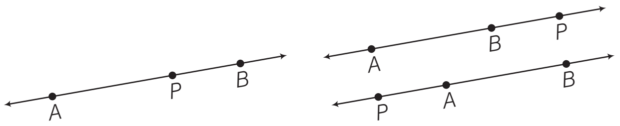
Session 3

Section Formula, Centroid of a Triangle, Incentre, Some Standard Results, Area of Triangle

Section Formula

Definition : If P be any point on the line AB between A and B then we say that P divides segment AB *internally* in the ratio $AP : PB$.



Also, if P be any point on the line AB but not between A and B (P may be to the right or the left of the points A, B) then P divides AB *externally* in the ratio $AP : PB$

Note

$\frac{AP}{PB} = \begin{cases} \text{Positive, in internally division} \\ \text{Negative, in externally division} \end{cases}$

(i) Formula for Internal Division

Theorem : If the point $P(x, y)$ divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m : n$, then prove that

$$x = \frac{mx_2 + nx_1}{m + n}$$

$$y = \frac{my_2 + ny_1}{m + n}$$

Proof : The given points are $A(x_1, y_1)$ and $B(x_2, y_2)$. Let us assume that the points A and B are both in 1st quadrant (for the sake of exactness). Since $P(x, y)$ divides AB internally in the ratio $m : n$ i.e. $AP : PB = m : n$. From A, B and P draw AL, BM and PN perpendiculars to X-axis. From A and P draw AH and PJ perpendiculars to PN and BM respectively, then

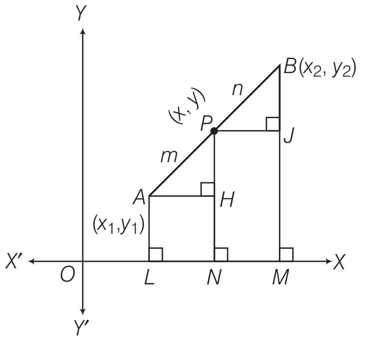
$OL = x_1, ON = x, OM = x_2, AL = y_1, PN = y$ and $BM = y_2$

$\therefore \quad AH = LN = ON - OL = x - x_1$

$PJ = NM = OM - ON = x_2 - x$

$PH = PN - HN = PN - AL = y - y_1$

and $\quad BJ = BM - JM = BM - PN = y_2 - y$



Clearly, the Δs AHP and PJB are similar and therefore, their sides are proportional

$$\therefore \quad \frac{AH}{PJ} = \frac{PH}{BJ} = \frac{AP}{PB}$$

$$\text{or} \quad \underset{\text{(i)}}{\frac{x - x_1}{x_2 - x}} = \underset{\text{(ii)}}{\frac{y - y_1}{y_2 - y}} = \underset{\text{(iii)}}{\frac{m}{n}}$$

From Eqs. (i) and (iii), we have

$$\frac{x - x_1}{x_2 - x} = \frac{m}{n}$$

$$\Rightarrow \quad nx - nx_1 = mx_2 - mx$$

$$\Rightarrow \quad (m + n)\, x = mx_2 + nx_1$$

$$\therefore \quad x = \frac{mx_2 + nx_1}{m + n}$$

and from Eqs. (ii) and (iii), we have

$$\frac{y - y_1}{y_2 - y} = \frac{m}{n}$$

$$\Rightarrow \quad ny - ny_1 = my_2 - my$$

$$\Rightarrow \quad (m + n)\, y = my_2 + ny_1$$

$$\therefore \quad y = \frac{my_2 + ny_1}{m + n}$$

Thus, the coordinates of P are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right).$$

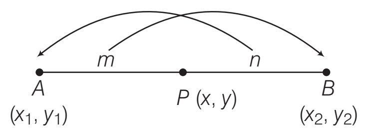
***Corollary* 1 :** The above section formula is true for all positions of the points (i.e. either point or both points are not in the 1st quadrant), keeping in mind, the proper signs of their coordinates.

***Corollary* 2 :** If P is the mid-point of AB then $m = n$, the coordinates of the middle-point of AB are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

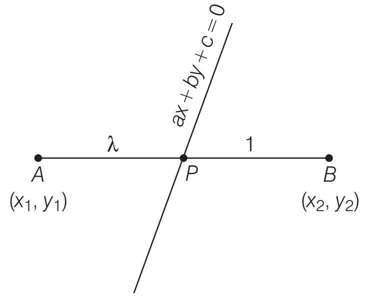
Remarks

1. If $P(\alpha, \beta)$ be the mid-point of AB and if coordinates of A are (λ, μ) then the coordinates of B are $(2\alpha - \lambda, 2\beta - \mu)$, i.e. (Double the x-co-ordinate of mid point – x-coordinate of given point, Double the y-co-ordinate of mid point – y-coordinate of given point).
2. The following diagram will help to remember the section formula.



3. For finding ratio, use ratio $\lambda : 1$, then coordinates of P are $\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}\right)$. If λ is positive then divides internally and if λ is negative, then divides externally.
4. The straight line $ax + by + c = 0$ divides the joint of points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio

$$\frac{AP}{PB} = \frac{\lambda}{1} = -\frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$$



If ratio is positive, then divides internally and if ratio is negative then divides externally.

Proof : Coordinates of P are $\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}\right)$

$\because P$ lies on the line $ax + by + c = 0$, then

$$a\left(\frac{x_1 + \lambda x_2}{1 + \lambda}\right) + b\left(\frac{y_1 + \lambda y_2}{1 + \lambda}\right) + c = 0$$

or $\quad (ax_1 + by_1 + c) + \lambda(ax_2 + by_2 + c) = 0$

or $\quad \frac{\lambda}{1} = -\frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$

5. The line joining the points (x_1, y_1) and (x_2, y_2) is divided by the X-axis in the ratio $-\frac{y_1}{y_2}$ and by Y-axis in the ratio $-\frac{x_1}{x_2}$.
6. In square, rhombus, rectangle and parallelogram diagonals bisect to each other.

Example 20. Find the coordinates of the point which divides the line segment joining the points $(5, -2)$ and $(9, 6)$ in the ratio $3 : 1$.

Sol. Let the required point be (x, y), then

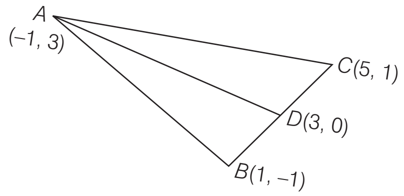
$$x = \left(\frac{3 \times 9 + 1 \times 5}{3 + 1}\right) = 8$$

and $$y = \left(\frac{3 \times 6 + 1 \times (-2)}{3 + 1}\right) = 4$$

Thus, the coordinates of the required point are (8, 4).

Example 21. Find the length of median through A of a triangle whose vertices are $A(-1, 3), B(1, -1)$ and $C(5, 1)$.

Sol. Let D be the mid-point of BC, then coordinates of D are $\left(\frac{1 + 5}{2}, \frac{-1 + 1}{2}\right)$ i.e. $(3, 0)$



$$\therefore \quad \text{Median } AD = \sqrt{(3 + 1)^2 + (0 - 3)^2}$$
$$= \sqrt{16 + 9} = \sqrt{25}$$
$$= 5 \text{ units}$$

Example 22. Determine the ratio in which $y - x + 2 = 0$ divides the line joining $(3, -1)$ and $(8, 9)$.

Sol. Suppose the line $y - x + 2 = 0$ divides the line segment joining $A(3, -1)$ and $B(8, 9)$ in the ratio $\lambda : 1$ at point P, then the coordinates of the point P are $\left(\frac{8\lambda + 3}{\lambda + 1}, \frac{9\lambda - 1}{\lambda + 1}\right)$.

But P lies on $y - x + 2 = 0$ therefore

$$\left(\frac{9\lambda - 1}{\lambda + 1}\right) - \left(\frac{8\lambda + 3}{\lambda + 1}\right) + 2 = 0$$

$$\Rightarrow \quad 9\lambda - 1 - 8\lambda - 3 + 2\lambda + 2 = 0$$

$$\Rightarrow \quad 3\lambda - 2 = 0 \quad \text{or} \quad \lambda = \frac{2}{3}$$

So, the required ratio is $\frac{2}{3} : 1$, i.e. 2 : 3 (internally) since here λ is positive.

Shortcut method

According to Remark 4 :

$$\lambda = -\left(\frac{-1 - 3 + 2}{9 - 8 + 2}\right) = \frac{2}{3}$$

or $\quad \lambda : 1 = 2 : 3$

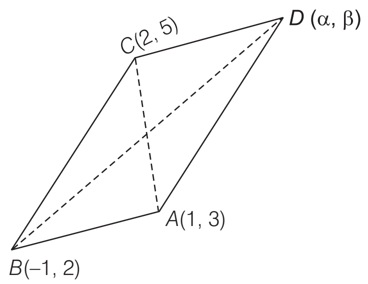
Example 23. The coordinates of three consecutive vertices of a parallelogram are $(1, 3), (-1, 2)$ and $(2, 5)$. Then find the coordinates of the fourth vertex.

Sol. Let the fourth vertex be $D(\alpha, \beta)$. Since $ABCD$ is a parallelogram, the diagonals bisect to each other.

i.e. mid-point of BD = mid-point of AC

$$\therefore \quad \left(\frac{\alpha - 1}{2}, \frac{\beta + 2}{2}\right) = \left(\frac{2+1}{2}, \frac{5+3}{2}\right)$$

$$\text{or} \quad \left(\frac{\alpha - 1}{2}, \frac{\beta + 2}{2}\right) = \left(\frac{3}{2}, 4\right)$$



On equating abscissaes and ordinates, we get

$$\frac{\alpha - 1}{2} = \frac{3}{2} \quad \text{or} \quad \alpha - 1 = 3 \quad \text{or} \quad \alpha = 4$$

$$\text{and} \quad \frac{\beta + 2}{2} = 4 \quad \text{or} \quad \beta + 2 = 8 \quad \text{or} \quad \beta = 6$$

Hence, the coordinates of the fourth vertex $D(\alpha, \beta)$ is $(4, 6)$.

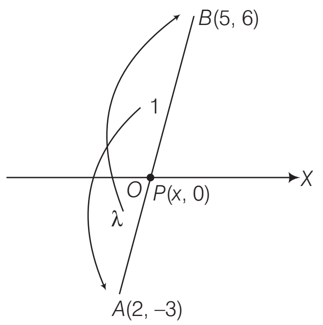
Example 24. In what ratio does X-axis divide the line segment joining $(2, -3)$ and $(5, 6)$?

Sol. Let the given points be $A(2, -3)$ and $B(5, 6)$. Let AB be divided by the X-axis at $P(x, 0)$ in the ratio $\lambda : 1$ internally. Considering the ordinate of P, then

$$0 = \frac{\lambda \times 6 + 1 \times (-3)}{\lambda + 1}$$

$$\text{or} \quad \lambda = \frac{1}{2}$$

$\therefore$ The ratio is $\frac{1}{2} : 1$ i.e. $1 : 2$ (Internally)



Shortcut Method

According to Remark 5 :

$$\frac{\lambda}{1} = -\frac{y_1}{y_2} = \frac{-(-3)}{6} = \frac{1}{2}$$

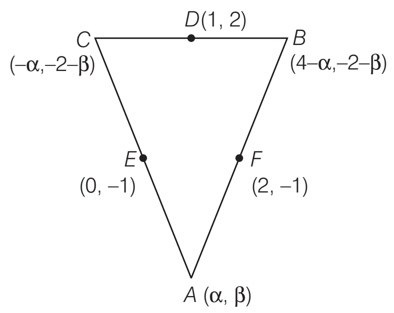
$\therefore$ The ratio is $\frac{1}{2} : 1$ i.e. $1 : 2$ (internally)

Example 25. The mid-points of the sides of a triangle are $(1, 2), (0, -1)$ and $(2, -1)$. Find the coordinates of the vertices of a triangle with the help of two unknowns.

