Integration of some Functions

Q.1. Evaluate $\int x^2/(x^2 - 4)\, dx$.

Solution : 1

$\int x^{\mathbf{2}}/(x^{\mathbf{2}} - 4)\, dx = \int (x^{\mathbf{2}} - 4 + 4)/(x^{\mathbf{2}} - 4)\, dx = \int \{1 + 4/(x^{\mathbf{2}} - 4)\}dx$

$= x + 4 \times \{1/(2\times2)\} \log [(x - 2)/(x + 2)] + c$

$= x + \log [(x - 2)/(x+2)] + c$.

Q.2. Integrate the following : (2 sin x + 3 cos x)/(5 sin x + 12 cos x).

Solution : 2

Let 2 sin x + 3 cos x ≡ A(5 sin x + 12 cos x) + B.d/dx(5 sin x + 12 cos x)

≡ A(5 sin x + 12 cos x) + B(5 cos x – 12sin x)

≡ sin x (5A – 12B) + cos x (12A + 5B)

Equating the coefficients of sin x and cos x, we get

5A – 12B = 2, 12A + 5B = 3.

Solving, we get A = 46/169 and B = – 9/169.

Therefore, 2 sin x + 3 cos x = 46/169(5 sin x + 12 cos x) – 9/169(5 cos x – 12 sin x).

Therefore, $\int$[(2 sin x + 3 cos x)/(5 sin x + 12 cos x)].dx

= (46/169)$\int$[(5 sin x + 12 cos x)/(5 sin x + 12 cos x)].dx – (9/169)$\int$[(5 cos x – 12 sin x)/(5 sin x + 12 cos x)].dx

=(46/169)$\int$dx – (9/169)$\int$[(5 cos x – 12 sin x)/(5 sin x + 12 cos x)].dx

= (46/169)x – (9/169)$\int$dt/t, [where 5 sin x + 12 cos x = t] = (46/169)x – log $|t|$ + c

= (46/169)x – (9/169)log$|5 \sin x + 12 \cos x|$ + c.

Q.3. Evaluate : $\int[1/(1 + \tan x)].dx$.

Solution : 3

Let $I = \int[1/(1 + \tan x)].dx$

$1/(1 + \tan x) = 1/(1 + \sin x/\cos x) = \cos x/(\cos x + \sin x)$

$= 1/2\ [(\cos x + \sin x + \cos x - \sin x)/(\cos x + \sin x)]$

$= 1/2\ [1 + (\cos x - \sin x)/(\cos x + \sin x)]$

Therefore, $I = 1/2\ [\int 1.dx + \int[(\cos x - \sin x)/(\cos x + \sin x)].dx$

$= 1/2\ [x + \log|\sin x + \cos x|] + c$.

Q.4. Evaluate : $\int dx/(a \sin x + b \cos x)$.

Solution : 4

Let $I = \int dx/(a \sin x + b \cos x)$.

Put $a = r \cos \alpha$ and $b = r \sin \alpha$, so that $r^2 = a^2 + b^2$ and $\alpha = \tan - 1\ (b/a)$

Therefore, $I = \int dx/(a \sin x + b \cos x) = \int dx/(r \cos \alpha \sin x + r \sin \alpha \cos x)$

$= (1/r)\int dx/\sin (x + \alpha) = (1/r)\int \text{cosec}\ (x + \alpha).dx$

$= (1/r) \log|\tan \{(x + \alpha)/2\}| + c$

$= \{1/\sqrt{(a^2 + b^2)}\}\log|\tan \{x/2 + 1/2 \tan - 1\ (b/a)| + c$.

Q.5. Evaluate : $\int d\theta/(\sin^4 \theta + \cos^4 \theta)$.

Solution : 5

Let $I = \int d\theta/(\sin^4 \theta + \cos^4 \theta)$.

Dividing both num. and denom. by $\cos^4 \theta$, we get

$I = \int \sec^4 \theta d\theta/(1 + \tan^4 \theta)$; Put $\tan \theta = t \Rightarrow \sec^2 \theta d\theta = dt$

Therefore, I = $\int[(1 + t^2)/(1 + t^4)]dt$, Dividing Num. and Denom. By t^2 , we get

I = $\int[\{1 + (1/t^2)\}/\{t^2 + (1/t^2)\}]dt$ Put t – 1/t = z => $(1 + 1/t^2)dt = dz$

And t2 + $1/t^2 = (t - 1/t)^2 + 2 = 2 + z^2$,

Therefore, I = $\int dz/(2 + t^2)$ = 1/√2 tan −1 (z/√2) +c

= 1/√2 tan −1 $\{(t^2 - 1)t\sqrt{2}\}$+c = 1/√2 tan −1$\{(\tan^2 \theta - 1)/\tan \theta\sqrt{2}\}$ + c.

