21 Conic Section (Parabola, Ellipse, Hyperbola)

QUICK LOOK

Conic sections or conics in short, are geometrical figures obtained by the intersection of a plane with a three-dimensional double cone. In the following figure, the angle which the normal to the (variable) plane makes with the axis of the double cone is represented by while the semi-vertical angle of the cone is θ:

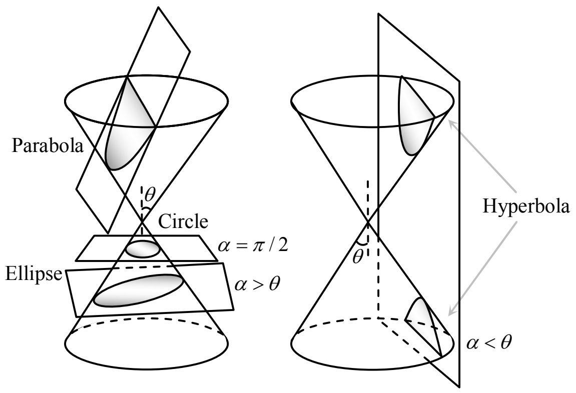


Figure: 21.1

Depending on the orientation of the intersecting plane, different types of conic sections will be generated from the double cone. Depending on the orientation of the intersecting plane, different types of conic sections will be generated from the double cone; the following four types of conics are obtained:

- **Circles:** Circles are a special kind of conics with the intersecting plane at inclination $\frac{\pi}{2}$.
- **Parabola:** The intersecting plane for parabola is parallel to the slant of the cone, i.e., at an angle θ.
- **Ellipse:** The intersecting plane is at an angle $\alpha > \theta (\alpha \neq \pi/2)$ since then a circle will be formed).
- **Hyperbola:** The intersecting plane is at an angle $\alpha < \theta$; in this case, the plane cuts both the top and bottom halves of the cone.

Recognisation of Conics

The equation of conics is represented by the general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \ldots (i)$$

and discriminant of above equation is represented by Δ, where

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

Case (i): When $\Delta = 0$

In this case equation (i) represents the degenerate conic whose nature is given in the following table.

Table 21.1: Condition and Nature of Conic

S. No.	Condition	Nature of Conic
1.	$\Delta = 0$ and $ab - h^2 = 0$`	A pair of coincident straight lines
2.	$\Delta = 0$ and $ab - h^2 < 0$	A pair of intersecting straight lines
3.	$\Delta = 0$ and $ab - h^2 > 0$	A point

Case (ii): When $\Delta \neq 0$

In this case equation (i) represents the non-degenerate conic whose nature is given in the following table.

Table 21.2: Condition and Nature of Conic

S. No.	Condition	Nature of Conic
1.	$\Delta \neq 0,\ h = 0,\ a = b$	A circle
2.	$\Delta \neq 0,\ ab - h^2 = 0$	A parabola
3.	$\Delta \neq 0,\ ab - h^2 > 0$	An ellipse
4.	$\Delta \neq 0,\ ab - h^2 < 0$	A hyperbola
5.	$\Delta \neq 0,\ ab - h^2 < 0$ and $a + b = 0$	A rectangular hyperbola

Method to find centre of a Conic

Let $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ be the given conic.

Find $\frac{\partial S}{\partial x}; \frac{\partial S}{\partial y}$

Solve $\frac{\partial S}{\partial x} = 0, \frac{\partial S}{\partial y} = 0$ for *x, y* we shall get the required centre (*x, y*)

$$(x, y) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

Intersection of a Right Circular Cone and a Plane

- The curve/lines of intersection of a right circular cone and a plane is a conic section. Conic sections are of different varieties for different orientation of the plane.
- The intersection of a right circular cone and a plane passing through the axis of the cone is a pair of lines.
- The intersection of a right circular cone and a plane perpendicular to the axis of the cone is a circle.
- The intersection of a right circular cone and a plane parallel to a generator of the cone is parabola.
- The intersection of a right circular cone and a plane cutting

the axis at an angle $\alpha\left(0<\alpha<\dfrac{\pi}{2}\right)$ is an ellipse.

- The intersection of a right circular cone and a plane parallel to the axis of the cone is hyperbola.

Conic Section as a Locus of a Point

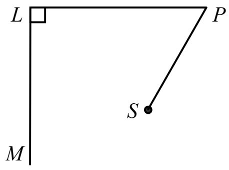


Figure :21.2

If a point moves in a plane such that its distances from a fixed point and a fixed line always bear a constant ratio l then the locus of a point is a conic section of the eccentricity e (focus-directrix property). The fixed point is the focus and the fixed line is the directrix.

- If e –1,it is a parabola.
- If e <1,1 it is an ellipse.
- If e >1,it is a hyperbola.

Equation of Conic Section by Focus-Directrix Property

- If the focus is (α,β) and the directrix is $ax+by+c=0$ then the equation of the conic section whose eccentricity $=e$, is

$$(x-\alpha)^2+(y-\beta)^2=e^2\cdot\frac{(ax+by+c)^2}{a^2+b^2}$$

- If the focus is (α,β) and the directrix is $ax+by+c=0$ then the equation of the parabola is

$$(x-\alpha)^2+(y-\beta)^2=\frac{(ax+by+c)^2}{a^2+b^2}$$

Standard Equation of a Parabola and its Parts

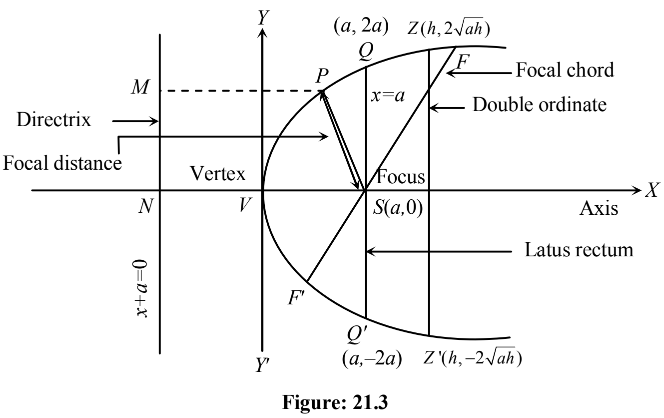


Figure: 21.3

For the parabola $y^2=4ax$,

Standard Equation of a Parabola and its Parts

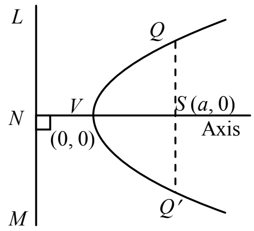


Figure :21.4

The standard equation of a parabola is $y^2=4ax$ for which vertex $V=(0,0)$, focus $S=(a,0)$

- The equation of the directrix MN is $x+a=0$ and that of the axis (or axis of symmetry) Vs of the parabola is $y=0$.
- Latus rectum $QSQ'=4a$ (= $4VS$).
- $VS=VN=a$.

Location of a Point in Relation to a Parabola

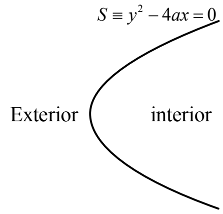


Figure :21.5

If $S\equiv y^2-4ax=0$ be a parabola and $P(x_1,y_1)$ be a point then

- P is in the interior of the parabola if $S_1<0$, i.e., $y_1^2-4ax_1<0$.
- P is on the parabola if $S_1=0$, i.e., $y_1^2-4ax_1=0$.
- P is in the exterior of the parabola if $S_1>0$ i.e., $y_1^2-4ax_1>0$.

Table 21.3: Important Terms of Parabola

Important terms	$y^2=4ax$	$y^2=-4ax$	$x^2=4ay$	$x^2=-4ay$
Coordinates of vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Coordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the directrix	$x=-a$	$x=a$	$y=-a$	$y=a$
Equation of the axis	$y=0$	$y=0$	$x=0$	$x=0$
Length of the latusrectum	$4a$	$4a$	$4a$	$4a$
Focal distance of a point $P(x,y)$	$x+a$	$a-x$	$y+a$	$a-y$

Condition for General Equation of the Second Degree to Represent a Parabola

The equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a parabola if $h^2=ab$, i.e., the second degree terms form a perfect square provided $\Delta\equiv abc+2fgh-af^2-bg^2-ch^2\neq 0$

Reduction of Equation of a Parabola into Standard Form: The equation $(y-\beta)^2 = 4a(x-\alpha)$ can be reduced to the standard form by the transformations $x-\alpha = X, y-\beta = \gamma$.

The equation becomes $\gamma^2 = 4aX$, which is the standard form in X,γ coordinates. $(y-\beta)^2 = 4a(x-\alpha)$ is the form of equation of a parabola whose axis of symmetry is parallel to the *x*-axis.

The equation $(ax+by+c)^2 = bx-ay+c'$ can be reduced to the standard form by the transformations

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = \gamma,\ \frac{bx-ay+c'}{\sqrt{b^2+a^2}} = X$$

Note

The lines $ax+by+c=0$ and $bx-ay+c'=0$ are perpendicular to each other. The equation becomes $\gamma^2 = \frac{1}{\sqrt{a^2+b^2}}X$ which is the standard form in X,γ coordinates.

Parametric Equations of a Parabola

- $x = at^2, y = 2at$ are the parametric equations of a parabola.
- Any point on the parabola $y^2 = 4ax$ has the coordinates $(at^2, 2at)$ or $\left(\frac{a}{m^2}, \frac{-2a}{m}\right)$ or $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Table 21.4: Parametric Equations of Parabola

Parabola	Parametric Coordinates	Parametric Equations
$y^2 = 4ax$	$(at^2, 2at)$	$x = at^2, y = 2at$
$y^2 = -4ax$	$(-at^2, 2at)$	$x = -at^2, y = 2at$
$x^2 = 4ay$	$(2at, at^2)$	$x = 2at, y = at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$x = 2at, y = -at^2$

The parametric equation of parabola $(y-k)^2 = 4a(x-h)$ are $x = h + at^2$ and $y = k + 2at$

Tangents and Normal's: Let the equation of a parabola be $y^2 = 4ax$.

- The equation of the tangent at (x_1, y_1) to the parabola is $yy_1 = 2a(x+x_1)$
- The equation of the tangent at $(at^2, 2at)$ is $ty = x + at^2$.
- The equation of the tangent at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is $y = mx + \frac{a}{m}$
- The line $y = mx + c$ touches the parabola if $c = \frac{a}{m}$ and so any tangent to the parabola can be taken as $y = mx + \frac{a}{m}$
- The equation of the normal at (x_1, y_1) is $\frac{x-x_1}{-2a} = \frac{y-y_1}{y_1}$.
- The equation of the normal at $(at^2, 2at)$ is $y + tx = 2at + at^3$,

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be a parabola.

- The equation of the tangent at (x_1, y_1) is $y - y_1 = \left(\frac{dy}{dx}\right)_{x_1 y_1} \cdot (x - x_1)$, i.e.,

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x+x_1) + f(y+y_1) + c = 0$$

- The equation of the normal at (x_1, y_1) is $y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{x_1, y_1}} \cdot (x - x_1)$

Chord of Contact, Polar Line, Pole: Let the equation of a parabola by $y^2 - 4ax = 0$

- The chord of contact of tangents from the exterior point $P(x_1, y_1)$ to the parabola is $T \equiv yy_1 - 2a(x+x_1) = 0$
- The polar line of the point $P(x_1, y_1)$ with respect to the parabola is $T \equiv yy_1 - 2a(x+x_1) = 0$
- The pole of a line $L = 0$ with respect to the parabola is the point (x_1, y_1) whose polar is the line $L = 0$.

Chord with Given Middle Point: The equation of a chord of a second degree curve S=0 whose middle point is (x_1, y_1) is $S_1 = T$. So, for the parabola $y^2 - 4ax = 0$ it is $y_1^2 - 4ax_1 = yy_2 - 2a(x+x_1)$

Diameter of a Parabola: The locus of the middle points of parallel chords of a parabola is a line which is called a diameter of the parabola.

Length of Tangent, Subtangent, Normal and Subnormal

Let the parabola $y^2 = 4ax$. Let the tangent and normal at $P(x_1, y_1)$ meet the axis of parabola at T and G respectively, and tangent at $P(x_1, y_1)$ makes angle ψ with the positive direction of x-axis.

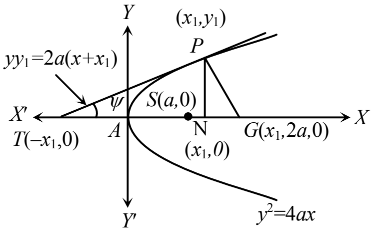


Figure :21.6

$A(0,0)$ is the vertex of the parabola and $PN = y$. Then,

- Length of tangent $= PT = PN \operatorname{cosec}\psi = y_1 \operatorname{cosec}\psi$
- Length of normal $= PG = PN\operatorname{cosec}(90^\circ - \psi) = y_1 \sec\psi$
- Length of sub-tangent $= TN = PN\cot\psi = y_1\cot\psi$
- Length of subnormal $= NG = PN\cot(90^\circ - \psi) = y_1\tan\psi$

where , $\tan\psi = \dfrac{2a}{y_1} = m$, [slope of tangent at $P(x, y)$]

Length of tangent, sub tangent, normal and subnormal to $y^2 = 4ax$ at $(at^2, 2at)$

- Length of tangent at $(at^2, 2at) = 2at\operatorname{cosec}\psi$

$= 2at\sqrt{(1+\cot^2\psi)} \quad = 2at\sqrt{1+t^2}$

- Length of normal at $(at^2, 2at) = 2at\sec\psi$

$= 2at\sqrt{(1+\tan^2\psi)}$

$= 2a\sqrt{t^2 + t^2\tan^2\psi} \quad = 2a\sqrt{(t^2+1)}$

- Length of subtangent at $(at^2, 2at) = 2at\cot\psi = 2at^2$
- Length of subnormal at $(at^2, 2at) = 2at\tan\psi = 2a$

Standard equation of the Ellipse

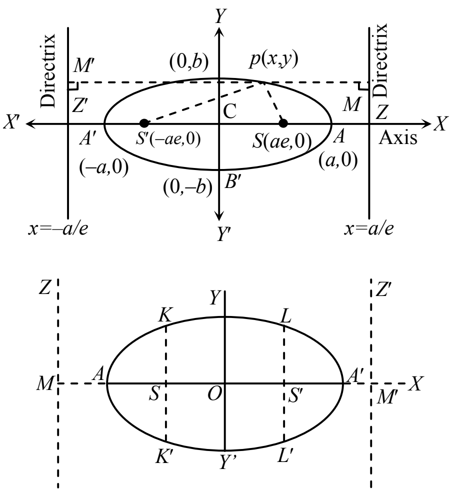


Figure :21.7

Let S be the focus, ZM be the directrix of the ellipse and $P(x, y)$ is any point on the ellipse, then by definition $\dfrac{SP}{PM} = e$

$\Rightarrow \quad (SP)^2 = e^2(PM)^2 \ (x-ae)^2 + (y-0)^2 = e^2\left(\dfrac{a}{e} - x\right)^2$

$\Rightarrow \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2(1-e^2)} = 1 \ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b^2 = a^2(1-e^2)$

Since $e < 1$, therefore $a^2(1-e^2) < a^2 \Rightarrow b^2 < a^2$.

Some terms related to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, a > b$:

The standard equation of an ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ for which

- centre $O = (0,0)$
- focus $S, S' = (\pm a, 0)$
- the equation of the major axis AA' is $y = 0$ the length of the major axis $= AA' = 2a$
- the equation of the minor axis BB' is $x = 0$ the length of the minor axis $= BB' = 2b$
- the relation between semimajor axis a, semiminor axis b and eccentricity e is $b^2 = a^2(1-e^2), e < 1$
- the equation of directrices are $x \pm \dfrac{a}{e} = 0$
- latus rectum LL' (or KK') $= \dfrac{2b^2}{a}$.

Focal Distances of a Point: The distance of a point from the focus is its focal distance. The sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis of the ellipse.

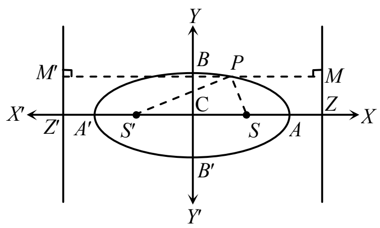


Figure: 21.8

Let $P(x_1, y_1)$ be any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$SP = ePM = e\left(\frac{a}{e} - x_1\right) = a - ex_1$$

and $\quad S'P = ePM' = e\left(\dfrac{a}{e} + x_1\right) = a + ex_1$

$\therefore \quad SP + S'P = (a - ex_1) + (a + ex_1) = 2a = AA' =$ major axis.