Sol. Let $D(1, 2)$, $E(0, -1)$ and $F(2, -1)$ be the mid-points of BC, CA and AB respectively.

Let the coordinates of A be (α, β) then coordinates of B and C are $(4 - \alpha, -2 - \beta)$ and $(-\alpha, -2 - \beta)$ respectively (see note 1)

$\because$ D is the mid-point of B and C



$$\text{then} \quad 1 = \frac{4 - \alpha - \alpha}{2}$$

$$\Rightarrow \quad 1 = 2 - \alpha \quad \text{or} \quad \alpha = 1$$

$$\text{and} \quad 2 = \frac{-2 - \beta - 2 - \beta}{2}$$

$$\Rightarrow \quad 2 = -2 - \beta \quad \text{or} \quad \beta = -4$$

Hence, coordinates of A, B and C are $(1, -4)$, $(3, 2)$ and $(-1, 2)$ respectively.

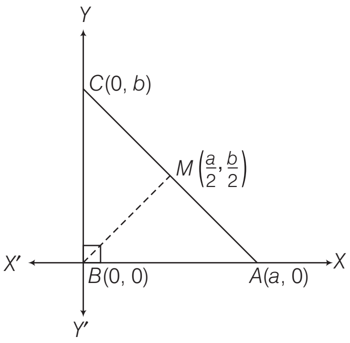
Example 26. Prove that in a right angled triangle the mid-point of the hypotenuse is equidistant from its vertices.

Sol. Let the given right angled triangle be ABC, with right angled at B. We take B as the origin and BA and BC as the X and Y-axes respectively.

Let $BA = a$ and $BC = b$

then $A \equiv (a, 0)$ and $C \equiv (0, b)$

Let M to be the mid-point of the hypotenuse AC, then coordinates of M are $\left(\frac{a}{2}, \frac{b}{2}\right)$



$$\therefore \quad |AM| = \sqrt{\left(a - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2} = \frac{\sqrt{(a^2 + b^2)}}{2} \quad \text{... (i)}$$

$$|BM| = \sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2} = \frac{\sqrt{(a^2 + b^2)}}{2} \quad \text{...(ii)}$$

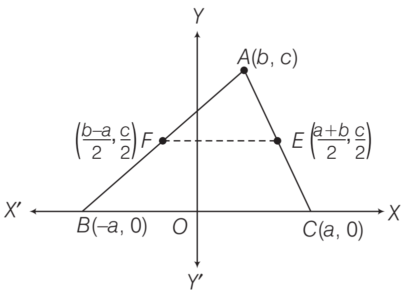
$$\text{and} \quad |CM| = \sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(b - \frac{b}{2}\right)^2} = \frac{\sqrt{(a^2 + b^2)}}{2} \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$|AM| = |BM| = |CM|$$

Example 27 Show that the line joining the mid-points of any two sides of a triangle is half the third side.

Sol. We take O as the origin and OC and OY as the X and Y-axes respectively.



Let $BC = 2a$, then $B \equiv (-a, 0)$, $C \equiv (a, 0)$

Let $A \equiv (b, c)$, if E and F are the mid-points of sides AC and AB respectively.

Then, $E \equiv \left(\frac{a+b}{2}, \frac{c}{2}\right)$ and $F \equiv \left(\frac{b-a}{2}, \frac{c}{2}\right)$

Now, $$FE = \sqrt{\left(\frac{a+b}{2} - \frac{b-a}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2} = a$$

$$= \frac{1}{2}(2a) = \frac{1}{2}(BG)$$

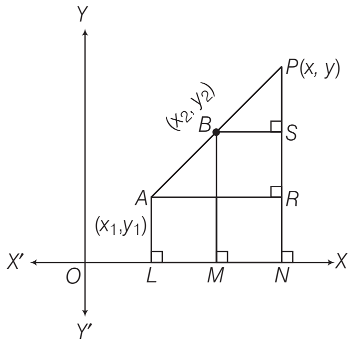
Hence, the line joining the mid-points of any two sides of a triangle is half the third side.

(ii) Formula for External Division

Theorem : If the point $P(x, y)$ divides the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m : n$ then prove that

$$x = \frac{mx_2 - nx_1}{m - n}, \; y = \frac{my_2 - ny_1}{m - n}$$

Proof : The given points are $A(x_1, y_1)$ and $B(x_2, y_2)$. Let us assume that the points A and B are both in the 1st quadrant (for the sake of exactness). Let $P(x, y)$ be the point which divides AB externally in the ratio $m : n$, so that $\frac{AP}{BP} = \frac{m}{n}$.



From A, B and P draw AL, BM and PN perpendiculars on X-axis. Also, from A and B draw AR and BS perpendiculars on PN,

then $$AR = LN = ON - OL = x - x_1$$

$$BS = MN = ON - OM = x - x_2$$

$$PR = PN - RN = PN - AL = y - y_1$$

and $$PS = PN - SN = PN - BM = y - y_2$$

Clearly, the Δs APR and BPS are similar and therefore their sides are proportional.

$$\therefore \quad \frac{AP}{PB} = \frac{AR}{BS} = \frac{PR}{PS}$$

$$\text{or} \quad \underset{\text{(i)}}{\frac{m}{n}} = \underset{\text{(ii)}}{\frac{x - x_1}{x - x_2}} = \underset{\text{(iii)}}{\frac{y - y_1}{y - y_2}}$$

From Eqs. (i) and (ii), we have

$$\frac{m}{n} = \frac{x - x_1}{x - x_2}$$

$$\Rightarrow \quad mx - mx_2 = nx - nx_1$$

$$\Rightarrow \quad (m - n)x = mx_2 - nx_1$$

$$\text{or} \quad x = \frac{mx_2 - nx_1}{m - n}$$

Also, from Eqs. (i) and (iii), we have

$$\frac{m}{n} = \frac{y - y_1}{y - y_2}$$

$$\Rightarrow \quad my - my_2 = ny - ny_1$$

$$\Rightarrow \quad (m - n)\, y = my_2 - ny_1$$

or
$$y = \frac{my_2 - ny_1}{m - n}$$

Thus, the coordinates of P are $\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right)$.

(Here, $m \neq n$)

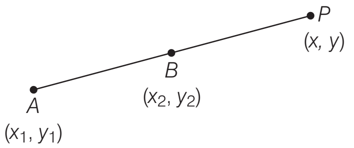
***Corollary* 1 :** The above formula is true for all positions of the points, keeping in mind, the proper signs of their coordinates.

***Corollary* 2 :** The above coordinates can also be expressed as

$$\left(\frac{mx_2 + (-n)\, x_1}{m + (-n)}, \frac{my_2 + (-n)\, y_1}{m + (-n)}\right)$$

and this can be thought of as the coordinates of the point dividing AB internally in the ratio $m : -n$

***Corollary* 3 :** $\because \quad \frac{AP}{PB} = \frac{m}{n}$



or
$$\frac{AP}{PB} - 1 = \frac{m}{n} - 1$$

or
$$\frac{AP - PB}{PB} = \frac{m - n}{n}$$

or
$$\frac{AB}{PB} = \frac{m - n}{n}$$

Now, we can say that B divides AP in the ratio $m - n : n$ internally.

i.e. $\quad x_2 = \frac{(m - n)\, x + nx_1}{(m - n) + n} \quad \Rightarrow \quad x = \frac{mx_2 - nx_1}{m - n}$

and $\quad y_2 = \frac{(m - n)\, y + ny_1}{(m - n) + n} \quad \Rightarrow \quad y = \frac{my_2 - ny_1}{m - n}$

***Corollary* 4 :** (for proving A, B and C are collinear)

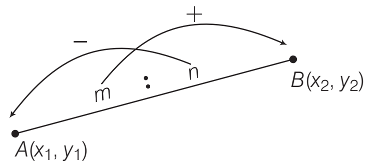
If A, B, C three points are collinear then let C divides AB in the ratio $\lambda : 1$ internally.

If $\lambda = +$ ve rational, then divide internally

and if $\lambda = -$ ve rational, then divide externally.

Remarks

1. The following diagram will help to remember the section formula



2. Let $\frac{m}{n} = \lambda$, then $\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right)$

or $\left(\frac{\frac{m}{n}x_2 - x_1}{\frac{m}{n} - 1}, \frac{\frac{m}{n}\, y_2 - y_1}{\frac{m}{n} - 1}\right)$ or $\left(\frac{\lambda x_2 - x_1}{\lambda - 1}, \frac{\lambda y_2 - y_1}{\lambda - 1}\right)$

Example 28. Find the coordinates of a point which divides externally the line joining $(1, -3)$ and $(-3, 9)$ in the ratio 1 : 3.

Sol. Let the coordinates of the required point be $P\,(x, y)$,

Then, $\quad x = \left(\frac{1 \times (-3) - 3 \times 1}{1 - 3}\right)$

and $\quad y = \left(\frac{1 \times 9 - 3 \times (-3)}{1 - 3}\right)$

i.e. $\quad x = 3 \quad$ and $\quad y = -9$

Hence, the required point is $(3, -9)$.

Example 29. The line segment joining $A\,(6, 3)$ to $B\,(-1, -4)$ is doubled in length by having its length added to each end. Find the coordinates of the new ends.

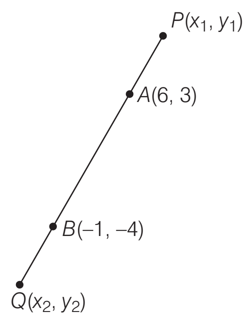
Sol. Let P and Q be the required new ends

Let the coordinates of P be (x_1, y_1)

Given, $\quad AB = 2AP$

$$\Rightarrow \quad \frac{AB}{AP} = \frac{2}{1}$$

i.e. A divides BP internally in the ratio 2 : 1.



Then, $\quad 6 = \frac{2 \times x_1 + 1 \times (-1)}{2 + 1}$

$\Rightarrow \qquad 19 = 2x_1$ or $x_1 = \dfrac{19}{2}$

and $\qquad 3 = \dfrac{2 \times y_1 + 1 \times (-4)}{2+1}$

$\Rightarrow \qquad 13 = 2y_1$ or $y_1 = \dfrac{13}{2}$

$\therefore$ Coordinates of P are $\left(\dfrac{19}{2}, \dfrac{13}{2}\right)$.

Also, let coordinates of Q be (x_2, y_2)

Given, $\qquad AB = 2BQ \Rightarrow \dfrac{AB}{BQ} = \dfrac{2}{1}$

i.e. B divides AQ internally in the ratio 2 : 1

Then $\qquad -1 = \dfrac{2 \times x_2 + 1 \times 6}{2+1}$

$\Rightarrow \qquad -9 = 2x_2$ or $x_2 = -\dfrac{9}{2}$

and $\qquad -4 = \dfrac{2 \times y_2 + 1 \times 3}{2+1}$

$\Rightarrow \qquad -15 = 2y_2$ or $y_2 = -\dfrac{15}{2}$

$\therefore$ Coordinates of Q are $\left(-\dfrac{9}{2}, -\dfrac{15}{2}\right)$

Aliter : $\because \qquad AB = 2AP$

$\Rightarrow \qquad \dfrac{AB}{AP} = \dfrac{2}{1} \Rightarrow \dfrac{AB}{AP} + 1 = \dfrac{2}{1} + 1$

$\Rightarrow \qquad \dfrac{AB + AP}{AP} = \dfrac{3}{1} \Rightarrow \dfrac{BP}{AP} = \dfrac{3}{1}$

$\therefore$ P divides AB externally in the ratio 1 : 3

Then, $\qquad x_1 = \dfrac{1 \times (-1) - 3 \times 6}{1-3} = \dfrac{19}{2}$

and $\qquad y_1 = \dfrac{1 \times (-4) - 3 \times 3}{1-3} = \dfrac{13}{2}$

$\therefore$ Coordinates of P are $\left(\dfrac{19}{2}, \dfrac{13}{2}\right)$

Also, $\qquad AB = 2BQ$

$\Rightarrow \qquad \dfrac{AB}{BQ} = \dfrac{2}{1} \Rightarrow \dfrac{AB}{BQ} + 1 = \dfrac{2}{1} + 1$

$\Rightarrow \qquad \dfrac{AB + BQ}{BQ} = \dfrac{3}{1} \Rightarrow \dfrac{AQ}{BQ} = \dfrac{3}{1}$

$\therefore$ Q divides AB externally in the ratio 3 : 1

then, $\qquad x_2 = \dfrac{3 \times (-1) - 1 \times 6}{3-1} = -\dfrac{9}{2}$

and $\qquad y_2 = \dfrac{3 \times (-4) - 1 \times 3}{3-1} = -\dfrac{15}{2}$

$\therefore$ Coordinates of Q are $\left(-\dfrac{9}{2}, -\dfrac{15}{2}\right)$.

