4 Determinants

BASIC CONCEPTS

1. **Determinant:** Every square matrix can be associated to an expression or a number which is known as its determinant.

   Determinant of square matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by

   $$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

   and determinant of a matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is given by

   $$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

   This is known as the expansion of $|A|$ along first row.

   In fact, $|A|$ can be expanded along any of its rows or columns.

2. **Singular and Non-singular Matrix:** A square matrix is a singular matrix if its determinant is zero. Otherwise, it is a non-singular matrix.

3. **(*i*) Minor:** Let $A = [a_{ij}]$ be a square matrix of order n. Then the minor M_{ij} of a_{ij} in A is the determinant of the sub-matrix of order $(n - 1)$ obtained by leaving ith row and jth column of A.

   For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{bmatrix}$, then

   $$M_{11} = \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} = 2,\ M_{12} = \begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix} = -7 \text{ and so on.}$$

   (*ii*) Cofactor: The cofactor C_{ij} of a_{ij} in $A = [a_{ij}]_{n\times n}$ is equal to $(-1)^{i+j}$ times M_{ij}.

   For example, if $A = \begin{vmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{vmatrix}$, then

   $C_{11} = (-1)^{1+1} M_{11} = M_{11} = 2$ and $C_{12} = (-1)^{1+2} M_{12} = -M_{12} = 7$ and so on

4. **Some Important Properties of Determinants:**

(*i*) Let $A = [a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with their cofactors is always equal to $|A|$ or, det (A), *i.e.*,

$$\sum_{j=1}^{n} a_{ij}C_{ij} = |A| \text{ and } \sum_{i=1}^{n} a_{ij}C_{ij} = |A|$$

(*ii*) Let $A = [a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with cofactors of the corresponding elements of some other row (column) is zero, *i.e.*,

$$\sum_{j=1}^{n} a_{ij}C_{kj} = 0 \text{ and } \sum_{i=1}^{n} a_{ij}C_{ik} = 0, i \neq k \text{ or } j \neq k$$

(*iii*) Let $A = [a_{ij}]$ be a square matrix of order n, then $|A| = |A^T|$.

In other words, we say that the value of a determinant remains unchanged, if its rows and columns are interchanged.

(*iv*) Let $A = [a_{ij}]$ be a square matrix of order $n(\geq 2)$ and B be a matrix obtained from A by interchanging any two rows (columns) of A, then $|B| = -|A|$.

(*v*) If any two rows (columns) of a square matrix $A = [a_{ij}]$ of order n (≥ 2) are identical, then value of its determinant is zero *i.e.*, $|A| = 0$.

(*vi*) Let $A = [a_{ij}]$ be a square matrix of order n, and let B be the matrix obtained from A by multiplying each element of a row (column) of A by a scalar k, then $|B| = k|A|$.

(*vii*) Let A be a square matrix such that each element of a row (column) of A is expressed as the sum of two or more terms. Then the determinant of A can be expressed as the sum of the determinants of two or more matrices of the same order.

(*viii*) Let A be a square matrix and B be a matrix obtained from A by adding to a row (column) of A a scalar multiple of another row (column) of A, then $|B| = |A|$.

(*ix*) Let A be a square matrix of order n (≥ 2) such that each element in a row (column) of A is zero, then $|A| = 0$.

(*x*) If $A = [a_{ij}]$ is a diagonal matrix of order n (≥ 2), then

$|A| = a_{11} . a_{22} . a_{33} .... a_{nn}$ *i.e.*, $|A|$ is the product of its diagonal elements.

(*xi*) If A and B are square matrices of the same order, then

$$|AB| = |A||B|$$

(*xii*) If $A = [a_{ij}]$ is a triangular matrix of order n, then

$|A| = a_{11} . a_{22} . a_{33} .... a_{nn}$ *i.e.*, $|A|$ is the product of its diagonal elements.

(*xiii*) If $A = [a_{ij}]$ is a square matrix of order n, then $|kA| = k^n|A|$, because k is common from each row (or column) of kA.

(*xiv*) We can take out any common factor from any one row or any one column of a given determinant.

5. Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \text{Numerical value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note: Since area is positive quantity therefore we take absolute value of Δ.

6. (*i*) If A is a skew-symmetric matrix of odd order, then $|A| = 0$.

(*ii*) The determinant of a skew-symmetric matrix of even order is a perfect square.

7. **Some Important Facts:**

(*i*) Only square matrices have determinants.

(*ii*) We cannot equate the corresponding elements of equal determinants like matrices

i.e., $$\begin{vmatrix} x & y \\ z & w \end{vmatrix} = \begin{vmatrix} l & m \\ n & p \end{vmatrix} \not\Rightarrow \begin{matrix} x = l, & y = m \\ z = n, & w = p \end{matrix}$$

(*iii*) In the case of matrices. We take out any common factor from each elements of matrix, while in the case of determinants we can take out common factor from any one row or any one column of the determinant.

(*iv*) If the value of determinant 'Δ' becomes zero by substituting $x = a$ then $(x - a)$ is factor of the determinant 'Δ'.

(*v*) If area is given then both positive and negative values of the determinant is taken for calculation.

(*vi*) To prove three points collinear, we show area of the triangle formed by these three points is zero.

Selected NCERT Questions

1. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$.

Sol. We have,

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$$

$$\therefore \quad \text{LHS} = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 8 - 32 = -24$$

$$\text{RHS} = 4|A| = 4\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 4(2 - 8) = 4 \times (-6) = -24$$

$$\therefore \quad \text{LHS} = \text{RHS}$$

Hence Proved

By using properties of determinant in problems 2 to 5 prove that:

2. $$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2.$$ **[*CBSE (Delhi) 2011*]**

Sol. $$\text{LHS} = \Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

Taking a, b and c common from R_1, R_2 and R_3 respectively, we get

$$\Delta = abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Taking a, b and c common from C_1, C_2 and C_3 respectively, we get

$$\Delta = a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Operating $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$, we get

$$\Delta = a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

Interchanging C_2 and C_3, we get

$$\Delta = (-1)\, a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

Since the determinant of a triangular matrix is product of its diagonal elements.

$$= (-1)\, a^2b^2c^2\, (-1) \times (2) \times (2) = 4\, a^2b^2c^2 = \text{RHS}$$

3. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$. **[CBSE (Delhi) 2012]**

Sol. LHS $= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$

Taking $(b-a)$ and $(c-a)$ common from C_2 and C_3, we get

$$= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & b^2+ba+a^2 & c^2+ca+a^2 \end{vmatrix}$$

Operating $C_3 \to C_3 - C_2$, we get

$$= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^3 & b^2+ba+a^2 & c^2+ca-b^2-ba \end{vmatrix}$$

Expanding along R_1, we get

$$= (b-a)(c-a)(c^2+ca-b^2-ba) = (b-a)(c-a)[c^2-b^2+a(c-b)]$$
$$= (b-a)(c-a)[(c-b)(c+b)+a(c-b)] = (b-a)(c-a)(c-b)[c+b+a]$$
$$= (a-b)(b-c)(c-a)(a+b+c) = \text{RHS.}$$

4. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$.

Sol. LHS $= \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$

Operating $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we get

$$= \begin{vmatrix} x-z & x^2-z^2 & yz-xy \\ y-z & y^2-z^2 & zx-xy \\ z & z^2 & xy \end{vmatrix} = \begin{vmatrix} (x-z) & (x-z)(x+z) & -y(x-z) \\ (y-z) & (y-z)(y+z) & -x(y-z) \\ z & z^2 & xy \end{vmatrix}$$

Taking $(x-z)$ and $(y-z)$ common from R_1 and R_2, we get

$$= (x-z)(y-z) \begin{vmatrix} 1 & x+z & -y \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$, and $R_3 \to R_3 - zR_1$, we get

$$= (x-z)(y-z)\begin{vmatrix} 1 & x+z & -y \\ 0 & y-x & y-x \\ 0 & -xz & xy+yz \end{vmatrix}$$

Expanding along R_1, we get

$$= (x-z)(y-z)[(y-x)(xy+yz)+xz(y-x)]$$
$$= (x-y)(y-z)(z-x)(xy+yz+zx) = \text{RHS}$$

5. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Sol. LHS $= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Operating $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a+b+c)$ common from first row, we get

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -c-a-b & 0 \\ 2c & 0 & -a-b-c \end{vmatrix}$$

Since determinant of a triangular matrix is equal to product of its diagonal elements

$\therefore \quad = (a+b+c)(a+b+c)(a+b+c) = (a+b+c)^3 = \text{RHS}$

6. By using properties of determinant, show that:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

[*CBSE Delhi 2008, 2009, (F) 2013; Guwahati 2015*]

Sol. LHS $= \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

Applying $C_1 \to C_1 - bC_3$, and $C_2 \to C_2 + aC_3$, we get

$$= \begin{vmatrix} (1+a^2+b^2) & 0 & -2b \\ 0 & (1+a^2+b^2) & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