Table 21.5: Basic Fundamentals of Ellipse

Basic fundamentals	**Ellipse** $\left\{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1\right\}$	
	For $a > b$	**For $b > a$**
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Relation in a, b and e	$b^2 = a^2(1-e^2)$	$a^2 = b^2(1-e^2)$

Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus-rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Parametric equations	$(a\cos\phi, b\sin\phi)$	$(a\cos\phi, b\sin\phi)$ $(0 \le \phi < 2\pi)$
Focal radii	$SP = a - ex_1$ and $S'P = a + ex_1$	$SP = b - ey_1$ and $S'P = b + ey_1$
Sum of focal radii $SP + S'P =$	$2a$	$2b$
Distance between foci	$2ae$	$2be$
Distance between directrices	$2a/e$	$2b/e$
Tangents at the vertices	$x = -a, x = a$	$y = b, y = -b$

Standard equation of the Hyperbola

Let S be the focus, ZM be the directrix and e be the eccentricity of the hyperbola, then by definition,

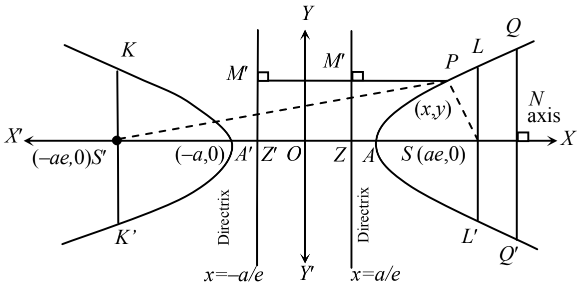


Figure :21.9

$$\Rightarrow \quad \frac{SP}{PM} = e \Rightarrow (SP)^2 = e^2(PM)^2$$

$$\Rightarrow \quad (x - a.e)^2 + (y - 0)^2 = e^2\left(x - \frac{a}{e}\right)^2$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \; b^2 = a^2(e^2-1)$$

The standard equation of a hyperbola is for which

- centre $O = (0,0)$
- focus, $S, S' = (\pm ae, 0)$
- vertices $A, A' = (\pm a, 0)$
- the equation of the transverse axis $A'A$ is $y = 0$ the length of the transverse axis $= A'A = 2a$
- the equation of the conjugate axis $B'B$ is $x = 0$
- the length of the conjugate axis $= B'B = 2b$
- the relation between semitransverse axis, a semiconjugate axis b and eccentricity e is $b^2 = a^2(e^2-1), e > 1$
- the equation of directrices are $x \pm \frac{a}{e} = 0$
- latus rectum LL' (or KK') $\frac{2b^2}{a}$.

Note

Comparing the results for ellipses and hyperbolas we find that for coordinates, lengths or equations results are the same, only difference being in the relation between a, b, e($-b^2$ takes place of b^2)

Equation of Ellipse/Hyperbola when one Directrix and the Corresponding Focus are given: If a directrix has the equation $ax + by + c = 0$ and the corresponding focus is (α, β) then the equation of the ellipse/hyperbola is

$$(x-\alpha)^2 + (y-\beta)^2 = e^2 \cdot \left(\frac{ax + by + c}{\sqrt{a^2 + b^2}}\right)^2$$

being given eccentricity of the ellipse/hyperbola.

Transformation into Standard form

- If the equation of the curve is $\frac{(x-\alpha)^2}{a^2} \pm \frac{(y-\beta)^2}{b^2} = 1$ then by taking the equations of transformation $x = \alpha + X, y = \beta + Y$ the equation changes in the standard form $\frac{X^2}{a^2} \pm \frac{Y^2}{b^2} = 1$
- In case of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b.$ If the equation of the ellipse is $\frac{(x-\alpha)^2}{b^2} + \frac{(y-\beta)^2}{a^2} = 1$ then we substitute $y = \beta + X, x = \alpha + Y$
- If the equation of the curve is $\frac{(lx + my + n)^2}{a^2} \pm \frac{(mx + ly + p)^2}{a^2} = 1$ where $1x + my + n = 0$ and $mx - 1y + p = 0$ are perpendicular lines then we substitute $\frac{lx + my + n}{\sqrt{l^2 + m^2}} = X, \frac{mx - ly + p}{\sqrt{l^2 + m^2}} = Y$ to put the equation in the standard form.

Location of a Point

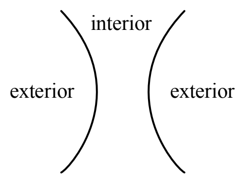


Figure :21.10

If the ellipse is $S \equiv \frac{x^2}{a} + \frac{y^2}{b^2} - 1 = 0$ and $P = (x_1, y_1)$ then

- P is in the interior of the ellipse if $S(x_1, y_1)$ i.e., $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 < 0$

- P is on the ellipse if $S(x_1, y_1) = 0$
- P is in the exterior of the ellipse if $S(x_1, y_1) > 0$.

If the hyperbola is $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ and $P = (x_1, y_1)$ then

- P is in the interior of the hyperbola if $S(x_1, y_1) < 0$
- P is on the hyperbola if $S(x_1, y_1) = 0$
- P is in the exterior of the hyperbola if $S(x_1, y_1) > 0$

Condition for the General Equation of the Second Degree in *x*, *y* to Represent an Ellipse/Hyperbola

The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents

- an ellipse if $h^2 - ab < 0, \Delta \neq 0$ and it is not a circle.
- a hyperbola if $h^2 - ab > 0, \Delta \neq 0$
- a rectangular (or equilateral) hyperbola {in a rectangular hyperbola, transverse axis = conjugate axis} if $h^2 - ab > 0$ and $a + b = 0$, $\Delta \neq 0$.

Standard Equation of a Rectangular Hyperbola: The standard equation of a rectangular hyperbola is

- $x^2 - y^2 = a^2$ whose eccentricity $e = \sqrt{2}$, the transverse axis and conjugate axis being the x and y axes respectively.
- $xy = c^2$ when the bisectors of the angles between the transverse and conjugate axes are taken as x and y axes.

Coordinates of any Point on an Ellipse/Hyperbola

- Any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has the coordinates $(a\cos\phi, b\sin\phi)$ where f is a parameter (called the eccentric angle of the point).
- Any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has the coordinates $(a\sec\phi, b\tan\phi)$.
- Any point on the rectangular hyperbola $x^2 - y^2 = a^2$ has the coordinates $(a\sec\phi, b\tan\phi)$.
- Any point on the rectangular hyperbola $xy = c^2$ has the coordinates $\left(ct, \frac{c}{t}\right)$.

Equation of the Chord Joining two Points of an Ellipse: The equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ joining the points 'ϕ_1' and 'ϕ_2' is $\frac{x}{a}\cos\frac{\phi_1 + \phi_2}{2} + \frac{y}{b}\sin\frac{\phi_1 + \phi_2}{2} = \cos\frac{\phi_1 - \phi_2}{2}$.

Equation of Tangent and Condition of Tangency

- The equation of the tangent at (x_1, y_1) to the curve $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} \pm \frac{yy_1}{b^2} = 1$
- The equation of the tangent at $(a\cos\phi, b\sin\phi)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1$
- The equation of the tangent at $(a\sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x\sec\phi}{a} - \frac{y\tan\phi}{b} = 1$
- The line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$. So, the line $y = mx \pm \sqrt{a^2m^2 + b^2}$ touches the ellipse for all real m.
- The line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 - b^2$. So, the line $y = mx \pm \sqrt{a^2m^2 - b^2}$ touches the hyperbola for all real m.

Equation of Normal: The equation of the normal at (x_1, y_1) to a curve is $y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{x_1, y_1}} \cdot (x - x_1)$.

- For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of the normal at (x_1, y_1) is $\frac{x - x_1}{x_1 / a^2} = \frac{y - y_1}{y_1 / b^2}$ and that at $(a\cos\phi, b\sin\phi)$ is $ax\sec\phi - by\,\mathrm{cosec}\,\phi = a^2 - b^2$.
- For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the equation of the normal at (x_1, y_1) is $\frac{x - x_1}{x_1 / a^2} = \frac{y - y_1}{-y_1 / b^2}$ and that at $(a\sec\phi, b\tan\phi)$ is $ax\cos\phi + by\cot\phi = a^2 + b^2$.

Chord of Contact, Polar line, Pole

- The chord of contact of tangents from (x_1, y_1) to the curve $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} \pm \frac{yy_1}{b^2} = 1$.
- The polar of the point (x_1, y_1) with respect to the curve $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} \pm \frac{yy_1}{b^2} = 1$.

- The pole of the line $L=0$ with respect to the ellipse or hyperbola $S=0$ is the point (x_1, y_1) whose polar is the line $L=0$.

Equation of Chords and the Pair of Tangents from a point

- The equation of the chord of $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ whose middle point is (x_1, y_1) is $T=S_1$ where

$$T=\frac{xx_1}{a^2}\pm\frac{yy_1}{b^2}-1, S_1 \equiv \frac{xx_1}{a^2}\pm\frac{yy_1}{b^2}-1$$

- The equation of the pair of tangents from (x_1, y_1) to the curve $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1$ is $S \cdot S_1=T^2$ where $S \equiv \frac{x^2}{a^2}\pm\frac{y^2}{b^2}-1$,

$$S_1 \equiv \frac{x_1^2}{a^2}\pm\frac{y_1^2}{b^2}-1,\ T=\frac{xx_1}{a^2}\pm\frac{yy_1}{b^2}-1.$$

Some Properties of Ellipse

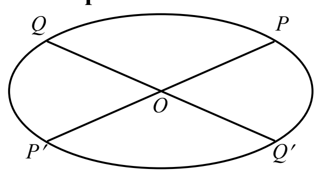


Figure: 21.11

- If S, S' are foci, major ax is = 2a then for any point P on the ellipse, $SP+S'P=2a$. A chord PP' passing through the centre O is a diameter of the ellipse. Two diameters PP' and QQ' are conjugate diameters if chords parallel to PP' are bisected by QQ' and chord parallel to QQ' are bisected by PP'.
- If the eccentric angle of P is ϕ then the other end P' of the diameter PP' will have the eccentric angle $\pi+\phi$ and the ends of the conjugate diameter have the eccentric angles $\phi \pm \frac{\pi}{2}$.
- Two diameters $y=m_1 x, y=m_2 x$ are conjugate diameters of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $m_1 m_2=-\frac{b^2}{a^2}$.

Some Properties of Hyperbola

- If S, S' are foci, transverse axis = 2a then for any point P on the hyperbola, $|SP-S'P|=2a$.
- The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is the conjugate hyperbola of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.
- The asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are the lines $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.

Table 21.6: Fundamentals of Hyperbola

Fundamentals	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **or** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x=\pm a/e$	$y=\pm b/e$
Eccentricity	$e=\sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e=\sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric co-ordinates	$(a\sec\phi, b\tan\phi)$, $0 \le \phi < 2\pi$	$(b\sec\phi, a\tan\phi)$, $0 \le \phi < 2\pi$
Focal radii	$SP=ex_1-a$ & $S'P=ex_1+a$	$SP=ey_1-b$ & $S'P=ey_1+b$
Difference of focal radii $(S'P-SP)$	$2a$	$2b$
Tangents at the vertices	$x=-a, x=a$	$y=-b,\ y=b$
Equation of the transverse axis	$y=0$	$x=0$
Equation of the conjugate axis	$x=0$	$y=0$

Rectangular or Equilateral Hyperbola

A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola. The eccentricity of rectangular hyperbola is always $\sqrt{2}$.

The general equation of second degree represents a rectangular hyperbola if $\Delta \neq 0,\ h^2>ab$ and coefficient of x^2 + coefficient of $y^2=0$. The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are given by $y=\pm\frac{b}{a}x$. The angle between these two asymptotes is given by

$$\tan\theta=\frac{\frac{b}{a}-\left(-\frac{b}{a}\right)}{1+\frac{b}{a}\left(\frac{-b}{a}\right)}=\frac{2b/a}{1-b^2/a^2}=\frac{2ab}{a^2-b^2}.$$

MULTIPLE CHOICE QUESTIONS

Conic Section

1. The equation $x^2 - 2xy + y^2 + 3x + 2 = 0$ represents:

a. A parabola **b.** An ellipse
c. A hyperbola **d.** A circle

2. The centre of $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ is:

a. (2, 3) **b.** (2, –3)
c. (–2, 3) **d.** (–2, –3)

Parabola

3. The equation of parabola whose focus is (5, 3) and directrix is $3x - 4y + 1 = 0$, is:

a. $(4x + 3y)^2 - 256x - 142y + 849 = 0$

b. $(4x - 3y)^2 - 256x - 142y + 849 = 0$

c. $(3x + 4y)^2 - 142x - 256y + 849 = 0$

d. $(3x - 4y)^2 - 256x - 142y + 849 = 0$

4. If the parabola $y^2 = 4ax$ passes through (–3, 2), then length of its latus rectum is:

a. 2/3 **b.** 1/3
c. 4/3 **d.** 4

Parametric Equations of a Parabola, Position of a Point and a Line with respect to a Parabola

5. $x - 2 = t^2, y = 2t$ are the parametric equations of the parabola:

a. $y^2 = 4x$ **b.** $y^2 = -4x$
c. $x^2 = -4y$ **d.** $y^2 = 4(x - 2)$

6. The equation of a parabola is $y^2 = 4x$. $P(1,3)$ and $Q(1,1)$ are two points in the xy-plane. Then, for the parabola:

a. P and Q are exterior points
b. P is an interior point while Q is an exterior point
c. P and Q are interior points
d. P is an exterior point while Q is an interior point

Point of intersection of Tangents at any two points on the Parabola and Equation of Pair of Tangents from a point to a Parabola

7. The straight line $y = 2x + \lambda$ does not meet the parabola $y^2 = 2x$, if:

a. $\lambda < \frac{1}{4}$ **b.** $\lambda > \frac{1}{4}$ **c.** $\lambda = 4$ **d.** $\lambda = 1$

8. If the tangent to the parabola $y^2 = ax$ makes an angle of 45° with x-axis, then the point of contact is:

a. $\left(\frac{a}{2}, \frac{a}{2}\right)$ **b.** $\left(\frac{a}{4}, \frac{a}{4}\right)$
c. $\left(\frac{a}{2}, \frac{a}{4}\right)$ **d.** $\left(\frac{a}{4}, \frac{a}{2}\right)$

9. The line $x - y + 2 = 0$ touches the parabola $y^2 = 8x$ at the point:

a. (2,–4) **b.** $(1, 2\sqrt{2})$ **c.** $(4, -4\sqrt{2})$ **d.** (2, 4)

Equations of Normal in Different forms

10. If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then k is:

a. 3 **b.** 9 **c.** –9 **d.** –3

11. The normals at three points P, Q, R of the parabola $y^2 = 4ax$ meet in (h, k), the centroid of triangle PQR lies on:

a. $x = 0$ **b.** $y = 0$ **c.** $x = -a$ **d.** $y = a$

Equations of Chord and Tangent in Different forms

12. If the points $(au^2, 2au)$ and $(av^2, 2av)$ are the extremities of a focal chord of the parabola $y^2 = 4ax$, then:

a. $uv - 1 = 0$ **b.** $uv + 1 = 0$
c. $u + v = 0$ **d.** $u - v = 0$

Diameter of a Parabola, Length of Tangent, Sub-tangent, Normal and Subnormal

13. Equation of diameter of parabola $y^2 = x$ corresponding to the chord $x - y + 1 = 0$ is:

a. $2y = 3$ **b.** $2y = 1$
c. $2y = 5$ **d.** $y = 1$

14. The length of the sub-tangent to the parabola $y^2 = 16x$ at the point, whose abscissa is 4, is:

a. 2 **b.** 4
c. 8 **d.** None of these

Pole and Polar

15. The pole of the line $2x = y$ with respect to the parabola $y^2 = 2x$ is:

a. $\left(0, \frac{1}{2}\right)$ **b.** $\left(\frac{1}{2}, 0\right)$
c. $\left(0, -\frac{1}{2}\right)$ **d.** None of these

16. A ray of light moving parallel to the x-axis gets reflected from a parabolic mirror whose equation is $(y-2)^2 = 4(x+1)$. after reflection, the ray must pass through the point:

a. (0, 2) **b.** (2, 0)
c. (0, –2) **d.** (–1, 2)

Ellipse

17. The equation of an ellipse whose focus is (–1, 1), whose directrix is $x-y+3=0$ and whose eccentricity is $\frac{1}{2}$, is given by:

a. $7x^2+2xy+7y^2+10x-10y+7=0$
b. $7x^2-2xy+7y^2-10x+10y+7=0$
c. $7x^2-2xy+7y^2-10x-10y-7=0$
d. $7x^2-2xy+7y^2+10x+10y-7=0$

18. If $P(x,y), F_1=(3,0)$, $F_2=(-3,0)$ and $16x^2+25y^2=400$, then PF_1+PF_2 equals:

a. 8 **b.** 6
c. 10 **d.** 12

Equation of Ellipse in other form and Parametric Equation

19. The equation of a directrix of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is:

a. $y=\frac{25}{3}$ **b.** $x=3$
c. $x=-3$ **d.** $x=\frac{3}{25}$

20. The distance of the point 'θ' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from a focus is:

a. $a(e+\cos\theta)$ **b.** $a(e-\cos\theta)$
c. $a(1+e\cos\theta)$ **d.** $a(1+2e\cos\theta)$

Position of a point with respect to an Ellipse and Intersection of a line

21. Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2+y^2=9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then:

a. Q lies inside C but outside E
b. Q lies outside both C and E
c. P lies inside both C and E
d. P lies inside C but outside E

Equations of Tangent in Different forms and Pair of Tangents

22. The number of values of 'c' such that the straight line $y=4x+c$ touches the curve $\frac{x^2}{4}+y^2=1$ is:

a. 0 **b.** 1
c. 2 **d.** Infinite

23. The area of the quadrilateral formed by the tangents at the end points of latus- rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$, is:

a. 27/4 *sq.* units **b.** 9 *sq.* units
c. 27/2 *sq.* units **d.** 27*sq.* units