Example 30. Using section formula show that the points $(1, -1)$, $(2, 1)$ and $(4, 5)$ are collinear.

Sol. Let $A \equiv (1, -1)$, $B \equiv (2, 1)$ and $C \equiv (4, 5)$

Suppose C divides AB in the ratio $\lambda : 1$ internally, then

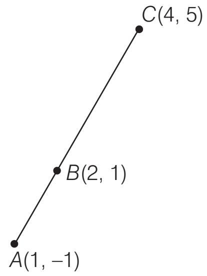
$$4 = \frac{\lambda \times 2 + 1 \times 1}{\lambda + 1}$$

$\Rightarrow \qquad 4\lambda + 4 = 2\lambda + 1$

or $\qquad \lambda = -\dfrac{3}{2}$

i.e. C divides AB in the ratio 3 : 2 (externally).

Hence, A, B, C are collinear.



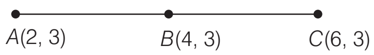
Example 31. Find the ratio in which the point $(2, y)$ divides the line segment joining $(4, 3)$ and $(6, 3)$ and hence find the value of y.

Sol. Let $\quad A \equiv (4, 3)$, $B \equiv (6, 3)$ $\quad$ and $\quad P \equiv (2, y)$

Let P divides AB internally in the ratio $\lambda : 1$

then, $\qquad 2 = \dfrac{6\lambda + 4}{\lambda + 1} \Rightarrow 2\lambda + 2 = 6\lambda + 4$

$\Rightarrow \qquad -4\lambda = 2$ or $\lambda = -\dfrac{1}{2}$



$\therefore$ P divides AB externally in the ratio 1 : 2 ($\because \lambda$ is negative)

Now, $\quad y = \dfrac{1 \times 3 - 2 \times 3}{1-2} = 3$

(iii) Harmonic Conjugates

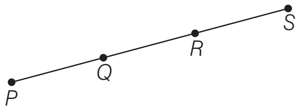
If four points in a line, then the system is said to form a range. Let four points say P, Q, R, S.

If the range (PQ, RS) has a cross ratio equal to -1, then it is called harmonic.

i.e. $\qquad \dfrac{PR}{RQ} \cdot \dfrac{SQ}{SP} = -1 \Rightarrow \dfrac{PR}{RQ} = -\dfrac{SP}{SQ} = \lambda \qquad$ (say)

$\therefore \qquad \dfrac{PR}{RQ} = \dfrac{\lambda}{1} \Rightarrow PR : RQ = \lambda : 1 \qquad$ (internally)

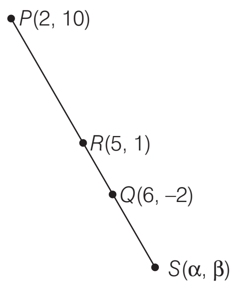
and $\qquad \dfrac{SP}{SQ} = -\dfrac{\lambda}{1} \Rightarrow PS : SQ = \lambda : 1 \qquad$ (externally)



Hence, R and S are called the **harmonic conjugates** to each other with respect to the points P and Q.

Example 32 Find the harmonic conjugates of the point $R(5,1)$ with respect to the points $P(2,10)$ and $Q(6,-2)$.

Sol. Let $S(\alpha, \beta)$ (be the harmonic conjugates of the point $R(5, 1)$). Suppose R divides PQ in the ratio $\lambda : 1$ internally, then S divides PQ in the ratio $\lambda : 1$ externally, then



$$5 = \frac{6\lambda + 2}{\lambda + 1} \Rightarrow 5\lambda + 5 = 6\lambda + 2$$

$\therefore \quad \lambda = 3$

Also, $\quad 1 = \frac{-2\lambda + 10}{\lambda + 1}$

$$\lambda + 1 = -2\lambda + 10 \Rightarrow 3\lambda = 9$$

$\therefore \quad \lambda = 3$

Now, $\quad \alpha = \frac{3 \times 6 - 1 \times 2}{3 - 1} = 8$

and $\quad \beta = \frac{3 \times (-2) - 1 \times 10}{3 - 1}$

$$= -8$$

Hence, harmonic conjugates of $R(5, 1)$ is $S(8, -8)$.

Centroid of a Triangle

Definition : The point of intersection of the medians of a triangle is called the centroid of the triangle and it divides the median internally in the ratio 2 : 1.

Theorem : Prove that the coordinates of the centroid of the triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

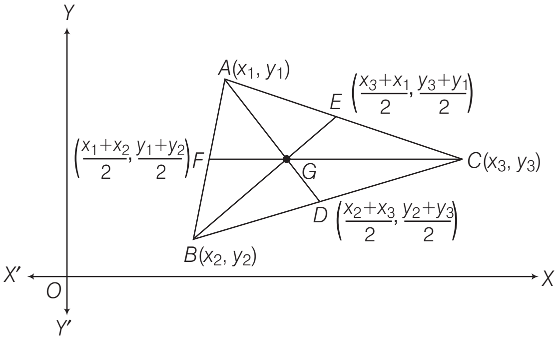
Also, deduce that the medians of a triangle are concurrent.

Proof : Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ be the vertices of the triangle ABC. Let us assume that the points A, B and C are in the 1st quadrant (for the sake of exactness) whose medians are AD, BE and CF respectively so D, E and F are respectively the mid-points of BC, CA and AB then the coordinates of D, E, F are

$$D \equiv \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

$$E \equiv \left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$$

and $\quad F \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$



The coordinates of a point dividing AD in the ratio $2:1$ are

$$\left(\frac{2\cdot\left(\frac{x_2 + x_3}{2}\right) + 1\cdot x_1}{2+1}, \frac{2\cdot\left(\frac{y_2 + y_3}{2}\right) + 1\cdot y_1}{2+1}\right)$$

or $\quad \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

and the coordinates of a point dividing BE in the ratio 2 : 1 are

$$\left(\frac{2\cdot\left(\frac{x_3 + x_1}{2}\right) + 1\cdot x_2}{2+1}, \frac{2\cdot\left(\frac{y_3 + y_1}{2}\right) + 1\cdot y_2}{2+1}\right)$$

or $\quad \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

Similarly the coordinates of a point dividing CF in the ratio $2:1$ are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

$\therefore$ The common point which divides AD, BE and CF in the ratio $2:1$ is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Hence, medians of a triangle are concurrent and the coordinates of the centroid are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Important Theorem

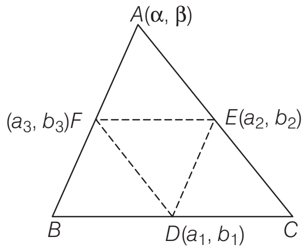
Centroid of the triangle obtained by joining the middle points of the sides of a triangle is the same as the centroid of the original triangle.

Or

If $(a_1, b_1), (a_2, b_2)$ and (a_3, b_3) are the mid-points of the sides of a triangle, then its centroid is given by

$$\left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right)$$

Proof : Let D, E, F are the mid-points of BC, CA and AB respectively now let coordinates of A are (α, β) then coordinates of B and C are $(2a_3 - \alpha, 2b_3 - \beta)$ and $(2a_2 - \alpha, 2b_2 - \beta)$ are respectively.



$\because$ $D(a_1, b_1)$ is the mid-point of B and C, then

$$2a_1 = 2a_3 - \alpha + 2a_2 - \alpha \Rightarrow \alpha = a_2 + a_3 - a_1$$

and $$2b_1 = 2b_3 - \beta + 2b_2 - \beta \Rightarrow \beta = b_2 + b_3 - b_1$$

Now, coordinates of B are $(2a_3 - \alpha, 2b_3 - \beta)$

or $$(a_3 + a_1 - a_2, b_3 + b_1 - b_2)$$

and coordinates of C are $(2a_2 - \alpha, 2b_2 - \beta)$

or $$(a_2 + a_1 - a_3, b_2 + b_1 - b_3)$$

Hence, coordinates of A, B and C are

$$A \equiv (a_2 + a_3 - a_1, b_2 + b_3 - b_1),$$
$$B \equiv (a_3 + a_1 - a_2, b_3 + b_1 - b_2)$$
and $$C \equiv (a_2 + a_1 - a_3, b_2 + b_1 - b_3)$$

$\therefore$ Coordinates of centroid of triangle ABC are

$$\left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right)$$

which is same as the centroid of triangle DEF.

Corollary 1 (Finger Rule) : If mid-points of the sides of a triangle are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), then coordinates of the original triangle are

$$(x_2 + x_3 - x_1, y_2 + y_3 - y_1),$$
$$(x_3 + x_1 - x_2, y_3 + y_1 - y_2)$$
and $$(x_1 + x_2 - x_3, y_1 + y_2 - y_3).$$

***Corollary* 2 :** If two vertices of a triangle are (x_1, y_1) and (x_2, y_2) and the coordinates of centroid are (α, β), then coordinates of the third vertex are

$$(3\alpha - x_1 - x_2, 3\beta - y_1 - y_2)$$

***Corollary* 3 :** According to important theorem $\Delta s\ ABC$ and DEF are similar

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{(BC)^2}{(EF)^2}$$

$$= \frac{4\{(a_2 - a_3)^2 + (b_2 - b_3)^2\}}{\{(a_2 - a_3)^2 + (b_2 - b_3)^2\}} = 4$$

$\therefore$ Area of $\Delta\ ABC = 4 \times$ Area of ΔDEF

i.e. Area of a triangle is four times the area of the triangle formed by joining the mid-points of its sides.

Example 33. Two vertices of a triangle are $(-1, 4)$ and $(5, 2)$. If its centroid is $(0, -3)$, find the third vertex.

Sol. Let the third vertex be (x, y) then the coordinates of the centroid of triangle are

$$\left(\frac{-1 + 5 + x}{3}, \frac{4 + 2 + y}{3}\right) \text{ i.e. } \left(\frac{4 + x}{3}, \frac{6 + y}{3}\right)$$

Now, $$\left(\frac{4 + x}{3}, \frac{6 + y}{3}\right) = (0, -3)$$

$$\Rightarrow \quad \frac{4 + x}{3} = 0 \quad \text{and} \quad \frac{6 + y}{3} = -3$$

$$\Rightarrow \quad 4 + x = 0 \quad \text{and} \quad y + 6 = -9$$

or $$x = -4 \quad \text{and} \quad y = -15$$

Hence, the third vertex is $(-4, -15)$.