Taking out $(1 + a^2 + b^2)$ from C_1 and C_2 column, we get

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

Applying $R_3 \to R_3 - bR_1$, we get

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

Expanding along first column, we get

$$= (1 + a^2 + b^2)^2 \, [1 - a^2 + b^2 + 2a^2]$$

$$= (1 + a^2 + b^2)^2 \, (1 + a^2 + b^2) = (1 + a^2 + b^2)^3 = \text{RHS}$$

7. By using properties of determinant, show that:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$$

[CBSE Delhi 2014; (F) 2009, 2013]

Sol. LHS $= \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$ [Multiplying and dividing by abc]

Multiplying a in C_1, b in C_2 and c in C_3, we get

$$= \frac{1}{abc}\begin{vmatrix} a^3+a & ab^2 & ac^2 \\ a^2b & b^3+b & bc^2 \\ a^2c & b^2c & c^3+c \end{vmatrix}$$

Taking a, b and c common from R_1, R_2 and R_3 respectively, we get

$$= \frac{1}{abc} \times abc \begin{vmatrix} 1+a^2 & b^2 & c^2 \\ a^2 & 1+b^2 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & 1+b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

Taking $(1+a^2+b^2+c^2)$ common from C_1, we get

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & 1+b^2 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we get

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Expanding along R_1, we get

$$(1 + a^2 + b^2 + c^2)[-1(-1)] = (1 + a^2 + b^2 + c^2) = \text{RHS}$$

8. Prove that: $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$ ***[CBSE (F) 2014; Allahabad 2015, 2019 (65/5/3)]***

Sol. LHS $= \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

$= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$ [Taking out a, b, c from C_1, C_2 and C_3]

$= abc\begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 - C_3$]

$= 2ab^2c\begin{vmatrix} 0 & c & a+c \\ 1 & b & a \\ 1 & b+c & c \end{vmatrix}$ [Taking out $2b$ from C_1]

$= 2ab^2c\begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ 1 & b+c & c \end{vmatrix}$ [Applying $R_2 \to R_2 - R_3$]

$= 2ab^2c.1.\begin{vmatrix} c & a+c \\ -c & a-c \end{vmatrix} = 2ab^2c(ac - c^2 + ac + c^2)$ [Expanding by I column]

$= 2ab^2c(2ac) = 4a^2b^2c^2 = \text{RHS}$

9. Prove: $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1 + pxyz)(x-y)(y-z)(z-x)$ ***[CBSE (AI) 2010]***

Sol. LHS $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & px^2 \\ 1 & y & py^2 \\ 1 & z & pz^2 \end{vmatrix}$ [Taking common x, y, z from R_1, R_2, R_3 respectively]

$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (xyz)\,p\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ [Taking p common from C_3]

By changing (transforming) column to column in first determinant, we get

$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (1 + pxyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we get

$$= (1 + pxyz)\begin{vmatrix} 0 & x - z & x^2 - z^2 \\ 0 & y - z & y^2 - z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Taking out $(x - z)$, $(y - z)$ from R_1 and R_2 respectively, we get

$$= (1 + pxyz)(x - z)(y - z)\begin{vmatrix} 0 & 1 & x + z \\ 0 & 1 & y + z \\ 1 & z & z^2 \end{vmatrix}$$

Expanding along C_1, we get

$$= (1 + pxyz)\ (x - z)\ (y - z)\ [y + z - x - z]$$

$$= (1 + pxyz)\ (x - y)\ (y - z)\ (z - x) = \text{RHS}.$$

10. If a, b and c are real numbers and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$ then show that either $a + b + c = 0$ or $a = b = c$. **[CBSE (AI) 2007C; (F) 2009]**

Sol. Given $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \qquad [\text{Applying } R_1 \to R_1 + R_2 + R_3]$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \qquad [\text{Taking common } 2(a+b+c) \text{ from } R_1]$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$= 2(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a-b & a-c & b+c \\ b-c & b-a & c+a \end{vmatrix}$$

Expanding along R_1, we get

$$= 2(a+b+c)[(a-b)(b-a) - (b-c)(a-c)]$$

$$= 2(a+b+c)[ab - a^2 - b^2 + ab - \{ab - bc - ac + c^2\}]$$

$$= 2(a+b+c)[ab - a^2 - b^2 + ab - ab + bc + ac - c^2]$$

$$= 2(a+b+c)[-a^2 - b^2 - c^2 + ab + bc + ca]$$

$$= -2(a+b+c)[a^2 + b^2 + c^2 - ab - bc - ca]$$

$$= -(a+b+c)[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$$

$$= -(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Now, given that $\Delta = 0$

$\Rightarrow \quad \Delta = (a + b + c)\ [(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$

So, either $(a + b + c) = 0$ or $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$ *i.e.*, $a = b = c$.

11. Show that points $A(a, b + c)$, $B(b, c + a)$, $C(c, a + b)$ are collinear.

Sol. We have,

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} a & a+b+c & 1 \\ b & b+c+a & 1 \\ c & a+b+c & 1 \end{vmatrix} \qquad (\text{Applying } C_2 \to C_2 + C_1)$$

$$= \frac{1}{2}(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} \qquad (\text{Taking } (a+b+c) \text{ common from } C_2)$$

$$= \frac{1}{2} \times (a+b+c) \times 0 \qquad (\because C_2 = C_3)$$

$\Rightarrow \quad ar(\Delta ABC) = 0$

Since area of ΔABC is zero, therefore points A, B and C are collinear.

Hence proved.

Multiple Choice Questions [1 mark]

Choose and write the correct option in the following questions.

1. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

(*a*) 6 (*b*) ± 6 (*c*) -6 (*d*) 0

2. The value of determinant $\begin{vmatrix} a-b & b+c & a \\ b-c & c+a & b \\ c-a & a+b & c \end{vmatrix}$ *[NCERT Exemplar]*

(*a*) $a^3 + b^3 + c^3$ (*b*) $3bc$ (*c*) $a^3 + b^3 + c^3 - 3abc$ (*d*) None of these

3. The area of a triangle with vertices (–3, 0),(3, 0) and (0, *k*) is 9 sq. units. The value of *k* will be

(*a*) 9 (*b*) 3 (*c*) –9 (*d*) 6

4. If *A*, *B* and *C* are angles of a triangle, then the determinant $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$ is equal to

(*a*) 0 (*b*) –1 (*c*) 1 (*d*) None of these

5. If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then *[NCERT Exemplar]*

(*a*) $f(a) = 0$ (*b*) $f(b) = 0$ (*c*) $f(0) = 0$ (*d*) $f(1) = 0$

6. If *x*, *y*, *z* are all different from zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$, then value of $x^{-1} + y^{-1} + z^{-1}$ is

(*a*) xyz (*b*) $x^{-1} y^{-1} z^{-1}$ (*c*) $-x - y - z$ (*d*) -1

7. There are two values of a which makes determinant $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$, then sum of these numbers is

(*a*) 4 (*b*) 5 (*c*) – 4 (*d*) 9

8. If A is a non-singular square matrix of order 3 such that $A^2 = 3A$, then value of $|A|$ is [*CBSE 2020 (65/2/1)*]

(*a*) –3 (*b*) 3 (*c*) 9 (*d*) 27

9. If $\begin{vmatrix} 2 & 3 & 2 \\ x & x & x \\ 4 & 9 & 1 \end{vmatrix} + 3 = 0$, then the value of x is [*CBSE 2020 (65/4/1)*]

(*a*) 3 (*b*) 0 (*c*) –1 (*d*) 1

10. The value of the determinant $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is [*NCERT Exemplar*]

(*a*) $9x^2(x + y)$ (*b*) $9y^2(x + y)$ (*c*) $3y^2(x + y)$ (*d*) $7x^2(x + y)$

11. If a, b, c are in AP, then the value of determinant $\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is

(*a*) 0 (*b*) 1 (*c*) x (*d*) $2x$

12. The value of $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ is

(*a*) $(a - b)(b - c)(c - a)$ (*b*) $(b - a)(c - b)(c - a)$

(*c*) $a(b - c)(c - a)$ (*d*) None of these

13. The value of $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$ is

(*a*) a (*b*) b (*c*) 0 (*d*) None of these

14. The value of $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{vmatrix}$ is

(*a*) 1 (*b*) –1 (*c*) 0 (*d*) ω

15. If area of triangle is 35 sq units with vertices (2, –6) (5, 4) and (k, 4), then k is

(*a*) 12 (*b*) –2 (*c*) –12, –2 (*d*) 12, –2

16. Let A be a square matrix of order 3 × 3, then $|KA|$ is equal to

(*a*) $K|A|$ (*b*) $K^2|A|$ (*c*) $K^3|A|$ (*d*) $3K|A|$

17. The value of $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ is

(*a*) 0 (*b*) 1 (*c*) –1 (*d*) None

18. **The value of** $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ **is**

(*a*) 1 (*b*) 0 (*c*) $a + b$ (*d*) $a - b$

19. **If** $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ **and** A_{ij} **is cofactors of** a_{ij}, **then value of** Δ **is given by**