Equations of Normal in Different forms Eccentric angles of the Co-normal points

24. The equation of normal at the point (0, 3) of the ellipse $9x^2+5y^2=45$ is:

a. $y-3=0$ **b.** $y+3=0$
c. x-axis **d.** y-axis

25. If the normal at any point P on the ellipse cuts the major and minor axes in G and g respectively and C be the centre of the ellipse, then:

a. $a^2(CG)^2+b^2(Cg)^2=(a^2-b^2)^2$
b. $a^2(CG)^2-b^2(Cg)^2=(a^2-b^2)^2$
c. $a^2(CG)^2-b^2(Cg)^2=(a^2+b^2)^2$
d. None of these

Chord of Contact, Equation with Mid Points and Chord Joining Points

26. What will be the equation of the chord of contact of tangents drawn from (3, 2) to the ellipse $x^2+4y^2=9$?

a. $3x+8y=9$ **b.** $3x+8y=25$
c. $3x+4y=9$ **d.** $3x+8y+9=0$

Pole and Polar

27. The pole of the straight line $x+4y=4$ with respect to ellipse $x^2+4y^2=4$ is:

a. (1, 4) **b.** (1, 1)
c. (4, 1) **d.** (4, 4)

Diameter of the Ellipse

28. If one end of a diameter of the ellipse $4x^2+y^2=16$ is $(\sqrt{3},2)$, then the other end is:

a. $(-\sqrt{3},2)$ **b.** $(\sqrt{3},-2)$ **c.** $(-\sqrt{3},-2)$ **d.** $(0,0)$

29. If θ and ϕ are eccentric angles of the ends of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\theta-\phi$ is equal to:

a. $\pm\frac{\pi}{2}$ **b.** $\pm\pi$

c. 0 **d.** None of these

Sub-tangent and Subnormal; Con-cyclic points and Reflection property of an Ellipse

30. Length of sub-tangent and subnormal at the point $\left(\frac{-5\sqrt{3}}{2},2\right)$ of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ are:

a. $\left(\frac{5\sqrt{3}}{2}-\frac{10}{\sqrt{3}}\right),\frac{8\sqrt{3}}{5}$ **b.** $\left(\frac{5\sqrt{3}}{2}+\frac{10}{\sqrt{3}}\right),\frac{8\sqrt{3}}{10}$

c. $\left(\frac{5\sqrt{3}}{2}+\frac{12}{\sqrt{3}}\right),\frac{16\sqrt{3}}{5}$ **d.** None of thee

Hyperbola

31. The equation of the conic with focus at (1, – 1), directrix along $x-y+1=0$ and with eccentricity $\sqrt{2}$ is:

a. $x^2-y^2=1$ **b.** $xy=1$

c. $2xy+4x-4y-1=0$ **d.** $2xy+4x-4y-1=0$

Position of a point with respect to a Hyperbola and Intersection of a Line

32. The number of tangents to the hyperbola $\frac{x^2}{4}-\frac{y^2}{3}=1$ through (4, 1) is:

a. 1 **b.** 2

c. 0 **d.** 3

Equations of Tangent in Different forms and Equation of Pair of Tangents

33. The points of contact of the line $i\ \ y=x-1$ with $3x^2-4y^2=12$ is:

a. (4, 3) **b.** (3, 4)

c. (4,–3) **d.** None of these

34. The locus of the point of intersection of tangents to the hyperbola $4x^2-9y^2=36$ which meet at a constant angle $\pi/4$, is:

a. $(x^2+y^2-5)^2=4(9y^2-4x^2+36)$

b. $(x^2+y^2-5)=4(9y^2-4x^2+36)$

c. $4(x^2+y^2-5)^2=(9y^2-4x^2+36)$

d. None of these

Equations of Normal in Different forms

35. The equation of the normal to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at the point $(8,3\sqrt{3})$ is:

a. $\sqrt{3}x+2y=25$ **b.** $x+y=25$

c. $y+2x=25$ **d.** $2x+\sqrt{3y}=25$

36. If the normal at 'ϕ' on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets transverse axis at G, then $AG.A'G=?$ (Where A and A' are the vertices of the hyperbola)

a. $a^2(e^4\sec^2\phi-1)$ **b.** $(a^2e^4\sec^2\phi-1)$

c. $a^2(1-e^4\sec^2\phi)$ **d.** None of these

Equation of Chord of Contact of Tangents drawn from a Point to a Hyperbola, Mid Points joining Two points on the Hyperbola

37. The equation of the chord of contact of tangents drawn from a point (2, –1) to the hyperbola $16x^2-9y^2=144$ is:

a. $32x+9y=144$

b. $32x+9y=55$

c. $32x+9y+144=0$

d. $32x+9y+55=0$

38. The point of intersection of tangents drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the points where it is intersected by the line $lx+my+n=0$ is:

a. $\left(\frac{-a^2l}{n},\frac{b^2m}{n}\right)$ **b.** $\left(\frac{a^2l}{n},\frac{-b^2m}{n}\right)$

c. $\left(-\frac{a^2n}{l},\frac{b^2n}{m}\right)$ **d.** $\left(\frac{a^2n}{l},\frac{-b^2n}{m}\right)$

Pole and Polar and Diameter of the Hyperbola

39. If the polar of a point *w.r.t.* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the hyperbola $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the locus of the point is:

a. Given hyperbola **b.** Ellipse
c. Circle **d.** None of these

40. If a pair of conjugate diameters meet the hyperbola and its conjugate in *P* and *D* respectively, then $CP^2-CD^2=?$

a. a^2+b^2 **b.** a^2-b^2
c. $\frac{a^2}{b^2}$ **d.** None of these

Sub-tangent and Subnormal of the Hyperbola, Reflection property of the Hyperbola and Asymptotes

41. From any point on the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ tangents are drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2$. The area cut-off by the chord of contact on the asymptotes is equal to:

a. $\frac{ab}{2}$ **b.** ab
c. $2ab$ **d.** $4ab$

42. The combined equation of the asymptotes of the hyperbola $2x^2+5xy+2y^2+4x+5y=0$?

a. $2x^2+5xy+2y^2=0$
b. $2x^2+5xy+2y^2-4x+5y+2=0=0$
c. $2x^2+5xy+2y^2+4x+5y-2=0$
d. $2x^2+5xy+2y^2+4x+5y+2=0$

Rectangular or Equilateral Hyperbola

43. If $5x^2+\lambda y^2=20$ represents a rectangular hyperbola, then λ equals:

a. 5 **b.** 4
c. – 5 **d.** None of these

44. If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy=c^2$ meets the curve again in t', then:

a. $t'=-\frac{1}{t^3}$ **b.** $t'=-\frac{1}{t}$
c. $t'=\frac{1}{t^2}$ **d.** $t'^2=-\frac{1}{t^2}$

Intersection of a Circle and a Rectangular Hyperbola

45. If a circle cuts a rectangular hyperbola $xy=c^2$ in *A*, *B*, *C*, *D* and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively. Then:

a. $t_1t_2=t_3t_4$ **b.** $t_1t_2t_3t_4=1$
c. $t_1=t_2$ **d.** $t_3=t_4$

46. If the circle $x^2+y^2=a^2$ intersects the hyperbola $xy=c^2$ in four points $P(x_1,y_1), Q(x_2,y_2), R(x_3,y_3), s(x_4,y_4)$ then:

a. $x_1+x_2+x_3+x_4=0$ **b.** $y_1+y_2+y_3+y_4=0$
c. $x_1x_2x_3x_4=c^4$ **d.** $y_1y_2y_3y_4=c^4$

NCERT EXEMPLAR PROBLEMS

More than One Answer

47. Equation of common tangent of $y=x^2$, $y=-x^2+4x-4$ is:

a. $y=4(x-1)$ **b.** $y=0$
c. $y=-4(x-1)$ **d.** $y=-30x-50$

48. Let $P(x_1,y_1)$ and $Q(x_2,y_2)$, $y_1<0$, $y_2<0$, be the end points of the latus rectum of the ellipse $x^2+4y^2=4$.The equations of parabolas with latus rectum *PQ* are:

a. $x^2+2\sqrt{3}y=3+\sqrt{3}$ **b.** $x^2-2\sqrt{3}y=3+\sqrt{3}$
c. $x^2+2\sqrt{3}y=3-\sqrt{3}$ **d.** $x^2-2\sqrt{3}y=3-\sqrt{3}$

49. The tangent *PT* and the normal *PN* to the parabola $y^2=4ax$ at a point *P* on it meet its axis at points *T* and *N*, respectively. The locus of the centroid of the triangle *PTN* is a parabola whose:

a. vertex is $\left(\frac{2a}{3},0\right)$ **b.** directrix is $x=0$
c. latus rectum is $\frac{2a}{3}$ **d.** focus is $(a, 0)$

50. Let *A* and *B* be two distinct points on the parabola $y^2=4x$. If the axis of the parabola touches a circle of radius *r* having *AB* as its diameter, then the slope of the line joining *A* and *B* can be:

a. $-\frac{1}{r}$ **b.** $\frac{1}{r}$
c. $\frac{2}{r}$ **d.** $-\frac{2}{r}$

51. Let *L* be a normal to the parabola $y^2=4x$. If *L* passes through the point (9, 6), then *L* is given by?

a. $y-x+3=0$ **b.** $y+3x-33=0$
c. $y+x-15=0$ **d.** $y-2x+12=0$

52. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3), S(x_4, y_4)$, then?

a. $x_1 + x_2 + x_3 + x_4 = 0$ **b.** $y_1 + y_2 + y_3 + y_4 = 0$

c. $x_1x_2x_3x_4 = c^4$ **d.** $y_1y_2y_3y_4 = c^4$

53. An ellipse intersects the hyperbola $2x^2 – 2y^2=1$ orthogonally. The eccentricity of the ellipse is along the coordinate axes, then:

a. Equation of ellipse is $x^2 + 2y^2 = 2$

b. The foci of ellipse are $(\pm 1,0)$

c. Equation of ellipse is $x^2 + 2y^2 = 4$

d. The foci of ellipse are $(\pm\sqrt{2},0)$

54. Let the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be reciprocal to that of the ellipse $x^2 = 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then:

a. the equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$

b. a focus of the hyperbola is (2, 0)

c. the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

d. the equation of the hyperbola is $x^2 – 3y^2 = 3$

55. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line $2x – y = 1$. The points of contacts of the tangents on the hyperbola are:

a. $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ **b.** $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

c. $(3\sqrt{3},-2\sqrt{2})$ **d.** $(-3\sqrt{3},2\sqrt{2})$

56. In the ellipse $25x^2 + 9y^2 + 150x – 190y + 225 = 0$?

a. foci are at (3,1), (3, 9) **b.** $e = 4/5$

c. centre is (5,3) **d.** major axis is

57. The points, where the normals to the ellipse $x^2 + 3y^2 = 37$ be parallel to the line $6x – 5y + 7 = 0$ is:

a. (5, 2) **b.** (2, 5)

c. (1, 3) **d.** (–5, –2)

58. If the tangent at the point $\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ to the ellipse $16x^2 + 11y^2 = 256$ is also a tangent to the circle $x_2 + y^2 –$ 256 is also a tangent to the circle $x^2 + y^2 - 2x = 15$, then θ equals:

a. $\frac{\pi}{3}$ **b.** $\frac{2\pi}{3}$ **c.** $-\frac{\pi}{3}$ **d.** $\frac{5\pi}{3}$

59. The product of eccentricities of two conics is unity, one of them can be a/an?

a. parabola **b.** ellipse

c. hyperbola **d.** circle

60. If m_1 and m_2 are the slopes of the tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ which pass through the point (6,2), then:

a. $m_1 + m_2 = \frac{24}{11}$ **b.** $m_1m_2 = \frac{20}{11}$

c. $m_1 + m_2 = \frac{48}{11}$ **d.** $m_1m_2 = \frac{11}{20}$

61. If the tangent at the point $(a\sec\alpha, b\tan\alpha)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the transverse axis at T, then the distance of T form a focus of the hyperbola is:

a. $a(e-\cos\alpha)$ **b.** $b(e+\cos\alpha)$

c. $a(e+\cos\alpha)$ **d.** $\sqrt{(a^2e^2+b^2+\cot^2\alpha)}$

Assertion and Reason

Note: Read the Assertion (A) and Reason (R) carefully to mark the correct option out of the options given below:

a. If both assertion and reason are true and the reason is the correct explanation of the assertion.

b. If both assertion and reason are true but reason is not the correct explanation of the assertion.

c. If assertion is true but reason is false.

d. If the assertion and reason both are false.

e. If assertion is false but reason is true.

62. Consider the two curves $C_1 : y^2 = 4x, C_2 : x^2 + y^2 - 6x + 1 = 0$

Assertion: C_1 and C_2 touch each other exactly at two points.

Reason: Equation of the tangent at (1,2) to C_1 and C_2 both is $x – y + 1 = 0$ and at (1,–2) is $x + y + 1 = 0$

63. **Assertion:** The curve $y = -\frac{x^2}{2} + x + 1$ is symmetrical with respect to the line $x = 1$.

Reason: A parabola is symmetric about its axis

64. **Assertion:** If the length of the latus rectum of an ellipse is 1/3 of the major axis, then the eccentricity of the ellipse is $\sqrt{2/3}$

Reason: If a focus of an ellipse is at the origin directrix is the line $x = 4$ and the eccentricity is $\sqrt{2/3}$, then the length of the semi major axis is $4.\sqrt{6}$

65. **Assertion:** A parabola has the origin as its focus and the line $y = 2$ as the directrix, then the vertex of the parabola is at the point (0,1)
Reason: Vertex of a parabola is equidistance form the focus and the directrix and lies on the line through the foucs perpendicular to the directrix.

66. **Assertion:** A equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.
Reason: If the line $y = mx + \frac{4\sqrt{3}}{m}, (m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$

67. **Assertion:** Two tangents drawn from any point on the hyperbola $x^2 - y^2 = a^2 - b^2$ to the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ make complementary angles with the axis of the ellipse
Reason: If two lines make complementary angles with the axis of x then the product of their slopes is 1.

68. **Assertion:** The tangent to the parabola $y^2 = 4x$ at any point P and perpendicular on it form the focus S meet on the directrix of the parabola.
Reason: Tangents and normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square whose area is $8a^2$sq. units

69. **Assertion:** If the vertex of a parabola lies at the point $(a, 0)$ and the directrix is y-axis then the focus of the parabola is at the point $(2a, 0)$.
Reason: Length of the common chord of the parabola $y^2 = 12x$ and the circle $x^2 + y^2 = 9$ is equal to the length of the latus rectum of the parabola.

70. **Assertion:** If the foci an hyperbola are at the points (4, 1) and (–6,1), eccentricity is 5/4 then the length of the transverse axis is 4.
Reason: Distance between the foci of a hyperbola is equal to the product of its eccentricity and the length of the transverse axis.

71. **Assertion:** If the normal at an end of a latusrectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis at G,O is the centre of the ellipse, then $OG = ae^3, e$ being the eccentricity of the ellipse
Reason: Equation of the normal at a point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{ax}{\cos\theta} + \frac{by}{\sin\theta} = a^2 + b^2$

Comprehension Based

Paragraph–I

Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a, a > 0$.

72. Length of chord PQ is:
a. $7a$ **b.** $5a$
c. $2a$ **d.** $3a$

73. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta$ is equal to:
a. $\frac{2}{3}\sqrt{7}$ **b.** $\frac{-2}{3}\sqrt{7}$ **c.** $\frac{2}{3}\sqrt{5}$ **d.** $\frac{-2}{3}\sqrt{5}$

Paragraph–II

Tangents are drawn from the point $P(3,4)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ touching the ellipse at points A and B.

74. The coordinates of A and B are:
a. (3, 0) and (0, 2) **b.** $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$
c. $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2) **d.** (3, 0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

75. The orthocentre of the triangle PAB is:
a. $\left(5, \frac{8}{7}\right)$ **b.** $\left(\frac{7}{5}, \frac{25}{8}\right)$
c. $\left(\frac{11}{5}, \frac{8}{5}\right)$ **d.** $\left(\frac{8}{25}, \frac{7}{5}\right)$

76. The equation of the locus of the point whose distance from the point P and the line AB are equal, is:
a. $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$
b. $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$
c. $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$
d. $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Paragraph–III

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

77. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is:
a. $2x - \sqrt{5}y - 20 = 0$ **b.** $2x - \sqrt{5}y + 4 = 0$
c. $3x - 4y + 8 = 0$ **d.** $4x - 3y + 4 = 0$

78. Equation of the circle with AB as its diameter is:

a. $x^2 + y^2 - 12x + 24 = 0$ **b.** $x^2 - y^2 + 12x + 24 = 0$

c. $x^2 + y^2 + 24x - 12 = 0$ **d.** $x^2 + y^2 - 24x - 12 = 0$

Paragraph–IV

The difference between the second degree curve and pair of asymptotes is constant. If second degree curve represented by a hyperbola $S = 0$, then the equation of its asymptotes is $S + \lambda = 0$, and if equation of conjugate which will be a pair of straight lines, then we get λ. Then equation of asymptotes is $A \equiv S + \lambda = 0$ and if equation of conjugate hyperbola of S represented by S_1, then A is the arithmetic means of S and S_1.