Q.6. Evaluate : $\int dx/(a^2 \sin^2 x + b^2 \cos^2 x)$.

Solution : 6

Let I = $\int dx/(a^2 \sin^2 x + b^2\cos^2 x)$ [Dividing Nr and Dr by $\cos^2 x$]

= $\int(\sec^2 x\, dx)/(a^2\tan^2 x + b^2)$

= $\int dt/(a^2 t^2 + b^2)$ [Putting tan x

= t so that $\sec^2 x\, dx = dt$] = $1/a^2\int dt/\{t^2 + (b/a)^2\}$

= $(1/a^2)$\{1/(b/a)\}tan −1 \{t/(b/a)\} + c = (1/ab) tan −1 (at/b) + c

= (1/ab) tan −1 \{(a tan x)/b\} + c.

Q.7. Evaluate : $\int dx/(2 + \cos x)$.

Solution : 7

$\int dx/(2 + \cos x) = \int dx/\{1 + (1 + \cos x)\}$

= $\int dx/\{1 + 2 \cos^2 (x/2)\}$

= $\int[\{\sec^2 (x/2)\}/\{\sec^2 (x/2) + 2\}].dx$

[Dividing Nr & Dr by $\cos^2 (x/2)$]

= $\int[\{\sec^2 (x/2)\}/\{3 + \tan^2 (x/2)\}].dx$

[Put tan (x/2) = t so that $[\{\sec^2 (x/2)\}/(1/2)]dx = dt$ = 2∫dt/(3 + t2)

$= 2\int dt/\{(\sqrt{3})^2 + t^2\} = 2.(1/\sqrt{3}) \tan - 1 (t/\sqrt{3}) + c$

$= (2/\sqrt{3})\tan - 1 [\{\tan (x/2)\}/\sqrt{3}] + c.$

Q.8. Evaluate : $\int dx/(5 + 4 \cos x)$.

Solution : 8

$\int dx/(5 + 4 \cos x) = \int dx/[5 + 4 \{1 - \tan^2 (x/2)\}/\{1 + \tan^2 (x/2)\}]$

$= \int[\{1 + \tan^2 (x/2)\}/\{9 + \tan2 (x/2)\}].dx$

[Put $\tan (x/2) = t$, then $\sec^2 (x/2)\ dx = dt$]

$= 2\int dt/(32 + t^2)$

$= (2/3) \tan - 1 (t/3) + c$

$= (2/3) \tan - 1 [\{\tan (x/2)\}/3] + c.$

Q.9. Evaluate : $\int[x/\sqrt{(4 - x^4)}].dx$.

Solution : 9

Let $I = \int[x/\sqrt{(4 - x^4)}].dx$, Put $x^2 = t$ then $2xdx = dt$ or, $xdx = 1/2\ dt$

Therefore, $I = \int[(1/2dt)/\sqrt{(4 - t^2)}$

$= (1/2) \int dt/\sqrt{(2^2 - t^2)}$

$= (1/2) \sin -1 (t/2) + c$

$= (1/2) \sin -1 (x^2/2) + c.$

Q.10. Evaluate : $\int[\sqrt{(a - x)}/\sqrt{(a + x)}].dx$.

Solution : 10

Let $I = \int[\sqrt{(a - x)}/\sqrt{(a + x)}].dx$

$= \int[\sqrt{(a - x)}/\sqrt{(a + x)}].[\sqrt{(a - x)}/\sqrt{(a - x)}].dx$

$= \int[(a - x)/\sqrt{(a^2 - x^2)}].dx = a\int dx/\sqrt{(a^2 - x^2)} - \int[x/\sqrt{(a^2 - x^2)}].dx$

$= a \sin - 1 (x/a) + 1/2 \int dt/\sqrt{(t)}$ [

Where $a^2 - x^2 = t$ so that $- 2xdx = dt$]

