4. DEFINITE INTEGRATION

Let us Study

- Definite integral as limit of sum.
- Fundamental theorem of integral calculus.
- Methods of evaluation and properties of definite integral.

4. 1 Definite integral as limit of sum :

In the last chapter, we studied various methods of finding the primitives or indefinite integrals of given function. We shall now interprete the definite integrals denoted by $\int_a^b f(x)\,dx$, read as the integral from a to b of the function $f(x)$ with respect to x. Here $a < b$, are real numbers and $f(x)$ is definited on $[a, b]$. At present, we assume that $f(x) \geq 0$ on $[a, b]$ and $f(x)$ is continuous.

$\int_a^b f(x)\,dx$ is defined as the area of the region bounded by $y = f(x)$, X-axis and the ordinates $x = a$ and $x = b$. If $g(x)$ is the primitive of $f(x)$ then the area is $g(b) - g(a)$.

The reason of the above definition will be clear from the figure 4.1. and the discussion that follows here. We are using the mean value theorem learnt earlier. Divide the interval $[a, b]$ into a equal parts by

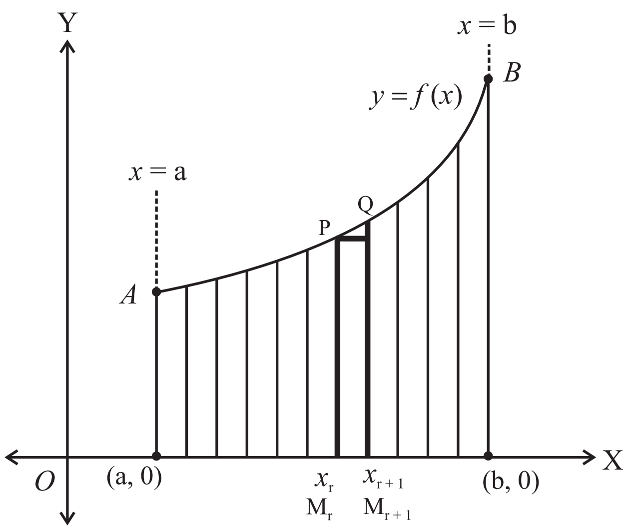

Fig. 4.1

$a = x_0 < x_1 < x_2 < . . . < x_{n-1} < x_n = b.$

Draw the curve $y = f(x)$ in $[a, b]$ and divide the interval $[a, b]$ into n equal parts by

$a = x_0 < x_1 < x_2 < . . . < x_{n-1} < x_n = b.$

Divide the region whose area is measured into their strips as above.

Note that, the area of each strip can be approximated by the area of a rectangle $M_r\,M_{r+1}$ QP as shown in the figure 4.1, which is $(x_{r-1} - x_r) \times f(\mathrm{T})$ where T is a point on the curve $y = f(x)$ between P and Q.

The mean value theorem states that if $g(x)$ is the primitive of $f(x)$,

$g(x_{r+1}) - g(x_r) = (x_{r+1} - x_r) \cdot f(t_r)$ where $x_r < t_r < x_{r+1}$.

Now we can replace $f(\mathrm{T})$ by $f(t_r)$ given here and express the approximation of the area of the shaded region as $\sum_{r=0}^{n=1} (x_{r+1} - x_r) \cdot f(t_r)$ where $x_r < t_r < x_{r+1}$.

Now we can replace $f(\mathrm{T})$ by $f(t_r)$ given here and express the approximation of the area of the shaed region as

$$\sum_{r=0}^{n=1} (x_{r+1} - x_r) \cdot f(t_r) = \sum_{r=0}^{n=1} g(x_{r+1}) - g(x_r) = g(b) - g(a)$$

Thus taking limit as $n \to \infty$

$$g(b) - g(a) = \lim_{n\to\infty} \sum (x_{r+1} - x_r) \cdot f(t_r)$$

$$= \lim_{n\to\infty} S_n$$

$$= \int_a^b f(x)\, dx$$

The word 'to integrate' means 'to find the sum of'. The technique of integration is very useful in finding plane areas, length of arcs, volume of solid revolution etc...