79. Pair of asymptotes of the hyperbola $xy - 3y - 2x = 0$ is:

a. $xy - 3y - 2x + 2 = 0$ **b.** $xy - 3y - 2x + 4 = 0$

c. $xy - 3y - 2x + 16 = 0$ **d.** $xy - 3y - 2x + 12 = 0$

80. The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$. If the hyperbola passes through the point (5,3) then its equation is:

a. $(2x + 3y - 3)(3x + 2y - 5) = 256$

b. $(2x + 3y - 7)(3x + 2y - 8) = 156$

c. $(2x + 3y - 5)(3x + 2y - 3) = 256$

d. $(2x + 3y - 8)(3x + 2y - 7) = 154$

81. If angle between the asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{\pi}{3}$ thant he centricity of conjugate hyperbola is:

a. $\sqrt{2}$ **b.** 2

c. $\frac{2}{\sqrt{3}}$ **d.** $\frac{4}{\sqrt{3}}$

82. A hyperbola passing through origin has $3x - 4y - 1 = 0$ and $4x - 3y - 6 = 0$ as its asymptotes. Then the equation of its tansverse and conjugate axes as:

a. $x - y - 5 = 0$ and $x + y + 1 = 0$

b. $x - y = 0$ and $x + y + 5 = 0$

c. $x + y - 5 = 0$ and $x - y - 1 = 0$

d. $x + y - 1 = 0$ and $x - y - 5 = 0$

83. The tangent at any point of a hyperbola $16x^2 - 25y^2 = 400$ cuts off a triangle form the asymptotes and that the portion of it intercepted between the asymptotes, then the area of this triangle is:

a. 10 aq. unit **b.** 20 sq. unit

c. 30 sq. unit **d.** 40 sq unit

Match the Column

84. Match the statement of Column with those in Column II:

Column I	Column II
(A) The minimum and maximum distance of a point (2, 6) from the ellipse $9x^2 + 8y^2 - 36x - 16y - 28 = 0$ are L and G, then	**1.** $L + G = 10$
(B) The minimum and maximum distance of a point (1, 2) from the ellipse $4x^2 + 9y^2 + 8x - 36y + 4 = 0$ are L and G, then	**2.** $L + G = 6$
(C) The minimum and maximum distance of a point $\left(\frac{9}{5}, \frac{12}{5}\right)$ from the ellipse $4(3x+4y)2 + 9(4x - 3y)^2 = 9000$ are L and G, then	**3.** $G - L = 6$
	4. $G - L = 4$
	5. $L^G - G^L = 6$

a. A→ 2,3,4; B→ 1,2; C→ 4,5

b. A→ 1,3; B→ 2,4,5; C→ 2,3

c. A→ 3,4,5; B→ 1,3; C→ 2,3

d. A→ 1,5; B→ 2,4,5; C→3,4

85. Normals at P, Q, R are drawn to $y^2 = 4x$ which intersect at (3, 0). Then:

Column I	Column II
(A) Area of ΔPQR	**1.** 2
(B) Radius of circumcircle of ΔPQR	**2.** 5/2
(C) Centroid of ΔPQR	**3.** (5/2, 0)
(D) Circumcentre of ΔPQR	**4.** (2/3, 0)

a. A→ 1; B→ 2; C→ 4; D→ 3

b. A→ 2; B→ 4; C→ 3; D→ 1

c. A→ 3; B→ 4; C→ 2; D→ 1

d. A→ 4; B→ 1; C→ 3; D→ 2

86. Match the statement of Column with those in Column II:

Column I	Column II
(A) Direction circles of $x^2 - 2y^2 = 2$ and $x^2 + 2y^2 = 2$ are	**1.** $x^2 + y^2 = 1$
(B) Direction circles of $3x^2 + 2y^2 = 6$ and	**2.** $x^2 + y^2 = 2$

$3x^2 - 2y^2 = 6$ are	
(C) Direction circles of $5x^2 - 9y^2 = 45$ and $x^2 + y^2 = 1$ are 0	**3.** $x^2 + y^2 = 3$
	4. $x^2 + y^2 = 4$
	5. $x^2 + y^2 = 5$

a. A→ 1,3; B→ 5; C→ 2,4 **b.** A→ 2,3; B→ 4,5; C→ 3
c. A→ 1,2; B→ 3; C→ 2,4 **d.** A→ 4,5; B→ 1,3; C→ 3

Integer

87. The normal to the parabola $y^2 = 8x$ at the point (2, 4) meets it again at (18, –12). If length of normal chord is λ, then the value of λ^2 must be:

88. From a point A common tangents are drawn to the circle $x^2 + y^2 = \frac{a^2}{2}$ and the parabola $y^2 = 4ax$. If the area of the quadrilateral formed by the common tangents, the chords of contact of the point A, w.r.t. the circle and the parabola is λ square unit, then the value of $\frac{256}{a^2}\lambda$ must be:

89. Three normals with slopes m_1, m_2 and m_3 are drawn form a point P not on the axis of parabola $y^2 = 4x$. If $m_1m_2 = \alpha$, results in the locus of P being a part of parabola, then the value of $(36)^\alpha$ must be:

90. TP and TQ are any two tangents to a parabola and the tangent at a third point R cuts then in P' and Q', then the value of $\frac{TP'}{TP} = \frac{TQ'}{TQ}$ must be:

91. A water jet from a fountain reaches its maximum height of 4 m at a distance 0.5 m from the vertical passing through the point O of water outlet. If λm be the height of the jet above the horizontal OX at a distance of 0.75 m from the point O, then the value of λ^6 must be:

92. If the normal at an end of a latus rectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through one extremity of the minor axis. If e be the eccentricity of the ellipse then the value of $625(2e^2 + 1)^2$ must be:

93. If e be the eccentricity of the ellipse $4(x - 2y + 1)^2 + 9$ $(2x + y + 2)^2 = 25,$ then the value of $2187e^2$ must be:

94. If the normals at the four points $(x_1, y_1),(x_2, y_2),(x_3, y_3)$ and (x_4,y_4) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are concurrent, then the value of $(x_1 + x_2 + x_3 + x_4) \times \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right)$ must be:

95. Tangents are drawn to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ at ends of latusrectum. If the area of an quadrilateral by λ sq unit, then the value of λ must be:

96. Let Δ_1 be the are of ΔPQR inscribed in an ellipse and Δ_2 be the area of the $\Delta P'Q'R'$ whose vertices are the points lying on the auxiliary circle corresponding to the points P, Q R, respectively. If the eccentricity of the ellipse is $\frac{4\sqrt{3}}{7}$ then the ratio $343\frac{\Delta_2}{\Delta_1}$ must be:

97. If four Points be taken on a rectangular hyperbola such that the chord joining any two is perpendicular to the chord joining the other two and if α, β, γ, δ be the inclinations to either asymptote of the straight line joining these points to the centre, then the value of tan α tan β × tan γ tan δ must be:

98. A triangle is inscribed in $xy = c^2$ and two to its sides are parallel to $y = m_1x$ and $y = m_2x$. If m_1, m_2 are two values of $x^2–6x+1=0$ and if third side envelopes the hyperbola $xy = c^2\lambda$, then the value of $16\lambda^2$ must be:

99. If a circle cuts a rectangular hyperbola $xy = c^2$ in A, B, C and D are the parameters of these four points be t_1, t_2, t_3 and t_4 respectively, then the value of $16t_1t_2t_3t_4$ must be:

100. The equation of the hyperbola whose asymptotes are $x + 2y + 3 = 0$ and $3x+4y+5=0$ which passes through the point (1,–1) is $3x^2 + 10xy + 8y^2 + 14x + 22y + \lambda = 0,$ then the equation of the conjugate hyperbola is $3x^2 + 10xy + 8y^2$ $+\mu x + 22y + v = 0,$ then the value of $\mu + v,$ must be:

ANSWER

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
a	a	a	c	d	d	b	d	d	b
11.	**12.**	**13.**	**14.**	**15.**	**16.**	**17.**	**18.**	**19.**	**20.**
b	b	b	c	a	a	a	c	a	c
21.	**22.**	**23.**	**24.**	**25.**	**26.**	**27.**	**28.**	**29.**	**30.**
d	c	d	d	a	a	b	c	a	a
31.	**32.**	**33.**	**34.**	**35.**	**36.**	**37.**	**38.**	**39.**	**40.**
c	c	a	a	d	a	a	a	a	b
41.	**42.**	**43.**	**44.**	**45.**	**46.**	**47.**	**48.**	**49.**	**50.**
d	d	c	a	b	All	a,b	b,c	a,d	c,d
51.	**52.**	**53.**	**54.**	**55.**	**56.**	**57.**	**58.**	**59.**	**60.**
a,b,d	All	a,b	b,d	a,b	a,b	a,d	a,c,d	a,b,c	a,b
61.	**62.**	**63.**	**64.**	**65.**	**66.**	**67.**	**68.**	**69.**	**70.**
a,c	a	a	b	a	a	a	d	c	d
71.	**72.**	**73.**	**74.**	**75.**	**76.**	**77.**	**78.**	**79.**	**80.**
c	b	d	d	c	a	b	a	c	d
81.	**82.**	**83.**	**84.**	**85.**	**86.**	**87.**	**88.**	**89.**	**90.**
b	c	b	b	a	a	c	960	1296	1
91.	**92.**	**93.**	**94.**	**95.**	**96.**	**97.**	**98.**	**99.**	**100.**
729	3125	1215	4	27	2401	1	1296	16	37

SOLUTION

Multiple Choice Questions

1. **(a)** Comparing the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here, $a = 1$, $b = 1$, $h = -1$, $g = \frac{3}{2}$, $f = 0$, $c = 2$

Now, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$$\Rightarrow \quad \Delta = (1)(1)(2) + 2\left(\frac{3}{2}\right)(0)(-1) - (1)(0)^2 - 1\left(\frac{3}{2}\right)^2 - 2(-1)^2$$

$$\Rightarrow \quad \Delta = \frac{-9}{4} \ \ i.e., \ \Delta \neq 0 \ \text{ and } \ h^2 - ab = 1 - 1 = 0 \ \ i.e., \ h^2 = ab$$

So, given equation represents a parabola.

2. **(a)** Centre of conic is $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

Here, $a = 14$, $h = -2$

$b = 11$

$g = -22$

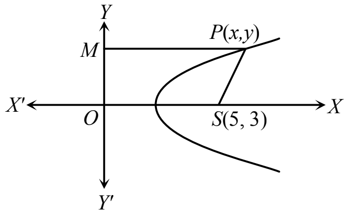
$f = -29$

$c = 71$

$$\text{Centre} \equiv \left(\frac{(-2)(-29) - (11)(-22)}{(14)(11) - (-2)^2}, \frac{(-22)(-2) - (14)(-29)}{(14)(11) - (-2)^2}\right)$$

Centre $\equiv (2, 3)$.

3. **(a)**



$$PM^2 = PS^2 \Rightarrow (x-5)^2 + (y-3)^2 = \left(\frac{3x - 4y + 1}{\sqrt{9+16}}\right)^2$$

$$\Rightarrow \quad 25(x^2 + 25 - 10x + y^2 + 9 - 6y)$$

$$= 9x^2 + 16y^2 + 1 - 12xy + 6x - 8y - 12xy$$

$$\Rightarrow \quad 16x^2 + 9y^2 - 256x - 142y + 24xy + 849 = 0$$

$$\Rightarrow \quad (4x + 3y)^2 - 256x - 142y + 849 = 0$$

4. **(c)** The point $(-3, 2)$ will satisfy the equation $y^2 = 4ax$

$\Rightarrow \quad 4 = -12a \Rightarrow$ Latus rectum $= 4|a| = 4 \times \left|-\frac{1}{3}\right| = \frac{4}{3}$

5. **(d)** Here $\frac{y}{2} = t$ and $x - 2 = t^2$

$$\Rightarrow \quad (x-2) = \left(\frac{y}{2}\right)^2$$

$$\Rightarrow \quad y^2 = 4(x-2)$$

6. **(d)** Here, $S \equiv y^2 - 4x = 0$; $S(1,3) = 3^2 - 4.1 > 0$

$\Rightarrow \quad P(1,3)$ is an exterior point.

$S(1,1) = 1^2 - 4.1 < 0 \Rightarrow q(1,1)$ is an interior point.

7. **(b)** $y = 2x + \lambda$ does not meet the parabola $y^2 = 2x$, If

$$\lambda > \frac{a}{m} = \frac{1}{2.2} = \frac{1}{4} \Rightarrow \lambda > \frac{1}{4}$$

8. **(d)** Parabola is $y^2 = ax$ i.e. $y^2 = 4\left(\frac{a}{4}\right)x$ . . . (i)

Let point of contact is (x_1, y_1).

$\therefore$ Equation of tangent is $y - y_1 = \frac{2(a/4)}{y_1}(x - x_1)$

$$\Rightarrow \quad y = \frac{a}{2y_1}(x) - \frac{ax_1}{2y_1} + y_1$$

Here, $m = \frac{a}{2y_1} = \tan 45^\circ$

$$\Rightarrow \quad \frac{a}{2y_1} = 1 \Rightarrow y_1 = \frac{a}{2}.$$

From (i), $x_1 = \frac{a}{4}$, So point is $\left(\frac{a}{4}, \frac{a}{2}\right)$.

9. **(d)** The line $x-y+2=0$

i.e. $x=y-2$ meets parabola $y^2=8x$

$\Rightarrow$ $y^2=8(y-2)=8y-16$

$\Rightarrow$ $y^2-8y+16=0$

$\Rightarrow$ $(y-4)^2=0 \Rightarrow y=4,4$

$\because$ Roots are equal,

$\therefore$ Given line touches the given parabola.

$\therefore$ $x=4-2=2,$ Thus the required point is (b, d).

10. **(b)** Any normal to the parabola $y^2=12x$ is $y+tx=6t+3t^3$. It is identical with $x+y=k$ if $\frac{t}{1}=\frac{1}{1}=\frac{6t+3t^3}{k}$

$\therefore$ $t=1$ and $1=\frac{6+3}{k} \Rightarrow k=9$

11. **(b)** Since the centroid of the triangle formed by the co-normal points lies on the axis of the parabola.

12. **(b)** Equation of focal chord for the parabola $y^2=4ax$ passes through the point $(au^2,2au)$ and $(av^2,2av)$

$\Rightarrow$ $y-2au=\frac{2av-2au}{av^2-au^2}(x-au^2)$

$\Rightarrow$ $y-2au=\frac{2a(v-u)}{a(v-u)(v+u)}(x-au^2)$

$\Rightarrow$ $y-2au=\frac{2}{v+u}(x-au^2)$

It this is focal chord, so it would passes through focus $(a, 0)$

$\Rightarrow$ $0-2au=\frac{2}{v+u}(a-au^2)$

$\Rightarrow$ $-uv-u^2=1-u^2$

$\therefore$ $uv+1=0$

Given points $(au^2, 2au)$ and $(av^2, 2av)$, then $t_1=u$ and $t_2=v$, we know that $t_1t_2=-1$.

Hence $uv+1=0$.

13. **(b)** Equation of diameter of parabola is $y=\frac{2a}{m}$,

Here $a=\frac{1}{4}, m=1 \Rightarrow y=\frac{2.\frac{1}{4}}{1}$

$\Rightarrow$ $2y=1$

14. **(c)** Since the length of the sub-tangent at a point to the parabola is twice the abscissa of the point. Therefore, the required length is 8.

15. **(a)** Let (x_1,y_1) be the pole of line $2x=y$ *w.r.t.* parabola $y^2=2x$ its polar is $yy_1=x+x_1$

Also polar is $y=2x,$

$\therefore$ $\frac{y_1}{1}=\frac{1}{2}=\frac{x_1}{0}$

$\therefore$ $x_1=0, y_1=\frac{1}{2}$. So Pole is $\left(0,\frac{1}{2}\right)$

16. **(a)** The equation of the axis of the parabola is $y-2=0,$ which is parallel to the x-axis. So, a ray parallel to x-axis is parallel to the axis of the parabola. We know that any ray parallel to the axis of a parabola passes through the focus after reflection. Here (0, 2) is the focus.

17. **(a)** Let any point on it be (x,y) then by definition,

$$\sqrt{(x+1)^2+(y-1)^2}=\frac{1}{2}\left|\frac{x-y+3}{\sqrt{1^2+1^2}}\right|$$

Squaring and simplifying, we get

$7x^2+2xy+7y^2+10x-10y+7=0,$ which is the required ellipse.

18. **(c)** We have $16x^2+25y^2=400 \Rightarrow \frac{x^2}{25}+\frac{y^2}{16}=1$

or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$ where $a^2=25$ and $b^2=16$

This equation represents an ellipse with eccentricity given by $e^2=1-\frac{b^2}{a^2}=1-\frac{16}{25}=\frac{9}{25} \Rightarrow e=3/5$

So, the coordinates of the foci are $(\pm ae,0)$ i.e. $(3,0)$ and $(-3,0)$, Thus, F_1 and F_2 are the foci of the ellipse.

Since, the sum of the focal distance of a point on an ellipse is equal to its major axis,

$\therefore$ $PF_1+PF_2=2a=10$

19. **(a)** From the given equation of ellipse $a^2=16, b^2=25$ (since $b>a$)

So, $a^2=b^2(1-e^2),$

$\therefore$ $16=25(1-e^2)$

$\Rightarrow$ $1-e^2=\frac{16}{25} \Rightarrow e^2=\frac{9}{25}$

$\Rightarrow$ $e=\frac{3}{5}$

$\therefore$ One directrix is $y=\frac{b}{e}=\frac{5}{3/5}=\frac{25}{3}$

20. **(c)** Focal distance of any point $P(x, y)$ on the ellipse is equal to $SP = a + ex$. Here $x = \cos\theta$.

