MATHEMATICS

FUNDAMENTALS OF MATHEMATICS

INTRODUCTION TO MODULUS

NOTES

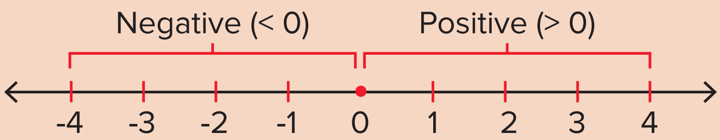
What you will learn

- Algebraic and Geometric Interpretation of Modulus
- Solving Modulus Equations
- Modulus Functions
- Modulus Functions Graph

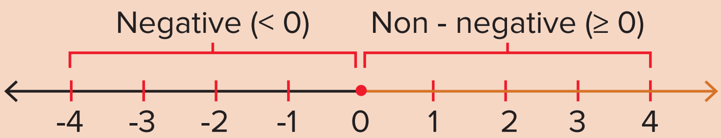
What you already know

- Real Numbers
- Sets and their representation
- Plotting x and y coordinates

For any real number, one of three cases holds. Negative, positive, or zero.

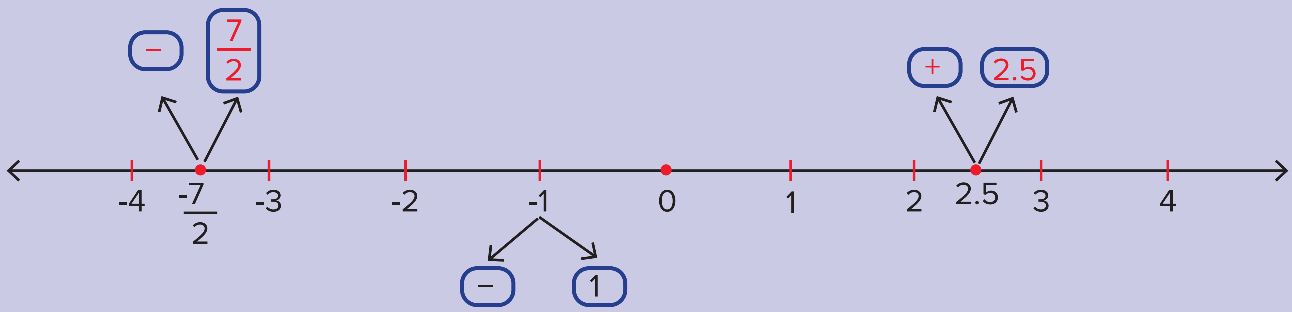


Any real number can be categorised in one of the two cases.



Algebraic Perspective

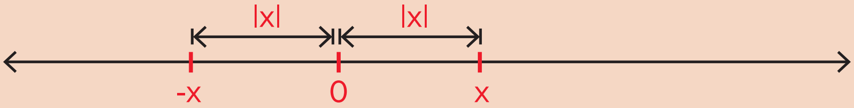
Any real number has a sign and numeric value. For example $\frac{-7}{2}$ has a negative sign and $\frac{7}{2}$ as its numeric value . Similarly, 2.5 has a positive sign and 2.5 as its numeric value.



The numeric value or the magnitude is always non-negative.
Magnitude of x = Absolute Value of x = Modulus of x = $|x|$

Geometric Perspective

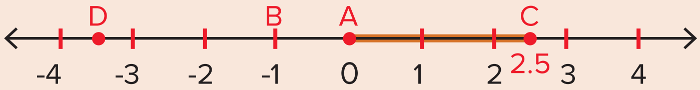
|x| is the distance of x from zero along the number line.



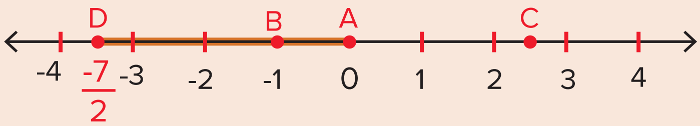
As distance cannot be negative, |x| is always non-negative. Therefore |x| is non-negative

$|-1|$ = Distance of -1 from the origin = |AB| = 1

$|2.5|$ = Distance of 2.5 from the origin = |AC| = 2.5



$\left|\frac{-7}{2}\right|$ = Distance of $\frac{-7}{2}$ from the origin = |AD| = $\frac{7}{2}$



Solve: |x| = 3

Step: 1

Geometrically interpret the equation

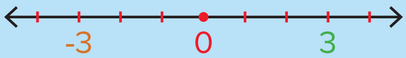
|x| = 3

⇒ |x - 0| =3

Distance between x and 0 = 3

Step: 2

Plot the number line



Step: 3

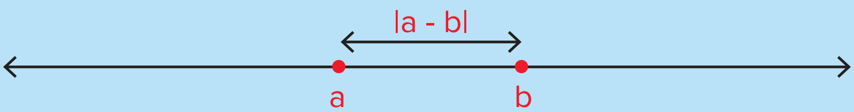
Arrive at solutions

On RHS, x = 3 is at 3 distance unit from 0 on LHS x = -3 is at 3 unit distance from 0 Hence x = {-3, 3 } is the solution set

Let a, b ∈ ℝ, then,

|a − b|= Distance between a and b along the number line

|x| = |x − 0|= Distance of x from zero



Concept Check 1

Solve |x - 2| = 1

Important Results

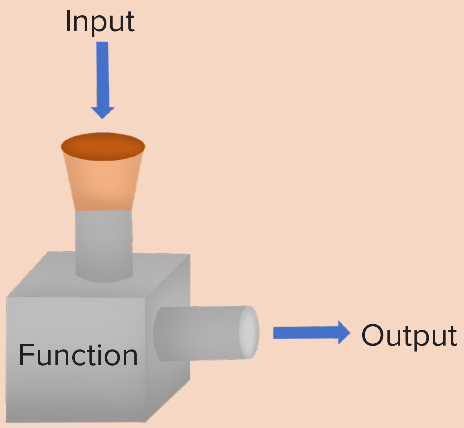
- $|x| = a \Rightarrow x = \pm a;\ a \geq 0$ — For example, $|x| = 5 \Rightarrow x = \pm 5$
- $|x| = a;\ a < 0 \Rightarrow x \in \Phi$ — For example, $|x| = -5$, has no solution
- $\sqrt{(x^2)} = |x|$ — For example, $\sqrt{49} = \sqrt{7 \times 7} = \sqrt{(-7) \times (-7)} = |7|$
- $|x| = |-x|;\ x \in \mathbb{R}$ — For example, $|7| = |-7|$

Quick Query 2

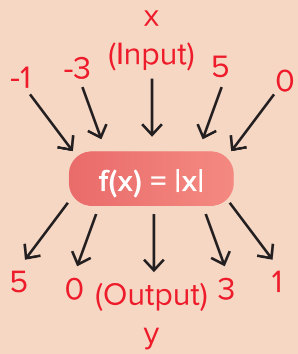
Find the solution for $x = \sqrt{25}$ and $x^2 = 25$

Modulus Function

Function takes an input and returns an output.