(*a*) $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$ (*b*) $a_{11} A_{11} + a_{12} A_{21} + a_{13} A_{31}$

(*c*) $a_{21} A_{11} + a_{22} A_{12} + a_{23} A_{13}$ (*d*) $a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$

20. **If** A **is a 3 × 3 matrix such that** $|A| = 8$, **then** $|3A|$ **equals** *[CBSE 2020 (65/5/1)]*

(*a*) 8 (*b*) 24 (*c*) 72 (*d*) 216

Answers

1. (*b*)	**2.** (*c*)	**3.** (*b*)	**4.** (*a*)	**5.** (*c*)	**6.** (*d*)
7. (*c*)	**8.** (*d*)	**9.** (*c*)	**10.** (*b*)	**11.** (*a*)	**12.** (*a*)
13. (*c*)	**14.** (*c*)	**15.** (*d*)	**16.** (*c*)	**17.** (*a*)	**18.** (*b*)
19. (*d*)	**20.** (*d*)				

Solutions of Selected Multiple Choice Questions

1. We have,

$$\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$$

$\Rightarrow \quad x^2 - 36 = 36 - 36$

$\Rightarrow \quad x^2 - 36 = 0$

$\Rightarrow \quad x^2 = 36$

$\Rightarrow \quad x = \pm 6$

2. We have,

$$\begin{vmatrix} a-b & b+c & a \\ b-c & c+a & b \\ c-a & a+b & c \end{vmatrix} = \begin{vmatrix} a+c & b+c+a & a \\ b+a & c+a+b & b \\ c+b & a+b+c & c \end{vmatrix}$$ [Applying $C_1 \to C_1 + C_2$ and $C_2 \to C_2 + C_3$]

$$= (a+b+c)\begin{vmatrix} a+c & 1 & a \\ b+a & 1 & b \\ c+b & 1 & c \end{vmatrix}$$ [Taking $(a + b + c)$ common from C_2]

$$= (a+b+c)\begin{vmatrix} a-b & 0 & a-c \\ a-c & 0 & b-c \\ c+b & 1 & c \end{vmatrix}$$ [Applying $R_2 \to R_2 - R_3$ and $R_1 \to R_1 - R_3$]

$= (a + b + c) [-(a - b)(b - c) + (a - c)^2]$ [Expanding along R_2]

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = (a^3 + b^3 + c^3 - 3abc)$

3. We know that, area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}\right|$$

Expanding along R_1, we get

$$9 = \frac{1}{2} |[-3(-k) - 0 + 1(3k)]| \quad \Rightarrow \quad 18 = |3k + 3k| = |6k|$$

$$\therefore \quad k = \pm\frac{18}{6} = \pm 3 = 3, -3$$

5. We have, $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$

$$\Rightarrow \quad f(a) = \begin{vmatrix} 0 & 0 & a-b \\ 2a & 0 & a-c \\ a+b & a+c & 0 \end{vmatrix} = [(a-b)\{2a.(a+c)\}] \neq 0$$

and $f(b) = \begin{vmatrix} 0 & b-a & 0 \\ b+a & 0 & b-c \\ 2b & b+c & 0 \end{vmatrix} = -(b-a)[-2b(b-c)] = 2b(b-a)(b-c) \neq 0$

and $f(0) = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = a(bc) - b(ac) = abc - abc = 0$

7. We have, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$

$\Rightarrow \quad 1(2a^2 + 4) - 2(-4a - 20) + 0 = 86$ [Expanding along first column]

$\Rightarrow \quad 2a^2 + 4 + 8a + 40 = 86$

$\Rightarrow \quad 2a^2 + 8a + 44 - 86 = 0$

$\Rightarrow \quad a^2 + 4a - 21 = 0$

$\Rightarrow \quad a^2 + 7a - 3a - 21 = 0$

$\Rightarrow \quad (a+7)(a-3) = 0 \quad \Rightarrow \quad a = -7$ and 3

$\therefore$ Required sum $= -7 + 3 = -4$

8. We have,

$A^2 = 3A \quad \Rightarrow \quad |A^2| = |3A|$

$\Rightarrow \quad |A|.|A| = 3^3 |A|$ ($\because$ order of matrix A is 3 and $|A|$ is not equal to zero)

$\Rightarrow \quad |A| = 3^3 = 27 \quad \Rightarrow \quad |A| = 27$

9. We have,

$$\begin{vmatrix} 2 & 3 & 2 \\ x & x & x \\ 4 & 9 & 1 \end{vmatrix} + 3 = 0$$

$\Rightarrow \quad x\begin{vmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 4 & 9 & 1 \end{vmatrix} + 3 = 0$ (Taking out x from R_2)

$\Rightarrow \quad x\{2(1-9) - 3(1-4) + 2(9-4)\} + 3 = 0$

$\Rightarrow \quad x(-16 + 9 + 10) + 3 \quad \Rightarrow \quad 3x + 3 = 0$

$\Rightarrow \quad 3x = -3 \quad \Rightarrow \quad x = -1$

20. We have,

$|3A| = 3^3 |A| = 27 \times 8 = 216$

Fill in the Blanks **[1 mark]**

1. The value of the determinant $\begin{vmatrix} \sin^2 23° & \sin^2 67° & \cos 180° \\ -\sin^2 67° & -\sin^2 23° & \cos^2 180° \\ \cos 180° & \sin^2 23° & \sin^2 67° \end{vmatrix}$ is ______________ .

2. The cofactor of element a_{12} in the matrix $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$ is ______________ .

3. If A is a skew-symmetric matrix of order 3, then the value of $|A|$ = ______________ .

4. If $\cos 2\theta = 0$, then $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$ = ______________ . [*NCERT Exemplar*]

5. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are ______________ . [*NCERT Exemplar*]

Answers

1. 0 **2.** 46 **3.** 0 **4.** $\frac{1}{2}$ **5.** $x = 2, x = 7$

Solutions of Selected Fill in the Blanks

1. Applying $C_1 \to C_1 + C_2 + C_3$, we have

$$\begin{vmatrix} \sin^2 23° + \sin^2 67° + \cos 180° & \sin^2 67° & \cos 180° \\ -\sin^2 67° - \sin^2 23° + \cos^2 180° & -\sin^2 23° & \cos^2 180° \\ \cos 180° + \sin^2 23° + \sin^2 67° & \sin^2 23° & \sin^2 67° \end{vmatrix}$$

$$= \begin{vmatrix} 1 + (-1) & \sin^2 67° & -1 \\ -1 + 1 & -\sin^2 23° & 1 \\ -1 + 1 & \sin^2 23° & \sin^2 67° \end{vmatrix} \qquad \left(\begin{array}{c} \because \ \sin^2 67° = \cos^2 23° \\ \cos 180° = -1 \\ \text{and } \sin^2\theta + \cos^2\theta = 1 \end{array} \right)$$

$$= \begin{vmatrix} 0 & \sin^2 67° & -1 \\ 0 & -\sin^2 23° & 1 \\ 0 & \sin^2 23° & \sin^2 67° \end{vmatrix} = 0$$

3. Since matrix A is a skew-symmetric of odd order *i.e.* 3

$\therefore \quad |A| = 0$

Very Short Answer Questions **[1 mark]**

1. If A and B are square matrices of the same order 3, such that $|A| = 2$ and $AB = 2I$, write the value of $|B|$. [*CBSE Delhi 2019*]

Sol. $|A| = 2$ and $AB = 2I$

$\Rightarrow \quad |AB| = |2I| = 8$

$\Rightarrow \quad |A||B| = 8 \Rightarrow 2|B| = 8 \Rightarrow |B| = 4$

2. Find $|AB|$, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$ [*CBSE 2019 (65/2/2)*]

Sol. We have, $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \qquad |AB| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$

3. Let A be a square matric of order 3 × 3. Write the value of $|2A|$, where $|A| = 4$. **[*CBSE Delhi 2012*]**

Sol. $\because \; |2A| = 2^n \, |A|$, where n is order of matrix A.

Here $\quad |A| = 4$ and $n = 3$

$|2A| = 2^3 \times 4 = 32$

4. If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, then write the value of x. **[*CBSE Delhi 2013*]**

Sol. Given $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$

$$\begin{aligned}
\Rightarrow \quad (x+1)(x+2) - (x-1)(x-3) &= 12 + 1 \\
\Rightarrow \quad x^2 + 2x + x + 2 - x^2 + 3x + x - 3 &= 13 \\
\Rightarrow \quad 7x - 1 &= 13 \\
\Rightarrow \quad 7x &= 14 \\
\Rightarrow \quad x &= 2
\end{aligned}$$

5. If $A = [a_{ij}]$ is a matrix of order 2 × 2 , such that $|A| = -15$ and C_{ij} represents the cofactor of a_{ij}, then find $a_{21}C_{21} + a_{22}C_{22}$. **[*CBSE Sample Paper 2018*]**

Sol. Given, $\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \therefore \qquad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

Expanding along R_2

$\Rightarrow \quad -15 = a_{21} \, . \, C_{21} + a_{22} \, . \, C_{22} \qquad [C_{ij} = \text{Cofactor of } a_{ij}]$

$\Rightarrow \quad a_{21} . \, C_{21} + a_{22} \, . \, C_{22} = -15$

6. Write the value of the following determinant:

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

[*CBSE (AI) 2009; (East) 2016*]

Sol. Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0 \qquad [\because \text{ All elements of } C_1 \text{ are zero}]$$

7. Show that the points (1, 0), (6, 0), (0, 0) are collinear. **[*CBSE (AI) 2008*]**

Sol. Since $\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$

Hence, (1, 0), (6, 0) and (0, 0) are collinear.

