1

Logarithms

KEY FACTS

1. Definition: If a and n are positive real numbers such that $a \neq 1$ and x is real, then $a^x = n \Rightarrow x = \log_a n$. Here x is said to be the logarithm of the number n to the base a.

Ex. $4^3 = 64 \Rightarrow \log_4 64 = 3$, $10^{-1} = \frac{1}{10} = 0.1 \Rightarrow \log_{10} 0.1 = -1$, $5^x = 4 \Rightarrow x = \log_5 4$,

$a^0 = 1 \Rightarrow \log_a 1 = 0$, $a^1 = a \Rightarrow \log_a a = 1$.

2. Some Important Facts about Logarithms

- $\log_a n$ is real if $n > 0$
- $\log_a n$ is imaginary if $n < 0$
- $\log_a n$ is not defined if $n = 0$
- The logarithm of 1 to any base ***a***, $a > 0$ and $a \neq 1$ is zero. $\boxed{\log_a 1 = 0}$
- The logarithm of any number ***a***, $a > 0$ and $a \neq 1$, to the same base is 1. $\boxed{\log_a a = 1}$
- If ***a*** and ***x*** are positive real numbers, where $a \neq 1$, then $\boxed{a^{\log_a x} = x}$

Proof. Let $\log_a x = p$. Then, $x = a^p$ (By def.) $\Rightarrow x = a^{\log_a x}$ (Substituting the value of p)

Ex. $3^{\log_3 7} = 7$, $2^{\log_2 9} = 9$, $5^{\log_5 x} = x$

- For $a > 0$, $a \neq 1$, $\mathbf{\log_a x_1 = \log_a x_2 \Rightarrow x_1 = x_2\ (x_1, x_2 > 0)}$
- If $a > 1$ and $x > y$, then $\log_a x > \log_a y$.
- If $0 < a < 1$ and $x > y$, then $\log_a x < \log_a y$

3. Laws of Logarithms

For $x > 0$, $y > 0$ and $a > 0$ and $a \neq 1$, any real number n

- $\mathbf{\log_a xy = \log_a x + \log_a y}$ **Ex.** $\log_2(15) = \log_2(5 \times 3) = \log_2 5 + \log_2 3$
- $\mathbf{\log_a (x/y) = \log_a x - \log_a y}$ **Ex.** $\log_2\left(\frac{3}{7}\right) = \log_2 3 - \log_2 7$
- $\mathbf{\log_a (x)^n = n \log_a x}$ **Ex.** $\log (2)^5 = 5 \log 2$,

  $\log\left(\frac{a^3}{b^3}\right) = \log a^3 - \log b^3 = 3 \log a - 3 \log b$
- $\mathbf{\log_a x = \frac{1}{\log_x a}}$ **Ex.** $\log_5 2 = \frac{1}{\log_2 5}$
- $\mathbf{\log_{a^n} x = \frac{1}{n} \log_a x}$ **Ex.** $\log_8 7 = \log_{2^3}(7) = \frac{1}{3} \log_2 7$, $\log_{\sqrt{5}} 3 = \log_{(5)^{\frac{1}{2}}} (3) = \log_{5^{1/2}} (3) = \frac{1}{1/2} \log_5 3 = 2 \log_5 3$
- $\mathbf{\log_{a^n} x^m = \frac{m}{n} \log_a x}$ **Ex.** $\log_{2^5} 5^4 = \frac{4}{5} \log_2 5$

Base changing formula

- $\log_a x = \log_b x . \log_a b$ **Ex.** $\log_{\underset{\underset{\textbf{Old base}}{\uparrow}}{12}} 32 = \log_{\underset{\underset{\textbf{New base}}{\uparrow}}{16}} 32 . \log_{12} 16.$ (The base has been changed from 12 to 16)
- $x^{\log_a y} = y^{\log_a x}$ **Ex.** $3^{\log 7} = 7^{\log 3}$ (It being understood that base is same)

[**Proof**. $x^{\log_a y} \xrightarrow{} = x^{\log_x y \, . \, \log_a x}$ (Base changing formula)

$$= \left(x^{\log_x y}\right)^{\log_a x} \quad \text{(Using } n \log_a x = (\log_a x)^n\text{)}$$

$$= y^{\log_a x} \quad \text{(Using } x^{\log_x y} = y.\text{)}$$

- $\log_a b = \dfrac{\log b}{\log a}$ (It being understood that base is same)
- If $\log_a b = x$ for all $a > 0$, $a \neq 1$, $b > 0$ and $x \in R$, then $\log_{1/a} b = -x$, $\log_a 1/b = -x$ and $\log_{1/a} 1/b = x$

4. Some Important Properties of Logarithms

- a, b, c are in G.P. $\Leftrightarrow \log_a x, \log_b x, \log_c x$ are in H.P.
- a, b, c are in G.P. $\Leftrightarrow \log_x a, \log_x b, \log_x c$ are in A.P.

5. Natural or Naperian logarithm **is denoted by $\log_e N$, where the base is e.**

Ex. $\log_e 7$, $\log_e \left(\dfrac{1}{64}\right)$, $\log_e b$, etc.

- **Common or Brigg's logarithm** is denoted by $\mathbf{\log_{10} N}$, where the base is **10.**

  Ex. $\log_{10} 5$, $\log_{10}\left(\dfrac{1}{81}\right)$, etc.
- $\log_a x$ is a decreasing function if $0 < a < 1$
- $\log_a x$ is an increasing function if $a > 1$.

6. Characteristic and Mantissa

- **Characteristic:** The integral part of the logarithm is called characteristic.

(*i*) If the number is greater than unity and there are n digits in integral part, then its characteristic = $(n - 1)$

(*ii*) When the number is less than 1, the characteristic is one more than the number of *zeroes* between the decimal point and the first significant digit of the number and is negative. It is written as $\overline{(n+1)}$ or Bar $(n + 1)$.

Ex.

Number	Characteristic	Number	Characteristic
4.1456	0	0.823	$\overline{1}$
24.8920	1	0.0234	$\overline{2}$
238.1008	2	0.000423	$\overline{4}$

7. Arithmetic Progression. **A sequence** $a_1, a_2, a_3, ..., a_n$ is said to be in arithmetic progression, when $a_2 - a_1 = a_3 - a_2 = ... = a_n - a_{n-1}$, *i.e.*, when the terms in the sequence increase or decrease by a constant quantity called the ***common difference***.

Ex. 1, 3, 5, 7, 9, ... 6, 11, 17, 23, ... – 5, –2, 1, 4, 7, ...

- **Sum of first '*n*' terms of an Arithmetic Progression**

  $$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l],$$

  where a = first term, n = number of terms, d = common difference, l = last term.
- **Sum of first "*n*" natural numbers.**

  $$S_n = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}.$$

Also, written as $\Sigma \boldsymbol{n} = \dfrac{\boldsymbol{n(n+1)}}{\mathbf{2}}$

- Also, **if *a, b, c* are in A.P.** then $\mathbf{2}\boldsymbol{b} = \boldsymbol{a + c}$

8. Geometric Progression : A sequence $a_1, a_2, a_3, \ldots\ldots\ldots\ldots, a_n$ is said to be in Geometric Progression when,

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots\ldots\ldots\ldots = \frac{a_n}{a_{n-1}} = r(\text{say})$$

where $a_1, a_2, a_3, \ldots\ldots\ldots$ are all non zero numbers and $\boldsymbol{r}$ is called the **common ratio.**

Ex. 3, 6, 12, 24, ... $\boldsymbol{r = 2;}$

$64, 16, 4, 1, \dfrac{1}{4}, \dfrac{1}{16} . \dfrac{1}{64}, \ldots\ldots\ldots\ldots\ r = \dfrac{1}{4}$

- **Sum of first *n* terms of a G.P.** $S_n = \dfrac{a(r^n - 1)}{(r-1)}$ **if** $r > 1 = \dfrac{a(1-r^n)}{(1-r)}$ **if** $r < 1 = \dfrac{lr - a}{r - 1}$

  where, a = first term, r = common ratio, l = last term

- **Sum of an infinite G.P.** $S_\infty = \dfrac{a}{1-r}$, where a = first term, r = common ratio.

- For three terms a, b, c to be in G.P., $\boldsymbol{b^2 = ac}$

9. Harmonic Progression : A series of quantities $a_1, a_2, a_3, \ldots\ldots\ldots\ldots, a_n$ are said to be in H.P. when their reciprocals $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \ldots\ldots\ldots\ldots, \dfrac{1}{a_n}$ are in A.P.

- When three quantities a, b, c are in H.P., then, $b = \dfrac{2ac}{a+c}$.

SOLVED EXAMPLES

Ex. 1. If $\log_a 5 + \log_a 25 + \log_a 125 + \log_a 625 = 10$, then find the value of *a*.

Sol. $\log_a 5 + \log_a 25 + \log_a 125 + \log_a 625 = 10$

$\Rightarrow \log_a (5 \times 25 \times 125 \times 625) = 10$

$\Rightarrow \log_a (5^1 \times 5^2 \times 5^3 \times 5^4) = 10$

$\Rightarrow \log_a 5^{10} = 10 \Rightarrow a^{10} = 5^{10} \Rightarrow \boldsymbol{a = 5.}$ [Using $\log_a x = n \Rightarrow x = a^n$]

Ex. 2. Solve for *x* : $\log_{10} [\log_2 (\log_3 9)] = x$.

Sol. $\log_{10} [\log_2 (\log_3 9)] = x$

$\Rightarrow \log_2 (\log_3 9) = 10^x$

$\Rightarrow \log_2 (\log_3 3^2) = 10^x$

$\Rightarrow \log_2 (2 \log_3 3) = 10^x$

$\Rightarrow \log_2 2 = 10^x \Rightarrow 10^x = 1 = 10^0 \Rightarrow \boldsymbol{x = 0}.$

Ex. 3. Find the value of $\log_x x + \log_x x^3 + \log_x x^5 + \ldots\ldots\ldots + \log_x x^{2n-1}$.

Sol. $\log_x x + \log_x x^3 + \log_x x^5 + \ldots\ldots\ldots + \log_x x^{2n-1} = \log_x x + 3 \log_x x + 5 \log_x x + \ldots\ldots\ldots + (2n-1) \log_x x$

$= 1 + 3 + 5 + \ldots\ldots\ldots\ldots + (2n-1) = \dfrac{n}{2}[1 + (2n-1)] = \boldsymbol{n^2}$ $\left[\text{Using } \log_x x = 1 \text{ and for A.P. } S_n = \dfrac{n}{2}(a + l)\right]$

Ex. 4. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

Sol. $$f\left(\frac{2x}{1+x^2}\right) = \log\left[\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right] = \log\left[\frac{1+x^2+2x}{1+x^2-2x}\right] = \log\left[\frac{(1+x)^2}{(1-x)^2}\right] = 2\log\left[\frac{1+x}{1-x}\right] = \mathbf{2f(x)}.$$

Ex. 5. If $a = \log_{24}12$, $b = \log_{36}24$, $c = \log_{48}36$, then prove that $1 + abc = 2bc$.