Hence, $SP = a + ae\cos\theta = a(1 + e\cos\theta)$

21. **(d)** The given ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$. The value of the expression $\frac{x^2}{9}+\frac{y^2}{4}-1$ is positive for $x=1,\ y=2$ and negative for $x=2,\ y=1$. Therefore *P* lies outside *E* and *Q* lies inside *E*. The value of the expression x^2+y^2-9 is negative for both the points *P* and *Q*. Therefore *P* and *Q* both lie inside *C*. Hence *P* lies inside *C* but outside *E*.

22. **(c)** We know that the line $y = mx + c$ touches the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *iff* $c^2 = a^2m^2 + b^2$

Here, $a^2 = 4, b^2 = 1, m = 4$

$\therefore \quad c^2 = 64 + 1 \Rightarrow c = \pm\sqrt{65}$

23. **(d)** By symmetry the quadrilateral is a rhombus. So area is our times the area of the right angled triangle formed by the tangents and axes in the 1st quadrant.

Now $ae = \sqrt{a^2 - b^2} \Rightarrow ae = 2 \Rightarrow$ Tangent (in the first quadrant) at one end of latus rectum $\left(2, \frac{5}{3}\right)$ is $\frac{2}{9}x + \frac{5}{3}.\frac{y}{5} = 1$

i.e. $\frac{x}{9/2}+\frac{y}{3}=1$. Therefore area $= 4.\frac{1}{2}.\frac{9}{2}.3 = 27sq.$ units.

24. **(d)** For $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, equation of normal at point (x_1, y_1), is $\frac{(x-x_1)a^2}{x_1}=\frac{(y-y_1)b^2}{y_1}$

Here, $(x_1, y_1) = (0, 3)$ and $a^2 = 5,\ b^2 = 9$.

Therefore $\frac{(x-0)}{0}.5 = \frac{(y-3)}{3}.9$ or $x = 0$ *i.e.*, *y*-axis.

25. **(a)** Let at point (x_1, y_1) normal will be

$$\frac{(x-x_1)}{x_1}a^2 = \frac{(y-y_1)b^2}{y_1}$$

At $G,\ y = 0 \Rightarrow x = CG = \frac{x_1(a^2-b^2)}{a^2}$ and at $g,\ x = 0$

$\Rightarrow \quad y = Cg = \frac{y_1(b^2-a^2)}{b^2}$

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \Rightarrow a^2(CG)^2 + b(Cg)^2 = (a^2-b^2)^2.$$

26. **(a)** The required equation is $T = 0$

i.e., $3x + 4(2y) - 9 = 0$

or $3x + 8y = 9$.

27. **(b)** Equation of polar of (x_1, y_1) *w.r.t* the ellipse is

$xx_1 + 4yy_1 = 4$ ... (*i*)

Comparing with $x + 4y = 4$ ... (*ii*)

$\frac{x_1}{1}=\frac{4y_1}{4}=1$

$\Rightarrow \quad x_1 = 1,\ y_1 = 1.$

$\therefore$ Coordinates of pole $(x_1, y_1) = (1, 1)$

28. **(c)** Since every diameter of an ellipse passes through the centre and is bisected by it, therefore the coordinates of the other end are $(-\sqrt{3}, -2)$.

29. **(a)** Let $y = m_1x$ and $y = m_2x$ be a pair of conjugate diameter of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi,\ b\sin\phi)$ be ends of these two diameters. Then $m_1m_2 = \frac{-b^2}{a^2}$

$\Rightarrow \quad \frac{b\sin\theta - 0}{a\cos\theta - 0}\times\frac{b\sin\phi - 0}{a\cos\phi - 0} = \frac{-b^2}{a^2}$

$\Rightarrow \quad \sin\theta\sin\phi = -\cos\theta\cos\phi$

$\Rightarrow \quad \cos(\theta - \phi) = 0$

$\Rightarrow \quad \theta - \phi = \pm\pi/2.$

30. **(a)** Here $a^2 = 25, b^2 = 16, x_1 = \frac{-5\sqrt{3}}{2}$. Length of sub-tangent $= \left|\frac{a^2}{x_1} - x_1\right| = \left|\frac{25}{-5\sqrt{3}/2} + \frac{5\sqrt{3}}{2}\right| = \left|\frac{5\sqrt{3}}{2} - \frac{10}{\sqrt{3}}\right|$.

Length of subnormal $= \left|\frac{b^2}{a^2}x_1\right| = \left|\frac{16}{25}\left(\frac{-5\sqrt{3}}{2}\right)\right| = \left|\frac{8\sqrt{3}}{5}\right|$.

31. **(c)** Here, focus (*S*) = (1, –1), eccentricity (*e*) $= \sqrt{2}$

From definition, $SP = e\,PM$

$$\sqrt{(x-1)^2 + (y+1)^2} = \frac{\sqrt{2}.(x-y+1)}{\sqrt{1^2+1^2}}$$

$\Rightarrow \quad (x-1)^2 + (y+1)^2$

$= (x-y+1)^2$

$\Rightarrow \quad 2xy - 4x + 4y + 1 = 0$, which is the required equation of conic (Rectangular hyperbola)

32. **(c)** Since the point (4, 1) lies inside the hyperbola $\left[\therefore \frac{16}{4}-\frac{1}{3}-1>0\right]$;

$\therefore$ Number of tangents through (4, 1) is 0.

33. **(a)** The equation of line and hyperbola are $y=x-1$ . . . (*i*)

and $3x^2-4y^2=12$ . . . (*ii*)

From (*i*) and (*ii*), we get $3x^2-4(x-1)^2=12$

$\Rightarrow$ $3x^2-4(x^2-2x+1)=12$ or $x^2-8x+16=0$

$\Rightarrow$ $x=4$

From (*i*), $y=3$ so points of contact is (4, 3)

Points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$.

Here $a^2=4, b^2=3$ and $m=1$. So the required points of contact is (4, 3).

34. **(a)** Let the point of intersection of tangents be $P(x_1,y_1)$. Then the equation of pair of tangents from $P(x_1,y_1)$ to the given hyperbola is $(4x^2-9y^2-36)(4x_1^2-9y_1^2-36)$

$=[4x_1x-9y_1y-36]^2$ . . . (*i*)

From $SS_1=T^2$

or $x^2(y_1^2+4)+2x_1y_1xy+y^2(x_1^2-9)+...=0$ . . . (*ii*)

Since angle between the tangents is $\pi/4$.

$\therefore$ $\tan(\pi/4)=\frac{2\sqrt{[x_1^2y_1^2-(y_1^2+4)(x_1^2-9)]}}{y_1^2+4+x_1^2-9}$.

Hence locus of $P(x_1,y_1)$ is $(x^2+y^2-5)^2$

$=4(9y^2-4x^2+36)$.

35. **(d)** From $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$

Here $a^2=16, b^2=9$ and $(x_1,y_1)=(8,3\sqrt{3})$

$\Rightarrow$ $\frac{16x}{8}+\frac{9y}{3\sqrt{3}}=16+9$

i.e., $2x+\sqrt{3}y=25$.

36. **(a)** The equation of normal at $(a\sec\phi, b\tan\phi)$ to the given hyperbola is $ax\cos\phi+by\cot\phi=(a^2+b^2)$

This meets the transverse axis *i.e.,* *x*-axis at *G*. So the co-ordinates of *G* are $\left(\left(\frac{a^2+b^2}{a}\right)\sec\phi, 0\right)$ and the co-ordinates of the vertices A and A' are $A(a,0)$ and $A'(-a,0)$ respectively.

$\therefore$ $AG.A'G=\left(-a+\left(\frac{a^2+b^2}{a}\right)\sec\phi\right)\left(a+\left(\frac{a^2+b^2}{a}\right)\sec\phi\right)$

$=\left(\frac{a^2+b^2}{a}\right)^2\sec^2\phi-a^2=(ae^2)^2\sec^2\phi-a^2$

$=a^2(e^4\sec^2\phi-1)$

37. **(a)** From $T=0$ *i.e.,* $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

Here, $16x^2-9y^2=144$ *i.e.,* $\frac{x^2}{9}-\frac{y^2}{16}=1$

So, the equation of chord of contact of tangents drawn from a point (2, –1) to the hyperbola is $\frac{2x}{9}-\frac{(-1)y}{16}=1$

i.e., $32x+9y=144$

38. **(a)** Let (x_1,y_1) be the required point. Then the equation of the chord of contact of tangents drawn from (x_1,y_1) to the given hyperbola is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ . . . (*i*)

The given line is $lx+my+n=0$ . . . (*ii*)

Equation (*i*) and (*ii*) represent the same line

$\therefore$ $\frac{x_1}{a^2l}=-\frac{y_1}{b^2m}=\frac{1}{-h}$

$\Rightarrow$ $x_1=\frac{-a^2l}{n}$, $y_1=\frac{b^2m}{n}$;

Hence the required point is $\left(-\frac{a^2l}{n}, \frac{b^2m}{n}\right)$.

39. **(a)** Let (x_1,y_1) be the given point.

It's polar *w.r.t.* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

i.e., $y=\frac{b^2}{y_1}\left(1-\frac{xx_1}{a^2}\right)=-\frac{b^2x_1}{a^2y_1}x+\frac{b^2}{y_1}$

This touches u $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

If $\left(\frac{b^2}{y_1}\right)^2=a^2.\left(\frac{b^2x_1}{a^2y_1}\right)-b^2 \Rightarrow \frac{b^4}{y_1^2}=\frac{a^2b^4x_1^2}{a^4y_1^2}-b^2$

$\Rightarrow$ $\frac{b^2}{y_1^2}=\frac{b^2x_1^2}{a^2y_1^2}-1$

$\Rightarrow$ $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$

$\therefore$ Locus of (x_1,y_1) is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Which is the same hyperbola.

40. **(b)** Coordinates of P and D are $(a\sec\phi,\, b\tan\phi)$ and $(a\tan\phi,\, b\sec\phi)$ respectively.

Then $(CP)^2-(CD)^2=a^2\sec^2\phi+b^2\tan^2\phi-a^2\tan^2\phi-b^2\sec^2\phi$

$=a^2(\sec^2\phi-\tan^2\phi)-b^2(\sec^2\phi-\tan^2\phi)$

$=a^2(1)-b^2(1)=a^2-b^2.$

41. **(d)** Let $P(x_1, y_1)$ be a point on the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$

The chord of contact of tangent from P to the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=2$ . . . (i)

The equation of asymptotes are $\frac{x}{a}-\frac{y}{b}=0$ . . . (ii)

And $\frac{x}{a}+\frac{y}{b}=0$ . . . (iii)

The point of intersection of the asymptotes and chord are

$\left(\frac{2a}{x_1/a-y_1/b}, \frac{2b}{x_1/a-y_1/b}\right); \left(\frac{2a}{x_1/a+y_1/b}, \frac{-2b}{x_1/a+y_1/b}\right),$

$(0,0)$

$\therefore$ Area of triangle $=\frac{1}{2}|(x_1y_2-x_2y_1)|$

$=\frac{1}{2}\left|\left(\frac{-8ab}{x_1^2/a^2-y_1^2/b^2}\right)\right|=4ab.$

42. **(d)** Given, equation of hyperbola $2x^2+5xy+2y^2+4x+5y=0$ and equation of asymptotes

$2x^2+5xy+2y^2+4x+5y+\lambda=0$ . . . (i)

which is the equation of a pair of straight lines. We know that the standard equation of a pair of straight lines is $ax^2+2hxy+by^2+2gx+2fy+c=0$

Comparing equation (i) with standard equation, we get

$a=2,\, b=2,\, h=\frac{5}{2},\, g=2,\, f=\frac{5}{2}$ and $c=\lambda.$

We also know that the condition for a pair of straight lines is $abc+2fgh-af^2-bg^2-ch^2=0.$

Therefore, $4\lambda+25-\frac{25}{2}-8-\frac{25}{4}\lambda=0$

or $\frac{-9\lambda}{4}+\frac{9}{2}=0$ or $\lambda=2$

Substituting value of λ in equation (i),

we get, $2x^2+5xy+2y^2+4x+5y+2=0.$

43. **(c)** Since the general equation of second degree represents a rectagular hyperbola if $\Delta\neq 0$, $h^2>ab$ and coefficient of x^2 + coefficient of $y^2=0.$

Therefore the given equation represents a rectangular hyperbola if $\lambda+5=0$

i.e., $\lambda=-5.$

44. **(a)** The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ is

$ty=t^3x-ct^4+c$

If it passes through $\left(ct', \frac{c}{t'}\right)$ then

$\Rightarrow \frac{tc}{t'}=t^3ct'-ct^4+c$

$\Rightarrow t=t^3t'^2-t^4t'+t'$

$\Rightarrow t-t'=t^3t'(t'-t)$

$\Rightarrow t'=-\frac{1}{t^3}$

45. **(b)** Let the equation of circle be $x^2+y^2=a^2$ . . . (i)

Parametric equation of rectangular hyperbola is

$x=ct, y=\frac{c}{t}$

Put the values of x and y in equation (i) we get $c^2t^2+\frac{c^2}{t^2}=a^2$

$\Rightarrow c^2t^4-a^2t^2+c^2=0$

Hence product of roots $t_1t_2t_3t_4=\frac{c^2}{c^2}=1$

46. **(a,b,c,d)** Given, circle is $x^2+y^2=a^2$ . . . (i)

and hyperbola be $xy=c^2$ . . . (ii)

from (ii) $y=\frac{c^2}{x}.$

Putting in (i), we get $x^2+\frac{c^4}{x^2}=a^2$

$\Rightarrow x^4-a^2x^2+c^4=0$

$\therefore x_1+x_2+x_3+x_4=0,$

$x_1x_2x_3x_4=c^4$

Since both the curves are symmetric in x and y,

$\therefore y_1+y_2+y_3+y_4=0;\ y_1y_2y_3y_4=c^4.$

NCERT Exemplar Problems

More than One Answer

47. **(a, b)** The equation of tangent to $y = x^2$, be $y = mx - \dfrac{m^2}{4}$.

Putting in $y = -x^2 + 4x - 4$, we should only get one value of x *ie*, Discriminant must be zero.

$\therefore \quad mx - \dfrac{m^2}{4} = -x^2 + 4x - 4$

$\Rightarrow \quad x^2 + x(m-4) + 4 - \dfrac{m^2}{4} = 0$

$\Rightarrow \quad D = 0$

Now, $(m-4)^2 - (16 - m^2) = 0$

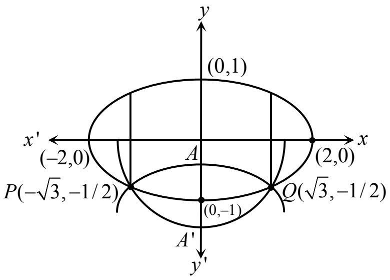
$\Rightarrow \quad 2m(m-4) = 0$

$\Rightarrow \quad m = 0, 4$

$\therefore$ $y = 0$ and $y = 4\,(x - 1)$ are the required tangents.

Hence, (a) and (b) are correct answers.

48. **(b, c)** The equation $x^2 + 4y^2 = 4$ represents an ellipse with 2 and 1 as semi-major and semi-minor axes and eccentricity $\dfrac{\sqrt{3}}{2}$.



Thus, the ends of latusrectum are $\left(\sqrt{3}, \dfrac{1}{2}\right)$ and $\left(\sqrt{3}, -\dfrac{1}{2}\right)$, $\left(-\sqrt{3}, \dfrac{1}{2}\right)$, and $\left(-\sqrt{3}, -\dfrac{1}{2}\right)$.

Now, $PQ = 2\sqrt{3}$

Thus, the coordinates of the vertex of the parabolas are and $A\left(0, \dfrac{-1+\sqrt{3}}{2}\right)$ and $A'\left(0, \dfrac{-1-\sqrt{3}}{2}\right)$ and corresponding equations are $(x-0)^2 = -4 \cdot \dfrac{\sqrt{3}}{2}\left(y + \dfrac{1-\sqrt{3}}{2}\right)$

and $\quad (x-0)^2 = 4 \cdot \dfrac{\sqrt{3}}{2}\left(y - \dfrac{-1-\sqrt{3}}{2}\right)$

ie, $\quad x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

and $\quad x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

49. **(a, d)** Equation of tangent and normal at point $P(at^2, 2at)$ is $ty = x = at^2$ and $y = -tx + 2at + at^2$

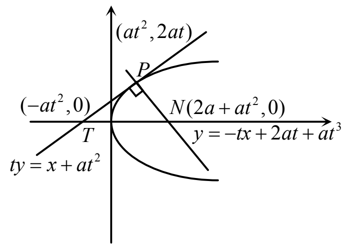
Let centroid of ΔPTN is $R(h, k)$

$\therefore \quad h = \dfrac{at^2 + (-at^2) + 2a + at^2}{3}$ and $k = \dfrac{2at}{3}$

$\Rightarrow \quad 3h = 2a + a \cdot \left(\dfrac{3k}{2a}\right)^2$

$\Rightarrow \quad 3h = 2a + \dfrac{9k^2}{4a}$

$\Rightarrow \quad 9k^2 = 4a(3h - 2a)$



$\therefore \quad$ Locus of centroid is $y^2 = \dfrac{4a}{3}\left(x - \dfrac{2a}{3}\right)$

$\therefore \quad$ Vertex $\left(\dfrac{2a}{3}, 0\right)$; directrix $x - \dfrac{2a}{3} = -\dfrac{a}{3}$

$\Rightarrow \quad x = \dfrac{a}{3}$

Latusrectum $= \dfrac{4a}{3}$

$\therefore \quad$ Focus $\left(\dfrac{a}{3} + \dfrac{2a}{3}, 0\right)$ *ie*, $(a, 0)$.

