VECTORS

TOTAL MARKS(12)

ONE MARK :

1. Define Zero vector

   A vector whose magnitude is equal to zero is called a zero vector

2. Define unit vector

   A vector whose magnitude is unity is called a unit vector

3. Define coinitial vectors

   Two or more vectors having the same initial point are called coinitial vectors

4. Define collinear vectors

   Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

5. Define equal vectors

   Two vectors are said to be equal if they have the same magnitude and direction.

6. Define negative vector

   A vector whose magnitude is the same as that of a given vector but direction is opposite to that of it, is called negative of the given vector.

7. Find unit vector in the direction of vector $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$

   Solution : $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$

   $$\therefore\ |\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

   $$\therefore\ \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{\imath}+3\hat{\jmath}+\hat{k}}{\sqrt{14}}$$

8. Find unit vector in the direction of vector $\vec{a} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$

   Solution : $\vec{a} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$

   $$\therefore\ |\vec{a}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$$

   $$\therefore\ \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{\imath}+\hat{\jmath}+2\hat{k}}{\sqrt{6}}$$

9. Find the vector joining the points $P(2,3,0)$ and $Q(-1,-2,-4)$ directed from P to Q

   Solution : Given $\overrightarrow{OP} = 2\hat{\imath} + 3\hat{\jmath}$ and $\overrightarrow{OQ} = -\hat{\imath} - 2\hat{\jmath} - 4\hat{k}$

   $$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \left(-\hat{\imath} - 2\hat{\jmath} - 4\hat{k}\right) - (2\hat{\imath} + 3\hat{\jmath}) = -3\hat{\imath} - 5\hat{\jmath} - 4\hat{k}$$

10. Find the direction cosines of the vector $\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

    Solution : $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

    $$\therefore\ |\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

    $\therefore$ direction cosines are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

11. If $\overrightarrow{AB} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\overrightarrow{OB} = 3\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$ find the position vector of $\overrightarrow{OA}$

    Solution : $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

    $$\overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{AB} = \left(3\hat{\imath} - 4\hat{\jmath} + 4\hat{k}\right) - \left(2\hat{\imath} - \hat{\jmath} + \hat{k}\right) = \hat{\imath} - 3\hat{\jmath} + 3\hat{k}$$

12. Show that the vectors $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $\vec{b} = -4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$ are collinear.

    Solution : Condition for collinear is $\vec{b} = \lambda\vec{a}$

$\vec{b} = -4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$

$= -2(2\hat{\imath} - 3\hat{\jmath} + 4\hat{k})$

$= -2\vec{a}$

$\therefore\ \vec{b} = \lambda\vec{a}$ where $\lambda = -2$

Hence the given vectors are collinear.

13. Find the angle between the two vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| = 1$, $|\vec{b}| = 1$ and $\vec{a}.\vec{b} = 1$

Solution : $\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{(1)(1)} = 1$

$\theta = \cos^{-1} 1 = 0^0$

14. Find the value of x for which $x(\hat{\imath} + \hat{\jmath} + \ddot{k})$ is a unit vector.

Solution : Given $x(\hat{\imath} + \hat{\jmath} + \ddot{k})$ is a unit vector

i.e $|(x\hat{\imath} + x\hat{\jmath} + x\hat{k})| = 1$

$\sqrt{x^2 + x^2 + x^2} = 1$

$\sqrt{3x^2} = 1 \Longrightarrow \sqrt{3}x = 1$

$x = {}^{1}/_{\sqrt{3}}$

15. Write the two different vectors having same magnitude

Solution: $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ and $\vec{b} = 2\hat{\imath} + \hat{\jmath} - 3\hat{k}$

TWO MARK :

1. Find a vector in the direction of vector $5\hat{\imath} - \hat{\jmath} + 2\hat{k}$ which has magnitude 8 units

Solution : $\vec{a} = 5\hat{\imath} - \hat{\jmath} + 2\hat{k}$

$\therefore\ |\vec{a}| = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{25 + 1 + 4} = \sqrt{30}$

$\therefore\ \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\hat{\imath} - \hat{\jmath} + 2\hat{k}}{\sqrt{30}}$