Sol. $1 + abc = 1 + \log_{24}12 . \log_{36}24 . \log_{48}36 = 1 + \log_{36}12 . \log_{48}36$

$= 1 + \log_{48}12 = \log_{48}48 + \log_{48}12$ $\quad$ [$\because \log_a x . \log_b a = \log_b x$]

$= \log_{48}(48 \times 12) = \log_{48}(24 \times 24)$

$= \log_{48}(24)^2 = 2\log_{48}24.$ $\quad$ ...(*i*)

Also, $\quad 2bc = 2\log_{36}24 . \log_{48}36 = 2\log_{48}24$ $\quad$ ...(*ii*)

From (*i*) and (*ii*), we have RHS = LHS.

Ex. 6. Solve $\log_{2x+3}(6x^2 + 23x + 21) = 4 - \log_{3x+7}(4x^2 + 12x + 9)$.

Sol. Given, $\log_{(2x+3)}(6x^2 + 23x + 21) = 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$

$\Rightarrow \log_{(2x+3)}(2x+3)(3x+7) = 4 - \log_{(3x+7)}(2x+3)^2$

$\Rightarrow \log_{(2x+3)}(2x+3) + \log_{(2x+3)}(3x+7) = 4 - 2\log_{(3x+7)}(2x+3)$

$\Rightarrow \log_{(2x+3)}(3x+7) + 2\log_{(3x+7)}(2x+3) = 4 - 1 = 3$ $\quad$ [Since $\log_{2x+3}(2x+3) = 1$]

$$\Rightarrow \log_{(2x+3)}(3x+7) + \frac{2}{\log_{(2x+3)}(3x+7)} = 3 \qquad \left[\text{Using } \log_a x = \frac{1}{\log_x a}\right]$$

Let $\log_{(2x+3)}(3x+7) = t$. Then,

$$t + \frac{2}{t} = 3 \Rightarrow t^2 - 3t + 2 = 0 \Rightarrow (t-1)(t-2) = 0 \Rightarrow t = 1, 2$$

$t = 1 \Rightarrow \log_{(2x+3)}(3x+7) = 1 \Rightarrow \log_{(2x+3)}(3x+7) = \log_{(2x+3)}(2x+3)$ [Replacing 1 by $\log_{(2x+3)}(2x+3)$]

$\Rightarrow 3x + 7 = 2x + 3 \Rightarrow x = -4.$

$t = 2 \Rightarrow \log_{(2x+3)}(3x+7) = 2 \Rightarrow \log_{(2x+3)}(3x+7) = \log_{(2x+3)}(2x+3)^2$

$\Rightarrow (3x+7) = (2x+3)^2 \Rightarrow 4x^2 + 9x + 2 = 0 \Rightarrow (4x+1)(x+2) = 0 \Rightarrow x = -1/4, -2$

But $x = -4$ and -2 are extraneous solutions, so $x = -\mathbf{\frac{1}{4}}$.

Ex. 7. If $\log_x(a - b) - \log_x(a + b) = \log_x(b/a)$, find $\frac{a^2}{b^2} + \frac{b^2}{a^2}$. ***(CAT 2012)***

Sol. Given, $\log_x(a - b) - \log_x(a + b) = \log_x(b/a) \Rightarrow \log_x\left[\frac{(a-b)}{(a+b)}\right] = \log_x\left(\frac{b}{a}\right)$

$\Rightarrow a(a - b) = b(a + b) \Rightarrow a^2 - ab = ab + b^2$

$\Rightarrow a^2 - b^2 = 2ab \Rightarrow a^2 - 2ab - b^2 = 0 \Rightarrow \left(\frac{a}{b}\right)^2 - 2\left(\frac{a}{b}\right) - 1 = 0$

This is a quadratic equation in $\frac{a}{b}$ and the product of the roots is -1 *i.e*, if a/b is a root, then $\left(-\frac{b}{a}\right)$ is the other root. Also, sum of its roots = 2

$$\therefore \left(\frac{a}{b}\right)^2 + \left(\frac{b}{a}\right)^2 = \frac{a^2}{b^2} + \frac{b^2}{a^2} = \left[\frac{a}{b} + \left(-\frac{b}{a}\right)\right]^2 + 2 = 2^2 + 2 = \mathbf{6}.$$

Ex. 8. If $\log_e 2.\ \log_b 625 = \log_{10} 16.\ \log_e 10$, then find the value of b.

Sol. Given, $\log_e 2.\ \log_b 625 = \log_{10} 16.\ \log_e 10 \Rightarrow \log_e 2.\ \log_b 5^4 = \log_{10} 2^4.\ \log_e 10$

$\Rightarrow\ \log_e 2.\ 4 \log_b 5 = 4 \log_{10} 2.\ \log_e 10$

$\Rightarrow\ \log_b 5. = \dfrac{\log_{10} 2. \log_e 10}{\log_e 2} = \dfrac{\log_e 2}{\log_e 2} = 1\ \Rightarrow\ b^1 = 5 \Rightarrow b = \mathbf{5}.$ $\quad [\because \log_a x. \log_x b = \log_a b]$

Ex. 9. If $(x^4 - 2x^2y^2 + y^2)^{a-1} = (x - y)^{2a}\ (x + y)^{-2}$, then the value of a is

(*a*) $x^2 - y^2$ (*b*) $\log(xy)$ (*c*) $\dfrac{\log(x-y)}{\log(x+y)}$ (*d*) $\log(x - y)$

Sol. Given, $(x^4 - 2x^2y^2 + y^2)^{a-1} = (x - y)^{2a}\ (x + y)^{-2}$

$\Rightarrow\ [(x^2 - y^2)^2]^{a-1} = (x - y)^{2a}\ (x + y)^{-2}$

$\Rightarrow\ (x - y)^{2(a-1)}\ (x + y)^{2(a-1)} = (x - y)^{2a}\ (x + y)^{-2}$

$\Rightarrow\ \dfrac{(x-y)^{2(a-1)}}{(x-y)^{2a}} . \dfrac{(x+y)^{2(a-1)}}{(x+y)^{-2}} = 1 \Rightarrow (x-y)^{-2}(x+y)^{2a} = 1$

$\Rightarrow\ \log\,[(x - y)^{-2}\ (x + y)^{2a}] = \log 1 \Rightarrow -2\log(x - y) + 2a \log(x + y) = \log 1$

$\Rightarrow\ 2a \log(x + y) = 2 \log(x - y) \Rightarrow a = \mathbf{\dfrac{\log(x-y)}{\log(x+y)}}.$ $\quad$ [Since log 1 = 0]

Ex. 10. If $\log_x a$, $a^{x/2}$ and $\log_b x$ are in GP, then x is

(*a*) $\log_a(\log_b a)$ (*b*) $\log_a(\log_e a) + \log_a(\log_e b)$
(*c*) $-\log_a(\log_a b)$ (*d*) $\log_a(\log_e b) - \log_a(\log_e a)$

Sol. If $\log_x a$, $a^{x/2}$ and $\log_b x$ are in GP, then $\left(a^{x/2}\right)^2 = (\log_b x) \times (\log_x a)$

$\Rightarrow\ a^x = \log_b a \Rightarrow \log a^x = \log(\log_b a) \Rightarrow x \log a = \log(\log_b a) \Rightarrow x \log_a a = \log_a(\log_b a)$

$\Rightarrow\ \boldsymbol{x = \log_a(\log_b a)}.$

Ex. 11. What is the least value of the expression $2 \log_{10} x - \log_x(1/100)$ for $x > 1$?

Sol. $2 \log_{10} x - \log_x \dfrac{1}{100} = 2 \log_{10} x - \dfrac{\log_{10} 10^{-2}}{\log_{10} x}$ $\quad \left[\text{Using } \log_a b = \dfrac{\log_x b}{\log_x a}\right]$

$= 2\log_{10} x + \dfrac{2}{\log_{10} x} = 2\left(\log_{10} x + \dfrac{1}{\log_{10} x}\right)$

Given, $x > 1 \Rightarrow \log_{10} x > 0$

But since AM ≥ GM

$$\therefore\ \left[\frac{\log_{10} x + \dfrac{1}{\log_{10} x}}{2}\right] \ge \sqrt{\log_{10} x \times \frac{1}{\log_{10} x}}$$

$\Rightarrow\ \log_{10} x + \dfrac{1}{\log_{10} x} \ge 2\ \Rightarrow 2\left[\log_{10} x + \dfrac{1}{\log_{10} x}\right] \ge 4$

For $x = 10$, $2[\log_{10} x + \log_{10} x] \ge 4$

Hence, the least value of $\left[\log_{10} x - \log_x \dfrac{1}{100}\right]$ is 4.

Ex. 12. If $\log_3 2$, $\log_3(2^x - 5)$ and $\log_3(2^x - 7/2)$ are in A.P., then what is the value of x ?

Sol. Given, $\log_3 2$, $\log_3(2^x - 5)$ and $\log_3(2^x - 7/2)$ are in A.P.

$\Rightarrow\ 2\left[\log_3(2^x - 5)\right] = \log_3 2 + \log_3\left(2^x - \dfrac{7}{2}\right)$

$\Rightarrow$ $\log_3 (2^x - 5)^2 = \log_3 [2 \times (2^x - 7/2)]$

$\Rightarrow$ $(2^x - 5)^2 = (2^{x+1} - 7) \Rightarrow 2^{2x} - 10.2^x + 25 = 2.2^x - 7$

$\Rightarrow$ $2^{2x} - 12.2^x + 32 = 0 \Rightarrow y^2 - 12y + 32 = 0$ [Let $y = 2^x$]

$\Rightarrow$ $(y - 8)(y - 4) = 0 \Rightarrow y = 8$ or $4 \Rightarrow 2^x = 8$ or $2^x = 4 \Rightarrow x =$ **3 or 2.**

Ex. 13. Let $u = (\log_2 x)^2 - 6(\log_2 x) + 12$, where x is a real number. Then the equation $x^u = 256$ has :

(*a*) No solution for *x* **(*b*) Exactly one solution for *x***

(*c*) Exactly two distinct solutions for *x* **(*d*) Exactly three distinct solutions for *x*** ***(CAT 2004)***

Sol. Given, $u = (\log_2 x)^2 - 6(\log_2 x) + 12 = p^2 - 6p + 12$ (where $p = \log_2 x$) ...(*i*)

Also, given, $x^u = 256$

Taking log to the base 2 of both the sides, we have

$$u \log_2 x = \log_2 256 = \log_2 2^8 = 8 \log_2 2 \Rightarrow u \log_2 x = 8 \Rightarrow u = \frac{8}{\log_2 x} = 8/p \quad ...(ii)$$

From (*i*) and (*ii*) $\frac{8}{p} = p^2 - 6p + 12$

$\Rightarrow$ $8 = p^3 - 6p^2 + 12 \Rightarrow p^3 - 6p^2 + 12p - 8 = 0$

$\Rightarrow$ $(p - 2)^3 = 0 \Rightarrow p = 2.$ [Using $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$]

$\therefore$ $\log_2 x = 2 \Rightarrow x = 2^2 = 4$

Hence the equation $u^4 = 256$ has **exactly one solution.**

Ex. 14. If $\log_y x = (a.\ \log_z y) = (b.\ \log_x z) = ab$, then which of the following pairs of values for (a, b) is not possible ?