50. **(c, d)** Here, coordinate $M = \left(\dfrac{t_1^2 + t_2^2}{2}, t_1 + t_2\right)$ *ie*, mid-point of chord AB. $MP = t_1 + t_2 = r \quad \ldots (i)$

Also, $m_{AB} = \dfrac{2t_2 - 2t_1}{t_2^2 - t_1^2} = \dfrac{2}{t_2 + t_1}$ (when AB is chord)

$\Rightarrow \quad m_{AB} = \dfrac{2}{r}$ [from equation. (i)]

Also, $m_{A'B'} = -\dfrac{2}{r}$ (when A'B' is chord)

Hence, (c, d) is the correct option.

51. **(a, b, d)** Normal to $y^2 = 4x$, is $y = mx - 2m - m^3$ which passes through (9, 6) $\Rightarrow \; 6 = 9m - 2m - m^3$

$\Rightarrow \quad m^3 - 7m + 6 = 0 \qquad \Rightarrow \; m = 1, 2, -3$

$\therefore \quad$ Equation of normals are, $y - x + 3 = 0, y + 3x - 33 = 0$

And $\; y - 2x + 12 = 0$

52. **(a, b, c, d)** It is given that $x^2 + y^2 = a^2$ ... (*i*)

And $xy = c^2$ ... (*ii*)

We obtain $x^2 + c^4/x^2 = a^2$

$\Rightarrow$ $x^4 - a^2x^2 + c^4 = 0$ ... (*iii*)

Now, x_1, x_2, x_3, x_4 will be roots of equation. (*iii*)

Therefore, $\Sigma x_1 = x_1 + x_2 + x_3 + x_4 = 0$ and product of the roots $x_1x_2x_3x_4 = c^4$

Similarly, $y_1 + y_2 + y_3 + y_4 = 0$

and $y_1\, y_2\, y_3\, y_4 = c^4$

Hence, all options are correct.

53. **(a, b)** Given, $2x^2 - 2y^2 = 1$

$\Rightarrow$ $\dfrac{x^2}{\left(\frac{1}{2}\right)} - \dfrac{y^2}{\left(\frac{1}{2}\right)} = 1$ ...(*i*)

Eccentricity of hyperbola $= \sqrt{2}$

So, eccentricity of ellipse $= 1/\sqrt{2}$

Let equation of ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 (a > b)$

$\Rightarrow$ $\dfrac{1}{\sqrt{2}} = \sqrt{1 - \dfrac{b^2}{a^2}}$

$\Rightarrow$ $\dfrac{b^2}{a^2} = \dfrac{1}{2}$

$\Rightarrow$ $a^2 = 2b^2$

$\therefore$ $x^2 + 2y^2 = 2b^2$ ...(*ii*)

Let ellipse and hyperbola intersect at $A\left(\dfrac{1}{\sqrt{2}}\sec\theta, \dfrac{1}{\sqrt{2}}\tan\theta\right)$

On differentiating equation (*i*) $4x - 4y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y}$

$\left.\dfrac{dy}{dx}\right|_{at\, A} = \dfrac{\sec\theta}{\tan\theta} = \cos ec\theta$ and differentiating equation (*ii*),

$2x + 4y\dfrac{dy}{dx} = 0$

$\left.\dfrac{dy}{dx}\right|_{at\, A} = -\dfrac{x}{2y} = -\dfrac{1}{2}\cos ec\theta$

Since, ellipse and hyperbola are orthogonal.

$\therefore$ $-\dfrac{1}{2}\cos ec^2\theta = -1$

$\Rightarrow$ $\text{cosec}^2\theta = 2$

$\Rightarrow$ $\theta = \pm\dfrac{\pi}{4}$

$\therefore$ $A\left(1, \dfrac{1}{\sqrt{2}}\right)$ or $\left(1, -\dfrac{1}{\sqrt{2}}\right)$

$\therefore$ Form equation (*i*), $1 + 2\left(\dfrac{1}{\sqrt{2}}\right)^2 = 2b^2$

$\Rightarrow$ $b^2 = 1$

Equation of ellipse is $x^2 + 2y^2 = 2$

Coordinate of foci $(\pm ae, 0) = \left(\pm\sqrt{2}\cdot\dfrac{1}{\sqrt{2}}, 0\right) = (\pm 1, 0)$

Hence, option (i) and (ii) are correct.

If major axis is along *y*-axis, then

$\dfrac{1}{\sqrt{2}} = \sqrt{1 - \dfrac{a^2}{b^2}} \Rightarrow b^2 = 2a^2$

$\therefore$ $2x^2 + y^2 = 2a^2 \Rightarrow y' = -\dfrac{2x}{y}$

$\Rightarrow$ $y'_{\left(\frac{1}{\sqrt{2}}\sec\theta, \frac{1}{\sqrt{2}}\tan\theta\right)} = \dfrac{-2}{\sin\theta}$

As ellipse and hyperbola are orthogonal

$\therefore$ $-\dfrac{2}{\sin\theta}\cdot\cos ec\theta = -1$

$\Rightarrow$ $\cos ec^2\theta = 1 \Rightarrow \theta = \pm\dfrac{\pi}{4}$

$\therefore$ $2x^2 + y^2 = 2a^2$

$\Rightarrow$ $2 + \dfrac{1}{2} = 2a^2 \Rightarrow a^2 = \dfrac{5}{4}$

$\therefore$ $2x^2 + y^2 = \dfrac{5}{2}$, corresponding foci are $(0, \pm 1)$.

54. **(b, d)** Here, equation of ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

$\Rightarrow$ $e^2 = 1 - \dfrac{b^2}{a^2} = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

$\therefore$ $e = \dfrac{\sqrt{3}}{2}$ and focus $(\pm ae, 0) \Rightarrow (\pm\sqrt{3}, 0)$

For hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1, e_1^2 = 1 + \dfrac{b^2}{a^2}$

where, $e_1^2 = \dfrac{1}{e^2} = \dfrac{4}{3} \Rightarrow 1 + \dfrac{b^2}{a^2} = \dfrac{4}{3}$

$\therefore$ $\dfrac{b^2}{a^2} = \dfrac{1}{3}$ ... (*i*)

and hyperbola passes through $(\pm\sqrt{3}, 0)$

$\Rightarrow$ $\dfrac{3}{a^2} = 1$

$\Rightarrow \quad a^2 = 3 \qquad \ldots (ii)$

From Eqs. (*i*) and (*ii*), we get $b^2 = 1 \qquad \ldots (iii)$

$\therefore$ Equation of hyperbola is $\dfrac{x^2}{3} - \dfrac{y^2}{1} = 1$ Focus is $(\pm ae_1, 0)$

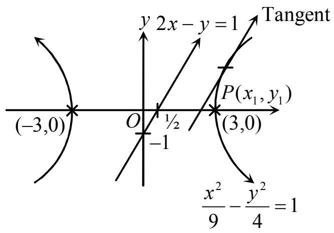
$\Rightarrow \left(\pm\sqrt{3}\cdot\dfrac{2}{\sqrt{3}}, 0\right) \Rightarrow (\pm 2, 0)$ (ii) and (iv) are correct answers.

55. **(a, b)** Equation of tangent, parallel to $y = 2x - 1$

$\Rightarrow \quad y = 2x \pm \sqrt{9(4) - 4} \qquad \ldots (i)$

$\therefore \quad y = 2x \pm \sqrt{32}$

The equation of tangent at (x_1, y_1) is $\dfrac{xx_1}{9} - \dfrac{yy_1}{4} = 1 \ldots (ii)$



From Eqs. (*i*) and (*ii*), we get $\dfrac{2}{\frac{x_1}{9}} = \dfrac{-1}{\frac{-y_1}{4}} = \dfrac{\pm\sqrt{32}}{1}$

$\Rightarrow \quad x_1 = -\dfrac{9}{2\sqrt{2}}$

And $y_1 = -\dfrac{1}{\sqrt{2}}$

or $x_1 = \dfrac{9}{2\sqrt{2}}, y_1 = \dfrac{1}{\sqrt{2}}$

56. **(a, b)** $25x^2 + 9y^2 - 150x - 90y + 225 = 0$

$\Rightarrow \quad 25(x^2 + 6x) + 9(y^2 - 10y) + 225 = 0$

$\Rightarrow \quad 25\{(x-3)^2 - 9\} + 9\{(y-5)^2 - 25\} + 225 = 0$

$\Rightarrow \quad 25(x-3)^2 + 9(y-5)^2 = 225$

$\Rightarrow \quad \dfrac{(x-3)^2}{3^2} + \dfrac{(y-5)^2}{5^2} \qquad \ldots (i)$

Let $x - 3 = X, y - 5 = Y$

Then, equation (*i*) becomes $\dfrac{X^2}{3^2} + \dfrac{Y^2}{5^2} = 1 \qquad \ldots (ii)$

Now, comparing equation (*ii*) with $\dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} = 1$

$\therefore \quad a = 3, b = 5$

$\Rightarrow \quad a^2 = b^2(1 - e^2)$

$\Rightarrow \quad 9 = 25(1 - e^2)$

$\Rightarrow \quad e = \dfrac{4}{5}$

Centre: $X = 0, Y = 0$

$\Rightarrow \quad x - 3 = 0, y - 5 = 0$

$\therefore$ Centre $\equiv (3, 5)$

Foci: $X = 0, Y = \pm be$

$\Rightarrow \quad x - 3 = 0, y - 5 \pm 4$

$\Rightarrow \quad y = 5 \pm 4$

$\therefore$ Foci $(3, 1)$ and $(3, 9)$; Major axis $2b = 10$

57. **(a, d)** Let (x_1, y_1) be a point then $x_1^2 + 3y_1^2 = 37 \quad \ldots (i)$

Equation of tangent at (x_1, y_1) is $xx_1 + 3yy_1 = 37$

Slope of tangent $= -\dfrac{x_1}{3y_1}$

Then, slope of normal $= \dfrac{3y_1}{x_1} = \dfrac{6}{5}$ (given)

$\therefore \quad x_1 = \dfrac{5y_1}{2} \qquad \ldots (ii)$

From equation (*i*) and (*ii*) $\dfrac{25y_1^2}{4} + 3y_1^2 = 37$

$\Rightarrow \quad y_1^2 = 4$

$\therefore \quad y_1 = \pm 5$

From equation and (*ii*), $x_1 = \pm 5$

$\therefore$ Required points (5, 2) and (–5, –2)

58. **(a, c, d)** Equation of tangent to $16x^2 + 11y^2 = 256$ at $\left(4\cos\theta, \dfrac{16}{\sqrt{11}}\sin\theta\right)$ is $4\cos\theta x + \sqrt{11}\sin\theta y = 16$

The perpendicular form centre (1, 0) is equal to radius $\sqrt{(1+15)} = 4$

or $\dfrac{|4\cos\theta - 16|}{\sqrt{(16\cos^2\theta + 11\sin^2\theta)}} = 4$

on simplification, we get $\cos\theta = \dfrac{1}{2}$

or $-\dfrac{5}{2}$ (not possible)

or $\theta = \pm\dfrac{\pi}{3}, \dfrac{5\pi}{3}$

59. **(a, b, c)** Since, the product of the two eccentricities e and e′ is 1. Either $e = e' = 1,$ in which case both the conics are parabolas of if $e > 1, e' < 1$ and vice-versa

So, one of them is an ellipse and the other is a hyperbola.

60. (a, b) $\frac{x^2}{25}-\frac{y^2}{16}=1$

Equation of tangent in terms of slope

$y=mx\pm\sqrt{(25m^2-16)}$

or $(y-mx)^2=25m^2-16$

It is passing through (6, 2), then $(2-6m)^2=25m^2-16$

$\Rightarrow$ $4+36m^2-24m=25m^2-16$

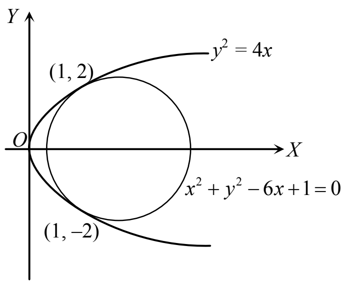
$\Rightarrow$ $11m^2-24m+20=0$

$\therefore$ $m_1+m_2=\frac{24}{44},\ m_1m_2=\frac{20}{11}$

61. (a,c) Equation of the tangent at $(a\sec\alpha, b\tan\alpha)$ to the hyperbola $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\sec\alpha-\frac{y}{b}\tan\alpha=1$ which meets the transverse axis $y=0$ at the point $T(a\cos\alpha, 0)$ whose distance from the focus $(ae,0)$ is $ae-a\cos\alpha$ and from the focus Note that $ae>a\cos\alpha$ Since, $e>1$.

Assertion and Reason

62. (a) Solving for the points of intersection we have $x^2+4x-6x+1=0$



$\Rightarrow$ $(x-1)^2=0$

$\Rightarrow$ $x=1$

$\Rightarrow$ $y=\pm 2$

Thus the two curves meet at $(1, 2)$ and $(1, -2)$

Tangent at $(1, 2)$ to $y^2=4x$ is $y(2)=2(x+1)$

$\Rightarrow$ $x-y+1=0$

Tangent at $(1, 2)$ to the circle C_2 is

$2x+1y-3(x+1)+1=0$

or $x-y+1=0$ same as the tangent to the curve C_1, Similarly the tangent at the point $(1,-2)$ to the two curves is $x+y+1=0$

$\Rightarrow$ Reason is true and hence Assertion is also true.

63. (a) Reason is true, equation in Assertion is $(x-1)^2=-2(y-3/2)$ which is a parabola with axis $x-1=0$, using Reason, Assertion is also True.

64. (b) In Assertion, if the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\frac{2b^2}{a}=\frac{1}{3}\times 2a$

$\Rightarrow$ $\frac{b^2}{a^2}=\frac{1}{3}$

$\Rightarrow$ $1-e^2=\frac{1}{3}$

$\Rightarrow$ $e^2=\frac{2}{3}$

So The Assertion is true. In Reason, equation of the ellipse is $x^2+y^2(2/3)(4-x)^2$ (by definition of ellipse)

$\Rightarrow$ $3(x^2+y^2)=2(16-8x+x^2)$

$\Rightarrow$ $x^2+16x+3y^2=32$

$\Rightarrow$ $(x+8)^2+3y^2=96$

$\Rightarrow$ $\frac{(x+8)^2}{96}+\frac{y^2}{32}=1$

Length of the semi-major axis $=\sqrt{96}=4\sqrt{6}$

So, The Reason is also true but does not lead to Assertion.

65. (a) Reason is true and using it in Assertion, the vertex is on the line $x=0$ at a distance 1 from the focus $(0, 0)$, So the vertex is at the point $(0, 1)$ and the Assertion is also true.

66. (a) Equation of a tangent to $y^2=16\sqrt{3}x$ is $y=mx\frac{4\sqrt{3}}{m}$ and to $\frac{x^2}{2}+\frac{y^2}{4}=1$ is $x=m_1y+\sqrt{4m_1^2+2}$

or $y=\frac{1}{m_1}x-\sqrt{4+\frac{2}{m_1^2}},\ m=\frac{1}{m_1}$

and $\left(\frac{4\sqrt{3}}{m}\right)^2=\left[-\sqrt{4+\frac{2}{m_1^2}}\right]$

$\Rightarrow$ $\frac{48}{m^2}=4+2m^2$

$\Rightarrow$ $m^4+2m^2-24=0$

$\Rightarrow$ $m^2=4$

$\Rightarrow$ $m=\pm 2$

Showing that both the are true and Reason is a correct explanation for Assertion.

67. **(a)** If the angles in reason are α and β s.t. $\alpha+\beta=\pi/2$ then the product of the slopes is $\tan\alpha\tan\beta=1$, and the reason is true. In assertion, any tangent to the ellipse is $y=mx+\sqrt{a^2m^2+b^2}$ which passes through $\left(\sqrt{a^2-b^2}\sec\theta, \sqrt{a^2-b^2}\tan\theta\right)$

$\Rightarrow$ $(a^2-b^2)(\tan\theta-m\sec\theta)^2=a^2m^2+b^2$

Product of the slopes

$$=\frac{(a^2-b^2)\tan^2\theta-b^2}{(a^2-b^2)\sec^2\theta-a^2}=\frac{a^2\sin^2\theta-b^2}{a^2\sin^2\theta-b^2}=1$$

So, by Reason, Assertion is also true,

68. **(d)** Assertion is false, Equation of any tangent to the parabola is $y=mx+\frac{1}{m}$ and equation of the perpendicular from the focus $S(1,0)$ on it is $y=-\frac{1}{m}x+\frac{1}{m}$ and these intersect at $x=0$, directrix is $x=-1$

Reason is true, tangents and normals at $(a,\pm 2a)$ are respectively $x\pm y+a=0$ and $x\pm y-3a=0$ which enclose a square, length of a side $=2\sqrt{2}a$.

69. **(c)** In assertion, focus is on the *x*-axis at a distance *a* from the vertex so assertion is true.