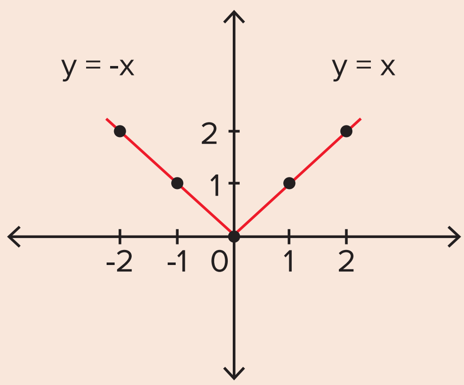


Modulus function takes an input and returns modulus value of the input.



$$y = f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

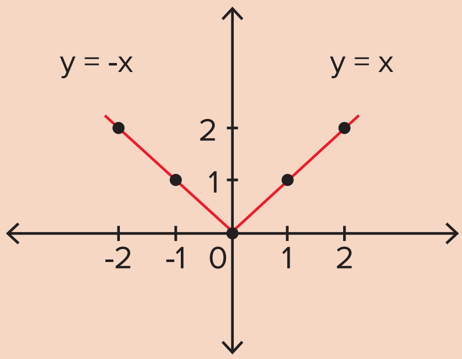
x	0	1	-1	2	-2
f(x)	0	1	1	2	2



y = |x| changes its nature at x = 0 ⇒ critical point

Graph of y = |x|

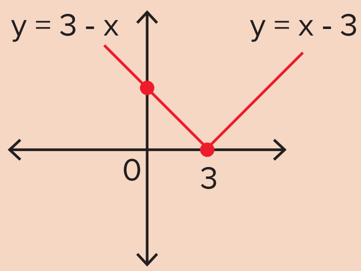
x < 0 ⇒ **Graph of y** = -x	x ≥ 0 ⇒ **Graph of y** = x



1. Critical points are points at which function changes its nature.
   For f(x) = |x|, x = 0 is the critical point
   For f(x) = |ax + b|, $x = -\frac{b}{a}$ is the critical point
2. To find the critical points of |**expression**|, equate the **expression** to zero and solve for variable.

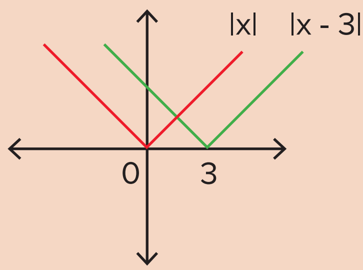
f(x) = |x − 3|

Critical Point :
⇒ **x − 3 = 0**
⇒ **x = 3**



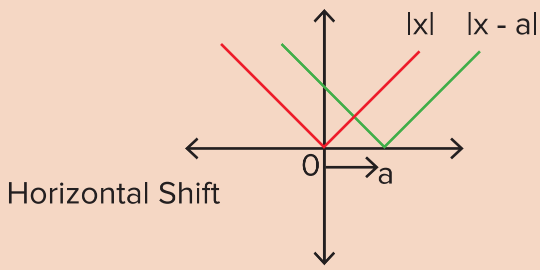
i.e. $|x - 3| = \begin{cases} x - 3; & x \geq 3 \\ -(x - 3); & x < 3 \end{cases}$

As |x| → |x − 3|,
The graph of |x| shifts towards the right by 3 units.



Horizontal Shift

When |x| transforms to |x − a|; a > 0 then the graph of |x| shifts by “a” units to wards the right along the x–axis.



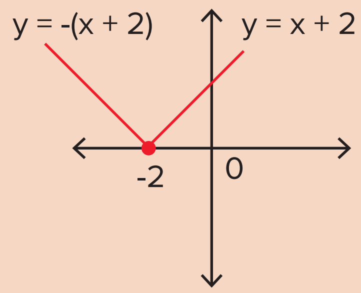
f(x) = |x + 2|

Critical Point :

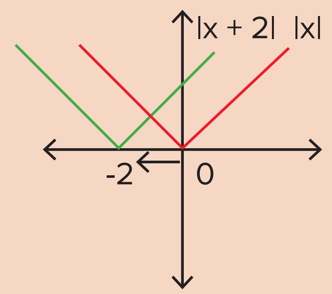
x + 2 = 0

⇒ x = -2

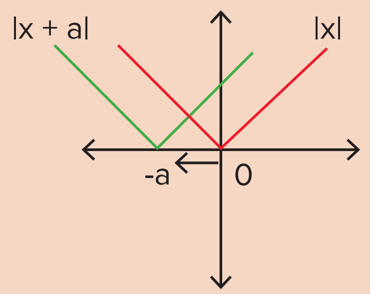
i.e, $|x + 2| = \begin{cases} x + 2; \; x \geq -2 \\ -(x + 2); \; x \leq -2 \end{cases}$



As |x|→|x + 2|, the graph of |x| shifts towards the left by 2 units along the x− axis

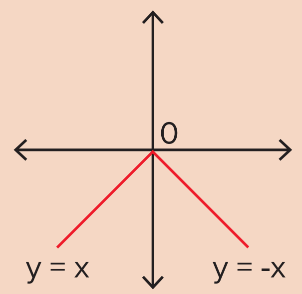


When |x| transforms to |x + a|; a > 0 then graph of |x| shifts by a units towards the left along x−axis



f(x) = -|x|

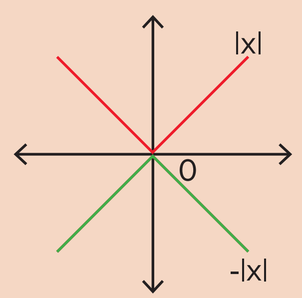
$-|x| = \begin{cases} -x; \; x \geq 0 \\ x; \; x < 0 \end{cases}$



Vertical flip

To draw the graph of −|x|, take mirror image of the graph of |x| w.r.t the x−axis

The graph of -|x| is vertical flip of |x| graph w.r.t the x-axis



Solve $|x - 5| + |x - 4| = 1$

Step 1

Find the critical points of $|x - 5|$ and $|x - 4|$,
$x = 4, 5$ are the critical points

Step 2

Observe regions, where $x < 4$ and $x > 5$
For $x < 4$ region, distance from 5 will be greater than 1. Similarly for $x > 5$ region, distance on the number line from 4 will be greater than 1.