(*a*) $\left(-2, \frac{1}{2}\right)$ **(*b*) (1, 1)** **(*c*) (0.4, 2.5)** **(*d*) $\left(\pi, \frac{1}{\pi}\right)$** **(*e*) (2, 2)**

(CAT 2004)

Sol. Given, $\log_y x = (a.\ \log_z y) = (b \log_x z) = ab$

$$\Rightarrow a = \frac{\log_y x}{\log_z y} \text{ and } b = \frac{\log_y x}{\log_x z}$$

$$\therefore a \times b = \frac{\log_y x}{\log_z y} \times \frac{\log_y x}{\log_x z} = \frac{\left[\frac{\log x}{\log y}\right]}{\left[\frac{\log y}{\log z}\right]} \times \frac{\left[\frac{\log x}{\log y}\right]}{\left[\frac{\log z}{\log x}\right]} = \left(\frac{\log x}{\log y}\right)^3 = (\log_y x)^3 = (ab)^3$$

$\Rightarrow ab - a^3b^3 = 0 \Rightarrow ab(1 - a^2b^2) = 0 \Rightarrow ab = \pm 1$

$\therefore$ Only option (*e*) does not satisfy the condition, sin(2, 2) is not a possible value of (*a, b*).

PRACTICE SHEET

LEVEL–1

1. (*i*) The solution of $\log_\pi (\log_2 (\log_7 x)) = 0$ is

(*a*) 2 (*b*) π^2 (*c*) 7^2 (*d*) None of these

(*WBJEE 2008*)

Similar questions

(*ii*) $\log_{27} (\log_3 x) = \frac{1}{3} \Rightarrow x =$

(*a*) 3 (*b*) 6 (*c*) 9 (*d*) 27

(*EAMCET 2004*)

(*iii*) The solution of $\log_{99} (\log_2 (\log_3 x)) = 0$

(*a*) 4 (*b*) 9 (*c*) 44 (*d*) 99

(*BCECE 2006*)

2. If $x = \log_b a$, $y = \log_c b$, $z = \log_a c$, then xyz is

(*a*) 0 (*b*) 1 (*c*) *abc* (*d*) *a* + *b* + *c*

(*UPSEE 2003*)

3. (*i*) $7^{2 \log_7 5}$ is equal to

(*a*) 5 (*b*) 25 (*c*) $\log_7 25$ (*d*) $\log_7 35$

(*KCET 2007*)

Similar question

(*ii*) The real roots of the equation $7^{\log_7 (x^2 - 4x + 5)}$

(*a*) 1 and 2 (*b*) 2 and 3 (*c*) 3 and 4 (*d*) 3 and 4

(*DCE 2001*)

4. $\left(\frac{1}{\log_3 12}+\frac{1}{\log_4 12}\right)$ is

(*a*) 0 (*b*) $\frac{1}{2}$ (*c*) 1 (*d*) 2

(*WBJEE 2009*)

5. If *a, b, c* do not belong to the set {0, 1, 2, 3 .... 9}, then $\log_{10}\left(\frac{a+10b+10^2c}{10^{-4}a+10^{-3}b+10^{-2}c}\right)$ is equal to

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

(*EAMCET 2005*)

6. Assuming that the base is 10, the value of the expression log 6 + 2 log 5 + log 4 – log 3 – log 2 is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

7. $\log\frac{a^2}{bc}+\log\frac{b^2}{ac}+\log\frac{c^2}{ab}$ equals

(*a*) –1 (*b*) *abc* (*c*) 3 (*d*) 0

8. If $\log_r 6 = m$ and $\log_r 3 = n$, then what is $\log_r (r/2)$ equal to ?

(*a*) $m - n + 1$ (*b*) $m + n - 1$ (*c*) $1 - m - n$ (*d*) $1 - m + n$

(*CDS 2009*)

9. The value of $25^{(-1/4\log_5 25)}$ is

(*a*) $\frac{1}{5}$ (*b*) $-\frac{1}{25}$ (*c*) –25 (*d*) None of these

10. If $\log_{10}x - \log_{10}\sqrt{x} = \frac{2}{\log_{10} x}$, find the value of *x*.

(*a*) 10 (*b*) –1 (*c*) $100, \frac{1}{100}$ (*d*) $\frac{1}{1000}$

(*CAT 2004*)

11. If $\log_4 2 + \log_4 4 + \log_4 x + \log_4 16 = 6$, then *x* is equal to

(*a*) 4 (*b*) 8 (*c*) 32 (*d*) 64

(*KCET 2006*)

12. If $2^x . 3^{x+4} = 7^x$, then *x* is equal to

(*a*) $\frac{3\log_e 4}{\log_e 7 - \log_e 6}$ (*b*) $\frac{4\log_e 3}{\log_e 6 - \log_e 7}$

(*c*) $\frac{3\log_e 4}{\log_e 6 - \log_e 7}$ (*d*) $\frac{4\log_e 3}{\log_e 7 - \log_e 6}$

(*MPPET 2009*)

13. (*i*) If *n* = 1000 !, then the value of

$$\frac{1}{\log_2 n}+\frac{1}{\log_3 n}+\ldots+\frac{1}{\log_{1000} n} \text{ is}$$

(*a*) 0 (*b*) 1 (*c*) 10 (*d*) 10^3

(*KCET 2009, Kerala PET 2006, DCE 2005*)

Similar question

(*ii*) If *x* = 1999!, then $\sum_{x=1}^{1999} \log_n x$ is equal to

(*a*) – 1 (*b*) 0 (*c*) 1 (*d*) $\sqrt[1999]{1999}$.

(*AMU 2003*)

14. If $\frac{\log x}{b-c}=\frac{\log y}{c-a}=\frac{\log z}{a-b}$, then the value of $x^{b+c} . y^{c+a} . z^{a+b}$ is

(*a*) 1 (*b*) 0 (*c*) *abc* (*d*) *xyz*

(*KCET 2011*)

15. If $\log_x 484 - \log_x 4 + \log_x 14641 - \log_x 1331 = 3$, then the value of *x* is

(*a*) 1 (*b*) 3 (*c*) 11 (*d*) None of these

(*DCE 2008*)

LEVEL–2

16. If $\log_{\sqrt{3}} 5 = a$ and $\log_{\sqrt{3}} 2 = b$, then $\log_{\sqrt{3}} 300$ is equal to

(*a*) $a + b + 1$ (*b*) $2(a + b + 1)$

(*c*) $2(a + b + 2)$ (*d*) $(a + b + 4)$ **(*Kerala 2007*)**

17. If $\log_7 2 = \lambda$, then the value of $\log_{49}(28)$ is

(*a*) $\frac{1}{2}(2\lambda+1)$ (*b*) $(2\lambda + 1)$

(*c*) $2\ (2\lambda + 1)$ (*d*) $\frac{3}{2}(2\lambda+1)$

(*WBJEE 2011*)

18. The value of *x* satisfying $\log_2(3x - 2) = \log_{\frac{1}{2}} x$ is

(*a*) – 1 (*b*) $-\frac{1}{3}$ (*c*) $\frac{1}{3}$ (*d*) 1

(*AMU 2011*)

19. $\log_3 2$, $\log_6 2$, $\log_{12} 2$ are in

(*a*) A.P. (*b*) G.P. (*c*) H.P. (*d*) None of these

(*Raj PET 2006, 2001*)

20. (*i*) The value of $\frac{\log_3 5\times\log_{25} 27\times\log_{49} 7}{\log_{81} 3}$ is

(*a*) 1 (*b*) $\frac{2}{3}$ (*c*) 3 (*d*) 6

(*WBJEE 2010*)

(*ii*) $\log_{4\sqrt[3]{4^2}}\left(\frac{1}{1024}\right)$ is equal to

(*a*) – 5 (*b*) – 3 (*c*) 3 (*d*) 5

(*COMEDK 2010*)

21. If $2^{\log_{10} 3\sqrt{3}} = 3^{k\log_{10} 2}$ then the value of *k* is :

(*a*) 1 (*b*) 1/2 (*c*) 2 (*d*) $\frac{3}{2}$

22. $\frac{1}{(\log_a bc)+1}+\frac{1}{(\log_b ac)+1}+\frac{1}{(\log_c ab)+1}$ is equal to :

(*a*) 1 (*b*) 2 (*c*) 0 (*d*) *abc*

$_{1/2}2)$ $(\log_{1/3}3)$ $(\log_{1/4}4)$ ..... $(\log_{1/1000}1000)$?

(*a*) 1 (*b*) –1 (*c*) 1 or –1 (*d*) 0

(***CDS 2007***)

24. The value of $\log_{10}\sqrt{10\sqrt{10\sqrt{10\sqrt{10}}}}$ .... to ∞ is

(*a*) 4 (*b*) 3 (*c*) 2 (*d*) 1

25. If $\log_a m = x$, then $\log_{1/a}\left(\frac{1}{m}\right)$ equals

(*a*) $\frac{1}{x}$ (*b*) $-x$ (*c*) $-\frac{1}{x}$ (*d*) x

26. Find the value of x if the base is 10 :

$5^{\log x} - 3^{\log x - 1} = 3^{\log x + 1} - 5^{\log x - 1}$

(*a*) 1 (*b*) 0 (*c*) 100 (*d*) 10

27. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, then $a^a b^b c^c$ equals :

(*a*) –1 (*b*) 0 (*c*) *abc* (*d*) 1

28. The value of $\log_{\sqrt{b}} a \log_{\sqrt[3]{c}} b \log_{\sqrt[4]{a}} c$ is :

(*a*) 1 (*b*) 10 (*c*) 24 (*d*) 0

29. Evaluate x if $\log_3 (3 + x) + \log_3 (8 - x) - \log_3 (9x - 8)$
$= 2 - \log_3 9$

(*a*) 2 (*b*) –2 (*c*) 4 (*d*) –4

30. If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$, then

(*a*) $xyz = x + y + z + 2$ (*b*) $xyz = x + y + z + 1$

(*c*) $x + y + z = 1$ (*d*) $xyz = 1$

LEVEL–3

31. Given, $\log_a x = \frac{1}{\alpha}, \log_b x = \frac{1}{\beta}, \log_c x = \frac{1}{\gamma}$, then $\log_{abc} x$ equals :

(*a*) $\alpha\beta\gamma$ (*b*) $\frac{1}{\alpha\beta\gamma}$

(*c*) $\alpha + \beta + \gamma$ (*d*) $\frac{1}{\alpha + \beta + \gamma}$

32. If $y = \frac{1}{a^{1-\log_a x}}$, $z = \frac{1}{a^{1-\log_a y}}$ and $x = a^k$, then $k =$

(*a*) $\frac{1}{a^{1-\log_a z}}$ (*b*) $\frac{1}{1-\log_a z}$ (*c*) $\frac{1}{1+\log_z a}$ (*d*) $\frac{1}{1-\log_z a}$

33. Solve for x if $a > 0$ and $2 \log_x a + \log_{ax} a + 3 \log_{a^2 x} a = 0$

(*a*) $a^{3/2}$ (*b*) $a^{1/2}$ (*c*) $a^{3/4}$ (*d*) $a^{-4/3}$

34. Find the value of x, if $\log_2 (5.2^x + 1)$, $\log_4(2^{1-x} + 1)$ and 1 are in A.P.

(*a*) $1 + \log_5 2$ (*b*) $1 - \log_2 5$ (*c*) $\log_2 10$ (*d*) $\log_2 5 + 1$

(***AIEEE 2002***)

35. If $\frac{1}{3}\log_3 M + 3\log_3 N = 1 + \log_{0.008} 5$, then

(*a*) $M^9 = \frac{9}{N}$ (*b*) $N^9 = \frac{9}{M}$

(*c*) $M^3 = \frac{3}{N}$ (*d*) $N^9 = \frac{3}{M}$

(***CAT 2003***)

