and the plane passes through point (2, 2, 1), then find the equation of the plane.

Solution:

Let the given plane make intercepts $3k$, $15k$, and $5k$ on the x, y, and z axes respectively.

Equation of plane in intercept is given by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ where a, b, and c are intercepts on the x, y, and z axes respectively.

Therefore, equation of the given planeis

$$\frac{x}{3k}+\frac{y}{15k}+\frac{z}{5k}=1$$
$$\Rightarrow \frac{5x+y+3z}{15k}=1$$
$$\Rightarrow 5x+y+3z=15k$$

Since the plane passes through the point (2, 2, 1), we have

$5(2) + 2 + 3(1) = 15k$

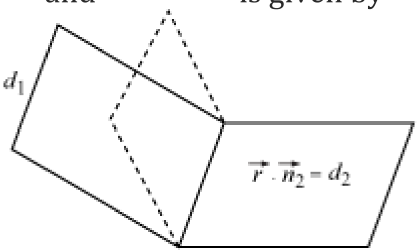
$\Rightarrow 15 = 15k$

$\Rightarrow k = 1$

Hence, the required equation of the plane is $5x + y + 3z = 15$

Plane passing through the Intersection of two given Planes

- The vector equation of a plane passing through the intersection of two planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is given by $\vec{r}\cdot\left(\vec{n}_1+\lambda\vec{n}_2\right) = d_1+\lambda d_2$, where λ is any real number.



- It can be noted that infinitely many planes can pass through the intersection of the two planes.

- The Cartesian equation of a plane passing through the intersection of two planes $A_1x+B_1y+C_1z=d_1$ and $A_2x+B_2y+C_2z=d_2$ is given by, $\left(A_1x+B_1y+C_1z-d_1\right)+\lambda\left(A_2x+B_2y+C_2z-d_2\right)=0$

Solved Examples

Example 1:

A plane, passing through the intersection of the planes $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=9$ and $\vec{r}\cdot\left(3\hat{i}-5\hat{j}-\hat{k}\right)=4$, passes through the point (−1, 6, 4). Find the equation of the plane.

Solution:

The equation of given planes are $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=9$ and $\vec{r}\cdot\left(3\hat{i}-5\hat{j}-\hat{k}\right)=4$

Comparing with $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$, we obtain

$\vec{n}_1 = 2\hat{i}+3\hat{j}-4\hat{k},\ d_1 = 9$

$\vec{n}_2 = 3\hat{i}-5\hat{j}-\hat{k},\ d_2 = 4$

Equation of the plane passing through the intersection of the given planes can be written as $\vec{r}\cdot\left[2\hat{i}+3\hat{j}-4\hat{k}+\lambda\left(3\hat{i}-5\hat{j}-\hat{k}\right)\right]=9+4\lambda$

$$\therefore \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \cdot \left[(2+3\lambda)\hat{i} + (3-5\lambda)\hat{j} + (-4-\lambda)\hat{k}\right] = 9 + 4\lambda \qquad \ldots(1)$$

$$\Rightarrow x(2+3\lambda) + y(3-5\lambda) + z(-4-\lambda) = 9 + 4\lambda$$

Since this plane passes through point (−1, 6, 4), we obtain

−1(2 + 3λ) + 6(3 − 5λ) + 4(−4 −λ) = 9 + 4λ

⇒ −2 − 3λ + 18 − 30λ − 16 − 4λ = 9 + 4λ

⇒ −9 = 41λ

$$\Rightarrow \lambda = \frac{-9}{41}$$

Putting this value of λ in equation (1), we obtain

$$\vec{r} \cdot \left[\left(2 - \frac{9\times 3}{41}\right)\hat{i} + \left(3 + 5\times\frac{9}{41}\right)\hat{j} + \left(-4 + \frac{9}{41}\right)\hat{k}\right] = 9 - \frac{36}{41}$$

$$\Rightarrow \vec{r} \cdot \left[\frac{55}{41}\hat{i} + \frac{168}{41}\hat{j} - \frac{155}{41}\hat{k}\right] = \frac{333}{41}$$

$$\Rightarrow \vec{r} \cdot \left[55\hat{i} + 168\hat{j} - 155\hat{k}\right] = 333, \text{ which is the required equation of the plane}$$

Example 2:

Find the Cartesian equation of the plane passing through the intersection of the planes

$2x - 5y + 3z = 4$ and $7x - 9y + z = 8$ that passes through the point (1, −2, −1).