Therefore, $I = a \sin -1 (x/a) + (1/2) .2\sqrt{(t)} + c$

$= a \sin -1 (x/a) + \sqrt{(a^2 - x^2)} + c.$

Q.11. Evaluate : $\int[(6x + 5)/\sqrt{(6 + x - 2x^2)}].dx.$

Solution : 11

Let $I = \int[(6x + 5)/\sqrt{(6 + x - 2x^2)}].dx$

Let $6x + 5 = A. d/dx (6 + x - 2x^2) + B = A (1 - 4x) + B.$

Equating coefficients of x and constants, we get

$- 4A = 6$ and $A + B = 5 \Rightarrow A = - 3/2, B = 13/2.$

Therefore, $I = \int[\{- 3/2(1 - 4x) + 13/2\}/\sqrt{(6 + x - 2x^2)}].dx$

$= - 3/2\int[(1 - 4x)/\sqrt{(6 + x - 2x^2)}].dx + 13/2\int dx/\sqrt{(6 + x - 2x^2)}$

$= - 3/2[\{\sqrt{(6 + x - 2x^2)}\}/(1/2)] + 13/2\int[1/\sqrt{\{2(3 + x/2 - x^2)\}}].dx$

$= - 3\sqrt{(6 + x - 2x^2)} + \{13/(2\sqrt{2})\}\int[1/\sqrt{\{(7/4)^2 - (x - 1/4)^2\}}].dx$

$= - 3\sqrt{(6 + x - 2x^2)} + \{13/(2\sqrt{2})\}\sin - 1 \{(x - 1/4)/(7/4)\} + c$

$= - 3\sqrt{(6 + x - 2x^2)} + \{13/(2\sqrt{2})\}\sin - 1 \{(4x - 1)/7\} + c.$

Q.12. Evaluate : $\int[\sqrt{\{(1 + x)/x\}}].dx.$

Solution : 12

Let I = $\int[\sqrt{\{(1 + x)/x\}}].dx$

= $\int[\sqrt{\{(1 + x)/x\}}].[\sqrt{\{(1 + x)/(1 + x)\}}].dx$

= $\int[(1 + x)/\sqrt{\{x(1 + x)\}}].dx = \int[(1 + x)/\sqrt{(x^2 + x)}].dx$

Let 1 + x = A. $d/dx(x^2 + x) + B = A (2x + 1) + B$

Equating coefficients of like terms, we get A = 1/2, B = 1/2.

Therefore, I = $\int[\{1/2 (2x + 1) + 1/2\}/\{\sqrt{(x^2 + x)}\}].dx$

= $1/2\int[(2x + 1)/\sqrt{(x^2 + x)}].dx + 1/2\int[1/\sqrt{(x^2 + x)}].dx$

= $I_1 + I_2$ I1 = $1/2\int[(2x + 1)/\sqrt{(x^2 + x)}].dx$ [Put $x^2 + x = t$ then $(2x + 1)dx = dt$]

Therefore, $I_1 = 1/2\int dt/\sqrt{t} = \sqrt{t}$;

And I2 = $1/2\int[1/\sqrt{(x^2 + x)}].dx$

= $1/2\int[1/\sqrt{\{(x + 1/2)^2 - (1/2)2\}}].dx$

= $1/2\log|(x + 1/2) + \sqrt{(x^2 + x)}|$

Therefore, I = I_1 + I2 = $\sqrt{(x^2 + x)} + 1/2 \log|(2x + 1)/2 + \sqrt{(x^2 + x)}| + c$.

Q.13. Evaluate : $\int[\sqrt{(2ax - x^2)}].dx$.

Solution : 13

Let I = $\int[\sqrt{(2ax - x^2)}].dx$

= $\int[\sqrt{\{a^2 - (x - a)^2\}}].dx$ [Put x – a = t then dx = dt]

Therefore, I = $\int \sqrt{(a^2 - t^2)}.dt$

= $(1/2)t \sqrt{(a^2 - t^2)} + (1/2) a^2 \sin - 1 (t/a) + c$

=$1/2 (x - a)\sqrt{\{a^2 - (x - a)^2\}} + (1/2) a^2 \sin -1 \{(x - a)/a\} + c$

= $1/2 (x - a)\sqrt{(2ax - x^2)} + (1/2)a^2 \sin -1 \{(x - a)/a\} + c$.

Q.14. Evaluate the following integral : $\int e^{2x}/(2 + e^x)\, dx$.

Solution : 14

Put 2 + e^x = t , then e^x d^x = dt

Therefore, ∫e2^x/(2 + e^x)d^x= ∫[(t – 2)2/t]. dt/(t – 2)

= ∫[(t – 2)/t] dt

= t – 2 log | t | + c

= e^x – 2 log | e^x + 2 | + c.