36. If $\frac{\log x}{a^2 + ab + b^2} = \frac{\log y}{b^2 + bc + c^2} = \frac{\log z}{c^2 + ca + a^2}$, then

$x^{a-b} . y^{b-c} . z^{c-a} =$

(*a*) 0 (*b*) –1 (*c*) 1 (*d*) 2

37. If x, y, z are distinct positive numbers different from 1, such that $(\log_y x . \log_z x - \log_x x) + (\log_x y . \log_z y - \log_y y) + (\log_x z . \log_y z - \log_z z) = 0$ then xyz equals

(*a*) 100 (*b*) –1 (*c*) 10 (*d*) 1

38. If a, b, c be the p^{th}, q^{th}, r^{th} terms of a GP, then the value of $(q - r) \log a + (r - p) \log b + (p - q) \log c$ is :

(*a*) 0 (*b*) 1 (*c*) –1 (*d*) *pqr*

39. If 1, $\log_9 (3^{1-x} + 2)$ and $\log_3 (4.3^x - 1)$ are in A.P., then x is equal to

(*a*) $\log_4 3$ (*b*) $\log_3 4$ (*c*) $1 + \log_3 4$ (*d*) $\log_3(3/4)$

40. What is the sum, of '*n*' terms in the series :

$\log m + \log\left(\frac{m^2}{n}\right) + \log\left(\frac{m^3}{n^2}\right) + \log\left(\frac{m^4}{n^3}\right) + ...$

(*a*) $\log\left[\frac{n^{(n-1)}}{m^{(n+1)}}\right]^{n/2}$ (*b*) $\log\left[\frac{m^m}{n^n}\right]^{n/2}$

(*c*) $\log\left[\frac{m^{(1-n)}}{n^{(1-m)}}\right]^{n/2}$ (*d*) $\log\left[\frac{m^{(1+n)}}{n^{(n-1)}}\right]^{n/2}$

(***CAT 2003***)

41. Find x, if $\log_{2x}\sqrt{x} + \log_{2\sqrt{x}} x = 0$:

(*a*) 1, $2^{-5/6}$ (*b*) 1, $2^{-6/5}$ (*c*) 4, –2 (*d*) None of these

42. The number of solutions satisfying the given equation

$x^{\left[(\log_3 x)^2 - \frac{9}{2}\log_3 x + 5\right]} = 3\sqrt{3}$ for $x \in R$ are :

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

43. Solve the following equations for x and y.

$\log_{100} |x + y| = \frac{1}{2}$, $\log_{10} y - \log_{10} |x| = \log_{100} 4$

(*a*) $\left(\frac{8}{3}, \frac{16}{3}\right)$, (–8, –16) (*b*) $\left(\frac{10}{3}, \frac{20}{3}\right)$, (+10, 20)

(*c*) $\left(-\frac{10}{3}, -\frac{20}{3}\right)$, (70, 20) (*d*) None of these

44. If $\log (a + c) + \log (a - 2b + c) = 2 \log (a - c)$, then a, b, c are in

(*a*) A.P. (*b*) G.P. (*c*) H.P. (*d*) None of these

45. If $5^{3x^2 \log_{10} 2} = 2^{(x + 1/2)\log_{10} 25}$, then the value of x is :

(*a*) –1 (*b*) 2 (*c*) $\frac{1}{2}$ (*d*) $-\frac{1}{3}$

46. The number $\log_2 7$ is :

(*a*) a prime number (*b*) a rational number

(*c*) an irrational number (*d*) an integer (***DCE 2000***)

47. If *x, y, z* are in G.P. and (log x – log $2y$), (log $2y$ – log $3z$) and (log $3z$ – log x) are in A.P., then *x, y, z* are the lengths of the sides of a triangle which is :

(*a*) acute angled (*b*) equilateral
(*c*) right angled (*d*) obtuse angled

(*Rajasthan PET 2006*)

48. In a right-angled triangle, the sides are *a, b* and *c* with *c* as hypotenuse and $c - b \neq 1$, $c + b \neq 1$. Then the value of

$$\left[\frac{\log_{c+b} a + \log_{c-b} a}{2\log_{c+b} a \times \log_{c-b} a}\right] \text{is}$$

(*a*) –1 (*b*) $\frac{1}{2}$ (*c*) 1 (*d*) 2

(*WBJEE 2010*)

49. The value of

$$6 + \log_{\frac{3}{2}}\left[\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}}}}} \ \ldots\right. \text{is}$$

(*a*) $\frac{8}{3\sqrt{2}}$ (*b*) $\frac{4}{3}$ (*c*) 8 (*d*) 4

(*IIT 2012*)

50. The equation $x^{3/4(\log_2 x)^2 + (\log_2 x) - 5/4} = \sqrt{2}$ has

(*a*) at least one real solution
(*b*) exactly one irrational solution
(*c*) exactly three real solutions
(*d*) all of the above. **(*IIT 1989*)**

ANSWERS

1. (*i*) (*c*) (*ii*) (*d*) (*iii*) (*b*) **2.** (*b*) **3.** (*i*) (*b*) (*ii*) (*b*) **4.** (*c*) **5.** (*d*) **6.** (*c*) **7.** (*d*) **8.** (*d*) **9.** (*a*)
10. (*c*) **11.** (*c*) **12.** (*d*) **13.** (*i*) (*b*) (*ii*) (*c*) **14.** (*a*) **15.** (*c*) **16.** (*b*) **17.** (*a*) **18.** (*d*) **19.** (*c*)
20. (*i*) (*c*) (*ii*) (*b*) **21.** (*d*) **22.** (*a*) **23.** (*b*) **24.** (*d*) **25.** (*d*) **26.** (*c*) **27.** (*d*) **28.** (*c*)
29. (*c*) **30.** (*a*) **31.** (*d*) **32.** (*b*) **33.** (*d*) **34.** (*b*) **35.** (*b*) **36.** (*c*) **37.** (*d*) **38.** (*a*)
39. (*d*) **40.** (*d*) **41.** (*b*) **42.** (*d*) **43.** (*b*) **44.** (*c*) **45.** (*d*) **46.** (*c*) **47.** (*d*) **48.** (*c*)
49. (*d*) **50.** (*c*)

HINTS AND SOLUTIONS

1. (*i*) $\log_\pi (\log_2 (\log_7 x)) = 0$

$\Rightarrow (\log_2 (\log_7 x) = \pi^0 = 1$ [Using $\log_a m = x \Rightarrow m = a^x$]

$\Rightarrow \log_7 x = 2^1 = 2 \Rightarrow x = 7^2$.

2. Hint. $x = \log_b a \Rightarrow x = \dfrac{\log_e a}{\log_e b}$, $y = \log_c b \Rightarrow y = \dfrac{\log_e b}{\log_e c}$

$z = \log_a c \Rightarrow z = \dfrac{\log_e c}{\log_e a}$.

3. (*i*) $7^{2\log_7 5} = 7^{\log_7 5^2} = 5^2$ $\left[\text{Using } a^{\log_a x} = x\right]$

(*ii*) $7^{\log_7 (x^2 - 4x + 5)} = x - 1$

$\Rightarrow x^2 - 4x + 5 = x - 1$. Now, solve.

4. Hint. $\dfrac{1}{\log_3 12} + \dfrac{1}{\log_4 12} = \log_{12} 3 + \log_{12} 4$

$$\left[\text{Using } \log_a x = \frac{1}{\log_x a}\right]$$

5. Hint. Given exp. = $\log_{10}\left\{10^4\left(\dfrac{a + 10b + 10^2 c}{a + 10b + 10^2 c}\right)\right\}$

6. Given exp. = log 6 + 2 log5 + log 4 – log 3 – log 2

= log 6 + log $(5)^2$ + log 4 – log 3 – log 2

= log 6 + log 25 + log 4 – (log 3 + log 2)

= log (6 × 25 × 4) – log (3 × 2)

$= \log\left(\dfrac{6 \times 25 \times 4}{3 \times 2}\right) = \log_{10} 100 = \log_{10} 10^2$

$= 2\log_{10} 10 = 2 \times 1 =$ **2.**

7. $\log\dfrac{a^2}{bc} + \log\dfrac{b^2}{ca} + \log\dfrac{c^2}{ab} = \log\left(\dfrac{a^2}{bc} \times \dfrac{b^2}{ac} \times \dfrac{c^2}{ab}\right)$

$= \log\left(\dfrac{a^2\, b^2\, c^2}{a^2\, b^2\, c^2}\right) = \log 1 = \mathbf{0}.$

8. Given, $\log_r 6 = m$ and $\log_r 3 = n$

Since, $\log_r 6 = \log_r (2 \times 3) = \log_r 2 + \log_r 3$

$\Rightarrow \log_r 2 + \log_r 3 = m$

$\Rightarrow \log_r 2 + n = m \Rightarrow \log_r 2 = m - n$

Now, $\log_r (r/2) = \log_r r - \log_r 2 = 1 - (m - n) = \mathbf{1 - m + n}$.

9. $25^{\left[\left(-\frac{1}{4}\log_5 25\right)\right]} = 5^{[2(-1/4)\log_5 25)]}$

$= 5^{(-1/2\log_5 25)} = 5^{\log_5 (25)^{-1/2}} = 25^{-1/2} = \dfrac{1}{5}$

$(\because\ a^{\log_a x} = x)$

10. Given, $\log_{10} x - \log_{10}\sqrt{x} = \dfrac{2}{\log_{10} x}$

$\Rightarrow \log_{10}\left(\dfrac{x}{\sqrt{x}}\right) = \dfrac{2}{\log_{10} x}$

$\Rightarrow \log_{10}\sqrt{x} = \dfrac{2}{\log_{10} x} \Rightarrow \dfrac{1}{2}\log_{10} x = \dfrac{2}{\log_{10} x}$

$\Rightarrow (\log_{10} x)^2 = 4 \Rightarrow \log_{10} x = \pm 2$

If $\log_{10} x = +2$ then $x = 10^2 =$ **100**

If $\log_{10} x = -2$ then $x = 10^{-2} =$ **1/100.**

11. Hint. Given, $\log_4 (2 \times 4 \times x \times 16) = 6 \Rightarrow \log_4 (128x) = 6$

$\Rightarrow 128x = 4^6$ [Using $\log_a x = n \Rightarrow x = a^n$]

12. Hint. $2^x \cdot 3^{x+4} = 7^x \Rightarrow \log_e (2^x \cdot 3^{x+4}) = \log_e 7^x$

Taking log to the same base on both sides

$\Rightarrow x \log 2 + (x + 4) \log 3 = x \log 7$

$\Rightarrow x (\log 7 - \log 2 - \log 3) = 4 \log 3$

13. (*i*) Given, $1000! = n$. Now, $\dfrac{1}{\log_2 n} + \dfrac{1}{\log_3 n} + ... + \dfrac{1}{\log_{1000} n}$

$= \log_n 2 + \log_n 3 + ... + \log_n 1000 \quad \left[\text{Using } \dfrac{1}{\log_b a} = \log_a b\right]$

$= \log_n (2 \times 3 \times 4 \times ... \times 1000) = \log_n (1000!) = \log_n n = 1.$

14. Hint. $\dfrac{\log x}{b-c} = \dfrac{\log y}{c-a} = \dfrac{\log z}{a-b} = k(\text{suppose})$

$\Rightarrow \log_e x = k (b - c) \Rightarrow x = e^{k (b - c)}$

$\log_e y = k (c - a) \Rightarrow y = e^{k (c - a)}$

$\log_e z = k (a - b) \Rightarrow z = e^{k (a - b)}$

$\therefore\ x^{b + c} \cdot y^{c + a} \cdot z^{a + b}$

$= e^{k (b - c) (b + c)} \cdot e^{k (c - a) (c + a)} \cdot e^{k (a - b) (a + b)}$

Now, complete.