Reason is false as the length of the latusrectum of the parabola is 12 which is greater than the diameter of the circle and the common chord is of length less than the diameter.

70. **(d)** Reason is true as the distance between the foci of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2es$ where e is the eccentricity and $2a$ the length of the transverse axis.

Using it in Assertion, length of transverse axis is $\frac{\sqrt{(4+6)^2}}{5/4}=\frac{10\times 4}{5}=8$, so the Assertion is false.

71. **(c)** Reason is false. Equation of the normal is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

In Assertion, $L(ae, b^2/a)=(a\cos\theta, b\sin\theta)$

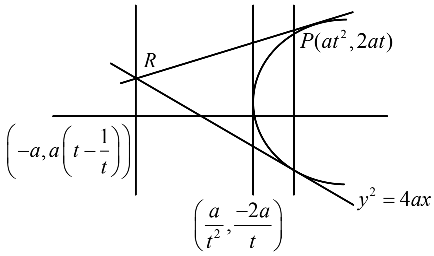
$\Rightarrow$ $\cos\theta=e$

So, normal at L, $\frac{ax}{e}-\frac{by}{\sqrt{1-e^2}}=a^2e^2$

Which meets the major axis $y=0$ at $x=ae^3$ and the assertion is true.

Comprehension Based

72. **(b)** Since, $R\left(-a, a\left(t-\frac{1}{t}\right)\right)$ lies on $y=2x+a$



$\Rightarrow$ $a\cdot\left(t-\frac{1}{t}\right)=-2a+a$

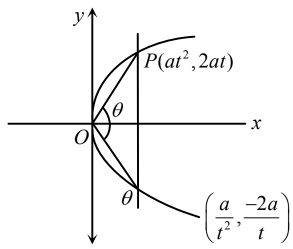
$\Rightarrow$ $t-\frac{1}{t}=-1$

Thus, length of focal chord

$$=a\left(t+\frac{1}{t}\right)^2=a\left\{\left(t-\frac{1}{t}\right)^2+4\right\}=5a$$

73. **(d)** $m_{OP}=\frac{2at-0}{at^2-0}=\frac{2}{t}$

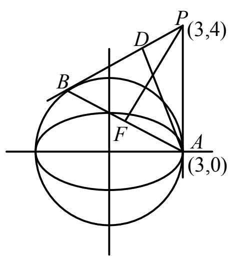
$$m_{OQ}=\frac{-2a/t}{a/t^2}=-2t$$



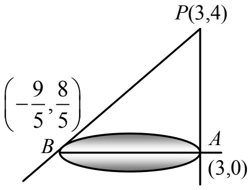
$\therefore$ $\tan\theta=\frac{\frac{2}{t}+2t}{1-\frac{2}{t}\cdot 2t}=\frac{2\left(t+\frac{1}{t}\right)}{1-4}$,

where $t+\frac{1}{t}=\sqrt{5}$ $=\frac{2\sqrt{5}}{-3}$

74. **(d)** Figure is self explanatory



75. **(c)** Equation of AB is $y-0=\dfrac{\frac{8}{5}}{-\frac{9}{5}-3}(x-3)=\dfrac{8}{-24}(x-3)$



$\Rightarrow \quad y=-\dfrac{1}{3}(x-3)$

$\Rightarrow \quad x+3y=3 \qquad \ldots(i)$

Equation of the straight line perpendicular to AB through P is $3x-y=5$. Equation of PA is $x-3=0$.

The equation of straight line perpendicular to PA through $B\left(\dfrac{-9}{5},\dfrac{8}{5}\right)$ is $y=\dfrac{8}{5}$

Hence, the orthocentre is $\left(\dfrac{11}{5},\dfrac{8}{5}\right)$.

76. **(a)** Equation of AB is $y-0=-\dfrac{1}{3}(x-3)$

$x+3y-3=0 \mid x+3y-3|^2=10[(x-3)^2+(y-4)^2]$

(Look at coefficient of x^2 and y^2 in the answers)

77. **(b)** Equation of tangent to hyperbola having slope m is

$y=mx+\sqrt{9m^2-4} \qquad \ldots(i)$

Equation of tangent to circle is

$y=m(x-4)+\sqrt{16m^2+16} \qquad \ldots(ii)$

Equation (i) and (ii) will be identical for $m=\dfrac{2}{\sqrt{5}}$ satisfy.

$\therefore$ Equation of common tangent is $2x-\sqrt{5}y+4=0$.

78. **(a)** The equation of the hyperbola is $\dfrac{x^2}{9}-\dfrac{y^2}{4}=1$ and that of circle is $x^2+y^2-8x=0$

For their points of intersection $\dfrac{x^2}{9}+\dfrac{x^2-8x}{4}=1$

$\Rightarrow \quad 4x^2+9x^2-72x=36$

$\Rightarrow \quad 13x^2-72x-36=0$

$\Rightarrow \quad 13x^2-78x+6x-36=0$

$\Rightarrow \quad 13x(x-6)+6(x-6)=0$

$\Rightarrow \quad x=6, x=-\dfrac{13}{6}$

$\Rightarrow \quad x=-\dfrac{13}{6}$ not acceptable

Now, for $x=6$,

$y=\pm 2\sqrt{3}$

Required equation is $(x-6)^2+(y+2\sqrt{3})(y-2\sqrt{3})=0$

$\Rightarrow \quad x^2-12x+y^2+24=0$

$\Rightarrow \quad x^2+y^2-12x+24=0$

79. **(c)** Pair of asymptotes is given by $xy-3y-2x+\lambda=0 \ldots(i)$

Where λ is any constant such that it represent two straight lines.

$\therefore \quad abc+2fgh-af^2-bg^2-ch^2=0$

$\Rightarrow \quad 0+2\times-\dfrac{3}{2}\times-1\times\dfrac{1}{2}-0-0-\lambda\times\left(\dfrac{1}{2}\right)^2=0$

$\therefore \quad \lambda=6$

From equation (i), asymptotes of given hyperbola are given by $xy-3y-2x+6=0$

80. **(d)** Let the asymptotes be $2x+3y+\lambda=0$ and $3x+2y+\mu=0$

Since, asymptotes passes through (1,2) then $\lambda=-8, \mu=-7$

Let the equation of hyperbola be

$(2x+3y-8)(2x+3y-7)+\gamma=0 \qquad \ldots(i)$

$\because$ It passes through (5,3), then $(10+9-8)(15+6-7)+\gamma=0$

$\Rightarrow \quad 11\times 14+\gamma=0$

$\therefore \quad \gamma-154$

Putting the value of γ in equation (i), then

$(2x+3y-8)(2x+3y-7)=154$

81. **(b)** $2\tan^{-1}\left(\dfrac{b}{a}\right)=\dfrac{\pi}{3}$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{b}{a}\right)=\dfrac{\pi}{6}$

$\Rightarrow \quad \dfrac{b}{a}=\dfrac{1}{\sqrt{3}}$

or $\quad a=\dfrac{b}{\sqrt{3}}$

Let e be an eccentricity of conjugate hyperbola, then $a^2=b^2(e^2-1)$

$\Rightarrow \quad 3b^2=b^2(e^2-1)$

$\Rightarrow \quad e^2=4$

$\therefore \quad e=2$

82. **(c)** The transverse axis is the bisector of the angle between asymptotes containing the origin and the conjugate axis is the other bisector. The bisectors of the angle between asymptotes are $\frac{(3x-4y-1)}{5}=\pm\frac{(4x-3y-6)}{5}$

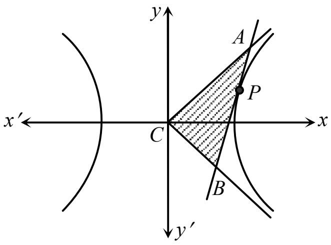
$\Rightarrow \quad (3x-4y-1)=\pm(4x-3y-6)$

$\Rightarrow \quad x+y-5=0$

and $\quad x-y-1=0$

Hence, transverse axis and conjugate axis are $x+y-5=0$ and $x-y-1=0$

83. **(b)** $16x^2-25y^2=400$



$\Rightarrow \quad \frac{x^2}{5^2}-\frac{x^2}{4^2}=1 \quad \ldots(i)$

Let $P(5\sec\phi, 4\tan\phi)$ be any point on the hyperbola (i)

Equation of tangent at P is $\frac{x}{5}\sec\phi-\frac{y}{4}\tan\phi=1 \quad \ldots(ii)$

and asymptotes of equation (i) are $y=\pm\frac{4}{5}x \quad \ldots(iii)$

Solving equation (ii) and (iii), then $\frac{x}{5}\sec\phi\mp\frac{y}{4}\tan\phi=1$

or $\quad x=\frac{5}{(\sec\phi\pm\tan\phi)}$

$=\frac{5(\sec\phi+\tan\phi)+(\sec\phi+\tan\phi)}{(\sec\phi\mp\tan\phi)}$

Then we get $A\equiv[5(\sec\phi+\tan\phi),\ 4(\sec\phi+\tan\phi)]$ and $B\equiv[5(\sec\phi-\tan\phi), -4(\sec\phi-\tan\phi)]$

$\therefore \quad$ Area of $\Delta ABC=\frac{1}{2}\begin{vmatrix}5(\sec\phi+\tan\phi) & 4(\sec\phi+\tan\phi) & 1\\ 5(\sec\phi-\tan\phi) & -4(\sec\phi-\tan\phi) & 1\\ 0 & 0 & 1\end{vmatrix}$

$=\frac{1}{2}|-20-20|=20$

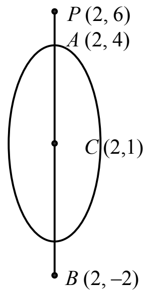
Match the Column

84. **(b)** **(A)** Let $S=9x^2+8y^2-36x-16y-28$

$\therefore \quad$ Value of S at is $(2, 6)$

$S_1=9(2)^2+8(6)^2-36(2)-16(6)-28$

$=36+288-72-96-28=128>0$



$\therefore \quad$ Point (2, 6) are outside the ellipse.

The equation of the given ellipse be rewritten as $9(x-2)^2+8(y-1)^2=72$

$\Rightarrow \quad \frac{(x-2)^2}{8}+\frac{(y-1)^2}{9}=1$

Centre of ellipse is (2, 1) and axis parallel to y-axis

$\therefore \quad$ Vertices are $x-2=0$ and $y-1=\pm 3$

Or $\quad$ (2, –2) and (2, 4)

$\therefore \quad$ Minimum distance $L=PA=2$ and maximum distance $G=PB=8$

Then, $L+G=10, G-L=6$

(B) Let $S=4x^2+9y^2+8x-36y+4$

$\therefore \quad$ Value of S at (1, 2) is $S_1=4(1)^2+9(2)^2+8(1)-36(2)+4$

$=4+36+8-72+4=-20<0$

$\therefore \quad$ Point (1, 2) are outside the ellipse.

The equation of the given ellipse be rewritten as $4(x+1)^2+9(y-2)^2=36$

$\Rightarrow \quad \frac{(x+1)^2}{9}+\frac{(y-2)^2}{4}=1$

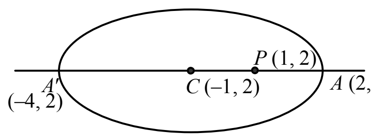
Centre of ellipse is (–1, 2) and axis parallel to x-axis

$\therefore \quad$ Vertices are $x+1=\pm 3$ and $y-2=0$ or (–4, 2) and (2, 2)

$\therefore \quad$ Minimum distance $L=PA=1$ and maximum distance $G=PA'=AA'-PA=6-1=5$

$\therefore \quad L+G=6, G-L=4, L^G+G^L=6$

(C) Here $3x+4y=0$ and $4x-3y=0$ are mutually perpendicular lines, then substituting $\frac{3x+4y}{\sqrt{3^2+4^2}}=X$ and $\frac{4x-3y}{\sqrt{(4)^2+(-3)^2}}=Y$



Then, the given equation can be written as $4X^2+9Y^2=36$

$\Rightarrow \quad \frac{X^2}{9}+\frac{Y^2}{4}=4$

$\therefore$ Vertices, $X=\pm 3, Y=0$

Or $\frac{3x+4y}{5}\pm 3, \frac{4x-3y}{5}=0$

Or $3x+4y\pm 15, y=\frac{4x}{3}$

Vertices are $\left(\frac{9}{5},\frac{12}{5}\right)$ and $\left(-\frac{9}{5},-\frac{12}{5}\right)$

$\because$ Given point is a vertex.

$\therefore$ Minimum distance $L=0$ and maximum distance $G=$ Length of major axis $=2\times 3=6$

Then $L+G=6, G-L=6$

85. **(a)** Since, equation of normal to the parabola $y^2=4ax$ is $y+xt=2at+at^3$ passes through (3, 0).

$\Rightarrow \quad 3t=2t+t^3 \quad (\because a=1)$

$\Rightarrow \quad t=0,1-1$

$\therefore$ Coordinates of the normals are $P(1, 2)$, $Q(0, 0)$, $R(1, -2)$.

Thus, (1) Area of $\Delta PQR=\frac{1}{2}\times 1\times 4=2$

(3) Centroid of $\Delta PQR=\left(\frac{2}{3},0\right)$

Equation of circle passing through P, Q, R is

$(x-1)(x-1)+(y-2)(y+2)+\lambda(x-1)=0$

$\Rightarrow \quad 1-4-\lambda=0 \Rightarrow \lambda=-3$

$\therefore$ Required equation of circle is $x^2+y^2-5x=0$

$\therefore$ Centre $\left(\frac{5}{2},0\right)$ and radius $\frac{5}{2}$.

86. **(a) (A)** $x^2-2y^2=2$

$\Rightarrow \quad \frac{x^2}{2}-\frac{y^2}{1}=1$

$\therefore$ Director circle is $x^2+y^2=2-1=1$

i.e., $x^2+y^2=1$ and $x^2+2y^2=2$

$\Rightarrow \quad \frac{x^2}{2}+\frac{y^2}{1}=1$

Director circle is $x^2+y^2=2+1=3$ i.e., $x^2+y^2=3$

(B) $3x^2+2y^2=6$

$\Rightarrow \quad \frac{x^2}{2}+\frac{y^2}{3}=1$

$\therefore$ Director circle is $x^2+y^2=2+3=5$

i.e., $x^2+y^2=5$ and $3x^2-2y^2=6$

$\Rightarrow \quad \frac{x^2}{2}-\frac{y^2}{3}=1$

Director circle is $x^2+y^2=2-3=-1$ (not defined)

(C) $5x^2-9y^2=45$

$\Rightarrow \quad \frac{x^2}{9}-\frac{y^2}{5}=1$

$\therefore$ Director circle is $x^2+y^2=9-5=4$

i.e., $x^2+y^2=4$ and director circle of $x^2+y^2=1$ is $x^2+y^2=2$

Integer

87. **(c)** Comparing the given parabola (i.e., $y^2=8x$) with $y^2=4ax$

$\therefore \quad 4a=8$

$\therefore \quad a=2$

Since, normal at (x_1, y_1) to the parabola $y^2=4ax$ is

$y-y_1=\frac{y_1}{2a}(x-x_1)$

Here, $x_1=2$ and $y_1=4$

$\therefore$ Equation of normal is $y-4=-\frac{4}{4}(x-2)$

$\Rightarrow \quad y-4=-x+2$

$\Rightarrow \quad x+y-6=0 \quad \quad \ldots (i)$

Solve equation (i) and $y^2=8x$ then $y^2=8(6-y)$

$\Rightarrow \quad y^2+8y-48=0$

$\Rightarrow \quad (y+12)(y-4)=0$

$\therefore \quad y=-12$ and $y=4$

Then $x=18$ and $x=2$

Hence, point of intersection of normal and parabola are (18,–12) and (2, 4) therefore normal meets the parabola at (18,–12) and length of normal chord is distance between their points $=PQ=\sqrt{(18,-12)^2+(-12-4)^2}=16\sqrt{2}$

$=\lambda$ (given)

$\therefore \quad \lambda^2=5/2 \quad (2.5\approx 3)$.