Shortcut Method

According to corollary 2

$$(x, y) = (3 \times 0 - (-1) - 5, 3 \times (-3) - 4 - 2)$$
$$= (-4, -15)$$

Example 34. ***The vertices of a triangle are*** $(1, 2), (h, -3)$ and $(-4, k)$. ***Find the value of*** $\sqrt{\{(h + k)^2 + (h + 3k)^2\}}$. ***If the centroid of the triangle be at the point*** $(5, -1)$.

Sol. Here, $\frac{1 + h - 4}{3} = 5$ and $\frac{2 - 3 + k}{3} = -1$

then, we get $h = 18, k = -2$

$$\therefore \quad = \sqrt{(h + k)^2 + (h + 3k)^2}$$
$$= \sqrt{(18 - 2)^2 + (18 - 6)^2}$$
$$= \sqrt{(16^2 + 12^2)} = 20$$

Example 35. If $D(-2, 3)$, $E(4, -3)$ and $F(4, 5)$ are the mid-points of the sides BC, CA and AB of triangle ABC, then find $\sqrt{(|AG|^2 + |BG|^2 - |CG|^2)}$ where, G is the centroid of ΔABC.

Sol. Let the coordinates of A be (α, β)

then coordinates of B are $(8-\alpha, 10-\beta)$

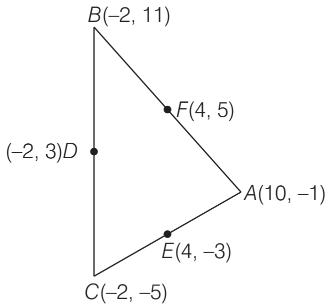
and coordinates of C are $(8-\alpha, -6-\beta)$

$\because$ D is the mid-point of BC, then

$$\frac{8-\alpha+8-\alpha}{2} = -2$$

and $$\frac{10-\beta-6-\beta}{2} = 3$$

i.e. $\alpha = 10$ and $\beta = -1$



$\therefore$ Coordinates of A, B, C are $(10, -1)$, $(-2, 11)$ and $(-2, -5)$ respectively.

Now, coordinates of centroid

$$G \equiv \left(\frac{10-2-2}{3}, \frac{-1+11-5}{3}\right)$$

i.e. $$G \equiv \left(2, \frac{5}{3}\right)$$

$\therefore$ $$AG = \sqrt{(10-2)^2 + \left(-1-\frac{5}{3}\right)^2}$$

$$= \sqrt{\left(64 + \frac{64}{9}\right)} = \frac{8}{3}\sqrt{(10)}$$

$$BG = \sqrt{(-2-2)^2 + \left(11-\frac{5}{3}\right)^2}$$

$$= \sqrt{16 + \frac{(28)^2}{9}} = \frac{4}{3}\sqrt{(58)}$$

and $$CG = \sqrt{(-2-2)^2 + \left(-5-\frac{5}{3}\right)^2}$$

$$= \sqrt{\left(16 + \frac{400}{9}\right)} = \frac{4}{3}\sqrt{(34)}$$

Hence, $\sqrt{(|AG|^2 + |BG|^2 - |CG|^2)}$

$$= \sqrt{\left(\frac{64}{9}\times 10 + \frac{16}{9}\times 58 - \frac{16}{9}\times 34\right)}$$

$$= \sqrt{\frac{32}{9}(20 + 29 - 17)}$$

$$= \sqrt{\left(\frac{32}{9}\times 32\right)} = \frac{32}{3}$$

Example 36. If G be the centroid of the ΔABC and O be any other point in the plane of the triangle ABC, then show that
$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$.

Sol. Let G be the origin and GO be X-axis.

$O \equiv (a, 0)$, $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$

and $C \equiv (x_3, y_3)$

Now, LHS $= OA^2 + OB^2 + OC^2$

$$= (x_1 - a)^2 + y_1^2 + (x_2 - a)^2 + y_2^2 + (x_3 - a)^2 + y_3^2$$

$$= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2) - 2a(x_1 + x_2 + x_3) + 3a^2$$

$$= \sum x_1^2 + \sum y_1^2 - 0 + 3a^2 \qquad \left(\because \frac{x_1 + x_2 + x_3}{3} = 0 \text{ i.e., } x_1 + x_2 + x_3 = 0\right)$$

$$= \sum x_1^2 + \sum y_1^2 + 3a^2 \qquad \text{...(i)}$$

and RHS $= GA^2 + GB^2 + GC^2 + 3GO^2$

$$= x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 + 3\{(a-0)^2\}$$

$$= \sum x_1^2 + \sum y_1^2 + 3a^2 \qquad \text{... (ii)}$$

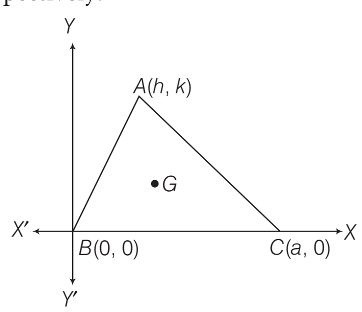
Hence, from Eqs. (i) and (ii), we get

$$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2.$$

Example 37. If G be the centroid of ΔABC, show that

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

Sol. We take B as the origin and BC and BY as the X and Y-axes respectively.



Let $BC = a$, then $B \equiv (0, 0)$ and $C \equiv (a, 0)$
and let $A \equiv (h, k)$
then, coordinates of G will be

$$\left(\frac{h+0+a}{3}, \frac{k+0+0}{3}\right), \text{ i.e. } \left(\frac{h+a}{3}, \frac{k}{3}\right)$$

Take ΔABC as in 1st quadrant (for the sake of exactness).

Now, $\text{LHS} = (AB)^2 + (BC)^2 + (CA)^2$

$$= (h-0)^2 + (k-0)^2 + a^2 + (h-a)^2 + (k-0)^2$$

$$= 2h^2 + 2k^2 - 2ah + 2a^2 \quad \text{... (i)}$$

$$\text{RHS} = 3((GA)^2 + (GB)^2 + (GC)^2)$$

$$= 3\left\{\left(\frac{a+h}{3} - h\right)^2 + \left(\frac{k}{3} - k\right)^2 + \left(\frac{a+h}{3} - 0\right)^2 + \left(\frac{k}{3} - 0\right)^2 + \left(\frac{a+h}{3} - a\right)^2 + \left(\frac{k}{3} - 0\right)^2\right\}$$

$$= \frac{3}{9}\{(a-2h)^2 + (-2k)^2 + (a+h)^2 + k^2 + (h-2a)^2 + k^2\}$$

$$= \frac{1}{3}\{6a^2 + 6h^2 + 6k^2 - 6ah\}$$

$$= 2h^2 + 2k^2 - 2ah + 2a^2 \quad \text{... (ii)}$$

Hence, from Eqs. (i) and (ii), we get

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$$

Example 38. The vertices of a triangle are $(1, a), (2, b)$ and $(c^2, -3)$

(i) Prove that its centroid can not lie on the Y-axis.
(ii) Find the condition that the centroid may lie on the X-axis.

Sol. Centroid of the triangle is

$$G \equiv \left(\frac{1+2+c^2}{3}, \frac{a+b-3}{3}\right) \text{ i.e. } \left(\frac{3+c^2}{3}, \frac{a+b-3}{3}\right)$$

(i) $\because$ G will lie on Y-axis, then

$$\frac{3+c^2}{3} = 0$$

$\Rightarrow$ $c^2 = -3$

or $c = \pm i\sqrt{3}$

$\because$ Both values of c are imaginary.
Hence, G can not lie on Y-axis.

(ii) $\because$ G will lie on X-axis, then $\dfrac{a+b-3}{3} = 0$

$\Rightarrow$ $a + b - 3 = 0$

or $a + b = 3$

Incentre

Definition : The point of intersection of internal angle bisectors of triangle is called the incentre of the triangle.

Theorem : Prove that the coordinates of the incentre of a triangle whose vertices are $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ are

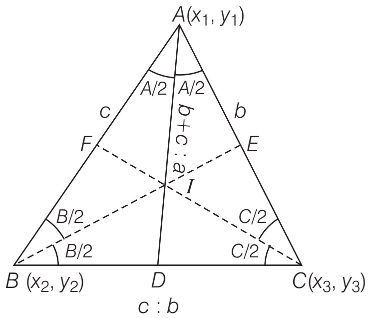
$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

where, a, b, c are the lengths of sides BC, CA and AB respectively.

Also, prove that the internal bisectors of the angles of a triangle are concurrent.

Proof : Given $A \equiv (x_1, y_1), B(x_2, y_2), C \equiv (x_3, y_3)$ be the vertices of ΔABC and $BC = a, CA = b$ and $AB = c$. Let AD be the bisector of A. We know that the bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the triangle.

$$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b} \quad \text{... (i)}$$



Hence, D divides BC in the ratio $c : b$

$\therefore$ Coordinates of D are $\left(\dfrac{cx_3 + bx_2}{c+b}, \dfrac{cy_3 + by_2}{c+b}\right)$

From Eq. (i), $\dfrac{DC}{BD} = \dfrac{b}{c}$ or $\dfrac{DC}{BD} + 1 = \dfrac{b}{c} + 1$

or $\dfrac{DC + BD}{BD} = \left(\dfrac{b+c}{c}\right)$ or $\dfrac{a}{BD} = \left(\dfrac{b+c}{c}\right)$

$$\therefore \quad BD = \frac{ac}{(b+c)}$$

Also, in ΔABD, BI is the bisector of B.

Then, $\dfrac{AI}{ID} = \dfrac{AB}{BD} = \dfrac{c}{\left(\dfrac{ac}{b+c}\right)} = \dfrac{b+c}{a}$

$\therefore$ I divides AD in the ratio $b+c:a$

$\therefore$ Coordinates of I are

$$\left(\frac{(b+c)\cdot\frac{cx_3+bx_2}{c+b}+a\cdot x_1}{b+c+a}, \frac{(b+c)\cdot\frac{cy_3+by_2}{c+b}+b\cdot y_1}{b+c+a}\right)$$

i.e. $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

Similarly we can show that the coordinates of the point which divides BE internally in the ratio $c+a:b$ and the coordinates of the point which divides CF internally in the ratio $a+b:c$ will be each

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

and $CE=\frac{ab}{(c+a)}, AE=\frac{bc}{(c+a)},$

$$AF=\frac{bc}{a+b}, BF=\frac{ac}{a+b}$$

Thus, the three internal bisectors of the angles of a triangle meet in a point I.

$$I \equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

***Corollary* 1 :** If ΔABC is equilateral, then $a=b=c$

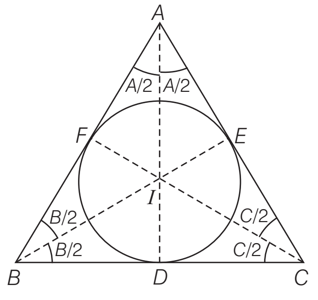
incentre $=\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)=$ centroid

i.e. incentre and centroid coincide in equilateral, triangle.

***Corollary* 2 :** $AE=AF=s-a$

$$BD=BF=s-b$$

$$CD=CE=s-c$$



where, $s=\frac{a+b+c}{2}$

and $|BC|=a, |CA|=b, |AB|=c$

Proof : Let $AE=AF=\alpha$

($\because$ Lengths of tangents are equal from a point to a circle)

$$BD=BF=\beta$$

$$CD=CE=\gamma$$

Also, $a=BC=BD+DC=\beta+\gamma$ ....(i)

$b=CA=CE+AE=\gamma+\alpha$ ...(ii)

and $c=AB=AF+BF=\alpha+\beta$ ...(iii)

Adding all, we get

$$a+b+c=2(\alpha+\beta+\gamma)$$

or $2s=2(\alpha+\beta+\gamma)$

$\therefore$ $s=\alpha+\beta+\gamma$

From Eqs. (i), (ii) and (iii), we get

$$\alpha=s-a, \beta=s-b, \gamma=s-c$$

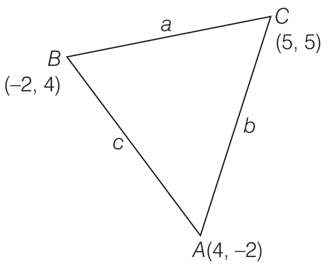
Example 39. Find the coordinates of incentre of the triangle whose vertices are $(4,-2)$, $(-2,4)$ and $(5,5)$.