15. Hint. $\log_x 484 - \log_x 4 + \log_x 14641 - \log_x 1331 = 3$

$\Rightarrow \log_x (2^2 \times 11^2) - \log_x (2^2) + \log_x (11^4) - \log_x (11^3) = 3$

$\Rightarrow 2 \log_x 2 + 2 \log_x 11 - 2 \log_x 2 + 4 \log_x 11 - 3 \log_x 11 = 3$

$\Rightarrow 3 \log_x 11 = 3 \Rightarrow \log_x 11 = 1 \Rightarrow x^1 = 11 \Rightarrow x = 11.$

16. Hint. $\log_{\sqrt{3}} 300 = \log_{\sqrt{3}} \left[(\sqrt{3})^2 .10^2\right]$

$= 2 \log_{\sqrt{3}} \sqrt{3} + 2 \log_{\sqrt{3}} 10 = 2 + 2 \log_{\sqrt{3}} (2 \times 5)$

17. $\log_{49} (28) = \log_{7^2} (7 \times 2^2) = \log_{7^2} 7 + \log_{7^2} 2^2$

$= \dfrac{1}{2} \log_7 7 + \dfrac{2}{2} \log_7 2$

$\left[\text{Using } \log_{a^n} x = \dfrac{1}{n} \log_a x, \log_{a^n} (x^m) = \dfrac{m}{n} \log_a x\right]$

$= \dfrac{1}{2} + \lambda = \dfrac{1}{2}(2\lambda + 1).$

18. Given, $\log_2 (3x - 2) = \log_{\frac{1}{2}} x \Rightarrow \log_2 (3x - 2) = \log_{2^{-1}} x$

$\Rightarrow \log_2 (3x - 2) = \dfrac{1}{-1} \log_2 x \quad \left[\text{Using } \log_{a^n} x = \dfrac{1}{n} \log_a x\right]$

$\Rightarrow \log_2 (3x - 2) = (-1) \log_2 x = \log_2 x^{-1}$

$[\text{Using } n \log_a x = \log_a x^n]$

$\Rightarrow (3x - 2) = x^{-1} = \dfrac{1}{x} \Rightarrow 3x^2 - 2x - 1 = 0$

$\Rightarrow (3x + 1) (x - 1) = 0 \Rightarrow x = -\dfrac{1}{3}$ or 1

$\Rightarrow x = 1$, since $\log_2 (3x - 2)$ is not defined when $x = -\dfrac{1}{3}$.

19. Since $\log_2 3 + \log_2 12 = \log_2 (3 \times 12) = \log_2 36 = \log_2 6^2$
$= 2 \log_2 6$, therefore,
$\log_2 3, \log_2 6$ and $\log_2 12$ in A.P.

$\Rightarrow \dfrac{1}{\log_2 3}, \dfrac{1}{\log_2 6}, \dfrac{1}{\log_2 12}$ and in H.P.

$\Rightarrow \log_3 2, \log_6 2, \log_{12} 2$ and in H.P.

$\left[\text{Using } \log_a x = \dfrac{1}{\log_x a}\right]$

20. (*i*) $\dfrac{\log_3 5 \times \log_{25} 27 \times \log_{49} 7}{\log_{81} 3} = \dfrac{\log_3 5 \times \log_{5^2} 3^3 \times \log_{7^2} 7}{\log_{3^4} 3}$

$= \dfrac{\log_3 5 \times \frac{3}{2} \log_5 3 \times \frac{1}{2} \log_7 7}{\frac{1}{4} \log_3 3}$

$\left[\text{Using } \log_{a^n} x^m = \dfrac{m}{n} \log_a x, \log_{a^n} x = \dfrac{1}{n} \log_a x\right]$

$= 3 (\log_3 5 \times \log_5 3) = 3 \times 1 = 3 \quad [\text{Using } \log_b a \times \log_a b = 1]$

(*ii*) **Hint.** Since $4.\sqrt[3]{4^2} = 2^2.(2^4)^{\frac{1}{3}} = 2^{2+\frac{4}{3}} = 2^{\frac{10}{3}}$ and $1024 = 2^{10}$, therefore,

$\log_{4.\sqrt[3]{4^2}} \left(\dfrac{1}{1024}\right) = \log_{2^{10/3}} (2^{-10}) = \dfrac{-10}{\frac{10}{3}} \log_2 2$

$= -3 \times 1 = -3. \qquad \left[\text{Using } \log_{a^n} x^m = \dfrac{m}{n} \log_a x\right].$

21. $2^{\log_{10} 3\sqrt{3}} = 3^{k \log_{10} 2} \Rightarrow 2^{\log_{10} (3^{3/2})} = 3^{k \log_{10} 2}$

$\Rightarrow 2^{\log_2 (3^{3/2}) \,.\, \log_{10} 2} = 3^{k \log_{10} 2} \quad [\text{Using } \log_a x = \log_b x \,.\, \log_a b]$

$\Rightarrow \left[2^{\log_2 (3^{3/2})}\right]^{\log_{10} 2} = (3^k)^{\log_{10} 2} \Rightarrow 2^{\log_2 3^{3/2}} = 3^k$

$\Rightarrow 3^{3/2} = 3^k \Rightarrow k = \mathbf{\dfrac{3}{2}} \qquad (\because a^{\log_a x} = x)$

22. $\dfrac{1}{\log_a bc + 1} + \dfrac{1}{\log_b ac + 1} + \dfrac{1}{\log_c ab + 1}$

$= \dfrac{1}{\log_a bc + \log_a a} + \dfrac{1}{\log_b ac + \log_b b} + \dfrac{1}{\log_c ab + \log_c c}$

$= \dfrac{1}{\log_a (abc)} + \dfrac{1}{\log_b (abc)} + \dfrac{1}{\log_c (abc)}$

$= \log_{abc}(a) + \log_{abc}(b) + \log_{abc}(c) = \log_{abc} abc = 1$

$\left(\text{Using } \log_a b = \dfrac{1}{\log_b a}\right)$

23. $(\log_{1/2}2) (\log_{1/3}3) (\log_{1/4}4) .... \left(\log_{\frac{1}{1000}} 1000\right)$

$= \left(\dfrac{\log 2}{\log \frac{1}{2}}\right)\left(\dfrac{\log 3}{\log \frac{1}{3}}\right)\left(\dfrac{\log 4}{\log \frac{1}{4}}\right) .... \left(\dfrac{\log 1000}{\log \frac{1}{1000}}\right)$

$= \left(\dfrac{\log 2}{-\log 2}\right)\left(\dfrac{\log 3}{-\log 3}\right)\left(\dfrac{\log 4}{-\log 4}\right) .... \left(\dfrac{\log 1000}{-\log 1000}\right)$

$(\because \log \dfrac{1}{2} = \log 1 - \log 2 = 0 - \log 2 = -\log 2$ and similarly for others)

$= (-1) \times (-1) \times (-1) \times .... \times (-1) = \mathbf{-1}$

$(\because$ Number of terms is odd $)$

24. Given exp. $= \log_{10}(10^{1/2}\ 10^{1/4}\ 10^{1/8} \text{ .... to } \infty)$

$$= \log_{10} 10^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{...........} \text{ to } \infty\right)}$$

$$= \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{........ to } \infty\right). \log_{10} 10$$

[Using $\log_a x^n = n \log_a x$]

$$= \frac{1/2}{(1-1/2)} \times 1 = 1$$

$\left(\text{Using sum of GP of infinite terms} = \dfrac{a}{1-r}\right)$

25. $\log_a m = x \Rightarrow a^x = m$

$\log_{1/a} 1/m = y \Rightarrow (1/a)^y = 1/m \Rightarrow m = a^y \Rightarrow a^y = a^x \Rightarrow y = x$

26. $5^{\log x} - 3^{\log x - 1} = 3^{\log x + 1} - 5^{\log x - 1}$

$\Rightarrow 5^{\log x} - 3^{\log x} \times 3^{-1} = 3^{\log x}. 3 - 5^{\log x}. 5^{-1}$

$$\Rightarrow 5^{\log x} - \frac{1}{3} \times 3^{\log x} = 3 \times 3^{\log x} - \frac{1}{5} \times 5^{\log x}$$

$$\Rightarrow \left(3+\frac{1}{3}\right)3^{\log x} = \left(1+\frac{1}{5}\right)5^{\log x} \Rightarrow \frac{10}{3} \times 3^{\log x} = \frac{6}{5} \times 5^{\log x}$$

$$\Rightarrow \frac{3^{\log x}}{5^{\log x}} = \frac{6}{5} \times \frac{3}{10} = \frac{9}{25} \Rightarrow \left(\frac{3}{5}\right)^{\log x} = \left(\frac{3}{5}\right)^2$$

$\Rightarrow \log_{10} x = 2 \Rightarrow x = 10^2 = \mathbf{100}$

27. Let $\dfrac{\log a}{b-c} = \dfrac{\log b}{c-a} = \dfrac{\log c}{a-b} = k$

$\Rightarrow \log a = k(b-c), \log b = k(c-a), \log c = k(a-b)$

Now let $a^a b^b c^c = p$. Then,

$\log p = \log a^a + \log b^b + \log c^c = a \log a + b \log b + c \log c$

$= a \times k(b-c) + b \times k(c-a) + c \times k(a-b)$

$= k(ab - ac + bc - ba + ca - cb) = 0$

$\Rightarrow \log p = \log 1$ (Putting log 1 for 0)

$\Rightarrow p = 1 \Rightarrow a^a b^b c^c = \mathbf{1}.$

28. Using the formula $\log_{a^n} x^m = \dfrac{m}{n} \log_a x$, we have

$$\log_{\sqrt{b}} a \log_{\sqrt[3]{c}} b \log_{\sqrt[4]{a}} c = \log_{b^{1/2}} a \log_{c^{1/3}} b \log_{a^{1/4}} c$$

$$= 2 \log_b a \times 3 \log_c b \times 4 \log_a c$$

$$= 24 \frac{\log a}{\log b} \times \frac{\log b}{\log c} \times \frac{\log c}{\log a} = \mathbf{24.}$$

29. $\log_3(3+x) + \log_3(8-x) - \log_3(9x-8) = 2 - \log_3 9$

$\Rightarrow \log_3(3+x) + \log_3(8-x) - \log_3(9x-8) + \log_3 9 = 2$

$$\Rightarrow \log_3\left[\frac{(3+x)(8-x)(9)}{(9x-8)}\right] = 2$$

$$\Rightarrow \frac{9(24+8x-3x-x^2)}{(9x-8)} = 3^2 = 9$$

$\Rightarrow -x^2 + 5x + 24 = 9x - 8 \Rightarrow x^2 + 4x - 32 = 0$

$\Rightarrow (x+8)(x-4) = 0 \Rightarrow x = -8, 4.$

Taking the positive value $x = \mathbf{4}.$

30. $x = \log_a bc \Rightarrow a^x = bc \Rightarrow a^{x+1} = abc \Rightarrow a = (abc)^{1/(x+1)}$