Step 3

For the region $4 <= x <= 5$, let's take any point A whose distance from 4 is 'a', so its distance from 5 will be '1-a'.
Now $|x - 4| + |x - 5| = a + 1 - a = 1$.
Therefore, solution set=[4,5]

Solve $2|x + 1|^2 - |x + 1| = 3$

Step 1

$|x + 1|^2 - |x + 1| - 3 = 0$
Put $|x + 1| = t$
$\Rightarrow 2t^2 - t - 3 = 0$

Step 2

$\Rightarrow 2t^2 - 3t + 2t - 3 = 0$
$\Rightarrow t(2t - 3) + 1(2t - 3) = 0$
$\Rightarrow (t + 1)(2t - 3) = 0$
$\Rightarrow t = \frac{3}{2}$ or $t = -1$

Step 3

$\Rightarrow |x + 1| = \frac{3}{2}$ or $|x + 1| = -1$
(rejected because $|y| \geq 0$)
$\Rightarrow |x + 1| = \frac{3}{2}$
$\Rightarrow x + 1 = \pm\frac{3}{2}$
$\Rightarrow x = \frac{1}{2}, -\frac{5}{2}$
The solution set $= \left\{\frac{1}{2}, -\frac{5}{2}\right\}$

Concept Check 2

Solve $|3x + 2| - |x - 3| = 1$

Summary sheet

Key Terms

- **Modulus:** Modulus is the absolute value of any number. Geometrically, it is the distance of a point from the origin.
- **Critical Point:** Points at which modulus function changes its behaviour.
  Example: $|x + 3| = 0$, critical point is at $x = 3$
- **Function:** Function takes an input and returns a output.
  Example: $f(x) = x + 5$, For input $x = 2$, output $f(x) = (2) + 5 = 7$

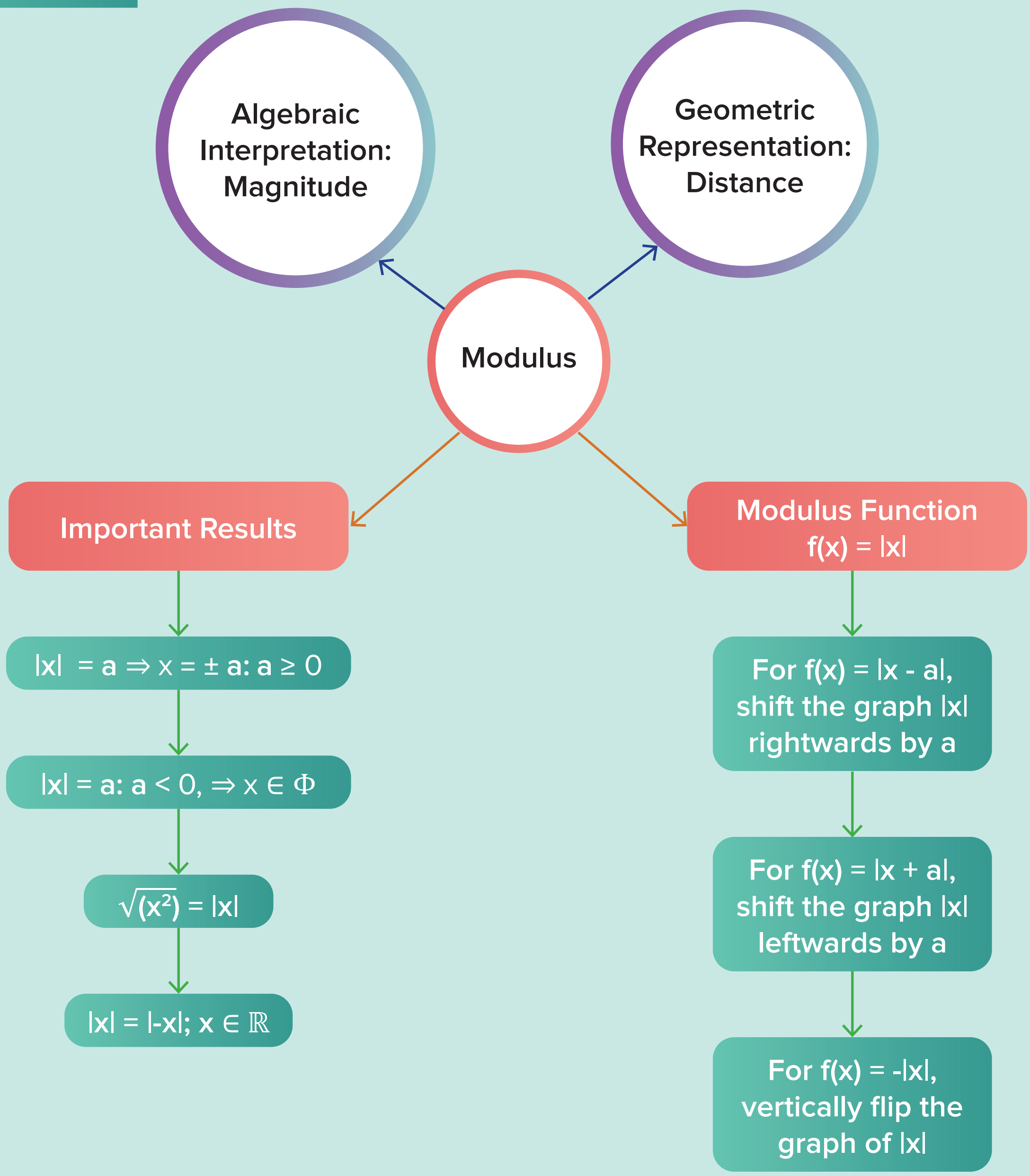
Key Takeaways

- When $|x|$ transforms to $|x + a|$; $a > 0$ then the graph of $|x|$ shifts by a units towards the left along the x-axis.
- When $|x|$ transforms to $|x - a|$; $a > 0$ then the graph of $|x|$ shifts by a units towards the right along the x–axis.
- The graph of $-|x|$ is vertical flip of $|x|$ graph w.r.t the x-axis.