Required vector $= 8\hat{a} = 8\left(\frac{5\hat{\imath} - \hat{\jmath} + 2\hat{k}}{\sqrt{30}}\right) = \frac{40\hat{\imath} - 8\hat{\jmath} + 16\hat{k}}{\sqrt{30}}$

2. Find a vector in the direction of vector $\hat{\imath} - 2\hat{\jmath}$ which has magnitude 7 units

Solution : $\vec{a} = \hat{\imath} - 2\hat{\jmath}$

$\therefore\ |\vec{a}| = \sqrt{1^2 + (-2)^2} = \sqrt{1 + 4} = \sqrt{5}$

$\therefore\ \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{\imath} - 2\hat{\jmath}}{\sqrt{5}}$

Required vector $= 7\hat{a} = 7\left(\frac{\hat{\imath} - 2\hat{\jmath}}{\sqrt{5}}\right) = \frac{7\hat{\imath} - 14\hat{\jmath}}{\sqrt{5}}$

3. Show that the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ is equally inclined to the OX, OY and OZ.

Solution : Let $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$

Then $|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Therefore, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

4. If $\vec{a}$ is a unit vector such that $(\vec{x} + \vec{a})(\vec{x} - \vec{a}) = 8$ then find $|\vec{x}|$

Solution : $\vec{a}$ is a unit vector $\Longrightarrow |\vec{a}| = 1$

$(\vec{x} + \vec{a})(\vec{x} - \vec{a}) = 8$

$|\vec{x}|^2 - |\vec{a}|^2 = 8$

$|\vec{x}|^2 - 1 = 8$

$|\vec{x}|^2 = 9 \Longrightarrow |\vec{x}| = 3$

5. If $\vec{a}$ is a unit vector such that $(\vec{x} + \vec{a}).(\vec{x} - \vec{a}) = 12$ then find $|\vec{x}|$

Solution : $\vec{a}$ is a unit vector $\Longrightarrow |\vec{a}| = 1$

$(\vec{x}+\vec{a}).(\vec{x}-\vec{a}) = 12$

$|\vec{x}|^2 - |\vec{a}|^2 = 12$

$|\vec{x}|^2 - 1 = 12$

$|\vec{x}|^2 = 13 \implies |\vec{x}| = \sqrt{13}$

6. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$

Solution : $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{\imath}(1-1) - \hat{\jmath}(1+1) + \hat{k}(-1-1) = -2\hat{\jmath} - 2\hat{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\therefore$ Area of the parallelogram= $2\sqrt{2}$ sq.units

7. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$ and $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$

Solution : $\vec{a} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$ and $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = \hat{\imath}(1+4) - \hat{\jmath}(3-4) + \hat{k}(-3-1) = 5\hat{\imath} + \hat{\jmath} - 4\hat{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{25+1+16} = \sqrt{42} =$

$\therefore$ Area of the parallelogram= $\sqrt{42}$ sq.units

8. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{b} = 2\hat{\imath} - 7\hat{\jmath} + \hat{k}$

Solution : $\vec{a} \times \vec{b} = 20\hat{\imath} + 5\hat{\jmath} - 5\hat{k}$

Area of the parallelogram= $\sqrt{450} = 15\sqrt{2}$ sq.units

9. Find the projection of the vector $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ on the vector $\vec{b} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

Solution : $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

$\vec{a}.\vec{b} = (2)(1) + (3)(2) + (2)(1) = 2 + 6 + 2 = 10$

$|\vec{b}| = \sqrt{1+4+1} = \sqrt{6}$

$\therefore$ The projection of the vector $\vec{a}$ on the vector $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{10}{\sqrt{6}}$

10. Find the projection of the vector $\vec{a} = \hat{\imath} - \hat{\jmath}$ on the vector $\vec{b} = \hat{\imath} + \hat{\jmath}$

Solution : $\vec{a} = \hat{\imath} - \hat{\jmath}$ and $\vec{b} = \hat{\imath} + \hat{\jmath}$