Similarly, $b = (abc)^{1/(y+1)}$, $c = (abc)^{1/(z+1)}$

$$\therefore abc = (abc)^{\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}}$$

$$\Rightarrow \frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1} = 1 \Rightarrow (y+1)(z+1) + (x+1)$$

$(z+1) + (x+1)(y+1) = (x+1)(y+1)(z+1)$

$\Rightarrow yz + y + z + 1 + xz + x + z + 1 + xy + y + x + 1 = xyz + xy$
$+ yz + zx + x + y + z + 1$

$\Rightarrow \boldsymbol{x + y + z + 2 = xyz.}$

31. $\alpha = \dfrac{1}{\log_a x}, \beta = \dfrac{1}{\log_b x}, \gamma = \dfrac{1}{\log_c x}$

$\Rightarrow \alpha = \log_x a, \beta = \log_x b, \gamma = \log_x c$

$\Rightarrow \alpha + \beta + \gamma = \log_x a + \log_x b + \log_x c = \log_x(abc)$

$$\Rightarrow \frac{1}{\alpha+\beta+\gamma} = \frac{1}{\log_x(abc)} = \mathbf{\log_{abc}} \boldsymbol{x}.$$

32. $y = \dfrac{1}{a^{1-\log_a x}} = a^{-(1-\log_a x)}$

$$\Rightarrow \log_a y = \frac{1}{1-\log_a x} \text{ and } \log_a z = \frac{1}{1-\log_a y}$$

$$\therefore \log_a z = \frac{1}{1-\left(\dfrac{1}{1-\log_a x}\right)} = \frac{1-\log_a x}{-\log_a x}$$

$$\Rightarrow -\log_a z = -1 + \frac{1}{\log_a x} \Rightarrow \frac{1}{\log_a x} = 1 - \log_a z$$

$$\Rightarrow \log_a x = \frac{1}{1-\log_a z}$$

$$\Rightarrow x = a^{\frac{1}{1-\log_a z}} = a^k \Rightarrow k = \boldsymbol{\frac{1}{1-\log_a z}}.$$

33. Since $\log_{ax} a = \dfrac{1}{\log_a ax} = \dfrac{1}{\log_a a + \log_a x} = \dfrac{1}{1+\log_a x}$ and

$$\log_{a^2x} a = \frac{1}{\log_a a^2x} = \frac{1}{\log_a a^2 + \log_{ax} x} = \frac{1}{2\log_a a + \log_{ax} x}$$

$$= \frac{1}{2+\log_a x}$$

Given, $2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0$

$$\Rightarrow \frac{2}{\log_a x} + \frac{1}{1+\log_a x} + \frac{3}{2+\log_a x} = 0$$

Now let $\log_a x = t$, then $\dfrac{2}{t} + \dfrac{1}{1+t} + \dfrac{3}{2+t} = 0$

$\Rightarrow 2(1+t)(2+t) + t(2+t) + 3t(1+t) = 0$

$\Rightarrow 2(2 + 2t + t + t^2) + 2t + t^2 + 3t + 3t^2 = 0$

$\Rightarrow 4 + 6t + 2t^2 + 2t + t^2 + 3t + 3t^2 = 0$

$\Rightarrow 6t^2 + 11t + 4 = 0$

$\Rightarrow (3t+4)(2t+1) = 0 \Rightarrow t = -1/2, -4/3$

When $t = -1/2$, $\log_a x = -\dfrac{1}{2} \Rightarrow x = a^{-1/2}$

When $t = -\dfrac{4}{3}$, $\log_a x = -\dfrac{4}{3} \Rightarrow \boldsymbol{x = a^{-4/3}}$

34. Given, $\log_2(5.2^x + 1)$, $\log_4(2^{1-x} + 1)$, 1 are in A.P.

$\Rightarrow \log_2(5.2^x + 1) + 1 = 2\log_4(2^{1-x} + 1)$

$\Rightarrow \log_2(5.2^x + 1) + \log_2 2 = 2\log_{2^2}(2^{1-x} + 1)$

$\Rightarrow \log_2(5.2^x + 1).2 = 2 \times \frac{1}{2}\log_2(2^{1-x} + 1)$

$$\left(\because \log_{a^n} x = \frac{1}{n}\log_a x\right)$$

$\Rightarrow \log_2(10.2^x + 2) = \log_2(2^{1-x} + 1)$

$\Rightarrow 10.2^x + 2 = 2^{1-x} + 1 \Rightarrow 10.2^x + 2 = \frac{2}{2^x} + 1$

Let $2^x = a$, then

$10.\ a + 2 = \frac{2}{a} + 1 \Rightarrow 10a + 1 = \frac{2}{a} \Rightarrow 10a^2 + a - 2 = 0$

$\Rightarrow (5a - 2)(2a + 1) = 0 \Rightarrow a = \frac{2}{5} \Rightarrow 2^x = \frac{2}{5}$

$$\left(\because 2^x > 0, \text{ reject } a = -\frac{1}{2}\right)$$

$\Rightarrow \log 2^x = \log\frac{2}{5}$

$\Rightarrow x\log_2 2 = \log_2 2 - \log_2 5 \Rightarrow \boldsymbol{x = 1 - \log_2 5.}$

35. $\frac{1}{3}\log_3 M + 3\log_3 N = 1 + \log_{0.008} 5$

$\Rightarrow \log_3 M^{1/3} + \log_3 N^3 = 1 + \log_{0.008} 5$

$\Rightarrow \log_3 M^{1/3} N^3 = 1 + \log_{0.008} 5$

$\Rightarrow M^{1/3} N^3 = 3^{(1 + \log_{0.008} 5)}$

$\Rightarrow M^{1/3} N^3 = 3^1 \,.\, 3^{\log_{0.008} 5}$

$\Rightarrow N^9 = \frac{27}{M}(3^{3\log_{0.008} 5})$

$\Rightarrow N^9 = \frac{27}{M}\left(3^{\log_{(0.2)^3}(5^3)}\right)$

$\Rightarrow N^9 = \frac{27}{M}(3^{\log_{0.2} 5}) \qquad \left[\because \log_{a^n} x^m = \frac{m}{n}\log_a x\right]$

$\Rightarrow N^9 = \frac{27}{M}(3^{\log_{1/5} 5}) = \frac{1}{M}(27)(3^{-1}) = \boldsymbol{\frac{9}{M}}.$

36. Let each ratio = k and base = e

$\Rightarrow \log_e x = k(a^2 + ab + b^2)$

$\Rightarrow (a - b)\log_e x = k(a - b)(a^2 + ab + b^2)$

$\Rightarrow \log_e x^{a-b} = k(a^3 - b^3) \Rightarrow x^{a-b} = e^{k(a^3 - b^3)}$

Similarly, $y^{b-c} = e^{k(b^3 - c^3)}$, $z^{c-a} = e^{k(c^3 - a^3)}$

$\therefore x^{a-b} \,.\, y^{b-c} \,.\, z^{c-a} = e^{k(a^3 - b^3)} \,.\, e^{k(b^3 - c^3)} \,.\, e^{k(c^3 - a^3)}$

$= e^{k[a^3 - b^3 + b^3 - c^3 + c^3 - a^3]} = e^0 = \mathbf{1.}$

37. $\log_y x \,.\, \log_z x - \log_x x = \frac{\log x}{\log y}\cdot\frac{\log x}{\log z} - 1 = \frac{(\log x)^2}{\log y.\log z} - 1$

Similarly, $\log_x y \,.\, \log_z y - \log_y y = \frac{(\log y)^2}{\log x.\log z} - 1$ and

$\log_x z.\ \log_y z - \log_z z = \frac{(\log z)^2}{\log x.\log y} - 1$

$\therefore \text{LHS} = \frac{(\log x)^2}{\log y.\log z} - 1 + \frac{(\log y)^2}{\log z.\log x} - 1 + \frac{(\log z)^2}{\log x.\log y} - 1$

$= \frac{(\log x)^3 + (\log y)^3 + (\log z)^3 - 3\log x.\log y.\log z}{\log x.\log y.\log z} = 0$

(given)

$\Rightarrow (\log x)^3 + (\log y)^3 + (\log z)^3 - 3\log x.\log y.\log z = 0$

$\Rightarrow \log x + \log y + \log z = 0$

(if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$)

$\Rightarrow \log xyz = 0 \Rightarrow xyz = \mathbf{1.}$

38. Let h be the first term and k be the common ratio of a GP, then

$a = hk^{p-1}, \qquad b = hk^{q-1}, \qquad c = hk^{r-1}$

$\therefore (q - r)\log a + (r - p)\log b + (p - q)\log c$

$= \log[hk^{p-1}]^{q-r} + \log[hk^{q-1}]^{r-p} + \log[hk^{r-1}]^{p-q}$

$= \log(h^{q-r+r-p+p-q})(k^{p-1})^{q-r}(k^{q-1})^{r-p}(k^{r-1})^{p-q}$

$= \log(h^o k^o) = \log 1 = \mathbf{0.}$

39. 1, $\log_9(3^{1-x} + 2)$, $\log_3(4.3^x - 1)$ are in A.P.

$\Rightarrow \log_3 3$, $\log_3(3^{1-x} + 2)^{1/2}$, $\log_3(4.3^x - 1)$ are in A.P.

$\Rightarrow 3, (3^{1-x} + 2)^{1/2}, (4.3^x - 1)$ are in G.P.

$$\left[\text{Since } \log_9(3^{1-x} + 2) = \log_{3^2}(3^{1-x} + 2) = \frac{1}{2}\log_3(3^{1-x} + 2),\right.$$
$$\left.\text{using } \log_{a^n} x = \frac{1}{n}\log_a x = \log_3(3^{1-x} + 2)^{\frac{1}{2}}\right]$$

$\Rightarrow [[3^{1-x} + 2]^{1/2}]^2 = 3.(4.3^x - 1)$

$\Rightarrow 3^{1-x} + 2 = 4.3^{x+1} - 3$

$\Rightarrow 4.3^{x+1} - 3^{1-x} = 5 \Rightarrow 12.3^x - \frac{3}{3^x} = 5$

Let $3^x = y$, then $12y - \frac{3}{y} = 5 \Rightarrow 12y^2 - 5y - 3 = 0$

$\Rightarrow (3y + 1)(4y - 3) = 0 \Rightarrow y = -\frac{1}{3}, \frac{3}{4}$

$\therefore$ Rejecting the negative value, we have $3^x = \frac{3}{4}$

$\Rightarrow x = \boldsymbol{\log_3\frac{3}{4}}.$

40. $S = \log m + \log\frac{m^2}{n} + \log\frac{m^3}{n^2} + \ldots\ldots\ldots n$ terms

$= \log\left[m\cdot\frac{m^2}{n}\cdot\frac{m^3}{n^2}\ldots\ldots\ldots\frac{m^n}{n^{n-1}}\right] = \log\left[\frac{m^{(1+2+3+4+\ldots\ldots\ldots+n)}}{n^{(1+2+3+\ldots\ldots\ldots(n-1))}}\right]$

$= \log\left[\frac{m^{\frac{n(n+1)}{2}}}{n^{\frac{n(n-1)}{2}}}\right] = \boldsymbol{\log\left[\frac{m^{n+1}}{n^{n-1}}\right]^{n/2}}.$