SOLVED EXAMPLES

Ex. 1 : $\int_1^2 (2x+5)\, dx$

Solution : Given, $\int_1^2 (2x+5)\, dx = \int_a^b f(x)\, dx$

$f(x) = 2x + 5 \qquad a = 1 ; b = 2$

$$\Rightarrow \quad f(a + rh) = f(1 + rh) = 2(1 + rh) + 5 = 2 + 2rh + 5 = 7 + 2rh$$

and $h = \frac{b-a}{n}$

$h = \frac{2-1}{n}$

$\therefore \quad nh = 1$

We know $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot f(a + rh)$

$$\therefore \quad \int_1^2 (2x+5)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot (7+2rh)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (7h + 2rh^2)$$

$$= \lim_{n\to\infty} \left(7h\sum_{r=1}^{n} 1 + 2h^2 \sum_{r=1}^{n} r \right)$$

$$= \lim_{n\to\infty} \left[7h\cdot(n) + 2h^2 \left(\frac{n\,(n+1)}{2} \right) \right]$$

$$= \lim_{n\to\infty} \left[7nh + h^2n^2 \left(1 + \frac{1}{n} \right) \right]$$

$$= \lim_{n\to\infty} \left[7\,(1) + (1)^2 \left(1 + \frac{1}{n} \right) \right]$$

$$= 7 + 1\,(1+0) = 8$$

Ex. 2 : $\int_2^3 7^x \cdot dx$

Solution : Given, $\int_2^3 7^x \cdot dx = \int_a^b f(x)\,dx$

$f(x) = 7^x \qquad a = 2\ ;\ b = 3$

$\Rightarrow \quad f(a+rh) = f(1+rh)$ and $h = \dfrac{b-a}{n}$

$= 7^{2+rh} \qquad h = \dfrac{3-2}{n} \qquad \therefore \quad nh = 1$

$= 7^2 \cdot 7^{rh}$

We know $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot f(a+rh)$

$$\therefore \quad \int_1^3 7^x \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \left(7^2 \cdot 7^{r\cdot h}\right)$$

$$= \lim_{n\to\infty} 7^2 \cdot \sum_{r=1}^{n} h \cdot 7^{r\cdot h}$$

$$= \lim_{n\to\infty} 7^2 \cdot h \cdot \left[7^h + 7^{2h} + 7^{3h} + 7^{4h} + \ldots + 7^{nh}\right]$$

$$= \lim_{n\to\infty} 7^2 \cdot h \cdot \left(\frac{7^h\,[(7^h)^n - 1]}{7^h - 1} \right) = \lim_{n\to\infty} 7^2 \cdot \left(\frac{7^h\,(7^{nh} - 1)}{\frac{7^h - 1}{h}} \right)$$

$$= \lim_{n\to\infty} 7^2 \cdot \left(\frac{7^h\,(7^{(1)} - 1)}{\frac{7^h - 1}{h}} \right)$$

$$= \frac{7^2 \cdot 7^0 \cdot (7-1)}{\log 7} = \frac{(49)(1)(6)}{\log 7} = \frac{294}{\log 7}$$

Ex. 3 : $\int_0^4 (x - x^2) \cdot dx$

Solution :

$$\int_0^4 (x - x^2) \cdot dx = \int_a^b f(x)\, dx$$

$f(x) = x - x^2$ $\qquad a = 0 \,;\, b = 4$

$\Rightarrow \quad f(a + rh) = f(0 + rh)$ $\qquad$ and $\quad h = \dfrac{b - a}{n}$