Sol. Let $A(4,-2)$, $B(-2,4)$ and $C(5,5)$ be the vertices of the given triangle. Then

$$a=BC=\sqrt{(-2-5)^2+(4-5)^2}=\sqrt{50}=5\sqrt{2}$$

$$b=CA=\sqrt{(5-4)^2+(5+2)^2}=\sqrt{50}=5\sqrt{2}$$

and $c=AB=\sqrt{(4+2)^2+(-2-4)^2}=\sqrt{72}=6\sqrt{2}$



Let (x, y) be the coordinates of incentre of ΔABC. Then

$$x=\frac{ax_1+bx_2+cx_3}{a+b+c}$$

$$=\frac{5\sqrt{2}\times 4+5\sqrt{2}\times(-2)+6\sqrt{2}\times 5}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}$$

$$=\frac{20\sqrt{2}-10\sqrt{2}+30\sqrt{2}}{16\sqrt{2}}$$

$$=\frac{40}{16}=\frac{5}{2}$$

and $y=\frac{ay_1+by_2+cy_3}{a+b+c}$

$$=\frac{5\sqrt{2}\times(-2)+5\sqrt{2}\times 4+6\sqrt{2}\times 5}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}$$

$$=\frac{40}{16}=\frac{5}{2}$$

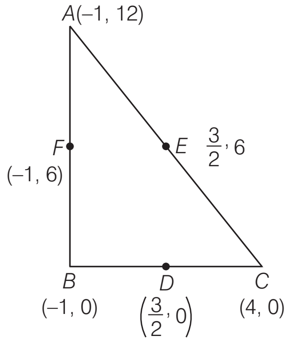
$\therefore$ The coordinates of the incentre are $\left(\frac{5}{2}, \frac{5}{2}\right)$

Example 40. ***If*** $\left(\frac{3}{2}, 0\right), \left(\frac{3}{2}, 6\right)$ ***and*** $(-1, 6)$ ***are mid-points of the sides of a triangle, then find***

(i) Centroid of the triangle

(ii) Incentre of the triangle

Sol. Let $A \equiv (\alpha, \beta)$, then coordinates of $B \equiv (-2-\alpha, 12-\beta)$ and coordinates of $C \equiv (3-\alpha, 12-\beta)$. But mid-point of BC is $\left(\frac{3}{2}, 0\right)$



then $\quad 3 = -2 - \alpha + 3 - \alpha$

$\Rightarrow \quad \alpha = -1$

and $\quad 0 = 12 - \beta + 12 - \beta$

$\Rightarrow \quad \beta = 12$

$\therefore$ Coordinates of vertices are

$A \equiv (-1, 12), B \equiv (-1, 0) \quad$ and $\quad C \equiv (4, 0)$

(i) Centroid : The centroid of ΔABC is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

or $$\left(\frac{-1-1+4}{3}, \frac{12+0+0}{3}\right) \text{ i.e. } \left(\frac{2}{3}, 4\right)$$

(ii) Incentre : We have

$$a = BC = \sqrt{(-1-4)^2 + (0-0)^2} = 5$$

$$b = CA = \sqrt{(4+1)^2 + (0-12)^2} = 13$$

and $$c = AB = \sqrt{(-1+1)^2 + (12-0)^2} = 12$$

$\therefore$ The incentre of ΔABC is

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

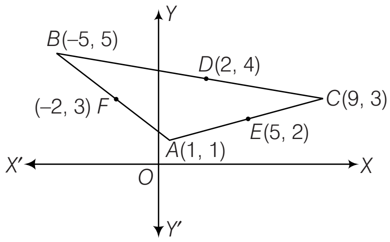
or $$\left(\frac{5 \times (-1) + 13 \times (-1) + 12 \times 4}{5 + 13 + 12}, \frac{5 \times 12 + 13 \times 0 + 12 \times 0}{5 + 13 + 12}\right)$$

i.e. $\qquad (1, 2)$

Example 41. If a vertex of a triangle be (1, 1) and the middle points of two sides through it be $(-2, 3)$ and $(5, 2)$, then find the centroid and the incentre of the triangle.

Sol. Let coordinate of A be (1, 1) and mid-points of AB and AC are F and E.

$\therefore \qquad F \equiv (-2, 3)$ and $E \equiv (5, 2)$



Hence, coordinates of B and C are $(2 \times (-2) - 1, (2 \times 3 - 1)$ and $(2 \times 2 - (-5), 2 \times 4 - 5)$ respectively.

i.e. $\quad B \equiv (-5, 5)$ and $C \equiv (9, 3)$

Then, centroid is $\left(\frac{1-5+9}{3}, \frac{1+5+3}{3}\right)$ i.e., $\left(\frac{5}{3}, 3\right)$

Also, $\quad a = |BC| = \sqrt{(-5-9)^2 + (5-3)^2} = \sqrt{200} = 10\sqrt{2}$

$\qquad b = |CA| = \sqrt{(9-1)^2 + (3-1)^2} = \sqrt{68} = 2\sqrt{17}$

and $\quad c = |AB| = \sqrt{(1+5)^2 + (1-5)^2} = \sqrt{52} = 2\sqrt{13}$

Then, incentre is

$$\left(\frac{10\sqrt{2} \times 1 + 2\sqrt{17} \times (-5) + 2\sqrt{13} \times 9}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}}, \frac{10\sqrt{2} \times 1 + 2\sqrt{17} \times 5 + 2\sqrt{13} \times 3}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}}\right)$$

i.e. $$\left(\frac{5\sqrt{2} - 5\sqrt{17} + 9\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}, \frac{5\sqrt{2} + 5\sqrt{17} + 3\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}\right)$$

Example 42. If G be the centroid and I be the incentre of the triangle with vertices $A(-36, 7), B(20, 7)$ and $C(0, -8)$ and $GI = \frac{25}{3}\sqrt{(205)}\,\lambda$, then find the value of λ.

Sol. Coordinates of centroid are

$$G \equiv \left(-\frac{16}{3}, 2\right)$$

and $\quad a = |BC| = \sqrt{(20-0)^2 + (7+8)^2}$

$\qquad = \sqrt{625} = 25$

$\qquad b = |CA| = \sqrt{(0+36)^2 + (-8-7)^2}$

$\qquad = \sqrt{1521} = 39$

$\qquad c = |AB| = \sqrt{(-36-20)^2 + (7-7)^2}$

$\qquad = \sqrt{(56)^2} = 56$

Therefore, the coordinates of incentre are

$$I \equiv \left(\frac{25 \times (-36) + 39 \times 20 + 56 \times 0}{25 + 39 + 56}, \frac{25 \times 7 + 39 \times 7 + 56 \times (-8)}{25 + 39 + 56}\right)$$

i.e., $I \equiv (-1, 0)$

$$\therefore \quad GI = \sqrt{\left(-\frac{16}{3}+1\right)^2 + (2-0)^2} = \frac{\sqrt{(205)}}{3}$$

but given $GI = \dfrac{25}{3}\sqrt{(205)}\,\lambda$

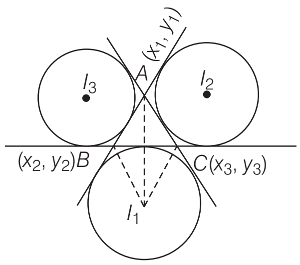
$$\therefore \quad \frac{1}{3}\sqrt{(205)} = \frac{25}{3}\sqrt{(205)}\,\lambda$$

$$\Rightarrow \quad \lambda = \frac{1}{25}$$

Some Standard Results

1. Excentres of a Triangle

This is the point of intersection of the external bisectors of the angles of a triangle.



The circle opposite to the vertex A is called the escribed circle opposite A or the circle escribed to the side BC. If I_1 is the point of intersection of internal bisector of $\angle BAC$ and external bisector of $\angle ABC$ and $\angle ACB$, then

$$I_1 \equiv \left(\frac{ax_1 - bx_2 - cx_3}{a-b-c}, \frac{ay_1 - by_2 - cy_3}{a-b-c}\right)$$

or $$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

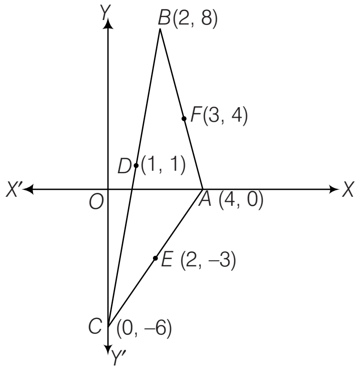
Similarly, $$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$$

and $$I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

where, $|BC| = a, |CA| = b$ and $|AB| = c$

Example 43. If the coordinates of the mid-points of sides BC, CA and AB of triangle ABC are (1, 1), $(2, -3)$ and $(3, 4)$, then find the excentre opposite to the vertex A.

Sol. Let $D(1, 1)$, $E(2, -3)$ and $F(3, 4)$ are the mid-points of the sides of the triangle BC, CA and AB respectively. Let $A \equiv (\alpha, \beta)$



then $B \equiv (6-\alpha, 8-\beta)$

and $C \equiv (4-\alpha, -6-\beta)$

Also, D is the mid-point of B and C, then

$$1 = \frac{6-\alpha + 4-\alpha}{2} \Rightarrow \alpha = 4$$

and $$1 = \frac{8-\beta - 6-\beta}{2} \Rightarrow \beta = 0$$

$\therefore$ $A \equiv (4, 0)$, $B \equiv (2, 8)$ and $C \equiv (0, -6)$, then

$$a = |BC| = \sqrt{(2-0)^2 + (8+6)^2} = \sqrt{200} = 10\sqrt{2}$$

$$b = |CA| = \sqrt{(0-4)^2 + (-6-0)^2} = \sqrt{52} = 2\sqrt{13}$$

and $$c = |AB| = \sqrt{(4-2)^2 + (0-8)^2} = \sqrt{68} = 2\sqrt{17}$$

Hence, the coordinates of the excentre opposite to A are

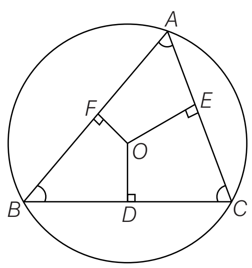
$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

i.e., $$\left(\frac{-10\sqrt{2}\times 4 + 2\sqrt{13}\times 2 + 2\sqrt{17}\times 0}{-10\sqrt{2} + 2\sqrt{13} + 2\sqrt{17}}, \frac{-10\sqrt{2}\times 0 + 2\sqrt{13}\times 8 + 2\sqrt{17}\times(-6)}{-10\sqrt{2} + 2\sqrt{13} + 2\sqrt{17}}\right)$$

or $$\left(\frac{-20\sqrt{2} + 2\sqrt{13}}{-5\sqrt{2} + \sqrt{13} + \sqrt{17}}, \frac{8\sqrt{13} - 6\sqrt{17}}{-5\sqrt{2} + \sqrt{13} + \sqrt{17}}\right)$$

2. Circumcentre of a Triangle

The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle (i.e., the lines through the mid-point of a side and perpendicular to it). Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and if angles of ΔABC are given, then coordinates of circumcentre



are $\left(\dfrac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C},\right.$

$\left.\dfrac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$

Or

$\left(\dfrac{ax_1 \cos A + bx_2 \cos B + cx_3 \cos C}{a\cos A + b\cos B + c\cos C},\right.$

$\left.\dfrac{ay_1 \cos A + by_2 \cos B + cy_3 \cos C}{a\cos A + b\cos B + c\cos C}\right)$

where, $|BC| = a, |CA| = b$ and $|AB| = c$

Example 44. In a ΔABC with vertices $A(1,2), B(2,3)$ and $C(3,1)$ and $\angle A = \angle B = \cos^{-1}\left(\dfrac{1}{\sqrt{10}}\right)$, $\angle C = \cos^{-1}\left(\dfrac{4}{5}\right)$, then find the circumcentre of ΔABC.