8. What positive value of x makes the following pair of determinants equal?

$$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}, \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$$

[*CBSE (AI) 2010*]

Sol. $\because \quad \begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix} = \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$

$\Rightarrow \quad 2x^2 - 15 = 32 - 15 \quad \Rightarrow \quad 2x^2 = 32$

$\Rightarrow \quad x^2 = 16 \quad \Rightarrow \quad x = \pm 4$

$\Rightarrow \quad x = 4$ (+ve value).

9. Evaluate: $\begin{vmatrix} \cos 15° & \sin 15° \\ \sin 75° & \cos 75° \end{vmatrix}$ **[*CBSE (AI) 2011*]**

Sol. Expanding the determinant, we get

$\cos 15°. \cos 75° - \sin 15°. \sin 75° = \cos(15° + 75°) = \cos 90° = 0$

[**Note :** $\cos(A + B) = \cos A. \cos B - \sin A. \sin B$]

10. Write the value of the following determinant:

$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$ **[*CBSE (F) 2012*]**

Sol. Let $\Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Applying $R_1 \to R_1 - 6R_3$, we get

$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$ [$\because$ Each element of R_1 is zero]

11. If A is a square matrix and $|A| = 2$, then write the value of $|AA'|$, where A' is the transpose of matrix A. **[*CBSE (F) 2013*]**

Sol. $|AA'| = |A|.|A'| = |A|.|A| = |A|^2 = 2^2 = 4$

[**Note:** $|AB| = |A|.|B|$ and $|A| = |A^T|$, where A and B are square matrices.]

12. If A is a 3×3 invertible matrix, then what will be the value of k if $\det(A^{-1}) = (\det A)^k$. **[*CBSE Delhi 2017*]**

Sol. Given, $\det(A^{-1}) = (\det A)^k$

$\Rightarrow |A^{-1}| = |A|^k \quad \Rightarrow \quad k = -1$

Short Answer Questions-I and II [2, 3 marks]

1. Find the equations of line joining (1,2) and (3,6) using determinants.

Sol. Let given points are $A(1, 2)$ and $B(3, 6)$ and $P(x, y)$ lies on the line joining points A and B.

$\therefore$ Points A,P and B are collinear

$\therefore$ Area of $\Delta APB = 0$

$\Rightarrow \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ x & y & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} 1 & 2 & 1 \\ x & y & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0$

$\Rightarrow 1(y - 6) - 2(x - 3) + 1(6x - 3y) = 0$

$\Rightarrow y - 6 - 2x + 6 + 6x - 3y = 0 \quad \Rightarrow \quad 4x - 2y = 0$

$\Rightarrow \quad 2(2x - y) = 0$

$\Rightarrow \quad 2x - y = 0$

$\therefore$ Equation of line be $2x - y = 0$

2. Evaluate the determinant: $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$ **[*NCERT Exemplar*]**

Sol. Let $\Delta = \begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

$= (x+1)\begin{vmatrix} x^2-x+1 & x-1 \\ 1 & 1 \end{vmatrix}$ [Taking common $(x+1)$ from R_2]

$= (x+1)\{x^2 - x + 1 - x + 1\} = (x+1)(x^2 - 2x + 2)$

$= x^3 - 2x^2 + 2x + x^2 - 2x + 2 = x^3 - x^2 + 2$

3. What is the value of the following determinant: **[*CBSE (F) 2010*]**

$$\begin{vmatrix} 4 & a & b+c \\ 4 & b & c+a \\ 4 & c & a+b \end{vmatrix}$$

Sol. Here, $\Delta = \begin{vmatrix} 4 & a & b+c \\ 4 & b & c+a \\ 4 & c & a+b \end{vmatrix} = \begin{vmatrix} 4 & a & a+b+c \\ 4 & b & a+b+c \\ 4 & c & a+b+c \end{vmatrix}$ [Applying $C_3 \to C_3 + C_2$]

$\Delta = 4(a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$ [Taking out common 4 from C_1 and $a+b+c$ from C_3]

$\Delta = 4(a+b+c) \,.\, 0 = 0$ $\left[\begin{array}{l} \because C_1 = C_3 \\ \therefore \Delta = 0 \end{array}\right]$

4. Write the value of $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$. **[*CBSE Allahabad 2015*]**

Sol. Here, $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2$, we get

$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$

Taking $(x+y+z)$ common from R_1, we get

$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$

Applying $R_3 \to R_3 + 3R_1$, we get

$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 0 & 0 & 0 \end{vmatrix} = 0$ [$\because$ Each element of R_3 is zero]

5. Without expanding evaluate the determinant: **[*HOTS*]**

$$\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (a^y+a^{-y})^2 & (a^y-a^{-y})^2 & 1 \\ (a^z+a^{-z})^2 & (a^z+a^{-z})^2 & 1 \end{vmatrix}, \text{ where } a > 0 \text{ and } x, y, z \in R$$

Sol. Here $\Delta = \begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (a^y + a^{-y})^2 & (a^y - a^{-y})^2 & 1 \\ (a^z + a^{-z})^2 & (a^z + a^{-z})^2 & 1 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} 4 & (a^x - a^{-x})^2 & 1 \\ 4 & (a^y - a^{-y})^2 & 1 \\ 4 & (a^z - a^{-z})^2 & 1 \end{vmatrix} \qquad [\text{Using } (a+b)^2 - (a-b)^2 = 4ab]$$

Taking out 4 from C_1, we get

$$\Delta = 4\begin{vmatrix} 1 & (a^x - a^{-x})^2 & 1 \\ 1 & (a^y - a^{-y})^2 & 1 \\ 1 & (a^z - a^{-z})^2 & 1 \end{vmatrix} \qquad \Rightarrow \qquad \Delta = 4 \times 0 = 0. \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

Long Answer Questions [5 marks]

1. If a, b, c are pth, qth and rth terms respectively of a G.P, then prove that

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$$

[*CBSE 2020 (65/5/1)*]

Sol. Let A be the first term and R be the common ratio of the G.P respectively.

$\therefore \quad a = AR^{p-1}, b = AR^{q-1}, c = AR^{r-1}$

Now, we have,

L.H.S. $\quad \Delta = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$

$$\Rightarrow \qquad \Delta = \begin{vmatrix} \log AR^{p-1} & p & 1 \\ \log AR^{q-1} & q & 1 \\ \log AR^{r-1} & r & 1 \end{vmatrix}$$

$$\Rightarrow \qquad \Delta = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

$\because \quad (\log(ab) = \log a + \log b \text{ and } \log a^m = m \log a)$

$$\Rightarrow \quad \Delta = \begin{vmatrix} \log A & p & 1 \\ \log A & q & 1 \\ \log A & r & 1 \end{vmatrix} + \log R \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}$$

apply $C_2 \to C_2 - C_1$

$$\Rightarrow \quad \Delta = \log A \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} + \log R \begin{vmatrix} p-1 & 1 & 1 \\ q-1 & 1 & 1 \\ r-1 & 1 & 1 \end{vmatrix}$$

$(C_1 = C_3) \qquad (C_2 = C_3)$

$\Delta = \log A \times 0 + \log \times 0$

$\therefore \quad \Delta = 0 + 0 = 0 = R.H.S$

$\therefore \quad \Delta = 0$

2. Using properties of determinants, find the value of x for which $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$.