$= f(rh)$ $\qquad h = \dfrac{4 - 0}{n}$

$= (rh) - (rh)^2$

$= rh - r^2h^2$ $\qquad \therefore \quad nh = 4$

We know $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot [f(a + rh)]$

$$\therefore \quad \int_0^4 (x - x^2)\cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \left(rh - r^2h^2\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(rh^2 - r^2h^3\right)$$

$$= \lim_{n\to\infty} \left(h^2 \cdot \sum_{r=1}^{n} r - h^3 \cdot \sum_{r=1}^{n} r^2\right)$$

$$= \lim_{n\to\infty} \left[h^2\left(\frac{n(n+1)}{2}\right) - h^3\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{h^2 \cdot n \cdot n\left(1 + \frac{1}{n}\right)}{2} - \frac{h^3 \cdot n \cdot n\left(1 + \frac{1}{n}\right) n\left(2 + \frac{1}{n}\right)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{(nh)^2\left(1 + \frac{1}{n}\right)}{2} - \frac{(nh)^3\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{(4)^2\left(1 + \frac{1}{n}\right)}{2} - \frac{(4)^3\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)}{6}\right]$$

$$= \frac{(4)^2 \cdot (1 + 0)}{2} - \frac{(4)^3(1 + 0)\,(2 + 0)}{6}$$

$$= 8 - \frac{(64)(2)}{6}$$

$$= -\frac{40}{3}$$

Ex. 4 : $\int_0^{\pi/2} \sin x \cdot dx$

Solution : $\int_0^{\pi/2} \sin x \cdot dx = \int_0^{\pi/2} f(x)\, dx$

$f(x) = \sin x \qquad a = 0 \; ; \; b = \frac{\pi}{2}$

$\Rightarrow \quad f(a + rh) = \sin(a + rh)$

$= \sin(0 + rh) \qquad$ and $\qquad h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{n}$

$= \sin rh \qquad \therefore \quad nh = \frac{\pi}{2}$

We know $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot [f(a + rh)]$

$$\therefore \quad \int_0^{\pi/2} \sin x \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \sin rh$$

$$= \lim_{n\to\infty} h \cdot \sum_{r=1}^{n} \sin rh$$

$$= \lim_{n\to\infty} h \cdot [\sin h + \sin 2h + \sin 3h + \ldots + \sin nh] \qquad \ldots \text{(I)}$$

Consider,

$$\sum_{r=1}^{n} \sin rh = \sin h + \sin 2h + \sin 3h + \ldots + \sin nh$$

$$= 2 \sin \frac{h}{2} \cdot \sin h + 2 \sin \frac{h}{2} \cdot \sin 2h + 2 \sin \frac{h}{2} \cdot \sin 3h + \ldots + 2 \sin \frac{h}{2} \cdot \sin nh$$

$\because \quad 2 \sin A \cdot \sin B = \cos(A - B) - \cos(A + B)$

$$2 \sin \frac{h}{2} \cdot \sum_{r=1}^{n} \sin rh = \left[\left(\cos\frac{h}{2} - \cos\frac{3h}{2}\right) + \left(\cos\frac{3h}{2} - \cos\frac{5h}{2}\right) + \left(\cos\frac{5h}{2} - \cos\frac{7h}{2}\right) + \ldots\right.$$

$$\left. + \ldots + \left(\cos\left(\frac{2n-1}{2}\right)h - \left(\cos\left(\frac{2n+1}{2}\right)h\right.\right.\right]$$

$$= \left[\cos\frac{h}{2} - \cos\left(\frac{2n+1}{2}\right)h\right]$$

$$= \left[\cos\frac{h}{2} - \cos\left(\frac{2nh}{2} + \frac{h}{2}\right)\right]$$

$$= \left[\cos\frac{h}{2} - \cos\left(\frac{\pi}{2} + \frac{h}{2}\right)\right] \qquad \because \quad nh = \frac{\pi}{2}$$

$$= \left(\cos\frac{h}{2} + \sin\frac{h}{2}\right)$$

$$\therefore \quad \sum_{r=1}^{n} \sin rh = \frac{\cos\frac{h}{2} + \sin\frac{h}{2}}{2\sin\frac{h}{2}}$$