Sol. Since, $\angle A = \angle B = \cos^{-1}\left(\dfrac{1}{\sqrt{10}}\right)$

$\Rightarrow \quad \cos A = \cos B = \dfrac{1}{\sqrt{10}}$

then, $\sin A = \sin B = \dfrac{3}{\sqrt{10}}$

$\therefore \quad \sin 2A = \sin 2B = 2 \times \dfrac{3}{\sqrt{10}} \times \dfrac{1}{\sqrt{10}} = \dfrac{3}{5}$

and $\quad \angle C = \cos^{-1}\left(\dfrac{4}{5}\right)$

$\Rightarrow \quad \cos C = \dfrac{4}{5}$ then, $\sin C = \dfrac{3}{5}$

$\therefore \quad \sin 2C = 2 \times \dfrac{3}{5} \times \dfrac{4}{5} = \dfrac{24}{25}$

Let the circumcenter be (x, y), then

$$x = \frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

$$= \frac{1 \times \frac{3}{5} + 2 \times \frac{3}{5} + 3 \times \frac{24}{25}}{\frac{3}{5} + \frac{3}{5} + \frac{24}{25}} = \frac{13}{6}$$

and $$y = \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

$$= \frac{2 \times \frac{3}{5} + 3 \times \frac{3}{5} + 1 \times \frac{24}{25}}{\frac{3}{5} + \frac{3}{5} + \frac{24}{25}} = \frac{11}{6}$$

Hence, coordinates of circumcenter are $\left(\dfrac{13}{6}, \dfrac{11}{6}\right)$.

Two Important Tricks for Circumcentre

(a) If angles of triangle ABC are not given and the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are given, then the circumcentre of the ΔABC is given by

$$\left(\frac{(x_1 + x_2) + \lambda(y_1 - y_2)}{2}, \frac{(y_1 + y_2) - \lambda(x_1 - x_2)}{2}\right)$$

Here, we observe that

$$P = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$

$$\therefore \qquad \lambda = \frac{\vec{\mathbf{R}}_1 \cdot \vec{\mathbf{R}}_2}{|P|}$$

(b) If the angle C is given instead of coordinates of the vertex C and the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ of ΔABC are given, then the circumcentre of ΔABC is given by

$$\left(\frac{(x_1 + x_2) \pm \cot C(y_1 - y_2)}{2}, \frac{(y_1 + y_2) \pm \cot C(x_1 - x_2)}{2}\right)$$

Remark

Circumcentre of the right angled triangle ABC, right angled at A is $\dfrac{B + C}{2}$.

Example 45. Find the circumcentre of the triangle whose vertices are (2, 2), (4, 2) and (0, 4).

Sol. Let the given points are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) respectively.

for the matrix

$$P = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix}$$

$$\therefore \quad \lambda = \frac{\vec{\mathbf{R}}_1 \cdot \vec{\mathbf{R}}_2}{|P|}$$

$$= \frac{2 \times 4 + (-2) \times (-2)}{2 \times (-2) - 4 \times (-2)} = \frac{12}{4} = 3$$

$\therefore$ Circumcentre of the triangle

$$\equiv \left(\frac{(x_1 + x_2) + \lambda(y_1 - y_2)}{2}, \frac{(y_1 + y_2) - \lambda(x_1 - x_2)}{2}\right)$$

$$\equiv \left(\frac{2 + 4 + 3(2 - 2)}{2}, \frac{2 + 2 - 3(2 - 4)}{2}\right)$$

$$\equiv (3, 5)$$

Example 46. Find the circumcentre of triangle ABC if $A \equiv (7, 4)$, $B \equiv (3, -2)$ and $\angle C = \dfrac{\pi}{3}$.

Sol. Here, $x_1 = 7, y_1 = 4, x_2 = 3, y_2 = -2$ and $\angle C = \dfrac{\pi}{3}$

$\therefore$ The circumcentre of ΔABC

$$\equiv \left(\frac{(x_1 + x_2) \pm \cot C(y_1 - y_2)}{2}, \frac{(y_1 + y_2) \mp \cot C(x_1 - x_2)}{2} \right)$$

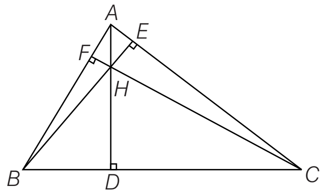
$$\equiv \left(\frac{(7+3) \pm \frac{1}{\sqrt{3}}(4+2)}{2}, \frac{(4-2) \mp \frac{1}{\sqrt{3}}(7-3)}{2} \right)$$

$$\equiv \left(5 + \sqrt{3}, 1 - \frac{2}{\sqrt{3}} \right) \text{ or } \left(5 - \sqrt{3}, 1 + \frac{2}{\sqrt{3}} \right)$$

3. Orthocentre of a Triangle

The orthocentre of a triangle is the point of intersection of altitudes
(i.e., the lines through the vertices and perpendicular to opposite sides).

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and if angles of ΔABC are given, then coordinates of orthocentre are



$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C} \right)$$

or $$\left(\frac{ax_1 \sec A + bx_2 \sec B + cx_3 \sec C}{a \sec A + b \sec B + c \sec C}, \frac{ay_1 \sec A + by_2 \sec B + cy_3 \sec C}{a \sec A + b \sec B + c \sec C} \right)$$

where, $|BC| = a, |CA| = b$ and $|AB| = c$

Important trick for orthocentre :

orthocentre of the triangle whose vertices are (0, 0), (x_1, y_1) is given by

$$(\lambda(y_2 - y_1), -\lambda(x_2 - x_1))$$

where, $$\lambda = \frac{x_1 x_2 + y_1 y_2}{x_1 y_2 - x_2 y_1}$$

Remarks

1. The orthocentre of a triangle having vertices (α, β), (β, α) and (α, α) is (α, α).
2. The orthocentre of a triangle having vertices is $\left(-\frac{1}{\alpha\beta\gamma}, -\alpha\beta\gamma \right)$
3. The orthocentre of right angled triangle *ABC*, right angled at *A* is *A*.

Example 47. Find the orthocentre of ΔABC if $A \equiv (0, 0), B \equiv (3, 5)$ and $C \equiv (4, 7)$.

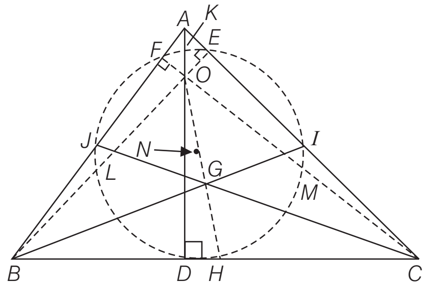
Sol. Here, $x_1 = 3, y_1 = 5, x_2 = 4, y_2 = 7$

$$\therefore \qquad \lambda = \frac{3 \times 4 + 5 \times 7}{3 \times 7 - 4 \times 5} = 47$$

$\Rightarrow$ Orthocentre of $\Delta ABC \equiv (47(7-5), -47(4-3))$
$\equiv (94, -47)$

4. Nine Point Centre of a Triangle

If a circle passing through the feet of perpendiculars (i.e., D, E, F) mid-points of sides BC, CA, AB respectively (i.e., H, I, J) and the n x
mid-points of the line joining the orthocentre O to the angular points A, B, C (i.e., K, L, M) thus the nine points $D, E, F, H, I, J, K, L, M$ all lie on a circle.



This circle is known as nine point circle and its centre is called the nine point centre. The nine-point centre of a triangle is collinear with the circumcentre and the orthocentre and bisects the segments joining them and radius of nine point circle of a triangle is half the radius of the circumcircle.

***Corollary* 1 :** The orthocentre, the nine point centre, the centroid and the circumcentre therefore all lie on a straight line.

Corollary* 2 :** If O is orthocentre, N is nine point centre, G is centroid and C is circumcentre, then to remember it see ***ONGC (i.e. Oil Natural Gas Corporation) in left of G are 2 and in right is 1, therefore G divides O and C in the ratio 2 : 1 (internally).

***Corollary* 3 :** N is the mid-point of O and C

***Corollary* 4 :** Radius of nine point circle $= \frac{1}{2} \times$ Radius of circumcircle

Remarks

1. The distance between the orthocentre and circumcentre in an equilateral triangle is zero.
2. If the circumcentre and centroid of a triangle are respectively (α, β) (γ, δ) then orthocentre will be $(3\gamma - 2\alpha, 3\delta - 2\beta)$.

Example 48. If a triangle has its orthocentre at (1,1) and circumcentre at $\left(\frac{3}{2},\frac{3}{4}\right)$, then find the centroid and nine point centre.

Sol. Since, centroid divides the orthocentre and circumcentre in the ratio 2 : 1 (internally) and if centroid $G(x, y)$, then

$$O(1,1)\overset{2}{\text{———}}G(x,y)\overset{1}{\text{———}}C\left(\frac{3}{2},\frac{3}{4}\right)$$

$$x=\frac{2\times\frac{3}{2}+1\times 1}{2+1}=\frac{4}{3}$$

and $$y=\frac{2\times\frac{3}{4}+1\times 1}{2+1}=\frac{5}{6}$$

$\therefore$ Centroid is $\left(\frac{4}{3},\frac{5}{6}\right)$ and nine point centre is the mid-point of orthocentre and circumcentre.

$\therefore$ Nine point centre is $\left(\frac{1+\frac{3}{2}}{2},\frac{1+\frac{3}{4}}{2}\right)$, i.e. $\left(\frac{5}{4},\frac{7}{8}\right)$.

Example 49. The vertices of a triangle are $A(a, a\tan\alpha)$, $B(b, b\tan\beta)$ and $C(c, c\tan\gamma)$. If the circumcentre of ΔABC coincides with the origin and $H(\bar{x},\bar{y})$ is the orthocentre, then show that $\frac{\bar{y}}{\bar{x}}=\left(\frac{\sin\alpha+\sin\beta+\sin\gamma}{\cos\alpha+\cos\beta+\cos\gamma}\right)$.