Key Results:

- $|x| = a \Rightarrow x = \pm a; a \geq 0$
- $|x| = a; a < 0 \Rightarrow x \in \Phi$
- $\sqrt{(x^2)} = |x|$
- $|x| = |-x|; x \in \mathbb{R}$

Mind map



Self-Assessment

1. Solve the equation: $2|3x + 2| - 12 = 0$
2. Solve: $2|x + 8| + 4 = 3|x + 8| + 10$

Answers

Quick Query:

1. There is a small difference between $x = \sqrt{25}$ and $x^2 = 25$. The first equation is linear, whereas the 2^{nd} equation is quadratic. So, we get one solution for the 1^{st} equation and two solutions of the 2^{nd} equation.

   For $x = \sqrt{25} \Rightarrow x = 5$ and for $x^2 = 25 \Rightarrow x = \pm\sqrt{25} \Rightarrow x = \pm 5$

Concept Check:

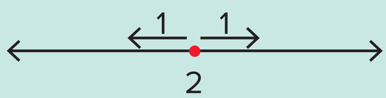
1. Solve $|x-2|=1$

Step 1

Geometrically interpret equation
$|x - 2| = 1$
$\Rightarrow |x - 0| = 3$
Distance between x and $2 = 1$

Step 2

Plot number line



Step 3

Arrive at solutions
On LHS, x = 1 is at 1 distance unit from 2.
On RHS, x = 3, is at one distance unit from 2
Hence $x = \{1, 3\}$ is the solution set

2.

Step 1

Find the critical points and divide the number line into different regions
The critical points of $|3x + 2|$ and $|x - 3|$, are $x = -\frac{2}{3}$ and $x = 3$

Step 2

Case: I when $x < -\frac{2}{3}$ i.e.
$x \in (-\infty, -\frac{2}{3})$
$|3x + 2| - |x - 3| = 1$
$\Rightarrow -(3x + 2) - (-(x - 3)) = 1$
$\Rightarrow x = -3$
(As $|x| = -x$, when $x < 0$)

Step 3

Case: II
when $-\frac{2}{3} \le x < 3$
i.e. $x \in [-\frac{2}{3}, 3)$
$|3x + 2| - |x - 3| = 1$ becomes
$(3x + 2) - (-(x - 3)) = 1$
$\Rightarrow 3x + 2 + x - 3 = 1$
$\Rightarrow x = \frac{1}{2}$

Step 4

Case III: when $x \ge 3$ i.e.
$x \in [3, \infty)$
$|3x + 2| - |x - 3| = 1$ becomes
$(3x + 2) - (x - 3) = 1$
$\Rightarrow x = -2$
But $x \ge 3$, so no solution exists in this region(III)

Step 5

Arrive at final solution
Solution Set = I U II U II
Therefore,
solution set = $\{\frac{1}{2}, -3\}$

Self Assessment:

1. $2|3x + 2| - 12 = 0$

Step 1

Simplify the equation

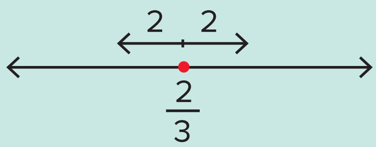
$2\,|3x + 2| = 12$

$\Rightarrow 6\left|x + \frac{2}{3}\right| = 12$

$\Rightarrow \left|x - (-\frac{2}{3})\right| = 2$

Step 2

Interpret geometrically, using number line



Step 3

Arrive at solutions

On LHS, $x = -\frac{8}{3}$ is at 2 distance unit from $-\frac{2}{3}$

On RHS, $x = \frac{4}{3}$ is at 2 unit distance from $-\frac{2}{3}$

Hence $x = \{-\frac{8}{3}, \frac{4}{3}\}$ is the solution set

2. Solve: $2|x + 8| + 4 = 3\,|x + 8| + 10$

Step 1

Simplify equation

$2\,|x + 8| + 4 = 3|\,x + 8| + 10$

$\Rightarrow 4 - 10 = 3|x + 8| - 2|x + 8|$

$\Rightarrow |x + 8| = -6$

Step 2

Use Result Number 2

$|\boldsymbol{x}| = \boldsymbol{a};\ \boldsymbol{a} < \mathbf{0} \Rightarrow x = \Phi$

Here $|x + 8| = -6 < 0$

But $|x + 8|$ is always non-negative,

Hence there is no real solution.

FUNDAMENTALS OF MATHEMATICS

MODULUS INEQUALITIES

What you already know

- Modulus function
- Inequalities representation
- Solving linear and polynomial inequalities

What you will learn

- Properties of modulus
- Triangle inequality
- Solving modulus inequality problems

Properties of Modulus

1. $|x| \geq 0 \ \forall \ x \in R$;
2. $|x| = 0 \Leftrightarrow x = 0$;
3. $|x| = a \Leftrightarrow x = \pm a$, where $a > 0$;
4. $\sqrt{x^2} = |x| = \pm x \ \forall \ x \in R$;
   Example: $\sqrt{5^2} = |5|$ and $\sqrt{(-5)^2} = |-5| = 5$
5. $|x| = |-x| \ \forall \ x \in R$;
6. $|x| = |y| \Rightarrow x = \pm y \ \forall \ x, y \in R$;
   For example: $|x| = |3| \Rightarrow x = \pm 3$
7. $|x\,y| = |x|\,|y| \ \forall \ x, y \in R$;
   Example: LHS $|2(-3)| = |-6| = 6$,
   RHS $= |2||-3| = 2(3) = 6$. LHS=RHS
8. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \ \forall \ x, y \in R$ and $y \neq 0$.
   Example: $\left|\frac{6}{(-7)}\right| = \frac{|6|}{|-7|} = \frac{6}{7}$

Property I (Triangle inequality)

$|x + y| \leq |x| + |y| \ \forall \ x, y \in R$,

Case I: When x and y are both of the same signs, then $|x + y| = |x| + |y|$.

Case II: When x and y are both of the opposite signs, then $|x + y| < |x| + |y|$.