Now from I,

$$\int_0^{\pi/2} \sin x \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \sin rh$$

$$= \lim_{n\to\infty} h \cdot \left[\frac{\cos\frac{h}{2} + \sin\frac{h}{2}}{2\sin\frac{h}{2}}\right]$$

$$\because \quad nh = \frac{\pi}{4} \text{ as } n\to\infty \Rightarrow h\to 0 \left(\frac{1}{n} \to 0\right)$$

$$= \lim_{\substack{n\to\infty \\ h\to 0}} \left[\frac{\cos\frac{h}{2} + \sin\frac{h}{2}}{\frac{2\cdot\sin\frac{h}{2}}{h}}\right]$$

$$= \frac{\cos 0 + \sin 0}{\left(\frac{1}{2}\right)}$$

$$= \frac{1+0}{2\cdot\frac{1}{2}} = 1$$

$$\therefore \quad \int_0^{\pi/2} \sin x \cdot dx = 1$$

EXERCISE 4.1

I. Evaluate the following integrals as limit of sum.

(1) $\int_1^3 (3x-4)\cdot dx$ (2) $\int_0^4 x^2\cdot dx$ (3) $\int_0^2 e^x\cdot dx$

(4) $\int_0^2 (3x^2-1)\cdot dx$ (5) $\int_1^3 x^3\cdot dx$

4.2 Fundamental theorem of integral calculus :

Let f be the continuous function defined on $[a, b]$ and if $\int f(x)\,dx = g(x) + c$

then $\int_a^b f(x)\,dx = \left[g(x) + c\right]_a^b$

$= [(g(b) + c) - (g(a) + c)]$

$= g(b) + c - g(a) - c$

$= g(b) - g(a)$

Thus $\int_a^b f(x)\,dx = g(b) - g(a)$

Ex. : $\int_2^5 (x^2 - x)\,dx = \left[\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\right]_2^5$

$= \left[\left(\frac{5^3}{3} - \frac{5^2}{2}\right) - \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\right]$

$= \frac{125}{3} - \frac{25}{2} - \frac{8}{3} + \frac{4}{2}$

$= \frac{117}{3} - \frac{21}{2} = \frac{234 - 83}{6}$

$\therefore \int_2^5 (x^2 - x)\,dx = \frac{151}{3}$

In $\int_a^b f(x)\,dx$ a is called as a lower limit and b is called as an upper limit.

Now let us discuss some fundamental properties of definite integration.

These properties are very useful in evaluation of the definite integral.

4.2.1

Property I : $\int_a^a f(x)\,dx = 0$

Let $\int f(x)\,dx = g(x) + c$

$\therefore \int_a^a f(x)\,dx = \left[g(x) + c\right]_a^a$

$= [(g(a) + c) - (g(a) + c)]$

$= 0$

Property II : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Let $\int f(x)\,dx = g(x) + c$

$\therefore \int_a^b f(x)\,dx = \left[g(x) + c\right]_a^b$

$= [(g(b) + c) - (g(a) + c)]$

$= g(b) - g(a)$

$= -[g(a) - g(b)]$

$= -\int_b^a f(x)\,dx$

Thus $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Ex. $\int_1^3 x\,dx = \left[\frac{x^2}{2}\right]_1^3$

$= \frac{3^2}{2} - \frac{1^2}{2} = \frac{9}{2} - \frac{1}{2} = 4$

Ex. $\int_3^1 x\,dx = \left[\frac{x^2}{2}\right]_3^1$

$= \frac{1^2}{2} - \frac{3^2}{2} = \frac{1}{2} - \frac{9}{2} = -4$

Property III : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Let $\int f(x)\,dx = g(x) + c$

L.H.S. : $\int_a^b f(x)\,dx = \left[g(x) + c\right]_a^b$

$= [(g(b) + c) - (g(a) + c)]$

$= g(b) - g(a) \quad \ldots\ldots \text{(i)}$

R.H.S. : $\int_a^b f(t)\,dt = \left[g(t) + c\right]_a^b$

$= [(g(b) + c) - (g(a) + c)]$

$= g(b) - g(a) \quad \ldots\ldots \text{(ii)}$

from (i) and (ii)

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

i.e. definite integration is independent of the variable.