Solution:

Equation of the plane passing through the intersection of the given planes

$2x - 5y + 3z = 4$ and $7x - 9y + z = 8$ can be written as:

$$(2x - 5y + 3z - 4) + \lambda(7x - 9y + z - 8) = 0$$

$$(2+7\lambda)x + (-5-9\lambda)y + (3+\lambda)z + (-4-8\lambda) = 0$$

Since this plane passes through point (1, −2, −1),

(2 + 7λ) (1) + (−5 − 9λ) (−2) + (3 + λ) (−1) + (−4 − 8λ) = 0

$\Rightarrow (2 + 7\lambda) + (10 + 18\lambda) - 3 - \lambda - 4 - 8\lambda = 0$

$\Rightarrow 5 + 16\lambda = 0$

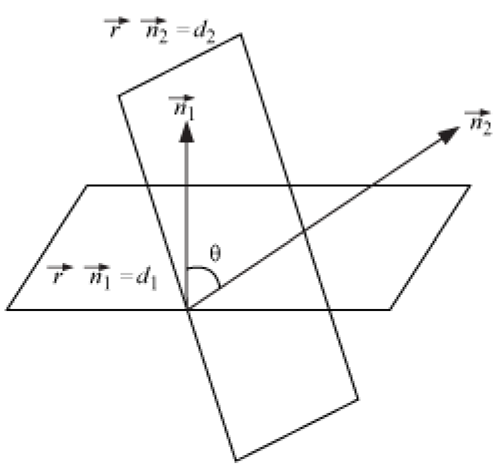
$\Rightarrow \lambda = \frac{-5}{16}$

Putting the value of λ, we obtain

$$\left[2+7\left(-\frac{5}{16}\right)\right]x+\left[-5-9\left(-\frac{5}{16}\right)\right]y+\left[3-\frac{5}{16}\right]z+\left[-4-8\left(-\frac{5}{16}\right)\right]=0$$
$$\Rightarrow\left(\frac{32-35}{16}\right)x+\left(\frac{-80+45}{16}\right)y+\left(\frac{43}{16}\right)z+\left[\frac{-64+40}{16}\right]=0$$
$$\Rightarrow\left(-\frac{3}{16}\right)x+\left(-\frac{35}{16}\right)y+\left(\frac{43}{16}\right)z+\left(-\frac{24}{16}\right)=0$$
$$\Rightarrow -3x-35y+43z-24=0$$
$$\Rightarrow 3x+35y-43z+24=0, \text{ which is the required equation of the plane}$$

Angle between Two Planes

- The angle between two planes is defined as the angle between their normals. There are always two angles between two planes. However, the acute angle between them is considered always.
- The angle θ between two planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by, $\cos\theta = \left|\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$



- The planes will be perpendicular to each other, if $\vec{n}_1.\vec{n}_2 = 0$; and parallel to each other, if $\vec{n}_1$ is parallel to $\vec{n}_2$.

- The angle θ between two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is given by,

$$\cos\theta = \left|\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

- The two planes will be at right angles to each other, if $A_1 A_2 + B_1 B_2 + C_1 C_2 = 0$; and two planes are parallel to each other, if $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

Solved Examples

Example 1:

Find the angle between two planes $2x - 3y + 6z = 4$ and $-4x + 7y + 4z = 9$

Solution:

The equation of given planes is

$2x - 3y + 6z = 4$

$-4x + 7y + 4z = 9$

Comparing with $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$, we obtain

$A_1 = 2, B_1 = -3, C_1 = 6, A_2 = -4, B_2 = 7, C_2 = 4$

$$\cos\theta = \left|\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

$$\cos\theta = \left|\frac{2(-4)+(-3)(7)+(6)(4)}{\sqrt{(2)^2+(-3)^2+(6)^2}\sqrt{(-4)^2+(7)^2+(4)^2}}\right|$$

$$\cos\theta = \left|\frac{-8-21+24}{(7)(9)}\right|$$

$$\cos\theta = \left|-\frac{5}{63}\right| = \frac{5}{63}$$

Example 2:

For what value of λ is the plane $\vec{r}\cdot\left(4\hat{i} - 3\hat{j} + 2\hat{k}\right) = 7$ perpendicular to the plane $\vec{r}\cdot\left(3\hat{i} + \lambda\hat{j} + 3\hat{k}\right) = 9$

Solution:

Comparing the given equation of planes by $\vec{r}.\vec{N}_1 = d_1$ and $\vec{r}.\vec{N}_2 = d_2$, we obtain

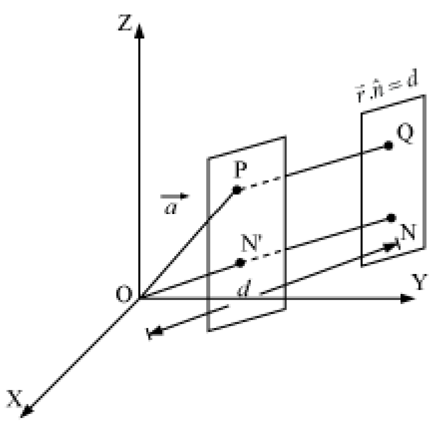
$$\vec{N}_1 = 4\hat{i} - 3\hat{j} + 2\hat{k}$$
$$\vec{N}_2 = 3\hat{i} + \lambda\hat{j} + 3\hat{k}$$

If two planes are perpendicular to each other, then $\vec{N}_1.\vec{N}_2 = 0$

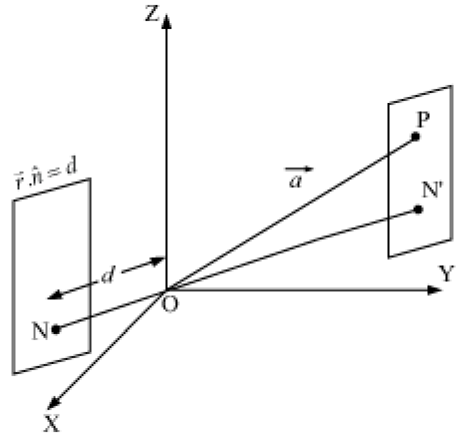
$$\left(4\hat{i} - 3\hat{j} + 2\hat{k}\right)\cdot\left(3\hat{i} + \lambda\hat{j} + 3\hat{k}\right) = 0$$
$$12 - 3\lambda + 6 = 0$$
$$\lambda = 6$$

Distance of a point from a Plane

- The distance of a point P, whose position vector is $\vec{a}$, from the plane $\vec{r}.\hat{n} = d$ is given by $\left|d - \vec{a}.\hat{n}\right|$.



- If the equation of plane is in the form $\vec{r}.\vec{N}=d$, then the distance of point P having its position vector as $\vec{a}$ will be given by $\left|\frac{d-\vec{a}.\vec{N}}{|\vec{N}|}\right|$.



- The length of perpendicular drawn from origin O to plane $\vec{r}.\vec{N}=d$ is $\left|\frac{d}{|\vec{N}|}\right|$.
- The distance of a point P (x_1, y_1 z_1) from plane $Ax + By + Cz = D$ is given as

$$\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|.$$

Solved Examples

Example 1:

Find the distance of a point P (2, −3, 4) from the plane $\vec{r}\cdot\left(2\hat{i}-6\hat{j}+3\hat{k}\right)=5$

Solution:

Comparing the equation of plane with $\vec{r}.\vec{N}=d$, we obtain $\vec{N}=2\hat{i}-6\hat{j}+3\hat{k}$

The distance of point (2, −3, 4) from plane $\vec{r}.\left(2\hat{i}-6\hat{j}+3\hat{k}\right)=5$ is

$$\frac{\left|\left(2\hat{i}-3\hat{j}+4\hat{k}\right)\cdot\left(2\hat{i}-6\hat{j}+3\hat{k}\right)-5\right|}{\left|\sqrt{(2)^2+(6)^2+(3)^2}\right|}$$

$$=\frac{\left|2(2)+(-3)(-6)+(4)(3)-5\right|}{\left|\sqrt{4+36+9}\right|}$$

$$=\frac{\left|4+18+12-5\right|}{\left|7\right|}$$

$$=\frac{29}{7}$$

Example 2:

Find the distance of point P (−6, 3, 4) from the plane $7x + 4y - 4z = 4$

Solution:

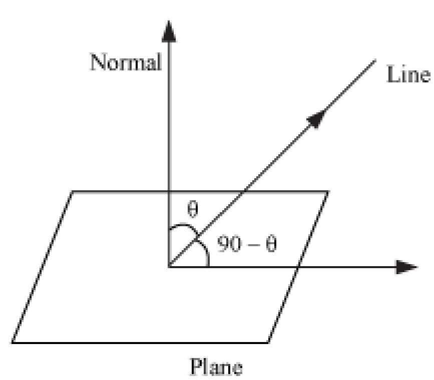
The distance of a point P $(x_1, y_1\ z_1)$ from plane $Ax + By + Cz = D$ is given by,

$$\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|$$

$$\therefore d=\frac{\left|(-6)(7)+3(4)+4(-4)-4\right|}{\left|\sqrt{(7)^2+(4)^2+(-4)^2}\right|}=\left|\frac{1-42+12-16-1}{\sqrt{81}}\right|=\frac{50}{9}$$

Angle Between a Line and a Plane

Let $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $a_2x+b_2y+c_2z+d=0$ be the equation of the line and plane.



The angle between a line and a plane is the complement of the angle between the line and normal to the plane.
Therefore, the angle between the line and plane is given by:

$$\cos\left(90^\circ - \theta\right) = \sin\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Now, let $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r}\cdot\vec{n} = d$ be the equations of the line and plane respectively in vector form. Then the angle between the line and the plane is given by $\cos\left(90^\circ - \theta\right) = \sin\theta = \frac{\left|\vec{b}\cdot\vec{n}\right|}{\left|\vec{b}\right|\left|\vec{n}\right|}$.

Angle Between a Line and a Plane

Find the angle between the line $\vec{r} = \left(-\hat{i} + 4\hat{j} + 18\hat{k}\right) + \lambda\left(3\hat{i} - 2\hat{j} + 6\hat{k}\right)$ and the plane $11x - 2y + 10z = 12$.

Solution: $\vec{r} = \left(-\hat{i} + 4\hat{j} + 18\hat{k}\right) + \lambda\left(3\hat{i} - 2\hat{j} + 6\hat{k}\right)$

On comparing the equation of line with $\vec{r} = \vec{a} + \lambda\vec{b}$, we obtain $\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$.

The equation of plane is $11x - 2y + 10z = 12$.

$\therefore \vec{n} = 11\hat{i} - 2\hat{j} + 10\hat{k}$

If θ is the angle between the given line and plane, then:

$$\sin\theta = \frac{\left|\vec{b}\cdot\vec{n}\right|}{\left|\vec{b}\right|\left|\vec{n}\right|}$$

$$\sin\theta = \left|\frac{\left(3\hat{i}-2\hat{j}+6\hat{k}\right)\cdot\left(11\hat{i}-2\hat{j}+10\hat{k}\right)}{\sqrt{9+4+36}\sqrt{121+4+100}}\right|$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{97}{105}\right)$$

Condition for a Line to Lie in a Plane

Let $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and $ax+by+cz+d=0$ be the equation of the line and plane. If the line lies in the plane, then the point $(x_1,\ y_1,\ z_1)$ will satisfy the equation of the plane and the line and the normal to the plane will be parallel. Therefore, we have the following condition for the line to lie in the plane:

- $ax_1+by_1+cz_1+d=0$
- $al+mb+cn=0$

If the equation of the line and plane are $\vec{r}=\vec{a}+\lambda\vec{b}$ and $\vec{r}.\vec{n}=\mathrm{d}$, then the conditions are $\vec{a}.\vec{n}=d$ and $\vec{b}.\vec{n}=0$.

Condition for Two Lines to be Coplanar and the Equation of the Plane Containing Them

Let $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ be two intersecting lines lying in a plane. Then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

And, the equation of the plane containing these lines is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$

Now, let $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\lambda\vec{b_2}$ be the equation of lines in vector form.

The condition for these lines to lie in a plane is $\left[\vec{a_1}\ \vec{b_1}\ \vec{b_2}\right]=\left[\vec{a_2}\ \vec{b_1}\ \vec{b_2}\right]$ and the equation of the plane containing the two lines is $\left[\vec{r}\ \vec{b_1}\ \vec{b_2}\right]=\left[\vec{a_1}\ \vec{b_1}\ \vec{b_2}\right]$.