In words: Modulus of the sum is less than or equal to the sum of individual modulus.

$|x^2 - 5x - 6| + |-x^2 + 5x - 4| = 10$

Step 1:

Let $a = x^2 - 5x - 6$ and $b = -x^2 + 5x - 4$

$\Rightarrow |a| + |b| = 10 \quad ...(1)$

$|a + b| = |(x^2 - 5x - 6) + (-x^2 + 5x - 4)| = |-10| = 10$

Step 2:

Use property

$|x + y| = |x| + |y|$

Here, $|a + b| = |a| + |b|$

This is possible when both a and b have the same sign i.e.,

$ab \geq 0 \Rightarrow (x^2 - 5x - 6)(-x^2 + 5x - 4) \geq 0$

Step 3:

Solve the inequality

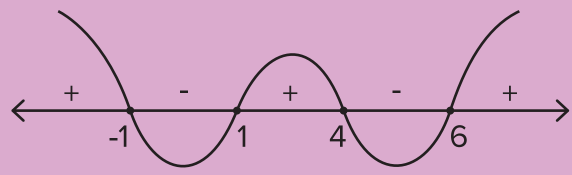
$(x^2 - 5x - 6)(-x^2 + 5x - 4) \geq 0$

$\Rightarrow (x^2 - 5x - 6)(x^2 - 5x + 4) \leq 0$

$\Rightarrow (x + 1)(x - 6)(x - 1)(x - 4) \leq 0$

Using wavy curve method.

$x \in [-1, 1] \cup [4, 6]$



Property II

$|x - y| \geq |x| - |y| \ \forall \ x, y \in R$,

Case I: When x and y are both of the same signs and $|x| > |y|$, then $|x - y| = |x| - |y|$.

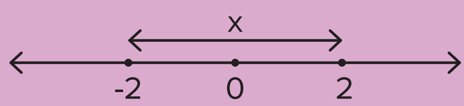
Case II: When (a) x and y are both of same signs but $|x| < |y|$ or (b) x and y are both of the opposite signs, then $|x - y| > |x| - |y|$.

In words: Modulus of the difference is greater than or equal to the difference between the modulus of first and the modulus of second.

Solve $|x| \leq 2$

Step 1:
Geometric Interpretation
$|x - 0| \leq 2 \Rightarrow$ Distance of x from 0 is less than or equal to 2

Step 2:
Plot number line.
Observe, x can sit anywhere between -2 and 2, including -2 and 2 such that $|x| \leq 2$.



Hence, solution set $x \in [-2, 2]$

Property III

(i) $|x| \leq a$, where $a > 0 \Leftrightarrow -a \leq x \leq a$ or $x \in [-a, a]$.

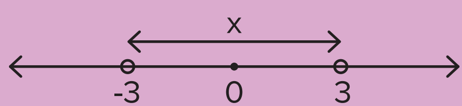
(ii) (trivial case) $|x| \leq b$, where $b < 0 \Rightarrow$ No possible solution exists as $|x| \geq 0 \ \forall \ x \in R$.

(iii) (trivial case) $|x| \leq 0 \Rightarrow |x| = 0$ (as $|x| \geq 0$) $\Rightarrow x = 0$.

Solve $|x| < 3$

Step 1:
Geometric interpretation
$|x - 0| < 3 \Rightarrow$ Distance of x from 0 is strictly less than 3.

Step 2:
Plot number line.



Observe, x can sit anywhere between -3 and 3, with end points excluded.
Hence, solution set is $x \in (-3, 3)$.

Property IV

$|x| < a$, where $a > 0 \Leftrightarrow -a < x < a$ or $x \in (-a, a)$.

What is the interval of real numbers 'x' that satisfies $|2x - 5| < 9$?

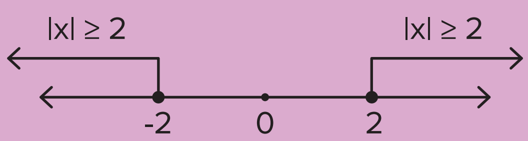
Solution

Step 1:
$|2x - 5| < 9 \Rightarrow -9 < 2x - 5 < 9 \Rightarrow -4 < 2x < 14 \Rightarrow -2 < x < 7$ ($|x| < a$, where $a > 0 \Leftrightarrow -a < x < a$)
$\Rightarrow x \in (-2, 7)$ is the solution set.

Solve $|x| \geq 2$

Step 1:
Geometric interpretation
$|x - 0| \geq 2 \Rightarrow$ Distance of x from 0 is greater than or equal to 2

Step 2:
Plot number line.

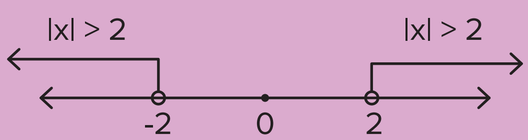


Observe, x can sit anywhere right of 2 and left of -2, including -2 and 2. Hence, the solution set is $x \in (-\infty, -2] \cup [2, \infty)$.

Solve $|x| > 2$

Step 1:
Geometric interpretation
$|x - 0| > 2 \Rightarrow$ Distance of x from 0 is strictly greater than 2

Step 2:
Plot number line



Observe x can sit anywhere right of 2 and left of -2, excluding -2 and 2. Hence, the solution set is $x \in (-\infty, -2) \cup (2, \infty)$.

Property V

(i) $|x| \geq a$, where $a > 0 \Leftrightarrow x \leq -a$ or $x \geq a \Leftrightarrow x \in (-\infty, -a] \cup [a, \infty)$

(ii) (trivial case) $|x| \geq 0 \Rightarrow x \in R$

(iii) (trivial case) $|x| > b$, where $b < 0 \Rightarrow x \in R$

Property VI

$|x| > a$, where $a > 0 \Leftrightarrow x < -a$ or $x > a \Leftrightarrow x \in (-\infty, -a) \cup (a, \infty)$

Solve $2 \leq |x| \leq 7$

Step 1:
Solve Case 1
$2 \leq |x| \Rightarrow x \in (-\infty, -2] \cup [2, \infty)$

Step 2:
Solve Case 2
$|x| \leq 7 \Rightarrow x \in [-7, 7]$

Step 3:
Final Solution Set
Case 1 $\cap$ Case 2
Solution set is
$x \in [-7, -2] \cup [2, 7]$