[CBSE 2019(65/4/3)]

Sol. We have, $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 12+x & 4+x & 4+x \\ 12+x & 4-x & 4+x \\ 12+x & 4+x & 4-x \end{vmatrix} = 0$$

$$\Rightarrow \quad (12+x)\begin{vmatrix} 1 & 4+x & 4+x \\ 1 & 4-x & 4+x \\ 1 & 4+x & 4-x \end{vmatrix} = 0 \qquad [\text{ Taking } (12+x) \text{ common from } C_1 \,]$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$(12+x)\begin{vmatrix} 1 & 4+x & 4+x \\ 0 & -2x & 0 \\ 0 & 0 & -2x \end{vmatrix} = 0$$

$\Rightarrow (x+12)(4x^2) = 0 \quad \Rightarrow \quad x = 0, -12$

3. Using properties of determinants, prove that ***[CBSE Delhi 2014]***

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3.$$

Sol. LHS = $\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$

Applying $R_2 \leftrightarrow R_3$, then $R_1 \leftrightarrow R_2$, we get

$$= \begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} x+y+z & y+z+x & z+x+y \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Taking out $(x+y+z)$ from first row, we get

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$= (x+y+z)\begin{vmatrix} 0 & 0 & 1 \\ 0 & (y+z+x) & 2y \\ (x+y+z) & z+x+y & z-x-y \end{vmatrix}$$

Expanding along first row, we get

$$= (x + y + z)\,(x + y + z)^2 = (x + y + z)^3 = \text{RHS}$$

4. If x, y, z are different and $\Delta = \begin{vmatrix} x & x^2 & x^3-1 \\ y & y^2 & y^3-1 \\ z & z^2 & z^3-1 \end{vmatrix} = 0$, then using properties of determinants, show that $xyz = 1$. **[*CBSE 2019 (65/5/1)*]**

Sol. We have $\begin{vmatrix} x & x^2 & x^3-1 \\ y & y^2 & y^3-1 \\ z & z^2 & z^3-1 \end{vmatrix} = 0$ $\Rightarrow$ $\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} + \begin{vmatrix} x & x^2 & -1 \\ y & y^2 & -1 \\ z & z^2 & -1 \end{vmatrix} = 0$

$$\Rightarrow xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & y^2 \end{vmatrix} + \begin{vmatrix} -1 & x^2 & x \\ -1 & y^2 & y \\ -1 & z^2 & z \end{vmatrix} = 0$$

[In det 1 taking x, y, z common from each row and in det 2 using $C_1 \longleftrightarrow C_3$ and applying $C_2 \longleftrightarrow C_3$ in det 2]

$$\Rightarrow xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} -1 & x & x^2 \\ -1 & y & y^2 \\ -1 & z & z^2 \end{vmatrix} = 0$$

$$\Rightarrow xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \quad \text{[Taking (-1) common from } C_1\text{]}$$

$$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(xyz - 1) = 0$$

If $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = 0$ $\begin{bmatrix} \text{Applying} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$

$$\Rightarrow \begin{vmatrix} y-x & y^2-x^2 \\ z-x & z^2-x^2 \end{vmatrix} = 0 \quad \Rightarrow \quad (y-x)(z-x)\begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix} = 0$$

$\Rightarrow (y-x)(z-x)(z+x-y-x) = 0$

$\Rightarrow (y-x)(z-x)(z-y) = 0$

$\Rightarrow x = y$ or $z = x$ or $y = z$, which is a contradiction.

Hence, $(xyz - 1) = 0 \Rightarrow xyz = 1$.

5. Prove that: $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$ **[*CBSE Delhi 2014; (F) 2010, 2011*]**

Sol. LHS $= \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking $2(a+b+c)$ common from C_1, we get

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

Taking $(a+b+c)$ common from R_2 and R_3, we get

$$= 2(a+b+c)^3\begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$= 2(a+b+c)^3[1-0] = 2(a+b+c)^3 = \text{RHS}$$

6. If $\Delta = \begin{vmatrix} 1 & a & a^2 \\ a & a^2 & 1 \\ a^2 & 1 & a \end{vmatrix} = -4$ then find the value of $\begin{vmatrix} a^3-1 & 0 & a-a^4 \\ 0 & a-a^4 & a^3-1 \\ a-a^4 & a^3-1 & 0 \end{vmatrix}$. [*CBSE Sample Paper 2018*]

Sol. We have $\Delta = \begin{vmatrix} 1 & a & a^2 \\ a & a^2 & 1 \\ a^2 & 1 & a \end{vmatrix}$

$C_{11} = a^3 - 1$; $C_{12} = 0$; $C_{13} = a - a^4$

$C_{21} = 0$; $C_{22} = a - a^4$; $C_{23} = a^3 - 1$

$C_{31} = a - a^4$; $C_{32} = a^3 - 1$; $C_{33} = 0$

Where C_{ij} = Co-factor of a_{ij} $(i, j)^{\text{th}}$ element of determinant Δ. Let Δ_1 be the determinant made by corresponding co-factor of each element of determinant Δ.

i.e., $\Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$

We know that $\Delta_1 = \Delta^2$ [$\because$ $\Delta_1 = \Delta^{n-1}$ where each element of Δ_1 is cofactor of corresponding element of Δ and n is order of the determinant]

$\therefore$ $\Delta_1 = (-4)^2 = 16$

$$\Rightarrow \begin{vmatrix} a^3-1 & 0 & a-a^4 \\ 0 & a-a^4 & a^3-1 \\ a-a^4 & a^3-1 & 0 \end{vmatrix} = 16$$

7. Using property of determinant, prove the following:

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$$

[*CBSE Delhi 2017; (AI) 2008, 2013*]

Sol. LHS $= \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$

$$= \begin{vmatrix} 3(a+b) & 3(a+b) & 3(a+b) \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$ [Applying $R_1 = R_1 + R_2 + R_3$]

$$= 3(a+b)\begin{vmatrix} 1 & 1 & 1 \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$ [Taking $3(a + b)$ common from R_1]

$$= 3(a+b)\begin{vmatrix} 0 & 0 & 1 \\ b & -b & a+b \\ b & 2b & a \end{vmatrix}$$ [Applying $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$]

Expanding along R_1, we get

$= 3(a + b) \{1 (2b^2 + b^2)\} = 9b^2(a + b) = \text{RHS}$

8. Using properties of determinants, find the value of k if $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = k(x^3 + y^3)$. **[*CBSE 2019 (65/4/2)*]**

Sol. We have $\quad k(x^3 + y^3) = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

$$= \begin{vmatrix} 2x+2y & y & x+y \\ 2x+2y & x+y & x \\ 2x+2y & x & y \end{vmatrix}$$ [Using $C_1 \to C_1 + C_2 + C_3$]

` $$= (2x+2y)\begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix}$$ [Taking $(2x + 2y)$ common from C_1]

$$= 2(x+y)\begin{vmatrix} 1 & y & x+y \\ 0 & x & -y \\ 0 & x-y & -x \end{vmatrix}$$ [Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$]

$$= 2(x+y)\begin{vmatrix} x & -y \\ x-y & -x \end{vmatrix}$$

$= 2(x + y)(-x^2 + xy - y^2) = -2(x + y)(x^2 - xy + y^2)$

$\Rightarrow \quad k(x^3 + y^3) = -2(x^3 + y^3)$

Comparing the coefficient of $(x^3 + y^3)$ on both the sides, we get

$k = -2$

9. Using properties of determinants show that

$$\begin{vmatrix} 1 & 1 & 1+x \\ 1 & 1+y & 1 \\ 1+z & 1 & 1 \end{vmatrix} = -(xyz + yz + zx + xy)$$ **[*CBSE (F) 2017*]**

Sol. LHS $= \begin{vmatrix} 1 & 1 & 1+x \\ 1 & 1+y & 1 \\ 1+z & 1 & 1 \end{vmatrix}$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= \begin{vmatrix} 1 & 1 & 1+x \\ 0 & y & -x \\ z & 0 & -x \end{vmatrix}$$

Expanding by R_1, we get

$= 1[-yx - 0] - 1[+zx] + (1 + x)(-zy) = -yx - zx - zy - xyz$

$= -(xy + xz + zy + xyz) = \text{RHS}$

10. Using properties of determinant, prove that:

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z)$$ **[*CBSE (F) 2014*]**

Sol. LHS $= \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

$= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$= (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix}$ [Taking out $(a + x + y + z)$ common from C_1]

$= (a+x+y+z)\begin{vmatrix} 0 & -a & 0 \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix}$ [Apply $R_1 \to R_1 - R_2$]

Expanding along R_1, we get

$= (a + x + y + z)\{0 + a(a + z - z)\}$

$= a^2(a + x + y + z) = \text{RHS}$

11. Using properties of determinant, prove the following:

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$ **[*CBSE (AI) 2014, 2009*]**

Sol. LHS $= \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$

$= x^2\begin{vmatrix} x+y & 1 & 1 \\ 5x+4y & 4 & 2 \\ 10x+8y & 8 & 3 \end{vmatrix}$ [Taking out x from C_2 and C_3]

$= x^2\begin{vmatrix} x+y & 1 & 1 \\ 3x+2y & 2 & 0 \\ 7x+5y & 5 & 0 \end{vmatrix}$ [Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$]

Expanding along C_3, we get

$= x^2[1\{(3x + 2y)5 - 2(7x + 5y)\} - 0 + 0] = x^2(15x + 10y - 14x - 10y)$

$= x^2(x) = x^3 = \text{RHS}$

12. Without expanding, show that: $\begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0$ **[*NCERT Exemplar*]**

Sol. Given, $\Delta = \begin{vmatrix} \text{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \text{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$

$= \begin{vmatrix} \text{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \text{cosec}^2\theta + 1 & \text{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix}$ [Applying $C_1 \to C_1 - C_2 - C_3$]

$= \begin{vmatrix} 1-1 & \cot^2\theta & 1 \\ -1+1 & \text{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix}$ [$\because$ $\text{cosec}^2\theta - \cot^2\theta = 1$]

$= \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \text{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0$ [$\because$ All elements of C_1 are 0]

13. Prove the following using properties of determinant:

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(3abc - a^3 - b^3 - c^3)$$ **[*CBSE (F) 2010*]**

Sol. LHS $= \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$ [Taking $2(a+b+c)$ common from R_1]

$= 2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$ [Applying $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$]

$= 2(a+b+c)\,[1(bc - b^2 - c^2 + bc - bc + ac + ab - a^2)]$ [Expanding along R_1]

$= 2(a+b+c)\,(bc + ac + ab - a^2 - b^2 - c^2)$

$= -2(a+b+c)\,(a^2 + b^2 + c^2 - ab - bc - ca) = -2(a^3 + b^3 + c^3 - 3abc)$

$= 2(3abc - a^3 - b^3 - c^3) =$ RHS

14. If $a+b+c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then using properties of determinants, prove that $a = b = c$.