41. $\log_{2x}\sqrt{x} + \log_{2\sqrt{x}} x = 0 \qquad \ldots(i)$

Let $\log_2 x = t$. Then,

$$\log_{2x}\sqrt{x} = \frac{\log_2\sqrt{x}}{\log_2 2x} = \frac{\frac{1}{2}\log_2 x}{\log_2 2 + \log_2 x} = \frac{t/2}{1 + t}$$

$$\log_{2\sqrt{x}} x = \frac{\log_2 x}{\log_2 2\sqrt{x}} = \frac{\log_2 x}{\log_2 2 + \frac{1}{2}\log_2 x} = \frac{t}{1+t/2}$$

$\therefore$ Substituting in (*i*), we get

$$\frac{t/2}{1+t} + \frac{t}{1+t/2} = 0 \Rightarrow \frac{t}{2+2t} + \frac{2t}{2+t} = 0$$

$\Rightarrow t(2+t) + 2t\,(2+2t) = 0 \Rightarrow 2t + t^2 + 4t + 4t^2 = 0$

$\Rightarrow 5t^2 + 6t = 0 \Rightarrow t(5t+6) = 0 \Rightarrow t = 0 \text{ or } -\frac{6}{5}$

$\Rightarrow \log_2 x = 0 \Rightarrow x = 2^0 = 1 \text{ and } \log_2 x = -\frac{6}{5} \Rightarrow x = 2^{-6/5}$

$\therefore x = \mathbf{1 \text{ or } 2^{-6/5}}$

42. Taking log of both the sides to base 3, we have,

$$\left[(\log_3 x)^2 - \frac{9}{2}\log_3 x + 5\right]\log_3 x = \log_3 3^{3/2} = \frac{3}{2} \quad (\because \log_3 3 = 1)$$

$\Rightarrow 2(\log_3 x)^3 - 9(\log_3 x)^2 + 10\log_3 x - 3 = 0$

$\Rightarrow 2y^3 - 9y^2 + 10y - 3 = 0$ (Take $\log_3 x = y$)

$\Rightarrow (y-1)(y-3)(2y-1) = 0$ (Factorising)

$\Rightarrow (\log_3 x - 1)(\log_3 x - 3)(2\log_3 x - 1) = 0$

$\Rightarrow \log_3 x = 1, \log_3 x = 3, 2\log_3 x = 1 \Rightarrow x = 3^1, x = 3^3, x^2 = 3^1$

$\Rightarrow x = (3, 27, \sqrt{3})$

$\therefore$ There are three solutions.

43. $\log_{100}|x+y| = \frac{1}{2} \Rightarrow |x+y| = 100^{\frac{1}{2}}$

$\Rightarrow |x+y| = 10$ as (-10 is inadmissible) ...(*i*)

$\log_{10} y - \log_{10}|x| = \log_{100} 4$

$\Rightarrow \log_{10}\frac{y}{|x|} = \log_{10^2} 2^2 = \log_{10} 2$

$$\left[\text{Using } \log_{a^n}(x^m) = \frac{m}{n}\log_a x\right.$$

$\Rightarrow \frac{y}{|x|} = 2 \Rightarrow y = 2|x|$ ...(*ii*)

Substituting the value of *y* from (*ii*) in (*i*), we get

$||x + 2|x|| = 10$

If $x > 0$, then $3x = 10 \Rightarrow x = \frac{10}{3}$

If $x < 0$, then $x = 10$.

$\therefore$ If $x = \frac{10}{3}$, then $y = \frac{20}{3}$ and if $x = 10, y = 20$.

44. Given, $\log(a+c) + \log(a-2b+c) = 2\log(a-c)$

$\Rightarrow \log(a+c)(a-2b+c) = \log(a-c)^2$

$\Rightarrow (a+c)(a-2b+c) = (a-c)^2$

$\Rightarrow \cancel{a^2} + ca - 2ba - 2bc + ac + \cancel{c^2} = \cancel{a^2} - 2ac + \cancel{c^2}$

$\Rightarrow 4ac = 2ba + 2bc \Rightarrow 2ac = b(a+c)$

$\therefore b = \frac{2ac}{a+c} \Rightarrow$ *a, b, c* are in H.P.

45. $5^{3x^2\log_{10}2} = 2^{\left(x+\frac{1}{2}\right)\log_{10}25}$

$\Rightarrow 5^{3x^2\log_{10}2} = 2^{\left(\frac{2x+1}{2}\right)\times 2\log_{10}5} = 2^{(2x+1)\log_{10}5}$

$\Rightarrow 5^{3x^2\log_{10}2} = 2^{(2x+1)\log_2 5.\log_{10}2}$

(using $\log_a x = \log_b x \,.\, \log_a b$)

$\Rightarrow 5^{3x^2\log_{10}2} = [2^{\log_2 5^{(2x+1)}}]^{\log_{10}2}$

$\Rightarrow (5^{3x^2})^{\log_{10}2} = (5^{2x+1})^{\log_{10}2}$ $\left[\text{Using } a^{\log_a x} = x\right]$

$\Rightarrow 3x^2 = 2x + 1 \Rightarrow 3x^2 - 2x - 1 = 0$

$\Rightarrow (x-1)(3x+1) = 0$

$\Rightarrow x = \mathbf{1, -\frac{1}{3}}.$

46. Let us assume $\log_2 7$ be a rational number. Then,

$\log_2 7 = \frac{p}{q}$, where $p, q \in I$ and $q \neq 0$

$\Rightarrow 2^{p/q} = 7 \Rightarrow 2^p = 7^q$

This is not true as 2 is even and 7 is odd.

$\therefore$ Hence our assumption that $\log_2 7$ is a rational number is wrong.

$\therefore$ $\log_2 7$ is an irrational number.

47. *x, y, z* are in G.P. $\Rightarrow y^2 = xz$ ...(*i*)

$(\log x - \log 2y)$, $(\log 2y - \log 3z)$ and $(\log 3z - \log x)$ are in A.P.

$\Rightarrow 2(\log 2y - \log 3z) = (\log x - \log 2y) + (\log 3z - \log x)$

$\Rightarrow 3\log 2y = 3\log 3z \Rightarrow \log 2y = \log 3z \Rightarrow \mathbf{y = \frac{3}{2}z.}$

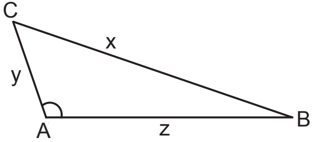
$\therefore$ Putting the value of *y* in (*i*), we have

$\left(\frac{3}{2}z\right)^2 = xz \Rightarrow x = \frac{9}{4}z.$

Now, by the cosine rule of triangles,

$\cos A = \frac{y^2 + z^2 - x^2}{2yz}$,

where *x* is the length of the side opposite $\angle A$.



$$\Rightarrow \cos A = \frac{\left(\frac{3}{2}z\right)^2 + z^2 - \left(\frac{9}{4}z\right)^2}{2\times\frac{3}{2}z\times z} = \frac{\frac{9}{4}z^2 + z^2 - \frac{81}{16}z^2}{3z^2}$$

$$= \frac{\frac{9}{4} + 1 - \frac{81}{16}}{3} = \frac{1}{3}\times\left[\frac{36+16-81}{16}\right] = -\frac{29}{48} < 0$$

$\because \cos A$ is less than 0, *i.e*, negative, $\angle A$ is obtused and the triangle is obtuse angled.

48. In a right angled triangle with *a*, *b* as sides and *c* as hypotenuse,

$c^2 = a^2 + b^2$ (Pythagoras' Theorem)

Now, given expression $= \frac{\log_{c+b} a + \log_{c-b} a}{2\times\log_{c+b} a\times\log_{c-b} a}$

$$= \frac{1}{2}\left[\frac{1}{\log_{c-b} a} + \frac{1}{\log_{c+b} a}\right]$$

$$= \frac{1}{2}[\log_a(c-b) + \log_a(c+b)] = \frac{1}{2}[\log_a[(c-b)(c+b)]$$

$$= \frac{1}{2}[\log_a(c^2 - b^2)] = \frac{1}{2}\log_a a^2 = \log_a a = \mathbf{1.}$$

49. Let

$$A = 6 + \log_{\frac{3}{2}}\left[\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}}}}} \;....\right]$$

$$\text{Let } p = \sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}}}}} \;....$$

$$\Rightarrow p = \sqrt{4-\frac{1}{3\sqrt{2}}p} \Rightarrow p^2 = 4 - \frac{1}{3\sqrt{2}}p \Rightarrow p^2 + \frac{1}{3\sqrt{2}}p - 4 = 0$$

$$\Rightarrow p = \frac{-\frac{1}{3\sqrt{2}} \pm \sqrt{\frac{1}{18}+16}}{2} = \frac{-\frac{1}{3\sqrt{2}} \pm \frac{17}{3\sqrt{2}}}{2}$$

(Applying the formula for roots of Q.E.)

$$\Rightarrow p = \frac{16}{3\times 2\sqrt{2}} \text{ or } \frac{-18}{3\times 2\sqrt{2}} = \frac{8}{3\sqrt{2}} \text{ or } -\frac{3}{\sqrt{2}}$$

Neglecting $p = \frac{-3}{\sqrt{2}}$ as $p \geq 0$, we have $p = \frac{8}{3\sqrt{2}}$

$$\therefore A = 6 + \log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}} \times \frac{8}{3\sqrt{2}}\right) = 6 + \log_{\frac{3}{2}}\left(\frac{4}{9}\right)$$

$$= 6 + \log_{\frac{3}{2}}\left(\frac{3}{2}\right)^{-2} = 6 - 2\log_{\frac{3}{2}} 3/2 = 6 - 2 = \mathbf{4.}$$

50. Given, $x^{3/4(\log_2 x)^2} + \log_2 x - 5/4 = \sqrt{2}$

Taking log to the base 2 of both the sides, we have

$$\left[\frac{3}{4}(\log_2 x)^2 + (\log_2 x) - 5/4\right] \log_2 x = \log_2 \sqrt{2}$$

$$= \log_2 2^{1/2} = \frac{1}{2}\log_2 2 = \frac{1}{2}$$

Let us assume $\log_2 x = a$. Then,

$$\left(\frac{3}{4}a^2 + a - \frac{5}{4}\right)a = \frac{1}{2} \Rightarrow 3a^3 + 4a^2 - 5a = 2$$

$\Rightarrow 3a^3 + 4a^2 - 5a - 2 = 0.$

Using hit and trial method check for $a = 1$.

$f(a) = 3a^3 + 4a^2 - 5a - 2 \Rightarrow f(1) = 3.1^3 + 4.1^2 - 5.1 - 2 = 0$

$\therefore$ $(a - 1)$ is a factor of $3a^3 + 4a^2 - 5a - 2$

$\therefore$ Now by dividing $3a^3 + 4a^2 - 5a - 2$ by $(a - 1)$, we get

$3a^3 + 4a^2 - 5a - 2 = (a - 1)(3a + 1)(a + 2) = 0$

$\Rightarrow a = 1$ or $a = -\frac{1}{3}$ or $a = -2$

$\Rightarrow \log_2 x = 1$ or $\log_2 x = -\frac{1}{3}$ or $\log_2 x = -2$

$\Rightarrow x = 2^1 = 2$ or $x = 2^{-1/3}$ or $x = 2^{-2} = \frac{1}{4}$

$\therefore$ The given equation has exactly three real solutions, wherein $x = 2^{-1/3}$ is irrational.