Sol. If R be the circumradius and O be the circumcentre

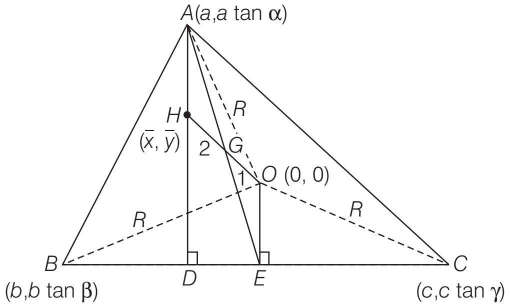
$\therefore$ $$OA=OB=OC=R$$

or $$\sqrt{(a^2+a^2\tan^2\alpha)}=\sqrt{(b^2+b^2\tan^2\beta)}=\sqrt{(c^2+c^2\tan^2\gamma)}=R$$

or $a\sec\alpha=b\sec\beta=c\sec\gamma=R$

or $a=R\cos\alpha,\ b=R\cos\beta,\ c=R\cos\gamma$

then, $a\tan\alpha=R\cos\alpha\cdot\frac{\sin\alpha}{\cos\alpha}=R\sin\alpha$



Similarly,

$$b\tan\beta=R\sin\beta \text{ and } c\tan\gamma=R\sin\gamma$$

$\therefore$ Coordinates of the vertices of a triangle are :

$A(R\cos\alpha, R\sin\alpha)$, $B(R\cos\beta, R\sin\beta)$

and $C(R\cos\gamma, R\sin\gamma)$

$\therefore$ Centroid

$$(G)\equiv\left(\frac{R(\cos\alpha+\cos\beta+\cos\gamma)}{3},\frac{R(\sin\alpha+\sin\beta+\sin\gamma)}{3}\right)$$

Since, G divides H and O in the ratio 2 : 1 (internally), then

$$\frac{R}{3}(\cos\alpha+\cos\beta+\cos\gamma)=\frac{2\cdot 0+1\cdot\bar{x}}{2+1}$$

or $$\frac{R}{3}(\cos\alpha+\cos\beta+\cos\gamma)=\frac{\bar{x}}{3} \quad \text{...(i)}$$

and $$\frac{R}{3}(\sin\alpha+\sin\beta+\sin\gamma)=\frac{2\cdot 0+1\cdot\bar{y}}{2+1}$$

or $$\frac{R}{3}(\sin\alpha+\sin\beta+\sin\gamma)=\frac{\bar{y}}{3} \quad \text{...(ii)}$$

Dividing Eqs. (ii) by (i), then we get

$$\frac{\bar{y}}{\bar{x}}=\left(\frac{\sin\alpha+\sin\beta+\sin\gamma}{\cos\alpha+\cos\beta+\cos\gamma}\right)$$

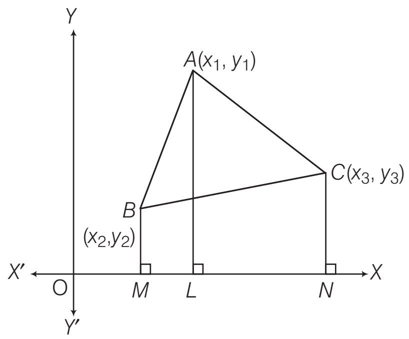
Area of a Triangle

Theorem : The area of a triangle, the coordinates of whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is

$$\frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|$$

or $$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}\right|$$

Proof : Let ABC be a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let us assume that the points A, B and C are in 1st quadrant (for the sake of exactness). Draw AL, BM and CN perpendicular on X-axis. Let Δ be the required area of the triangle ABC, then



Δ = Area of triangle ABC

= [Area of trapezium $ABML$ + Area of trapezium $ALNC$ − Area of trapezium $BMNC$]

[$\because$ Area of trapezium $= \frac{1}{2}$ (Sum of parallel sides) × (Distance between them)]

$$\therefore \Delta = \left[\frac{1}{2}(BM + AL)(ML) + \frac{1}{2}(AL + CN)(LN) - \frac{1}{2}(BM + CN)(MN)\right]$$

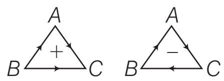
$$= \left[\frac{1}{2}(BM + AL)(OL - OM) + \frac{1}{2}(AL + CN)(ON - (OL) - \frac{1}{2}(BM + CN)(ON - OM)\right]$$

$$= \left[\frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)\right]$$

$$= \frac{1}{2}[x_1(y_2 + y_1 - y_1 - y_3) + x_2(-y_2 - y_1 + y_2 + y_3) + x_3(y_1 + y_3 - y_2 - y_3)]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

The area of triangle ABC will come out to be a positive quantity only when the vertices A, B, C are taken in anticlockwise direction and if points A, B, C are taken in clockwise direction then the area will be negative and if the points A, B, C are taken arbitrary then the area will be positive or negative, the numerical value being the same in all cases.

Thus in general (i.e., )

$$\text{Area of } \Delta ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

This expression can be written in determinant form as follows

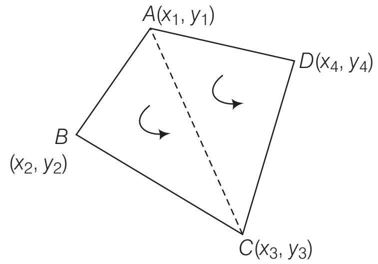
$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

***Corollary* 1 :** Area of triangle can also be found by easy method

$$\Delta = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}\right|$$

and **Area of quadrilateral *ABCD* :** The area of a quadrilateral can be found out by dividing the quadrilateral into two triangles.

$\therefore$ Area of quadrilateral $ABCD$



= Area of ΔABC + Area of ΔDAC

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}$$

$$= \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}\right| + \frac{1}{2}\begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix}$$

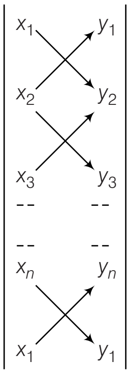
$$\left(\because \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} = -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}\right)$$

$\therefore$ **Area of polygon whose vertices are $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$ is**

$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \ldots\ldots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right|$$

Or

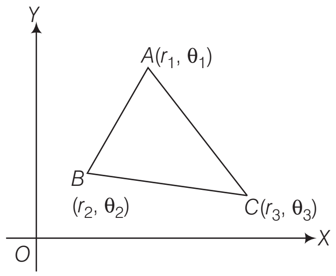
Stair Method Repeat first coordinates one time in last for down arrow use positive sign and for up arrow use negative sign.

$\therefore$ Area of polygon $= \frac{1}{2}\,|$  $|$

$$= \frac{1}{2}|\{(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (y_1x_2 + y_2x_3 + \ldots + y_nx_1)\}|$$

***Corollary* 2 :** If the coordinates of the vertices of the triangle are given in polar form i.e.,

$A(r_1, \theta_1), B(r_2, \theta_2), C(r_3, \theta_3)$.



Then, area of triangle

$$= \frac{1}{2} | r_1 r_2 \sin(\theta_1 - \theta_2) + r_2 r_3 \sin(\theta_2 - \theta_3) + r_3 r_1 \sin(\theta_3 - \theta_1) |$$

$$= \frac{1}{2} | \sum r_1 r_2 \sin(\theta_1 - \theta_2) |$$

***Corollary* 3 :** If $a_1 x + b_1 y + c_1 = 0$, $a_2 x + b_2 y + c_2 = 0$ and $a_3 x + b_3 y + c_3 = 0$ are the sides of a triangle, then the area of the triangle is given by (without solving the vertices)

$$\Delta = \frac{1}{2 | C_1 C_2 C_3 |} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where, C_1, C_2, C_3 are the cofactors of c_1, c_2, c_3 in the determinant

Here, $$C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)$$

$$C_2 = \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = (a_3 b_1 - a_1 b_3)$$

and $$C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = (a_1 b_2 - a_2 b_1)$$

and $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = c_1 C_1 + c_2 C_2 + c_3 C_3$$

Or

$$\text{Area of triangle} = \frac{\Delta^2}{2 | \Delta_1 \Delta_2 \Delta_3 |}$$

where, $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

and $$\Delta_3 = \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}$$

***Corollary* 4 :** Area of the triangle formed by the lines of the form $y = m_1 x + c_1$, $y = m_2 x + c_2$ and $y = m_3 x + c_3$ is

$$\Delta = \frac{1}{2} \left| \frac{(c_2 - c_3)^2}{(m_2 - m_3)} + \frac{(c_3 - c_1)^2}{(m_3 - m_1)} + \frac{(c_1 - c_2)^2}{(m_1 - m_2)} \right|$$

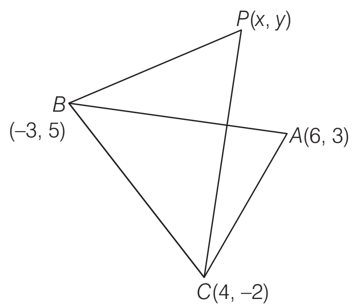
Remarks

1. If area of a triangle is given then, use ± sign.
2. The points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, then area of $(\Delta ABC) = 0$.
3. Four given points will be collinear, then area of the quadrilateral is zero.
4. Area of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\Delta = \frac{1}{2} \left| \begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix} \right|$
5. If one vertex (x_3, y_3) is at (0, 0) then, $\Delta = \frac{1}{2} | x_1 y_2 - x_2 y_1 |$

Example 50. The coordinates of A, B, C are $(6, 3), (-3, 5)$ and $(4, -2)$ respectively and P is any points (x, y). Show that the ratio of the areas of the triangles PBC and ABC is $\frac{|x + y - 2|}{7}$.

Sol. We have

$$\frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2} |\{x(5 + 2) - 3(-2 - y) + 4(y - 5)\}|}{\frac{1}{2} |\{6(5 + 2) - 3(-2 - 3) + 4(3 - 5)\}|}$$



$$= \frac{|7x + 7y - 14|}{|49|} = \frac{7 |x + y - 2|}{49} = \frac{|x + y - 2|}{7}$$

Example 51. Find the area of the pentagon whose vertices are $A(1, 1)$, $B(7, 21)$, $C(7 - 3)$, $D(12, 2)$ and $E(0, -3)$.

Sol. The required area

$$= \frac{1}{2} \left| \begin{vmatrix} 1 & 1 \\ 7 & 21 \end{vmatrix} + \begin{vmatrix} 7 & 21 \\ 7 & -3 \end{vmatrix} + \begin{vmatrix} 7 & -3 \\ 12 & 2 \end{vmatrix} + \begin{vmatrix} 12 & 2 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} |(21 - 7) + (-21 - 147) + (14 + 36) + (-36 - 0) + (0 + 3)|$$

$$= \frac{1}{2} |-137| = \frac{137}{2} \text{ sq units}$$

Example 52. Show that the points $(a, 0), (0, b)$ and $(1, 1)$ are collinear, if $\frac{1}{a}+\frac{1}{b}=1$

Sol. Let $A \equiv (a, 0)$, $B \equiv (0, b)$ and $C \equiv (1, 1)$

Now, points A, B, C will be collinear, if area of $\Delta ABC = 0$

or $$\frac{1}{2}\left|\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}+\begin{vmatrix} 0 & b \\ 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 \\ a & 0 \end{vmatrix}\right|=0$$

$$\Rightarrow \quad |(ab-0)+(0-b)+(0-a)|=0$$

or $$ab-a-b=0$$

or $$a+b=ab \text{ or } \frac{1}{a}+\frac{1}{b}=1$$

Example 53. Prove that the coordinates of the vertices of an equilateral triangle can not all be rational.

Sol. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC. If possible let $x_1, y_1, x_2, y_2, x_3, y_3$ be all rational.

Now, area of $\Delta ABC = \frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)$
$+x_3(y_1-y_2)|$
$= \text{Rational}$ ...(i)

Since, ΔABC is equilateral

$\therefore$ Area of $\Delta ABC = \frac{\sqrt{3}}{4}(\text{side})^2$
$= \frac{\sqrt{3}}{4}(AB)^2$
$= \frac{\sqrt{3}}{4}\{(x_1-x_2)^2+(y_1-y_2)^2\}$
$= \text{Irrational}$ ...(ii)

From Eqs. (i) and (ii),

Rational = Irrational

which is contradiction.

Hence, $x_1, y_1, x_2, y_2, x_3, y_3$ cannot all be rational.

Example 54. The coordinates of two points A and B are $(3, 4)$ and $(5, -2)$ respectively. Find the coordinates of any point P if $PA = PB$ and area of ΔAPB is 10.

Sol. Let coordinates of P be (h, k).