Ex. $\int_{\pi/6}^{\pi/3} \cos x \cdot dx = \left[\sin x\right]_{\pi/6}^{\pi/3}$

$= \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}$

$= \frac{\sqrt{3} - 1}{2}$

Ex. $\int_{\pi/6}^{\pi/3} \cos t \cdot dt = \left[\sin t\right]_{\pi/6}^{\pi/3}$

$= \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}$

$= \frac{\sqrt{3} - 1}{2}$

Property IV : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$ i.e. $c \in [a, b]$

Let $\int f(x)\,dx = g(x) + c$

Consider R.H.S. : $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$= \left[g(x) + c\right]_a^c + \left[g(x) + c\right]_c^b$

$= [(g(c) + c) - (g(a) + c)] + [(g(b) + c) - (g(c) + c)]$

$= g(c) + c - g(a) - c + g(b) + c - g(c) - c$

$= g(b) - g(a)$

$= \left[g(x) + c\right]_a^b$

$= \int_a^b f(x)\,dx$: L.H.S.

Thus $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$

Ex. : $\int_{-1}^{5}(2x+3)\cdot dx = \int_{-1}^{3}(2x+3)\cdot dx + \int_{3}^{5}(2x+3)\cdot dx$

L.H.S. : $\int_{-1}^{5}(2x+3)\cdot dx$

$= \left[2\frac{x^2}{2}+3x\right]_{-1}^{5}$

$= \left[x^2+3x\right]_{-1}^{5}$

$= [(5)^2+3(5)]-[(-1)^2+3(-1)]$

$= (25+15)-(1-3)$

$= 40+2 = 42$

R.H.S. : $\int_{-1}^{3}(2x+3)\cdot dx + \int_{3}^{5}(2x+3)\cdot dx$

$= \left[x^2+3x\right]_{-1}^{3} + \left[x^2+3x\right]_{3}^{5}$

$= [((3)^2+3(3))-((-1)^2+3(-1))] +$
$[((5)^2+3(5))-((3)^2+3(3))]$

$= [(9+9)-(1-3)]+[(25+15)-(9-9)]$

$= 18+2+40-18$

$= 42$

Property V : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Let $\int f(x)\,dx = g(x)+c$

Consider R.H.S. : $\int_a^b f(a+b-x)\,dx$

put $a+b-x=t$ i.e. $x=a+b-t$

$\therefore$ $-dx=dt \Rightarrow dx=-dt$

As $x\to a \Rightarrow t\to b$ and $x\to b\Rightarrow t\to a$

therefore $= \int_b^a f(t)(-dt)$

$= -\int_b^a f(t)\,dt$

$= \int_a^b f(t)\,dt \ldots \left(\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$

$= \int_a^b f(x)\,dx$ . . . as definite integration is independent of the variable.

$=$ L. H. S.

Thus $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Ex. :

$\int_{\pi/6}^{\pi/3} \sin^2 x\cdot dx$

$I = \int_{\pi/6}^{\pi/3} \sin^2 x\cdot dx$ . . . (i)

$= \int_{\pi/6}^{\pi/3} \sin^2\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)$

$= \int_{\pi/6}^{\pi/3} \sin^2\left(\frac{\pi}{2}-x\right)$

$I = \int_{\pi/6}^{\pi/3} \cos^2 x\cdot dx$ . . . (ii)

adding (i) and (ii)