$\vec{a}.\vec{b} = (1)(1) + (-1)(1) = 1 - 1 = 0$

$|\vec{b}| = \sqrt{1+1} = \sqrt{2}$

The projection of the vector $\vec{a}$ on the vector $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{0}{\sqrt{2}} = 0$

11. Find the projection of the vector $\vec{a} = \hat{\imath} + 3\hat{\jmath} + 7\hat{k}$ on the vector $\vec{b} = 7\hat{\imath} - \hat{\jmath} + 8\hat{k}$

Solution : The projection of the vector $\vec{a}$ on the vector $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{60}{\sqrt{114}}$

12. If $\vec{a} = 5\hat{\imath} - \hat{\jmath} - 3\hat{k}$ and $\vec{b} = \hat{\imath} + 3\hat{\jmath} - 5\hat{k}$ then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular

Solution : . $\vec{a} = 5\hat{\imath} - \hat{\jmath} - 3\hat{k}$ and $\vec{b} = \hat{\imath} + 3\hat{\jmath} - 5\hat{k}$

$\vec{a} + \vec{b} = (5\hat{\imath} - \hat{\jmath} - 3\hat{k}) + (\hat{\imath} + 3\hat{\jmath} - 5\hat{k}) = 6\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$

$\vec{a} - \vec{b} = (5\hat{\imath} - \hat{\jmath} - 3\hat{k}) - (\hat{\imath} + 3\hat{\jmath} - 5\hat{k}) = 4\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$

$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = (6)(4)+(2)(-4)+(-8)(2) = 24-8-16 = 0$

Hence $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors

13. Show that $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{b})$

Solution : LHS $= (\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$

$= (\vec{a}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \vec{b}\times\vec{b})$

$= 0 + \vec{a}\times\vec{b} + \vec{a}\times\vec{b} - 0 \qquad (\because \vec{b}\times\vec{a} = -\vec{a}\times\vec{b})$

$= 2(\vec{a}\times\vec{b}) = RHS \qquad (\because \vec{a}\times\vec{a} = \vec{b}\times\vec{b} = 0)$

14. Find $|\vec{a}|$ and $|\vec{b}|$ if $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$

Solution : $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 8$

$|\vec{a}|^2 - |\vec{b}|^2 = 8$

$[8|\vec{b}|]^2 - |\vec{b}|^2 = 8$

$64|\vec{b}|^2 - |\vec{b}|^2 = 8$

$63|\vec{b}|^2 = 8$

$|\vec{b}|^2 = \frac{8}{63} \Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}} = \frac{2\sqrt{2}}{\sqrt{63}}$

We have $|\vec{a}| = 8|\vec{b}| = 8\left(\frac{2\sqrt{2}}{\sqrt{63}}\right) = \frac{16\sqrt{2}}{\sqrt{63}}$

15. Find the angle θ between the vectors $\vec{a} = \hat{\imath}+\hat{\jmath}-\hat{k}$ and $\vec{b} = \hat{\imath}-\hat{\jmath}+\hat{k}$

Solution : $\vec{a} = \hat{\imath}+\hat{\jmath}-\hat{k}$ and $\vec{b} = \hat{\imath}-\hat{\jmath}+\hat{k}$

$\vec{a}.\vec{b} = (1)(1)+(1)(-1)+(-1)(1) = 1-1-1 = -1$

$|\vec{a}| = \sqrt{1+1+1} = \sqrt{3}$ and $|\vec{b}| = \sqrt{1+1+1} = \sqrt{3}$

Angle between the vector $= \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-1}{\sqrt{3}\sqrt{3}} = \frac{-1}{3}$

$\therefore \theta = \cos^{-1}\left(\frac{-1}{3}\right)$

THREE MARK :

1. Find the area of a triangle having the points $A(1,1,1)$, $B(1,2,3)$ and $C(2,3,1)$ as its vertices using vector method.