[*NCERT Exemplar, CBSE Bhubaneswar 2015*]

Sol. We have $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} (a+b+c) & b & c \\ (a+b+c) & c & a \\ (a+b+c) & a & b \end{vmatrix} = 0$$

Taking $(a + b + c)$ common from C_1, we get

$$(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0 \quad [\because\ a+b+c \neq 0]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = 0$$

Expanding along C_1, we get

$$1\{(c-b)(b-c) - (a-c)(a-b)\} - 0 + 0 = 0 \quad \Rightarrow \quad -(b-c)^2 - (a-c)(a-b) = 0$$

$$\Rightarrow -b^2 - c^2 + 2bc - a^2 + ab + ac - bc = 0 \quad \Rightarrow \quad a^2 + b^2 + c^2 - bc - ab - ac = 0$$

$$\Rightarrow \frac{1}{2}[2a^2 + 2b^2 + 2c^2 - 2bc - 2ab - 2ac] = 0 \Rightarrow \quad (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

$$\Rightarrow (a-b)^2 = 0;\ (b-c)^2 = 0;\ (c-a)^2 = 0 \quad \Rightarrow \quad a-b = 0;\ b-c = 0;\ c-a = 0;$$

$$\Rightarrow a = b = c$$

15. Using properties of determinants, show that ΔABC is an isosceles if:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$ **[*CBSE (Central) 2016, NCERT Exemplar, HOTS*]**

Sol. We have $\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1+\cos C \\ \cos^2 A + \cos A - \cos^2 C - \cos C & \cos^2 B + \cos B - \cos^2 C - \cos C & \cos^2 C + \cos C \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1+\cos C \\ (\cos A - \cos C)(\cos A + \cos C + 1) & (\cos B - \cos C)(\cos B + \cos C + 1) & \cos^2 C + \cos C \end{vmatrix} = 0$$

Taking common $(\cos A - \cos C)$ from C_1 and $(\cos B - \cos C)$ from C_2, we get

$$\Rightarrow \quad (\cos A - \cos C)(\cos B - \cos C)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\cos C \\ \cos A + \cos C + 1 & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_2$, we get

$$\Rightarrow (\cos A - \cos C)(\cos B - \cos C)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1+\cos C \\ \cos A - \cos B & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

Expanding along R_1, we get

$\Rightarrow (\cos A - \cos C)(\cos B - \cos C)(\cos B - \cos A) = 0$

$\Rightarrow \cos A - \cos C = 0$ *i.e.*, $\cos A = \cos C$

or, $\cos B - \cos C = 0$ *i.e.*, $\cos B = \cos C$

or, $\cos B - \cos A = 0$ *i.e.*, $\cos B = \cos A$

$\Rightarrow A = C$ or $B = C$ or $B = A$

Hence, ΔABC is an isosceles triangle.

16. Using properties of determinants, prove the following:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$ **[*CBSE Delhi 2011*]**

OR

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2$$ **[*CBSE (F) 2014*]**

Sol. LHS $= \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$ [Taking $(5x+4)$ common from R_1]

$= (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix}$ [Applying $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$]

$= (5x+4)\,[1\,\{(4-x)^2 - 0\} + 0 + 0]$ [Expanding along R_1]

$= (5x+4)\,(4-x)^2 =$ RHS

OR

Solve as above by putting λ instead of 4.

17. Using properties of determinants, prove that **[*CBSE Delhi 2012*]**

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

OR

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$ **[*CBSE (AI) 2014*]**

Sol. LHS $= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

$= \begin{vmatrix} a+b & p+q & x+y \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix}$ [Applying $R_1 \leftrightarrow R_3$ and $R_3 \leftrightarrow R_2$]

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix} = 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix}$$

$$= 2\begin{vmatrix} a & p & x \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix}$$ [Applying $R_1 \to R_1 - R_2$]

$$= 2\begin{vmatrix} a & p & x \\ b+c & q+r & y+z \\ c & r & z \end{vmatrix}$$ [Applying $R_3 \to R_3 - R_1$]

Again applying $R_2 \to R_2 - R_3$, we get

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{RHS}$$

18. Using properties of determinant, prove the following: **[*CBSE Delhi 2012; (AI) 2014*]**

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = ab + bc + ca + abc$$

OR

If a, b and c are all non-zero and $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0$**, then prove that** $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1 = 0$

[*CBSE (F) 2016*]

Sol. LHS $= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$$= abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$ [Taking out a, b, c common from R_1, R_2 and R_3]

$$= abc\begin{vmatrix} \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$, we get

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix} = abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)\times\{1(1-0)-0+0)\}$$

$$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right) = ab+bc+ca+abc \;= \text{RHS}$$

OR

$$\because \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0 \;\Rightarrow\; abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)=0 \;\Rightarrow\; \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1=0 \quad [a, b, c \text{ are non-zero}]$$

19. Using properties of determinants, show the following:

$$\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$$ **[*CBSE Delhi 2010*]**

Sol. LHS $= \begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix}$

Multiplying R_1, R_2 and R_3 by a, b and c respectively, we get

$$= \frac{1}{abc}\begin{vmatrix} a(b+c)^2 & ba^2 & a^2c \\ ab^2 & b(a+c)^2 & b^2c \\ ac^2 & bc^2 & c(a+b)^2 \end{vmatrix}$$

$$= \frac{1}{abc}abc\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$ [Taking common a, b and c from C_1, C_2 and C_3 respectively]

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$ [Taking common $(a+b+c)$ from C_1 and C_2]

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$ $[R_3 \to R_3-(R_1+R_2)]$

$$= \frac{(a+b+c)^2}{ab}\begin{vmatrix} ab+ac-a^2 & 0 & a^2 \\ 0 & bc+ba-b^2 & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix}$$ [Multiplying a in C_1 and b in C_2]

$$= \frac{(a+b+c)^2}{ab}\begin{vmatrix} ab+ac & a^2 & a^2 \\ b^2 & bc+ba & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$ $[C_1 \to C_1 + C_3 \text{ and } C_2 \to C_2 + C_3]$

$$= \frac{(a+b+c)^2}{ab} . ab . 2ab \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix}$$ $\left[\begin{matrix}\text{Taking } a, b \text{ and } 2ab \text{ common} \\ \text{from } R_1, R_2 \text{ and } R_3 \text{ respectively}\end{matrix}\right]$

$$= 2ab\,(a+b+c)^2 \begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix}$$

$$= 2ab\,(a+b+c)^2\,\{(b+c)\,(c+a) - ab\}$$

$$= 2abc\,(a+b+c)^3 = \text{RHS}$$

20. Using properties of determinant, show that: **[*CBSE (AI) 2012*]**

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc$$

Sol. LHS $= \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 2(b+c) & 2(c+a) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$$= 2\begin{vmatrix} (b+c) & (c+a) & (a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$ [Taking 2 common from R_1]

$$= 2\begin{vmatrix} (b+c) & (c+a) & (a+b) \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$ [Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$= 2\begin{vmatrix} 0 & c & b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$ [Applying $R_1 \to R_1 + R_2 + R_3$]

Expanding along R_1, we get

$$= 2\,[0 - c\,(0 - ab) + b\,(ac - 0)] = 2\,[abc + abc] = 4abc = \text{RHS}$$

21. Using properties of determinants, prove that **[*CBSE Ajmer 2015*]**

$$\begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix} = 2(a-b)\,(b-c)\,(c-a)\,(a+b+c).$$

Sol. LHS $= \begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix}$

$$= \begin{vmatrix} a^3 & 2 & a \\ b^3 - a^3 & 0 & b-a \\ c^3 - a^3 & 0 & c-a \end{vmatrix}$$ [Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$= (b-a)(c-a)\begin{vmatrix} a^3 & 2 & a \\ b^2+a^2+ab & 0 & 1 \\ c^2+a^2+ac & 0 & 1 \end{vmatrix}$$ $\begin{bmatrix}\text{Taking common } (b-a) \\ \text{from } R_2 \text{ and } (c-a) \text{ from } R_3\end{bmatrix}$

$$= (b-a)(c-a)\begin{vmatrix} a^3 & 2 & a \\ a^2+b^2+ab & 0 & 1 \\ c^2-b^2+ac-ab & 0 & 0 \end{vmatrix}$$ [Applying $R_3 \to R_3 - R_2$]