Property VII

$a \leq |x| \leq b$, where $a, b > 0 \Leftrightarrow x \in [-b, -a] \cup [a, b]$

Solve $2 < |x| < 7$

Step 1:
Solve Case 1
$2 < |x| \Rightarrow x \in (-\infty, -2) \cup (2, \infty)$

Step 2:
Solve Case 2
$|x| < 7 \Rightarrow x \in (-7, 7)$

Step 3:
Final Solution Set
Case 1 $\cap$ Case 2
Solution set is
$x \in (-7, -2) \cup (2, 7)$

Property VIII

$a < |x| < b$, where $a, b > 0 \Leftrightarrow x \in (-b, -a) \cup (a, b)$

Solve $||x| - 1| \geq 3$

Solution

Step 1:
Case 1
$|x| - 1 \geq 3$
$\Rightarrow |x| \geq 4$
$\Rightarrow x \geq 4$ or $x \leq -4$
$\Rightarrow x \in (-\infty, -4] \cup [4, \infty)$

Step 2:
Case 2
$|x| - 1 \leq -3$
$\Rightarrow |x| \leq -2$ which is not possible as $|x| \geq 0$.

Step 3:
Final solution
Case 1 $\cup$ Case 2
Hence the solution is
$x \in (-\infty, -4] \cup [4, \infty)$.
Because $A \cup \phi = A$

Solve $|x^2 - x - 2| = 2 + x - x^2$

Solution

Step 1:
$|x^2 - x - 2| = -(x^2 - x - 2)$. Let $y = x^2 - x - 2 \Rightarrow |y| = -y$. This is possible only when $y \leq 0$.

Step 2:
$y \leq 0 \Rightarrow x^2 - x - 2 \leq 0 \Rightarrow (x + 1)(x - 2) \leq 0 \Rightarrow x \in [-1, 2]$
The solution set for the question $|x^2 - x - 2| = x^2 - x - 2$ is $x \in [-1, 2]$.

Concept Check 1

Total number of integral solutions of x such that $x^2 - 2|x| - 15 \leq 0$

a. 11 b. 6 c. 5 d. 10

If $f(x) = \sqrt{(x^2 + 6x + 9)} - \sqrt{(x^2 - 6x + 9)}$, then plot the graph of f(x).

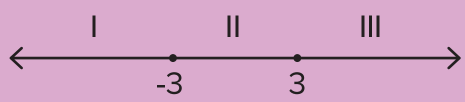
Step 1:
$f(x) = \sqrt{(x^2 + 6x + 9)} - \sqrt{(x^2 - 6x + 9)}$ $f(x) = \sqrt{(x + 3)^2} - \sqrt{(x - 3)^2} \Rightarrow f(x) = |x + 3| - |x - 3|$

Step 2:

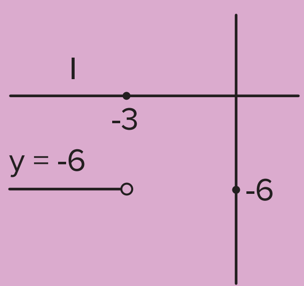
$$|x - 3| = \begin{cases} -(x - 3) : & x < 3 \\ x - 3 : & x \geq 3 \end{cases}$$

$$|x + 3| = \begin{cases} -(x + 3) : & x < -3 \\ x + 3 : & x \geq -3 \end{cases}$$

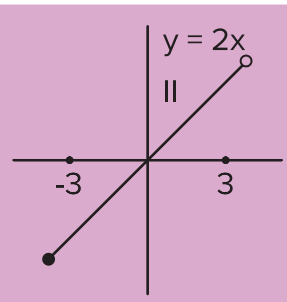
Step 3:
Region I: $x < -3$; Region II: $-3 \leq x < 3$;
Region III: $x \geq 3$.



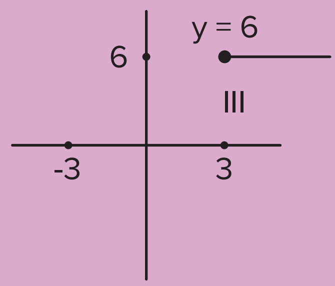
Step 4:
In region I $(x < -3)$: $|x + 3| = -(x + 3)$ and $|x - 3| = -(x - 3)$
Therefore,
$f(x) = |x + 3| - |x - 3| = -(x + 3) - (-(x - 3)) = -(x + 3) + (x - 3) = -6$



Step 5:
In region II ($-3 \leq x < 3$): $|x + 3| = (x + 3)$ and $|x - 3| = -(x - 3)$
Therefore,
$f(x) = |x + 3| - |x - 3| = (x + 3) - (-(x - 3)) = (x + 3) + (x - 3) = 2x$

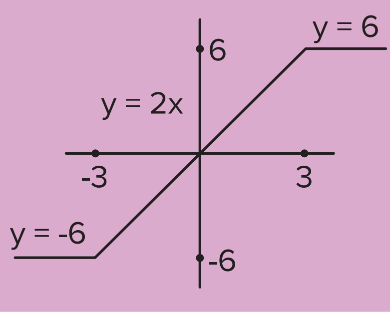


Step 6:
In region III ($x \geq 3$) : $|x + 3| = (x + 3)$ and $|x - 3| = (x - 3)$
Therefore,
$f(x) = |x + 3| - |x - 3| = (x + 3) - (x - 3) = 6$



Step 7:

Plot the graph of f(x) $y = \begin{cases} 6; x > 3 \\ 2x; -3 \leq x \leq 3 \\ -6; x < -3 \end{cases}$



If x satisfies $|x - 1| + |x - 2| + |x - 3| \geq 6$, then

(a) $0 \leq x \leq 4$ (b) $x \leq -2$ or $x \geq 3$ (c) $x \leq 0$ or $x \geq 4$ (d) None of these

Step 1:
Observe that **Solution Set 1:** $|x - 1| + |x - 2| + |x - 3| \geq 6$ and
Solution Set 2: $|x - 1| + |x - 2| + |x - 3| < 6$ are complements. We can solve Solution Set 2 and then at final stage take complement of the solution to arrive at Solution Set 1.