Solution : Given $\overrightarrow{OA} = \hat{\imath}+\hat{\jmath}+\hat{k}$, $\overrightarrow{OB} = \hat{\imath}+2\hat{\jmath}+3\hat{k}$ and $\overrightarrow{OC} = 2\hat{\imath}+3\hat{\jmath}+\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = (\hat{\imath}+2\hat{\jmath}+3\hat{k})-(\hat{\imath}+\hat{\jmath}+\hat{k}) = \hat{\jmath}+2\hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = (2\hat{\imath}+3\hat{\jmath}+\hat{k})-(\hat{\imath}+\hat{\jmath}+\hat{k}) = \hat{\imath}+2\hat{\jmath}$

Now $\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = \hat{\imath}(0-4)-\hat{\jmath}(0-2)+\hat{k}(0-1) = -4\hat{\imath}+2\hat{\jmath}-\hat{k}$

$\therefore |\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{16+4+1} = \sqrt{21}$

$\therefore$ Area of triangle $= \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = \frac{1}{2}\sqrt{21}$ sq.units

2. Find the area of a triangle having the points $A(1,1,2)$, $B(2,3,5)$ and $C(1,5,5)$ as its vertices using vector method.

Solution : Given $\overrightarrow{OA} = \hat{\imath}+\hat{\jmath}+2\hat{k}$, $\overrightarrow{OB} = 2\hat{\imath}+3\hat{\jmath}+5\hat{k}$ and $\overrightarrow{OC} = \hat{\imath}+5\hat{\jmath}+5\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = (2\hat{\imath}+3\hat{\jmath}+5\hat{k})-(\hat{\imath}+\hat{\jmath}+2\hat{k}) = \hat{\imath}+2\hat{\jmath}+3\hat{k}$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (\hat{\imath} + 5\hat{\jmath} + 5\hat{k}) - (\hat{\imath} + \hat{\jmath} + 2\hat{k}\ ) = 4\hat{\jmath} + 3\hat{k}$$

Now $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = \hat{\imath}(6-12) - \hat{\jmath}(3-0) + \hat{k}(4-0) = -6\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$

$\therefore\ |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{36+9+16} = \sqrt{61}$

$\therefore$ Area of triangle $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{61}$ sq.units

3. Prove that $[\vec{a}, \vec{b}, \vec{c} + \vec{d}] = [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{d}]$

Solution : $[\vec{a}, \vec{b}, \vec{c} + \vec{d}] = \vec{a}.(\vec{b} \times (\vec{c} + \vec{d}))$ $\qquad \because [\vec{a}, \vec{b}, \vec{c}] = \vec{a}.(\vec{b} \times \vec{c})$

$= \vec{a}.(\vec{b} \times \vec{c} + \vec{b} \times \vec{d})$

$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}.(\vec{b} \times \vec{d})$

$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{d}]$

4. Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a} + \vec{b} + \vec{c} = 0$. Evaluate the quantity $\mu = \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$, if $|\vec{a}| = 1, |\vec{b}| = 4$ and $|\vec{c}| = 2$

Solution : $\vec{a} + \vec{b} + \vec{c} = 0$

Squaring on both sides

$|\vec{a} + \vec{b} + \vec{c}|^2 = 0$

$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

$1 + 16 + 4 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

$21 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

$(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = -\frac{21}{2}$ $\qquad \therefore\ \mu = -\frac{21}{2}$

5. Find a vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

Solution : Given $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

$\vec{c}(say) = \vec{a} + \vec{b} = 4\hat{\imath} + 4\hat{\jmath}$ and $\vec{d} = \vec{a} - \vec{b} = 2\hat{\imath} + 4\hat{k}$

$\vec{c} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = \hat{\imath}(16-0) - \hat{\jmath}(16-0) + \hat{k}(0-8)$

$= 16\hat{\imath} - 16\hat{\jmath} - 8\hat{k}$

$\therefore\ |\vec{c} \times \vec{d}| = \sqrt{16^2 + (-16)^2 + (-8)^2} = \sqrt{256 + 256 + 64}$

$= \sqrt{576}$

$= 24$

$\therefore\ \hat{n} = \frac{\vec{c} \times \vec{d}}{|\vec{c} \times \vec{d}|} = \frac{16\hat{\imath} - 16\hat{\jmath} - 8\hat{k}}{24}$