$\because \quad PA = PB \Rightarrow (PA)^2 = (PB)^2$

$\Rightarrow \quad (h-3)^2+(k-4)^2=(h-5)^2+(k+2)^2$

$\Rightarrow \quad (h-3)^2-(h-5)^2+(k-4)^2-(k+2)^2=0$

$\Rightarrow \quad (2h-8)(2)+(2k-2)(-6)=0$

$\Rightarrow \quad (h-4)-3(k-1)=0$

$\Rightarrow \quad h-3k-1=0$ ...(i)

Now, Area of $\Delta PAB = \frac{1}{2}\left|\begin{vmatrix} h & k & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix}\right| = 10$

or $\quad 6h+2k-26=\pm 20$

$\Rightarrow \quad 6h+2k-46=0$ or $6h+2k-6=0$

$\Rightarrow \quad 3h+k-23=0$ or $3h+k-3=0$

Solving $\quad h-3k-1=0$ and $3h+k-23=0$,

we get $\quad h=7, k=2$

Solving $\quad h-3k-1=0$ and $3h+k-3=0$,

we get $\quad h=1, k=0$

Hence, the coordinates of P are $(7, 2)$ or $(1, 0)$.

Example 55. Find the area of the triangle formed by the straight lines $7x-2y+10=0$, $7x+2y-10=0$ and $9x+y+2=0$ (without solving the vertices of the triangle).

Sol. The given lines are :

$$7x-2y+10=0$$
$$7x+2y-10=0$$
$$9x+y+2=0$$

$\therefore$ Area of triangle $\Delta = \frac{1}{2|C_1C_2C_3|}\begin{vmatrix} 7 & -2 & 10 \\ 7 & 2 & -10 \\ 9 & 1 & 2 \end{vmatrix}^2$ ...(i)

where, $\quad C_1 = \begin{vmatrix} 7 & 2 \\ 9 & 1 \end{vmatrix} = 7-18=-11,$

$C_2 = \begin{vmatrix} 9 & 1 \\ 7 & -2 \end{vmatrix} = -18-7=-25$

and $\quad C_3 = \begin{vmatrix} 7 & -2 \\ 7 & 2 \end{vmatrix} = 14+14=28,$

and $\quad \begin{vmatrix} 7 & -2 & 10 \\ 7 & 2 & -10 \\ 9 & 1 & 2 \end{vmatrix} = 10C_1-10C_2+2C_3$

$= 10\times(-11)-10\times(-25)+2\times 28 = 196$

$\therefore$ From Eq. (i), $\Delta = \frac{1}{2|-11\times(-25)\times 28|}\times(196)^2$

$= \frac{196\times 196}{2\times 11\times 25\times 28} = \frac{686}{275}$ sq units

Example 56. If Δ_1 is the area of the triangle with vertices $(0, 0)$, $(a\tan\alpha, b\cot\alpha)$, $(a\sin\alpha, b\cos\alpha)$; Δ_2 is the area of the triangle with vertices (a, b), $(a\sec^2\alpha, b\operatorname{cosec}^2\alpha)$, $(a+a\sin^2\alpha, b+b\cos^2\alpha)$ and Δ_3 is the area of the triangle with vertices $(0, 0)$, $(a\tan\alpha, -b\cot\alpha)$, $(a\sin\alpha, b\cos\alpha)$. Show that there is no value of α for which Δ_1, Δ_2 and Δ_3 are in GP.

Sol. We have, $\Delta_1 = \frac{1}{2}|(a\tan\alpha)(b\cos\alpha) - (a\sin\alpha)(b\cot\alpha)|$

(∵ one vertex is (0,0))

$$= \frac{1}{2}|ab||\sin\alpha - \cos\alpha| \quad \text{... (i)}$$

and $\Delta_2 = \frac{1}{2}\left|\begin{vmatrix} a-(a+a\sin^2\alpha) & a\sec^2\alpha-(a+a\sin^2\alpha) \\ b-(b+b\cos^2\alpha) & b\,\text{cosec}^2\,\alpha-(b+b\cos^2\alpha)\end{vmatrix}\right|$

(See remark 4)

$$= \frac{1}{2}\left|\begin{vmatrix} -a\sin^2\alpha & a(\tan^2\alpha-\sin^2\alpha) \\ -b\cos^2\alpha & b(\cot^2\alpha-\cos^2\alpha)\end{vmatrix}\right|$$

$$= \frac{1}{2}|ab|\times\left|\begin{vmatrix} -\sin^2\alpha & \sin^2\alpha(\sec^2\alpha-1) \\ -\cos^2\alpha & \cos^2\alpha(\text{cosec}^2\alpha-1)\end{vmatrix}\right|$$

$$= \frac{1}{2}|ab|\times\left|\begin{vmatrix} -\sin^2\alpha & \sin^2\alpha\tan^2\alpha \\ -\cos^2\alpha & \cos^2\alpha\cot^2\alpha\end{vmatrix}\right|$$

$$= \frac{1}{2}|ab|\times|-\sin^2\alpha\cos^2\alpha\cot^2\alpha + \sin^2\alpha\cos^2\alpha\tan^2\alpha|$$

$$= \frac{1}{2}|ab|\times|-\cos^4\alpha + \sin^4\alpha|$$

$$= \frac{1}{2}|ab|\times|\sin^2\alpha + \cos^2\alpha|\times|\sin^2\alpha - \cos^2\alpha|$$

$$= \frac{1}{2}|ab|\times|1|\times|-\cos 2\alpha|$$

$$= \frac{1}{2}|ab|\times|\cos 2\alpha| \quad \text{... (ii)}$$

and $\Delta_3 = \frac{1}{2}|(a\tan\alpha)(b\cos\alpha) - (-b\cot\alpha)(a\sin\alpha)|$

$$= \frac{1}{2}|ab||\sin\alpha + \cos\alpha| \quad \text{... (iii)}$$

Since, $\Delta_1, \Delta_2, \Delta_3$ are in GP, then $\Delta_1\Delta_3 = \Delta_2^2$

$\Rightarrow \quad \frac{1}{2}|ab||\sin\alpha - \cos\alpha|\times\frac{1}{2}|ab||\sin\alpha + \cos\alpha|$

$= \frac{1}{4}|ab|^2|\cos 2\alpha|^2$ [from Eqs. (i), (ii) and (iii)]

$\Rightarrow \quad |\sin^2\alpha - \cos^2\alpha| = |\cos 2\alpha|^2$

$\Rightarrow \quad |-\cos 2\alpha| = |\cos 2\alpha|^2$

$\Rightarrow \quad |\cos 2\alpha| = |\cos 2\alpha|^2$

$\Rightarrow \quad |\cos 2\alpha|(1 - |\cos 2\alpha|) = 0$

$\therefore \quad 1 - |\cos 2\alpha| = 0 \quad (\because |\cos 2\alpha| \neq 0)$

$\Rightarrow \quad |\cos 2\alpha| = 1$

or $\quad \cos 2\alpha = \pm 1$ or $\cos 2\alpha = 1$

and $\quad \cos 2\alpha = -1$

or $\quad 2\alpha = 2n\pi, 2\alpha = (2p+1)\pi$

or $\quad \alpha = n\pi, \alpha = p\pi + \frac{\pi}{2}; n, p \in I$

For these values of α the vertices of the given triangles are not defined. Hence Δ_1, Δ_2 and Δ_3 cannot be in GP for any value of α.

Exercise for Session 3

1. The coordinates of the middle points of the sides of a triangle are (4, 2), (3, 3) and (2, 2), then coordinates of centroid are

(a) (3, 7 / 3) (b) (3, 3) (c) (4, 3) (d) (3, 4)

2. The incentre of the triangle whose vertices are (–36, 7), (20, 7) and (0, –8) is

(a) (0, – 1) (b) (–1, 0) (c) (1, 1) (d) $\left(\frac{1}{2}, 1\right)$

3. If the orthocentre and centroid of a triangle are (–3, 5) and (3, 3) then its circumcentre is

(a) (6, 2) (b) (3, – 1) (c) (–3, 5) (d) (–3, 1)

4. An equilateral triangle has each side equal to a. If the coordinates of its vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) then the square of the determinant $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ equals

(a) $3a^4$ (b) $\frac{3a^4}{2}$ (c) $\frac{3}{4}a^4$ (d) $\frac{3}{8}a^4$

5. The vertices of a triangle are $A(0, 0), B(0, 2)$ and $C(2, 0)$. The distance between circumcentre and orthocentre is

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 2 (d) $\frac{1}{2}$

6. $A(a, b)$, $B(x_1, y_1)$ and $C(x_2, y_2)$ are the vertices of a triangle. If a, x_1, x_2 are in GP with common ratio r and b, y_1, y_2 are in GP with common ratio s, then area of ΔABC is

(a) $ab(r-1)(s-1)(s-r)$
(b) $\frac{1}{2}ab(r+1)(s+1)(s-r)$
(c) $\frac{1}{2}ab(r-1)(s-1)(s-r)$
(d) $ab(r+1)(s+1)(r-s)$

7. The points $(x+1, 2)$, $(1, x+2)$, $\left(\frac{1}{x+1}, \frac{2}{x+1}\right)$ are collinear, then x is equal to

(a) -4
(b) -8
(c) 4
(d) 8

8. The vertices of a triangle are $(6, 0)$, $(0, 6)$ and $(6, 6)$. Then distance between its circumcentre and centroid, is

(a) $2\sqrt{2}$
(b) 2
(c) $\sqrt{2}$
(d) 1

9. The nine point centre of the triangle with vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$ is

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$
(b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$
(c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$
(d) $\left(1, \frac{1}{\sqrt{3}}\right)$

10. The vertices of a triangle are $(0, 0)$, $(1, 0)$ and $(0, 1)$. Then excentre opposite to $(0, 0)$ is

(a) $\left(1-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$
(b) $\left(1+\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$
(c) $\left(1+\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)$
(d) $\left(1-\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)$

11. If α, β, γ are the real roots of the equation $x^3 - 3px^2 + 3qx - 1 = 0$, then find the centroid of the triangle whose vertices are $\left(\alpha, \frac{1}{\alpha}\right)$, $\left(\beta, \frac{1}{\beta}\right)$ and $\left(\gamma, \frac{1}{\gamma}\right)$.

12. If centroid of a triangle be $(1, 4)$ and the coordinates of its any two vertices are $(4, -8)$ and $(-9, 7)$, find the area of the triangle.

13. Find the centroid and incentre of the triangle whose vertices are $(1, 2)$, $(2, 3)$ and $(3, 4)$.

14. Show that the area of the triangle with vertices $(\lambda, \lambda-2)$, $(\lambda+3, \lambda)$ and $(\lambda+2, \lambda+2)$ is independent of λ.

15. Prove that the points $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ are collinear.

16. Prove that the points (a, b), (c, d) and $(a-c, b-d)$ are collinear, if $ad = bc$.

17. If the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, show that $\sum\left(\frac{y_1 - y_2}{x_1 x_2}\right) = 0$, i.e.

$$\frac{y_1 - y_2}{x_1 x_2} + \frac{y_2 - y_3}{x_2 x_3} + \frac{y_3 - y_1}{x_3 x_1} = 0$$

18. The coordinates of points A, B, C and D are $(-3, 5)$, $(4, -2)$, $(x, 3x)$ and $(6, 3)$ respectively and $\frac{\Delta ABC}{\Delta BCD} = \frac{2}{3}$, find x.

19. Find the area of the hexagon whose vertices taken in order are $(5, 0)$, $(4, 2)$, $(1, 3)$, $(-2, 2)$, $(-3, -1)$ and $(0, -4)$.

Answers

Exercise for Session 3

1. (a) 2. (b) 3. (a) 4. (c) 5. (a) 6. (c)
7. (a) 8. (c) 9. (d) 10. (b) 11. (p, q)

12. $\frac{333}{2}$ sq units
13. Centroid = incentre $\equiv (2, 3)$

18. $-\frac{3}{67}, \frac{45}{17}$
19. 34 sq units