Expanding along R_3, we get

$= (b-a)(c-a)(c^2-b^2+ac-ab)2 \qquad = 2(b-a)(c-a)(c-b)(c+b+a)$

$= 2(a-b)(b-c)(c-a)(a+b+c) = \text{RHS}$

22. Using properties of determinant, prove that: **[*CBSE (F) 2012*]**

$$\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$$

Sol. LHS $= \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Expanding along C_1, we get

$= a[7a^2 + 3ab - 6a^2 - 3ab]$

$= a \times a^2 = a^3 = \text{RHS}$

23. Using properties of determinant, prove the following: **[*CBSE (F) 2015*]**

$$\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (1-a^3)^2$$

Sol. LHS $= \begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1+a^2+a & a+1+a^2 & a^2+a+1 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$$= (1+a+a^2)\begin{vmatrix} 1 & 1 & 1 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$$ [Taking out $(1+a+a^2)$ from first row]

$$= (1+a+a^2)\begin{vmatrix} 0 & 1 & 1 \\ a^2-1 & 1 & a \\ a-a^2 & a^2 & 1 \end{vmatrix}$$ [Applying $C_1 \to C_1 - C_2$]

$$= (1+a+a^2)\begin{vmatrix} 0 & 0 & 1 \\ a^2-1 & 1-a & a \\ a-a^2 & a^2-1 & 1 \end{vmatrix}$$ [Applying $C_2 \to C_2 - C_3$]

Expanding along R_1, we have

$= (1 + a + a^2) [(a^2 - 1)^2 - a (1 - a)^2] = (1 + a + a^2) [(a + 1)^2 (a - 1)^2 - a (a - 1)^2]$

$= (1 + a + a^2) (a - 1)^2 [a^2 + 1 + a] = (1 + a + a^2) (a - 1)^2 [a^2 + 1 + a]$

$= (a - 1)^2 (1 + a + a^2)^2 = (1 - a)^2 (1 + a + a^2)^2$

$= [(1 - a) (1 + a + a^2)]^2 = (1 - a^3)^2 = \text{RHS}$

24. Prove that $\begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - z^2 & zx - y^2 \end{vmatrix}$ **is divisible by $(x + y + z)$, and hence find the quotient.**

[*CBSE Delhi 2016*]

Sol. We have $\Delta = \begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} xy + yz + zx - x^2 - y^2 - z^2 & zx - y^2 & xy - z^2 \\ xy + yz + zx - x^2 - y^2 - z^2 & xy - z^2 & yz - x^2 \\ xy + yz + zx - x^2 - y^2 - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$$

Taking $(xy + yz + zx - x^2 - y^2 - z^2)$ common from C_1, we get

$$= (xy + yz + zx - x^2 - y^2 - z^2) \begin{vmatrix} 1 & zx - y^2 & xy - z^2 \\ 1 & xy - z^2 & yz - x^2 \\ 1 & yz - x^2 & zx - y^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= (xy + yz + zx - x^2 - y^2 - z^2) \begin{vmatrix} 1 & zx - y^2 & xy - z^2 \\ 0 & xy - z^2 - zx + y^2 & yz - x^2 - xy + z^2 \\ 0 & yz - x^2 - zx + y^2 & zx - y^2 - xy + z^2 \end{vmatrix}$$

$$= (xy + yz + zx - x^2 - y^2 - z^2) \begin{vmatrix} 1 & zx - y^2 & xy - z^2 \\ 0 & x(y - z) + (y^2 - z^2) & y(z - x) + (z^2 - x^2) \\ 0 & z(y - x) + (y^2 - x^2) & x(z - y) + (z^2 - y^2) \end{vmatrix}$$

$$= (xy + yz + zx - x^2 - y^2 - z^2) \begin{vmatrix} 1 & zx - y^2 & xy - z^2 \\ 0 & (y - z) \cdot (x + y + z) & (z - x) \cdot (x + y + z) \\ 0 & (y - x) \cdot (x + y + z) & (z - y) \cdot (x + y + z) \end{vmatrix}$$

Taking $(x + y + z)$ common from R_2 and R_3, we get

$$= (xy + yz + zx - x^2 - y^2 - z^2)(x + y + z)^2 \begin{vmatrix} 1 & zx - y^2 & xy - z^2 \\ 0 & y - z & z - x \\ 0 & y - x & z - y \end{vmatrix}$$

$= (xy + yz + zx - x^2 - y^2 - z^2) (x + y + z)^2 \{1. (yz - y^2 - z^2 + zy - yz + xy + xz - x^2)\}$

$= (xy + yz + zx - x^2 - y^2 - z^2)^2 (x + y + z)^2$

Hence, $\begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - z^2 & zx - y^2 \end{vmatrix}$ is divisible by $(x + y + z)$

and quotient is $(xy + yz + zx - x^2 - y^2 - z^2)^2 (x + y + z)$.

25. If *a, b, c* are real numbers, then prove that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

where ω is a complex number and cube root of unity. **[*HOTS*]**

Sol. LHS $= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$ [Taking out $(a + b + c)$ from C_1]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$ [Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix}$ [Expanding along C_1]

$= (a + b + c) \{-(b - c)^2 - (a - c)(a - b)\}$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ and

RHS $= -(a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$= -(a + b + c)(a^2 + ab\omega^2 + ac\omega + ab\omega + b^2\omega^3 + bc\omega^2 + ac\omega^2 + bc\omega^4 + c^2\omega^3)$

$= -(a + b + c)[(a^2 + b^2 + c^2 + ab(\omega^2 + \omega) + bc(\omega^2 + \omega^4) + ca(\omega + \omega^2)]$ [$\because \omega^3 = 1$]

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) =$ LHS [$\because \omega^2 + \omega + 1 = 0$ and $\omega^4 = \omega^3 . \omega = \omega$]

26. Let $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$**, then find** $\lim_{t \to 0} \frac{f(t)}{t^2}$. **[*NCERT Exemplar, HOTS*]**

Sol. Given, $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix} = \begin{vmatrix} \cos t & t & 1 \\ 0 & -t & 0 \\ \sin t & t & t \end{vmatrix}$ [Applying $R_2 \to R_2 - 2R_3$]

$= t\begin{vmatrix} \cos t & 1 & 1 \\ 0 & -1 & 0 \\ \sin t & 1 & t \end{vmatrix}$

Expanding along R_2, we get

$t[(-1)(t \cos t - \sin t)] = -t^2 \cos t + t \sin t$

$\therefore \lim_{t \to 0} \frac{f(t)}{t^2} = \lim_{t \to 0} \frac{-t^2 \cos t + t \sin t}{t^2} = \lim_{t \to 0}\left(\frac{-t^2 \cos t}{t^2} + \frac{t \sin t}{t^2}\right)$

$= \lim_{t \to 0}\left(-\cos t + \frac{\sin t}{t}\right) = -1 + \lim_{t \to 0} \frac{\sin t}{t} = -1 + 1 = 0$

PROFICIENCY EXERCISE

■ **Objective Type Questions** **[1 mark each]**

1. Choose and write the correct option in each of the following questions.

(*i*) The maximum value of $\Delta = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{bmatrix}$ is (θ is real number)

(*a*) $\frac{1}{2}$ (*b*) $\frac{\sqrt{3}}{2}$ (*c*) $\sqrt{2}$ (*d*) $\frac{2\sqrt{3}}{4}$

(*ii*) The value of $\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ is

(*a*) 0 (*b*) 5^2 (*c*) 5^9 (*d*) 5^{13}

(*iii*) Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, when $0 \le \theta \le 2\pi$. Then

(*a*) Det $(A) = 0$ (*b*) Det $(A) \in (2, \infty)$ (*c*) Det $(A) \in (2, 4)$ (*d*) Det $(A) \in [2, 4]$

(*iv*) If $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$, then the value x is

(*a*) 0 (*b*) ± 2 (*c*) 2 (*d*) -2

(*v*) If $f(x) = \begin{vmatrix} 0 & a-x & x-b \\ x+a & 0 & b-x \\ x+b & x+c & 0 \end{vmatrix}$, then

(*a*) $f(a) = 0$ (*b*) $f(b) = 0$ (*c*) $f(0) = 0$ (*d*) $f(1) = 0$

(*vi*) If $A + B + C = \pi$, then the value of $\begin{vmatrix} \sin(A+B+C) & \sin(A+C) & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & \tan(B+C) & 0 \end{vmatrix}$ is equal to

(*a*) 0 (*b*) 1 (*c*) $2\tan A \sin B \cos C$ (*d*) none of these

(*vii*) The determinant $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ equals

(*a*) $abc\,(b-c)\,(c-b)\,(a-b)$ (*b*) $(b-c)\,(c-b)\,(a-b)$

(*c*) $(a+b+c)\,(b-c)\,(c-a)\,(a-b)$ (*d*) None of these

2. Fill in the blanks.

(*i*) If $\begin{vmatrix} 2x & -9 \\ -2 & x \end{vmatrix} = \begin{vmatrix} -4 & 8 \\ 1 & -2 \end{vmatrix}$, then value of x is ______________ . [CBSE (2020) 65/2/2]

(*ii*) $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} =$ ______________ .