Step 2:
We know $|a + b + c| \leq |a| + |b| + |c|$ from triangle inequality
$\Rightarrow |(x - 1) + (x - 2) + (x - 3)| < |x - 1| + |x - 2| + |x - 3|$

Step 3:
From transitive property (if $p < q$ and $q < r$ implies $p < r$),
$|x - 1| + |x - 2| + |x - 3| < 6$
$\Rightarrow |3x - 6| < 6 \Rightarrow |3|\,|x - 2| < 6 \Rightarrow |x - 2| < 2 \Rightarrow -2 < x - 2 < 2 \Rightarrow 0 < x < 4 \Rightarrow x \in (0, 4)$

Step 4:
The required answer Solution Set 1 complements to Solution Set 2 = $x \in (0, 4)$
Therefore, the required answer is, $x \in (-\infty, 0] \cup [4, \infty)$

Concept Check 2

Solve $\left|1 - \frac{|x|}{(1 + |x|)}\right| \geq \frac{1}{2}$

Summary Sheet

Key Properties

$|x| \geq 0 \ \forall \ x \in R$;

$|x| = 0 \Leftrightarrow x = 0$;

$|x| = a \Leftrightarrow x = \pm a$, where $a > 0$;

$\sqrt{x^2} = |x| = \pm x \ \forall \ x \in R$;

$|x| = |-x| \ \forall \ x \in R$;

$|x| = |y| \Rightarrow x = \pm y \ \forall \ x, y \in R$;

$|x \ y| = |x| \ |y| \ \forall \ x, y \in R$;

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \ \forall \ x, y \in R$ and $y \neq 0$

Key Inequality Properties

$|x + y| \leq |x| + |y| \ \forall \ x, y \in R$,

$|x - y| \geq |x| - |y| \ \forall \ x, y \in R$,

$|x| \leq a$, where $a > 0 \Leftrightarrow -a \leq x \leq a$ or $x \in [-a, a]$

$|x| < a$, where $a > 0 \Leftrightarrow -a < x < a$ or $x \in (-a, a)$

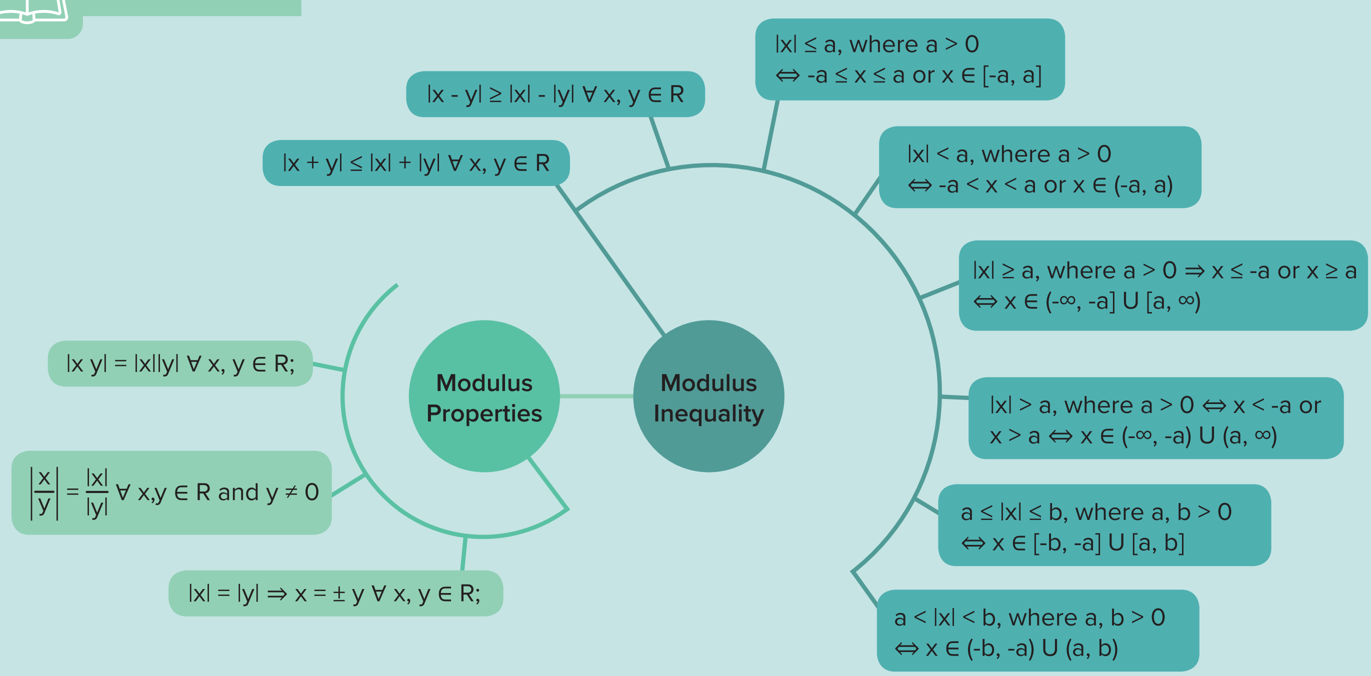
$|x| \geq a$, where $a > 0 \Leftrightarrow x \leq -a$ or $x \geq a \Leftrightarrow x \in (-\infty, -a] \cup [a, \infty)$

$|x| > a$, where $a > 0 \Leftrightarrow x < -a$ or $x > a \Leftrightarrow x \in (-\infty, -a) \cup (a, \infty)$

$a \leq |x| \leq b$, where $a, b > 0 \Leftrightarrow x \in [-b, -a] \cup [a, b]$

$a < |x| < b$, where $a, b > 0 \Leftrightarrow x \in (-b, -a) \cup (a, b)$

Mind map



Self-Assessment

1. Solve $|x - 1| \le |x^2 - 2x + 1|$

2. Solve $(x - 7)(|x| - 9) > 0$

Answers

Concept Check 1

Total number of integral solutions of x such that $x^2 - 2|x| - 15 \le 0$

a. 11 b. 6 c. 5 d. 10

Step 1:

Let $|x| = t$, then

$|x|^2 - 2|x| - 15 \le 0$

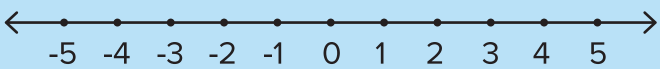
$t^2 - 2t - 15 \le 0 \Rightarrow (t - 5)(t + 3) \le 0 \Rightarrow t \in [-3, 5]$

Step 2:

$\Rightarrow |x| \in [-3, 5] \Rightarrow |x| \in [-3, 0) \cup [0, 5] \Rightarrow |x| \in [-3, 0)$ or $|x| \in [0, 5] \Rightarrow 0 \le |x| \le 5$ (|x| can't be negative)

Step 3:

$0 \le |x| \le 5 \Rightarrow -5 \le x \le 5$ ($|x| \le a \Rightarrow -a \le x \le a;\ a > 0$)



∴ Solution Set = [-5, 5], and there are 11 integral solutions.