6. If $\vec{a} = -4\hat{\imath} - 6\hat{\jmath} - \lambda\hat{k}$, $\vec{b} = -\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$ and $\vec{c} = -8\hat{\imath} - \hat{\jmath} + 3\hat{k}$ are coplanar find λ

Solution : Condition for coplanar is $\vec{a}.(\vec{b} \times \vec{c}) = 0$

$$\begin{vmatrix} -4 & -6 & -\lambda \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = 0$$

$-4(12+3) + 6(-3+24) - \lambda(1+32) = 0$

$-4(15) + 6(21) - \lambda(33) = 0$

$-60 + 126 - 33\lambda = 0$

$66 - 33\lambda = 0$

$$\therefore \lambda = \frac{66}{33} = 2$$

7. Prove that $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}, \vec{b}, \vec{c}]$

Solution : LHS $=[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$ $\qquad \because [\vec{a}, \vec{b}, \vec{c}] = \vec{a}.(\vec{b}\times\vec{c})$

$$= (\vec{a}+\vec{b}).\left((\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\right)$$

$$= (\vec{a}+\vec{b}).(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a})$$

$$= \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{c}) + \vec{a}.(\vec{c}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c}) + \vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{c}\times\vec{c}) + \vec{b}.(\vec{c}\times\vec{a})$$

$$= \vec{a}.(\vec{b}\times\vec{c}) + 0+0+0+0+0+0+\vec{b}.(\vec{c}\times\vec{a})$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{c}] \qquad \because \vec{a}.(\vec{b}\times\vec{c}) = \vec{b}.(\vec{c}\times\vec{a}) = \vec{c}.(\vec{a}\times\vec{b})$$

$$= 2[\vec{a}, \vec{b}, \vec{c}]$$

$$= RHS$$

8. Show that the points $A(-1,4,-3), B(3,2,-5), C(-3,8,-5)$ and $D(-3,2,1)$ are coplanar

Solution : Given $\overrightarrow{OA} = -\hat{\imath}+4\hat{\jmath}-3\hat{k}$, $\overrightarrow{OB} = 3\hat{\imath}+2\hat{\jmath}-5\hat{k}$, $\overrightarrow{OC} = -3\hat{\imath}+8\hat{\jmath}-5\hat{k}$

$$\overrightarrow{OD} = -3\hat{\imath}+2\hat{\jmath}+\hat{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = (3\hat{\imath}+2\hat{\jmath}-5\hat{k}) - (-\hat{\imath}+4\hat{\jmath}-3\hat{k}) = 4\hat{\imath}-2\hat{\jmath}-2\hat{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = (-3\hat{\imath}+8\hat{\jmath}-5\hat{k}) - (-\hat{\imath}+4\hat{\jmath}-3\hat{k}) = -2\hat{\imath}+4\hat{\jmath}-2\hat{k}$$

$$\overrightarrow{AD} = \overrightarrow{OD}-\overrightarrow{OA} = (-3\hat{\imath}+2\hat{\jmath}+\hat{k}) - (-\hat{\imath}+4\hat{\jmath}-3\hat{k}) = -2\hat{\imath}-2\hat{\jmath}+4\hat{k}$$

$$\therefore \quad \overrightarrow{AB}.(\overrightarrow{AC}\times\overrightarrow{AD}) = \begin{vmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} = 4(16-4)+2(-8-4)-2(4+8)$$

$$= 4(12)+2(-12)-2(12)$$

$$= 48-24-24$$

$$= 0$$

$\therefore$ The points are coplanar

9. If $\vec{a} = 2\hat{\imath}+2\hat{\jmath}+3\hat{k}$, $\vec{b} = -\hat{\imath}+2\hat{\jmath}+\hat{k}$ and $\vec{c} = 3\hat{\imath}+\hat{\jmath}$ such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ

Solution : Given $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$

$$i.e \;\; (\vec{a}+\lambda\vec{b}).\vec{c} = 0$$

$$\left[(2\hat{\imath}+2\hat{\jmath}+3\hat{k}) + \lambda(-\hat{\imath}+2\hat{\jmath}+\hat{k})\right].(3\hat{\imath}+\hat{\jmath}) = 0$$

$$\left[2\hat{\imath}+2\hat{\jmath}+3\hat{k}-\lambda\hat{\imath}+2\lambda\hat{\jmath}+\lambda\hat{k}\right].(3\hat{\imath}+\hat{\jmath}) = 0$$

$$(\,(2-\lambda)\hat{\imath}+(2+2\lambda)\hat{\jmath}+(3+\lambda)\hat{k}).(3\hat{\imath}+\hat{\jmath}) = 0$$

$$(2-\lambda)(3)+(2+2\lambda)(1)+(3+\lambda)(0) = 0$$

$$6-3\lambda+2+2\lambda = 0$$

$$8-\lambda = 0 \qquad \therefore \lambda = 8$$

10. Find the area of the triangle ABC where position vectors of A, B and C are $\hat{\imath}-\hat{\jmath}+2\hat{k}$, $2\hat{\jmath}+\hat{k}$ and $\hat{\jmath}+3\hat{k}$ respectively.

Solution : Given $\overrightarrow{OA} = \hat{\imath}-\hat{\jmath}+2\hat{k}$, $\overrightarrow{OB} = \hat{\imath}-\hat{\jmath}+2\hat{k}$ and $\overrightarrow{OC} = \hat{\jmath}+3\hat{k}$

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = (2\hat{\jmath}+\hat{k}) - (\hat{\imath}-\hat{\jmath}+2\hat{k}) = -\hat{\imath}+3\hat{\jmath}-\hat{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = (\hat{\jmath}+3\hat{k}) - (\hat{\imath}-\hat{\jmath}+2\hat{k}) = -\hat{\imath}+2\hat{\jmath}+\hat{k}$$

$$\text{Now } \overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 3 & -1 \\ -1 & 2 & 1 \end{vmatrix} = \hat{\imath}(3+2) - \hat{\jmath}(-1-1) + \hat{k}(-2+3)$$

$$= 5\hat{\imath} + 2\hat{\jmath} + \hat{k}$$

$$\therefore \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{25+4+1} = \sqrt{30}$$

$$\therefore \text{Area of triangle} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{30} \text{ sq.units}$$

11. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$ find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$

Solution: Given $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$$\vec{a} + \vec{b} + \vec{c} = 0$$

Squaring on both sides

$$\left|\vec{a} + \vec{b} + \vec{c}\right|^2 = 0$$

$$|\vec{a}|^2 + \left|\vec{b}\right|^2 + |\vec{c}|^2 + 2\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right) = 0$$

$$1 + 1 + 1 + 2\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right) = 0$$

$$3 + 2\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right) = 0$$

$$\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right) = -\frac{3}{2}$$

12. Find x such that the four points $A(3,2,1), B(4,x,5), C(4,2,-2)$ and $D(6,5,-1)$ are co-planar

Solution: Given $\overrightarrow{OA} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\overrightarrow{OB} = 4\hat{\imath} + x\hat{\jmath} + 5\hat{k}$, $\overrightarrow{OC} = 4\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

$$\overrightarrow{OD} = 6\hat{\imath} + 5\hat{\jmath} - \hat{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(4\hat{\imath} + x\hat{\jmath} + 5\hat{k}\right) - \left(3\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) = \hat{\imath} + (x-2)\hat{\jmath} + 4\hat{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(4\hat{\imath} + 2\hat{\jmath} - 2\hat{k}\right) - \left(3\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) = \hat{\imath} - 3\hat{k}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \left(6\hat{\imath} + 5\hat{\jmath} - \hat{k}\right) - \left(3\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) = 3\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$$

$$\therefore \quad \overrightarrow{AB}.\left(\overrightarrow{AC} \times \overrightarrow{AD}\right) = 0$$

$$\begin{vmatrix} 1 & x-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$$1(0+9) - (x-2)(-2+9) + 4(3-0) = 0$$

$$9 - (x-2)(7) + 12 = 0$$

$$9 - 7x + 14 + 12 = 0$$

$$-7x = -35$$

$$\therefore x = 5$$