SELF ASSESSMENT SHEET

1. If $\log_2[\log_7(x^2 - x + 37)] = 1$, then what could be the value of x ?

(*a*) 3 (*b*) 5 (*c*) 4 (*d*) None of these

(*CAT 1997*)

2. $\log_{\sqrt{3}} \sqrt{3\sqrt{3\sqrt{3\sqrt{3}}}} =$

(*a*) $\frac{31}{32}$ (*b*) $\frac{15}{16}$ (*c*) $\frac{7}{16}$ (*d*) $\frac{15}{8}$

3. Find the sum of 'n' terms of the series :

$$\log_2\left(\frac{x}{y}\right) + \log_4\left(\frac{x}{y}\right)^2 + \log_8\left(\frac{x}{y}\right)^3 + \log_{16}\left(\frac{x}{y}\right)^4 + ...$$

(*a*) $\log_2\left(\frac{x}{y}\right)^{4n}$ (*b*) $n\log_2\left(\frac{x}{y}\right)$

(*c*) $\log_2\left(\frac{x^{n-1}}{y^{n-1}}\right)$ (*d*) $\frac{1}{2}\log_2\left(\frac{x}{y}\right)^{n(n+1)}$

4. If $\log_x a$, $a^{x/2}$ and $\log_b x$ are in *GP*, then x is equal to :

(*a*) $\log_a(\log_b a)$ (*b*) $\log_a(\log_e a) - \log_a(\log_e b)$

(*c*) $-\log_a(\log_a b)$ (*d*) both (*a*) and (*b*)

5. If $\frac{\log x}{l+m-2n} = \frac{\log y}{m+n-2l} = \frac{\log z}{n+l-2m}$, then xyz is equal to :

(*a*) 0 (*b*) 1 (*c*) lmn (*d*) 2

6. If $a = \log_{12}m$ and $b = \log_{18}m$, then $\frac{a-2b}{b-2a}$ equals

(*a*) $\log_3 2$ (*b*) $\log_2 3$ (*c*) 0 (*d*) 1

7. If $x = \log_{2a}a$, $y = \log_{3a}2a$, $z = \log_{4a}3a$, then $xyz - 2yz$ equals

(*a*) a^3 (*b*) 1 (*c*) 0 (*d*) –1

8. The sum of n terms of the series $\sum_{x=1}^{n} \log\frac{2^x}{3^{x-1}}$ is :

(*a*) $\log\left(\frac{3^{n-1}}{2^{n+1}}\right)^{n/2}$ (*b*) $\log\left(\frac{2^{n-1}}{3^{n+1}}\right)^{n/2}$

(*c*) $\log\left(\frac{3^{n+1}}{2^{n-1}}\right)^{n/2}$ (*d*) $\log\left(\frac{2^{n+1}}{3^{n-1}}\right)^{n/2}$

(*UPSEE 2011*)

9. The number of meaningful solutions of

$\log_4(x - 1) = \log_2(x - 3)$ is

(*a*) zero (*b*) 1 (*c*) 2 (*d*) 3

(*IIT 2001*)

10. The value of $\left[0.16^{\log 2.5\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+......+\infty\right)}\right]^{1/2}$ is:

(*a*) –1 (*b*) 0 (*c*) 1 (*d*) None of these

(*AMU 2009*)

SELF ASSESSMENT SHEET

1. (*c*) **2.** (*d*) **3.** (*b*) **4.** (*b*) **5.** (*b*) **6.** (*a*) **7.** (*d*) **8.** (*d*) **9.** (*b*) **10.** (*d*)

HINTS AND SOLUTIONS

1. $\log_2 [\log_7(x^2 - x + 37)] = 1$

$\Rightarrow \log_7(x^2 - x + 37) = 2^1 = 2$

$\Rightarrow x^2 - x + 37 = 7^2 = 49 \Rightarrow x^2 - x - 12 = 0$

Now solve for x.

2. Given expression = $\log_{\sqrt{3}} 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}}$

$$= \log_{3^{1/2}} 3^{15/16} = \frac{15}{16} \times 2 \log_3 3 = \mathbf{\frac{15}{8}.}$$

3. Given series

$$= \log_2 \left(\frac{x}{y}\right) + \log_{2^2} \left(\frac{x}{y}\right)^2 + \log_{2^3} \left(\frac{x}{y}\right)^3 + \log_{2^4} \left(\frac{x}{y}\right)^4 + ...$$

$$= \log_2 \left(\frac{x}{y}\right) + \log_2 \left(\frac{x}{y}\right) + \log_2 \left(\frac{x}{y}\right) + \log_2 \left(\frac{x}{y}\right) + ...n \text{ terms}$$

$$= \log_2 \left(\frac{x}{y} . \frac{x}{y} . \frac{x}{y} . \frac{x}{y} . ... n \text{ terms}\right)$$

$$= \log_2 \left(\frac{x}{y}\right)^n = \boldsymbol{n \log_2 \left(\frac{x}{y}\right).}$$

4. $\log_x a$, $a^{x/2}$, $\log_b x$ are in GP $\Rightarrow [a^{x/2}]^2 = \log_x a \,.\, \log_b x$

$$\Rightarrow a^x = \frac{\log a}{\log x} . \frac{\log x}{\log b}$$

$$\Rightarrow a^x = \frac{\log a}{\log b} = \log_b a$$

$\Rightarrow x = \log_a (\log_b a)$

$$= \log a \left(\frac{\log_e a}{\log_e b}\right) = \mathbf{\log\, \boldsymbol{a}\, (\log_{\boldsymbol{e}} \boldsymbol{a}) - \log\, \boldsymbol{a}\, (\log_{\boldsymbol{e}} \boldsymbol{b})}$$

5. Let $\dfrac{\log x}{l + m - 2n} = \dfrac{\log y}{m + n - 2l} = \dfrac{\log z}{n + l - 2m} = k.$ Then

$\log x = k(l + m - 2n), \log y = k(m + n - 2l); \log z = k(n + l - 2m)$

$\Rightarrow \log x + \log y + \log z = k(l + m - 2n) + k(m + n - 2l) + k(n + l - 2m)$

$\Rightarrow \log(xyz) = 0 \Rightarrow \log(xyz) = \log 1 \Rightarrow xyz = \mathbf{1.}$

6. $$\frac{a - 2b}{b - 2a} = \frac{\log_{12} m - 2\log_{18} m}{\log_{18} m - 2\log_{12} m}$$

$$= \frac{\dfrac{\log m}{\log 12} - 2\dfrac{\log m}{\log 18}}{\dfrac{\log m}{\log 18} - 2\dfrac{\log m}{\log 12}} = \frac{\log m \log 18 - 2 \log m \log 12}{\log m \log 12 - 2 \log m \log 18}$$

$$= \frac{\log 18 - 2 \log 12}{\log 12 - 2 \log 18} = \frac{\log (3^2 \times 2) - 2 \log(2^2 \times 3)}{\log(2^2 \times 3) - 2 \log (3^2 \times 2)}$$

$$= \frac{2 \log 3 + \log 2 - 4 \log 2 - 2 \log 3}{2 \log 2 + \log 3 - 4 \log 3 - 2 \log 2} = \frac{-3 \log 2}{-3 \log 3} = \frac{\log 2}{\log 3} = \mathbf{\log_3 2.}$$

7. $x = \log_{2a} a = \dfrac{\log a}{\log 2a}$, $\quad y = \log_{3a} 2a = \dfrac{\log 2a}{\log 3a}$

$$z = \log_{4a} 3a = \frac{\log 3a}{\log 4a}$$

$$\therefore xyz - 2yz = \frac{\log a}{\log 2a} . \frac{\log 2a}{\log 3a} . \frac{\log 3a}{\log 4a} - 2 \frac{\log 2a}{\log 3a} . \frac{\log 3a}{\log 4a}$$

$$= \frac{\log a}{\log 4a} - 2 \frac{\log 2a}{\log 4a} = \frac{\log a - 2 \log 2a}{\log 4a}$$

$$= \frac{\log a - \log (2a)^2}{\log 4a} = \frac{\log a / 4a^2}{\log 4a} = \frac{\log (4a)^{-1}}{\log (4a)} = \frac{-1. \log 4a}{\log 4a} = \mathbf{-1.}$$

8. $$\sum_{x=1}^{n} \log \frac{2^x}{3^{x-1}} = \log \left(\frac{2^1}{3^0}\right) + \log \left(\frac{2^2}{3^1}\right) + \log \left(\frac{2^3}{3^2}\right) + ... + \log \left(\frac{2^n}{3^{n-1}}\right)$$

$$= \log \left(\frac{2^1}{3^0} . \frac{2^2}{3^1} . \frac{2^3}{3^2} . ... \frac{2^n}{3^{n-1}}\right)$$

$$= \log \left(\frac{2^{1+2+3+.......+n}}{3^{1+2+3+......+(n-1)}}\right) = \log \left[\frac{2^{\frac{n(n+1)}{2}}}{3^{\frac{n(n-1)}{2}}}\right] = \mathbf{\log} \left[\frac{\mathbf{2}^{\boldsymbol{n}+\mathbf{1}}}{\mathbf{3}^{\boldsymbol{n}-\mathbf{1}}}\right]^{\boldsymbol{n}/\mathbf{2}}$$

9. $\log_4(x - 1) = \log_2(x - 3) \Rightarrow \log_{2^2} (x - 1) = \log_2(x - 3)$

$$\Rightarrow \frac{1}{2} \log_2 (x - 1) = \log_2 (x - 3) \Rightarrow \log_2 (x - 1) = 2 \log_2 (x - 3)$$

$$\left[\text{Using } \log_{a^m} (b^n) = \frac{n}{m} \log_a b\right]$$

$\Rightarrow \log_2(x - 1) = \log_2(x - 3)^2$

$\Rightarrow (x - 1) = (x - 3)^2 \Rightarrow x - 1 = x^2 - 6x + 9$

$\Rightarrow x^2 - 7x + 10 = 0 \Rightarrow (x - 2)(x - 5) = 0 \Rightarrow x = 2$ or 5

Neglecting $x = 2$ as $\log_2(x - 3)$ is defined when $x > 2$.

$\Rightarrow$ There is only one meaningful solution of the given equation.

10. $$\left[(0.16)^{\log_{2.5}\left[\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ...\infty\right]}\right]^{1/2} = \left[(0.4)^{2 \log_{2.5}\left[\frac{1/3}{1 - 1/3}\right]}\right]^{1/2}$$

$$\left[\because \text{Sum of infinite GP} = \frac{\text{First term}}{1 - \text{common ratio}}\right]$$

$$= \left[(0.4)^{\log_{5/2}\left(\frac{1}{2}\right)}\right] = \left[(0.4)^{\log_{5/2}\left(\frac{1}{2}\right)}\right] = \left[0.4^{\log_{(2/5)^{-1}} (2)^{-1}}\right]$$

$$\left[\because \log_a m^{b^n} = \frac{n}{m} \log_a b\right]$$

$$= 0.4^{\log_{2/5} 2} = 0.4^{\log_{0.4} 2} = \mathbf{2.} \qquad \left[\text{Using } a^{\log_a x} = x\right]$$