2. Solve $\left|1 - \dfrac{|x|}{1 + |x|}\right| \ge \dfrac{1}{2}$

Step 1:

Case 1

When $x \ge 0$

$$\left|1 - \frac{x}{1 + x}\right| \ge \frac{1}{2} \Rightarrow \left|\frac{1 + x - x}{1 + x}\right| \ge \frac{1}{2} \Rightarrow \frac{1}{|1 + x|} \ge \frac{1}{2}$$

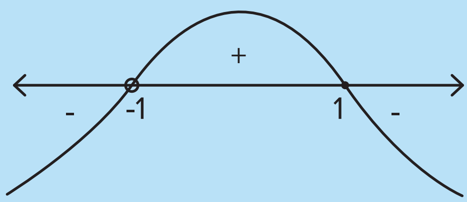
Step 2:

For $x \ge 0$, $(1 + x) > 0 \Rightarrow |1 + x| = 1 + x \Rightarrow \dfrac{1}{1 + x} \ge \dfrac{1}{2} \Rightarrow \dfrac{1}{1 + x} - \dfrac{1}{2} \ge 0 \Rightarrow \dfrac{(2 - 1 - x)}{(2(1 + x))} \ge 0 \Rightarrow \dfrac{(1 - x)}{(1 + x)} \ge 0$

Step 3:

Using wavy curve method

$$\frac{(1 - x)}{(1 + x)} \ge 0$$



$x \in (-1, 1]$

But in for case 1: we have assumed that $x \ge 0$

Therefore $x \in [0, 1]$

Step 4:
Case II
When $x < 0$

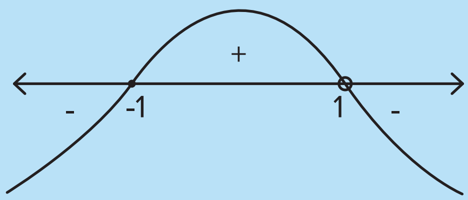
$$\left|1 - \frac{|x|}{1 + |x|}\right| \geq \frac{1}{2} \Rightarrow \left|1 - \frac{(-x)}{1 - x}\right| \geq \frac{1}{2} \Rightarrow \left|\frac{1 - x + x}{1 - x}\right| \geq \frac{1}{2} \Rightarrow \frac{1}{|1 - x|} \geq \frac{1}{2}$$

Step 5:
For $x < 0$, $(1 - x) > 0 \Rightarrow |1 - x| = 1 - x \Rightarrow \frac{1}{(1 - x)} \geq \frac{1}{2} \Rightarrow \frac{1}{(1 - x)} - \frac{1}{2} \geq 0 \Rightarrow \frac{(2 - 1 + x)}{(2(1 - x))} \geq 0$

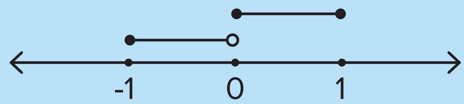
Step 6:
Using wavy curve method

$$\frac{(1 + x)}{(1 - x)} \geq 0$$

$x \in [-1, 1)$
But in our case, we have assumed that $x < 0$
Therefore, $x \in [-1,0)$



Step 7:
Final solution set: Case I ∪ Case II So, $x \in [-1, 1]$



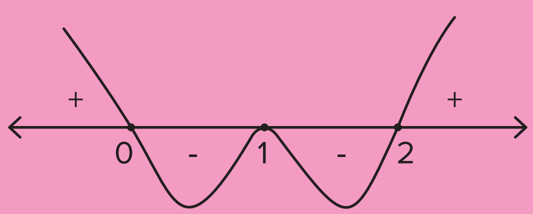
Self-Assessment

1. Solve $|x - 1| \leq |x^2 - 2x + 1|$

Step 1:
On squaring both sides, we get $\Rightarrow (x - 1)^2 \leq (x - 1)^4$ ($|x|^2 = x^2$ and $x^2 - 2x + 1 = (x - 1)^2$)

Step 2:
$(x - 1)^4 - (x - 1)^2 \geq 0 \Rightarrow (x - 1)^2 ((x - 1)^2 - 1) \geq 0 \Rightarrow (x - 1)^2 x(x - 2) \geq 0$

Step 3:
Using wavy curve method
$x \in (-\infty, 0] \cup \{1\} \cup [2,\infty)$



2. Solve $(x - 7)(|x| - 9) > 0$

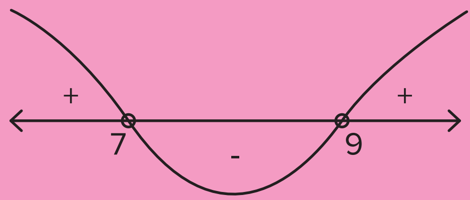
Step 1:

Case I When $x \geq 0$, We have $(x - 7)(|x| - 9) > 0 \Rightarrow (x - 7)(x - 9) > 0$

Step 2:

Using wavy curve method

$x \in (-\infty, 7) \cup (9, \infty)$ But we have already assumed in our case that $x \geq 0$ $\therefore x \in [0, 7) \cup (9, \infty)$



Step 3: Case II

When $x < 0$; $(x - 7)(|x| - 9) > 0 \Rightarrow (x - 7)(-x - 9) > 0 \Rightarrow (x - 7)(x + 9) < 0$

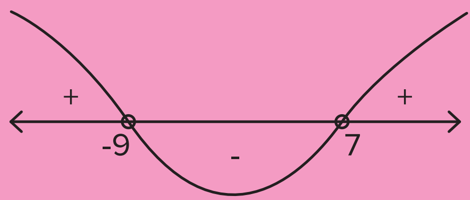
Step 4:

Using wavy curve method

$x \in (-9, 7)$

But, we have already assumed our case to be $x < 0$

$\Rightarrow x \in (-9, 0)$



Step 5:

Final solution set: Case I ∪ Case II

$\therefore x \in (-9, 7) \cup (9, \infty)$