(*iii*) If A and B are square matrices of order 3 and $|A| = 5$, $|B| = 3$, then the value of $|3AB|$ is ______________ . [CBSE (2020) 65/5/3]

(*iv*) The value of $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} =$ ______________ .

■ **Very Short Answer Questions:** **[1 mark each]**

3. For what value of x, the following matrix is singular? [CBSE Delhi 2011]

$$\begin{vmatrix} 5-x & x+1 \\ 2 & 4 \end{vmatrix} = 0$$

4. Write the value of the following determinant: $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ [CBSE Delhi 2009]

5. If A_{ij} is the cofactor of the element a_{ij} of the determinant

$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32} \cdot A_{32}$. [CBSE (AI) 2013]

6. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, then find the value of x. [CBSE (AI) 2014]

7. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then for any natural number n, find the value of det (A^n).
[CBSE Ajmer 2015]

8. If A is a square matrix of order 3 and $|3A| = k|A|$, then write the value of k. [CBSE Delhi 2010]

9. If $A = [a_{ij}]$ is a matrix of order 2×2, such that $|A| = -15$ and C_{ij} represents the cofactor of a_{ij}, then find $a_{21}C_{21} + a_{22}C_{22}$. [CBSE Sample Paper 2018]

10. Find the cofactors of all the elements of $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$. [CBSE 2020, (65/5/3)]

■ **Short Answer Questions–I and II:** **[2, 3 marks each]**

11. Using the properties of determinant, evaluate $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$ [NCERT Exemplar]

12. Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$, using properties of determinant.

13. Find the equation of line Joining (3, 1) and (9, 3) using determinant.

14. Using co-factors of elements of third column,

evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

■ **Long Answer Questions:** **[5 marks each]**

15. Using properties of determinant, solve for x:

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$ [CBSE (AI) 2011; (East) 2016]

16. In a triangle ABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0$$

then prove that ΔABC is an isosceles triangle. [NCERT Exemplar, HOTS]

17. Using properties of determinant, prove the following:

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$$ [CBSE Delhi 2011]

18. Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ is divisible by $(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$ and find the quotient.

19. If $\boldsymbol{a} \neq \boldsymbol{b} \neq \boldsymbol{c}$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then using properties of determinants, prove that $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=0$. *[CBSE Chennai 2015]*

20. Find the equation of the line joining $\boldsymbol{A}$ (1, 3) and $\boldsymbol{B}$ (0, 0) using determinants and find $\boldsymbol{k}$ if $\boldsymbol{D}$ $(\boldsymbol{k}, 0)$ is a point such that the area of $\Delta \boldsymbol{ABD}$ is 3 sq units.

21. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz + xy + yz + zx$$ *[CBSE (AI) 2009]*

22. Show that: $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3(x+y+z)(xy+yz+xz)$ *[CBSE (AI) 2013]*

23. Using properties of determinants, prove that :

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx)$$ *[CBSE Examination Paper 2018]*

24. Using properties of determinant, prove that: $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$ *[CBSE (AI) 2017]*

25. If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, using properties of determinants, find the value of $f(2x) - f(x)$. *[CBSE Delhi 2015]*

26. Using the properties of determinants, solve the following for x: $\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$ *[CBSE Panchkula 2015]*

27. Using the properties of determinants, prove the following:

$$\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(1-x) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix} = 6x^2(1-x^2)$$ *[CBSE Patna 2015]*

28. If x, y, z are in GP, then using properties of determinants, show that

$\begin{vmatrix} px+y & x & y \\ py+z & y & z \\ 0 & px+y & py+z \end{vmatrix} = 0$, where $x \neq y \neq z$ and p is any real number. *[CBSE Sample Paper 2015]*

29. Using properties of determinants, prove that $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 3 & 4+3p & 2+4p+3q \\ 4 & 7+4p & 2+7p+4q \end{vmatrix} = 1$. *[CBSE Sample Paper 2016]*

30. Without expanding the determinant at any stage, prove that $\begin{vmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{vmatrix} = 0$. *[CBSE Sample Paper 2016]*

31. Using properties of determinants, prove that:

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$ [CBSE 2020 (65/1/1), (South) 2016]

32. Using properties of determinants, prove the following:

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$$ [CBSE (AI) 2013]

33. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$ [CBSE 2019 (65/3/3)]

Answers

1. (*i*) (*a*) (*ii*) (*a*) (*iii*) (*d*) (*iv*) (*b*) (*v*) (*c*) (*vi*) (*a*) (*vii*) (*d*)

2. (*i*) ± 3 (*ii*) $(y-z)(z-x)(y-x+xyz)$ (*iii*) 405 (*iv*) $a^2+b^2+c^2+d^2$

3. $x = 3$ **4.** 0 **5.** 110 **6.** $x = -2$ **7.** $|A^n| = 1$ **8.** $k = 27$

9. –15 **10.** Cofactors of all the elements of given matrix are as follows:

$C_{11} = 3, C_{21} = 2$

$C_{12} = -4, C_{22} = 1$ where C_{ij} is the co-factors of ith row and jth co-factors

11. $a^2(a+x+y+z)$ **13.** $x - 3y = 0$ **14.** $(x - y)(y - z)(z - x)$ **15.** $x = 0, 3a$

18. $(3abc - a^3 - b^3 - c^3)$ **20.** $3x - y = 0; k = \pm 2$ **25.** $ax(2a + 3x)$ **26.** $x = -\frac{7}{3}$

SELF-ASSESSMENT TEST

Time allowed: 1 hour **Max. marks: 30**

1. **Choose and write the correct option in the following questions.** **(4 × 1 = 4)**

(*i*) If $x, y \in R$, then the determinant $\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ \cos(x+y) & -\sin(x+y) & 0 \end{vmatrix}$ lies in the interval

(*a*) $[-\sqrt{2}, \sqrt{2}]$ (*b*) $[-1, 1]$ (*c*) $[-\sqrt{2}, 1]$ (*d*) $[-1, -\sqrt{2}]$

(*ii*) The value of $\begin{vmatrix} \sin 10° & -\cos 10° \\ \sin 80° & \cos 80° \end{vmatrix}$ is

(*a*) 0 (*b*) 1 (*c*) –1 (*d*) $\frac{1}{2}$

(*iii*) The value(s) of k if area of triangle with vertices (–2, 0), (0, 4) and (0, k) is 4 sq. units is

(*a*) 0, 4 (*b*) – 8 (*c*) 0, 8 (*d*) 0 only

(*iv*) The value of x for which the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular, is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

2. **Fill in the blanks.** **(2 × 1 = 2)**

(*i*) If $A = \begin{bmatrix} 0 & i \\ i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then the value of $|A| + |B|$ is ____________ .

(*ii*) If A be a matrix of order 3 × 3 and $|A| = 10$, then the value of $|4A| =$ ____________ .

■ **Solve the following questions.** **(2 × 1 = 2)**

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then find the value of k if $|2A| = k|A|$.

4. If $\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, then write the positive value of x.

■ **Solve the following questions.** **(4 × 2 = 8)**

5. Using co-factors of elements of second row, evaluate

$$\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

6. Find the area of the triangle whose vertices area (3, 8), (–4, 2) and (5, 1).

7. Find the cofactors of all the elements of $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$.

8. Using co-factors of elements of third column, evaluate

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

■ **Solve the following questions.** **(3 × 3 = 9)**

9. Prove that: $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$

10. By using properties of determinant, prove the following:

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2$$

11. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

■ **Solve the following question.** **(1 × 5 = 5)**

12. If x, y, z are in GP, then using properties of determinants, show that

$\begin{vmatrix} px+y & x & y \\ py+z & y & z \\ 0 & px+y & py+z \end{vmatrix} = 0$, where $x \neq y \neq z$ and p is any real number. [*CBSE Sample Paper 2015*]

Answers

1. (*i*) (*a*) (*ii*) (*b*) (*iii*) (*c*) (*iv*) (*d*) **2.** (*i*) 0 (*ii*) 640

3. 4 **4.** –1, 2 **5.** 7 **6.** $\frac{61}{2}$ sq. units

7. Cofactors of all the elements of given matrix are as follows:

$C_{11} = 3, C_{21} = 2, C_{12} = -4, C_{22} = 1$ where C_{ij} is the co-factors of ith row and jth co-factors

8. $(x - y)(y - z)(z - x)$