88. **(960)** Here, centre of the circle is the vertex of the parabola and both circle and parabola are symmetrical about axis of parabola. In this case the point of intersection of common tangents must lie on the directrix and axis of the parabola. i.e., $A(-a, 0)$

Chord of contact of circle w.r.t., $A(-a,0)$ is

$x(-a)+y\cdot 0=\frac{a^2}{2}$

$\therefore \quad x = -\dfrac{a}{2}$

$\therefore$ Coordinates of R is $\left(-\dfrac{a}{2}, \dfrac{a}{2}\right)$ and chord of contact of parabola w.r.t. $A(-a, 0)$ is $y \cdot 0 = 2a(x - a)$

i.e., $x = a$

$\therefore$ Coordinates of P is $(a, 2a)$

$\therefore$ Area of quadrilateral $PQRS' = 2$ {Area of ΔPAS – Area of ΔRAN}

$$= 2\left\{\frac{1}{2}\cdot 2a \cdot 2a - \frac{1}{2}\cdot\frac{a}{2}\cdot\frac{a}{2}\right\} = 4a^2 - \frac{a^2}{4} = \frac{15}{4}a^2 \text{ sq unit}$$

$\therefore \quad \lambda = \dfrac{15a^2}{4}$ Then $\dfrac{256}{a^2}\lambda = 960$

89. **(1296)** Any normal of the parabola $y^2 = 4x$ with slope m is

$y = mx - 2m - m^3$

It is pass through P, then $k = mh - 2m - m^3$

$m^3 + (2 - h)m + k = 0 \quad \ldots (i)$

Thus, $m_1 m_2 m_3 = -k$

$\alpha m_3 = -k \quad (\because m_1 m_2 = \alpha)$

$\Rightarrow \quad m_3 = -\dfrac{k}{\alpha}$

$\because$ m_3 is a root of equation (i), then

$$\frac{-k^3}{\alpha^3} + (2 - h)\left(-\frac{k}{\alpha}\right) + k = 0 \quad k^3 + (2 - h)k\alpha^2 - k\alpha^3 = 0$$

$\therefore$ Locus of $P(h, k)$ is $y^3 + (2 - x)\, y\alpha^2 - y\alpha^3 = 0$

$\Rightarrow \quad y^3 + (2 - x)\,\alpha^2 - \alpha^3 = 0 \; (\because y \neq 0)$

(P does not lie on the axis of the parabola)

$\Rightarrow \quad y^2 = \alpha^2 x - 2\alpha^2 + \alpha^3$

If it is a part of parabola $y^2 = 4x$ then $\alpha^2 = 4$ and

$-2\alpha^2 + \alpha^3 = 0$

$\Rightarrow \quad \alpha^2(\alpha - 2) = 0$

$\Rightarrow \quad \alpha - 2 = 0, \alpha \neq 0$

$\therefore \quad \alpha = 2$

$\therefore \quad (36)^{\alpha} = (36)^2 = 1296$

90. **(1)** Let Parabola be $y^2 = 4ax$ and coordinates of P and Q on this parabola are $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$; T is the point of intersection of tangents at t_1 and t_2.

$\therefore$ Coordinates of $T \equiv \{at_1, t_2, a(t_1 + t_2)\}$

Similarly, $P' \equiv \{at_3, t_1, a(t_3 + t_1)\}$ $Q' \equiv \{at_2, t_3, a(t_2 + t_3)\}$

Let $TP' : TP = \lambda : 1$

$\therefore \quad \lambda = \dfrac{t_3 - t_2}{t_1 - t_2}$ or $\dfrac{TP'}{TP} = \dfrac{t_3 - t_2}{t_1 - t_2}$

Similarly, $\dfrac{TQ'}{TQ} = \dfrac{t_1 - t_3}{t_1 - t_2}$ or $\dfrac{TP'}{TP} = \dfrac{TQ'}{TQ} = 1$

91. **(729)** The path of the water jet is a parabola. Let the equation of the water jet being a parabola is

$y = ax^2 + bx + c \quad \ldots (i)$

the path is symmetrical to the lie AB, the maximum height, so, it strikes the x-axis at E such that $AE = OA = 0.5\ m$

i.e., $OE = 2OA = 2(0.5) = 1m$

Coordinates of B and E are (0.5, 4) and (1, 0) respectively since, O, B, E and on equation (i)

$\therefore \quad o = c,\ 4 = \dfrac{1}{4}a + \dfrac{1}{2}b + c,\ 0 = a + b + c$

Solving these we get $a = -16, b = 16, c = 0$

From equation (i), the equation of the parabola is

$y = -16x^2 + 16x \quad \ldots (ii)$

Let P be a point on the parabola (ii), such that P is at a distance 0.75 m from y-axis and let P is at a distance h from x-axis

$\therefore$ Coordinates of P is $(0.75, h)$

$\because$ P lies on equation. (ii) so we have

$h = -16(0.75)^2 + 16(0.75)$

$$= -16\left(\frac{9}{16}\right) + 16\left(\frac{3}{4}\right) = -9 + 12 = 3 = m$$

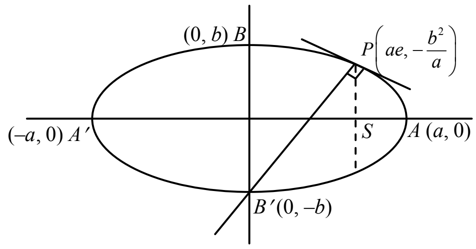
$\therefore \quad \lambda = 3$

Then, $\lambda^6 = 3^6 = 729$

92. **(3125)** The coordinates of an end of the latus rectum are $(ae, b^2/a)$. The equation of normal at $P(ae, b^2/a)$ is

$$\frac{a^2x}{ae} - \frac{b^2(y)}{b^2/a} = a^2 - b^2$$

Or $\quad \dfrac{ax}{e} - ay = a^2 - b^2$



It passes through on extremity of the minor axis whose coordinates are $(0, -b)$

$\therefore \quad 0 + ab = a^2 - b^2$

or $\quad (a^2b^2) = (a^2 - b^2)^2$

or $\quad a^2a^2(1 - e^2) = (a^2e^2)^2$

or $\quad 1-e^2=e^4$

or $\quad e^4+e^2-1=0$

or $\quad (e^2)^2+e^2-1=0$

$\therefore \quad e^2=\dfrac{-1\pm\sqrt{1-4}}{2}$

$\Rightarrow \quad e^2=\dfrac{\sqrt{5}-1}{2}$

(taking + ve sign) $2e^2+1=\sqrt{5}$

$\therefore \quad 625(2e^2+1)^2$

$=625\times 5=3125$

93. **(1215)** The equation of the ellipse can be written as

$$4\times 5\left(\frac{x-2y+1}{\sqrt{5}}\right)^2+9\times 5\left(\frac{2x+y+2}{\sqrt{5}}\right)^2=25$$

or $\quad \dfrac{\left(\dfrac{x-2y+1}{\sqrt{5}}\right)^2}{(5/4)}+\dfrac{\left(\dfrac{2x+y+2}{\sqrt{5}}\right)^2}{(5/9)}=1$

or $\quad \dfrac{X^2}{a^2}+\dfrac{Y^2}{b^2}=1$

Here $a>b$

$\therefore$ Equation of major axis is $Y=0$

i.e., $\quad 2x+y+2=0$

and equation of minor axis is $X=0$

i.e., $\quad x-2y+1=0$

Centre $X=0, Y=0$

$\Rightarrow \quad x-2y+1=0$

$2x+y+2=0$

We get $x=-1, y=0$

Length of latusrectum $=\dfrac{2b^2}{a}=\dfrac{2\times 5/9}{5/4}=\dfrac{8}{9}$

Eccentricity $b^2=a^2(1-e^2)$

$\Rightarrow \quad \dfrac{5}{9}=\dfrac{5}{9}(1-e^2)$

$\Rightarrow \quad \dfrac{4}{9}=1-e^2$

$\Rightarrow \quad e^2=\dfrac{5}{9}$

$\therefore \quad e=\dfrac{\sqrt{5}}{3}$

$\Rightarrow \quad e^2=5/9$

$\therefore \quad 2187e^2=2187\times\dfrac{5}{9}=1215$

94. **(4)** Let point of concurrent is (h, k)

Equation of normal at (x', y') is, $\dfrac{x-x'}{x'/a^2}=\dfrac{y_1-y'}{y'/b^2}$

It is passes through (h, k), then

$$y'^2\{a^2(h-x')+b^2x'\}^2=b^4k^2x'^2 \qquad \ldots(i)$$

But $\dfrac{x'^2}{a^2}=\dfrac{y'^2}{b^2}=1$

or $\quad y'^2=\dfrac{b^2}{a^2}(a^2-x'^2) \qquad \ldots(ii)$

Value of y'^2 from equation (*ii*), putting in equation (*i*), we get $\dfrac{b^2}{a^2}(a^2-x'^2)\{a^2h+(b^2-a^2)x'\}^2=b^4k^2x'^2$

$\Rightarrow \quad \dfrac{b^2}{a^2}(a^2-x'^2)\{a^4h^2+(b^2-a^2)^2x'^2+2a^2hx'(b^2-a^2)\}$

$=b^4k^2x'^2$

Arranging above as a fourth degree equation in x', we get

$\Rightarrow \quad -(a^2-b^2)^2x'^4+2ha^2(a^2-b^2)x'^3+x'^2(\ldots)$

$-2a^4h(a^2-b^2)x'+a^6h^2=0$

Above equation being of fourth degree in x', therefore roots of the above equation are x_1, x_2, x_3, x_4 then

$$(x_1+x_2+x_3+x_4)=-\frac{2ha^2(a^2-b^2)}{-(a^2-b^2)^2}=\frac{2ha^2}{(a^2-b^2)} \quad \ldots(iii)$$

$$\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=\frac{\Sigma x_1x_2x_3}{x_1\cdot x_2\cdot x_3\cdot x_4}$$

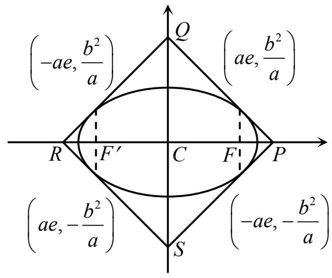
$$=\frac{\dfrac{2a^4h(a^2-b^2)}{-(a^2-b^2)^2}}{\dfrac{a^6h^2}{-(a^2-b^2)^2}}=\frac{2(a^2-b^2)}{a^2h} \qquad \ldots(iv)$$

Multiplying equation (*iii*) and (*iv*), we get

$$(x_1+x_2+x_3+x_4)\times\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$$

95. **(27)** $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$

$\Rightarrow \quad e^2=1-\dfrac{5}{9}=\dfrac{4}{5} \Rightarrow e=\dfrac{2}{3}$



One end of latusrectum is $\left(2, \frac{5}{3}\right)$

Equation of tangent at $\left(2, \frac{5}{3}\right)$ is $\frac{2x}{9} + \frac{y}{3} = 1$

F and F' be foci Area of $\Delta CPQ = \frac{1}{2} \times \frac{9}{2} \times 3 = \frac{27}{4}$

$\therefore$ Area of an quadrilateral $PQRS = 4 \times \frac{27}{4} = 27$ sq unit

$\therefore$ $\lambda = 27$

96. **(2401)** Let $P(a\cos\theta_1, b\sin\theta_1), Q(a\cos\theta_2, b\sin\theta_2)$ and $R(a\cos\theta_3, b\sin\theta_3)$ be the vertices of the triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The points on the auxiliary circle corresponding to these points $P'(a\cos\theta_1, a\sin\theta_1)$, $Q'(a\cos\theta_2, a\sin\theta_2)$ and $R'(a\cos\theta_3, a\sin\theta_3)$

$\therefore$ $\Delta_1 =$ Area of $\Delta PQR = \frac{1}{2}\begin{vmatrix} a\cos\theta_1 & b\sin\theta_1 & 1 \\ a\cos\theta_2 & b\sin\theta_2 & 1 \\ a\cos\theta_3 & b\sin\theta_3 & 1 \end{vmatrix}$

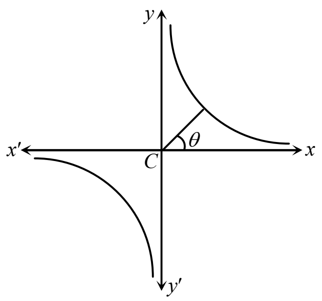
$= \frac{1}{2}ab\begin{vmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_3 & \sin\theta_3 & 1 \end{vmatrix}$

And $\Delta_2 =$ Area of $\Delta P'Q'R' = \frac{1}{2}\begin{vmatrix} a\cos\theta_1 & a\sin\theta_1 & 1 \\ a\cos\theta_2 & a\sin\theta_2 & 1 \\ a\cos\theta_3 & b\sin\theta_3 & 1 \end{vmatrix}$

$= \frac{1}{2}a^2\begin{vmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_3 & \sin\theta_3 & 1 \end{vmatrix}$ Clearly, $\frac{\Delta_1}{\Delta_2} = \frac{b}{a} = \sqrt{(1-e^2)} = \frac{1}{7}$

Then, $343 \frac{\Delta_1}{\Delta_2} 2401$

97. **(1)** Let the rectangular hyperbola is $xy = c^2$ . . .(i)



Since, the center of hyperbola (i) is origin (0, 0) and equation of an asymptotes are $x = 0$ and $y = 0$.

The equation of line through (0, 0) and makes an angle θ with an asymptote (x-axis) is $y = x\tan\theta$

It will meet the hyperbola, where $x(x\tan\theta) = c^2$.

i.e., $x = c\sqrt{\cot\theta}$

Putting $x = c\sqrt{\cot\theta}$ in equation (i) then $y = c\sqrt{(\tan\theta)}$

$\therefore$ The four points are $(c\sqrt{\cot\theta}, c\sqrt{(\tan\theta)}$ where $Q = \alpha, \beta, \gamma, \delta$

The line joining the points α and β is perpendicular to the line joining the points γ and δ.

Therefore, the product of their slopes $= -1$

i.e., $\frac{c\sqrt{\tan\beta} - c\sqrt{\tan\alpha}}{c\sqrt{\cot\beta} - c\sqrt{\cot\alpha}} \times \frac{c\sqrt{\tan\delta} - c\sqrt{\tan\gamma}}{c\sqrt{\cot\delta} - c\sqrt{\cot\gamma}} = -1$

$\Rightarrow$ $(-\sqrt{\tan\alpha}\sqrt{\tan\beta}) \times (-\sqrt{\tan\gamma}\sqrt{\tan\delta}) = -1$

or $\tan\alpha\tan\beta\tan\gamma\tan\delta = 1$

98. **(1296)** Let a triangle PQR be inscribed in $xy = c^2$

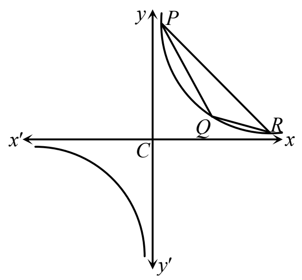
Let the coordinates of the vertices of the triangle be $P\left(ct_1, \frac{c}{t_1}\right), Q\left(ct_2, \frac{c}{t_2}\right)$ and $R\left(ct_3, \frac{c}{t_3}\right)$

Now, the equation of chord joining P and Q is

$x + yt_1t_2 = c(t_1 + t_2)$ . . . (i)

And the equation of chord joining Q and R is

$x + yt_2t_3 = c(t_2 + t_3)$ . . .(ii)



Let equation (i) be parallel to $y = m_1x$ and equation (ii) be parallel to $y = m_2x$

$\therefore$ $m_1 = -\frac{1}{t_1t_2}$ and $m_2 = -\frac{1}{t_2t_3}$

$\therefore$ $\frac{m_1}{m_2} = \frac{t_3}{t_1}$ i.e., $t_3 = \left(\frac{m_1}{m_2}\right)t_1$ . . . (iii)

Again the equation to the third side RP is

$x + yt_3t_1 = c(t_3 + t_1)$ $x + y\left(\frac{m_1}{m_2}\right)t_1^2 = c\left(\frac{m_1}{m_2}t_1 + t_1\right)$

or $ym_1t_1^2 - ct_1(m_1 + m_2) + xm_2 = 0$ . . . (iv)

t_1 being parameter. Since, t_1 is real the envelope of equation (iv) is given by the discrimination of equation (iv) = 0 i.e., $c^2(m_1+m_2)^2-4ym_1\cdot xm_2=0$

or $4m_1m_2xy=c^2(m_1+m_2)^2$ ...(v)

$\because$ m_1 and m_2 are roots of $x^2-6x+1=0$

$\therefore$ $m_1+m_2=6, m_1m_2=1$

Then from equation (v), $4xy=c^2(6)^2$

Then $xy=9c^2$

$\therefore$ $\lambda=9$

$\Rightarrow$ $16\lambda^2=16(9)^2=16\times 81=1296$

99. **(16)** Let the equation of circle

$x^2+y^2+2gx+2fy+k=0$ ...(i)

and the equation of the rectangular hyperbola is

$xy=c^2$ ...(ii)

Put $x=ct$ and $y=\frac{c}{t}$ in equation (i)

then $c^2t^2+\frac{c^2}{t^2}+2gct+\frac{2fc}{t}+k=0$

$\Rightarrow$ $c^2t^4+2gct^3+kt^2+2fct+c^2=0$

This equation being fourth degree in t, Let the roots be t_1, t_2, t_3, t_4 then $t_1t_2t_3t_4=1$

$\therefore$ $16t_1t_2t_3t_4=16\times 1=16$

100. **(37)** Combined equation of asymptotes is

$(x+2y+3)(3x+4y+5)=0$

or $3x^2+10xy+8y^2+14x+22y+15=0$ ...(i)

Also, we know that the equation of the hyperbola differs from that of asymptotes by a constant.

Let the equation of the hyperbola be

$3x^2+10xy+8y^2+14x+22y+\lambda=0$ ...(ii)

Since, it passes through (1, –1) then

$3(1)^2+10(1)(-1)+8(-1)^2+14(1)+22(-1)+\lambda=0$

$\Rightarrow$ $3-10+8+14-22+\lambda=0$

$\therefore$ $\lambda=7$

From equation (ii) equation of hyperbola is

$3x^2+10xy+8y^2+14x+22y+7=0$ ...(iii)

But we know the equation of conjugate hyperbola $=2$ (combined equation of an asymptotes) – (equation of hyperbola)

$\Rightarrow$ $6x^2+20xy+16y^2+28x+44y+30-3x^2$

$-10xy-8y^2-14x-22y-7=0$

or $3x^2+10xy+8y^2+14x+22y+23=0$

$\therefore$ $\mu=14$ and $v=23$

$\therefore$ $\mu+v=14+23=27$

* * *