$2I = \int_{\pi/6}^{\pi/3} \sin^2 x\cdot dx + \int_{\pi/6}^{\pi/3} \cos^2 x\cdot dx$

$2I = \int_{\pi/6}^{\pi/3} (\sin^2 x+\cos^2 x)\cdot dx$

$2I = \int_{\pi/6}^{\pi/3} 1\cdot dx = \left[x\right]_{\pi/6}^{\pi/3}$

$2I = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$ $\therefore$ $I = \frac{\pi}{12}$

$\int_{\pi/6}^{\pi/3} \sin^2 x\cdot dx = \frac{\pi}{12}$

Property VI : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Let $\int f(x)\,dx = g(x) + c$

Consider R.H.S. : $\int_0^a f(a-x)\,dx$

put $a - x = t$ i.e. $x = a - t$

$\therefore \quad -dx = dt \Rightarrow dx = -dt$

As x varies from 0 to a, t varies from a to 0

therefore I $= \int_a^0 f(t)\,(-dt)$

$= -\int_a^0 f(t)\,dt$

$= \int_0^a f(t)\,dt \ldots \left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$

$= \int_0^a f(x)\,dx$ . . . as definite integration is independent of the variable.

$=$ L. H. S.

Thus

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Ex. : $\int_0^{\pi/4} \log(1+\tan x)\cdot dx$

Let $\int_0^{\pi/4} \log(1+\tan x)\cdot dx$ . . . (i)

$$\text{I} = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]$$

$$= \int_0^{\pi/4} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}\right]\cdot dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]\cdot dx$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]\cdot dx$$

$$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan x}\right]\cdot dx$$

$$= \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]\cdot dx$$

$$= \int_0^{\pi/4} (\log 2)\cdot dx - \int_0^{\pi/4} \log(1+\tan x)\cdot dx$$

$$\text{I} = (\log 2)\int_0^{\pi/4} 1\cdot dx - \text{I} \qquad \ldots\text{by eq. (i)}$$

$$\text{I} + \text{I} = (\log 2)\Big[\,x\,\Big]_0^{\pi/4}$$

$$2\text{I} = (\log 2)\left[\frac{\pi}{4} - 0\right]$$

$$\therefore \quad \text{I} = \frac{\pi}{8}(\log 2)$$

Thus

$$\int_0^{\pi/4} \log(1+\tan x)\cdot dx = \frac{\pi}{8}(\log 2)$$

Property VII :

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

R.H.S. : $\int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

$= I_1 + I_2 \qquad \ldots \text{(i)}$

Consider $I_2 = \int_0^a f(2a-x)\,dx$

put $2a - x = t$ i.e. $x = 2a - t$

$\therefore \quad -1\,dx = 1\,dt \Rightarrow dx = -\,dt$

As x varies from 0 to $2a$, t varies from $2a$ to 0

$$\begin{aligned} I &= \int_{2a}^{a} f(t)\,(-\,dt) \\ &= -\int_{2a}^{a} f(t)\,dt \\ &= \int_0^{2a} f(t)\,dt \ldots \left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right) \\ &= \int_0^{2a} f(x)\,dx \ldots \left(\int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right) \end{aligned}$$

$$\therefore \int_0^a f(x)\,dx = \int_0^{2a} f(x)\,dx$$

from eq. (i)

$$\int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = \int_0^a f(x)\,dx + \int_0^{2a} f(x)\,dx$$

$$= \int_0^{2a} f(x)\,dx : \text{L.H.S}$$

Thus,

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Property VIII :

$$\int_{-a}^{a} f(x)\,dx = 2 \cdot \int_0^a f(x)\,dx \text{ , if } f(x) \text{ even function}$$

$$= 0 \text{ , if } f(x) \text{ is odd function}$$

$f(x)$ even function if $f(-x) = f(x)$

and $f(x)$ odd function if $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad \ldots \text{(i)}$$

Consider $\int_{-a}^{0} f(x)\,dx$

put $x = -\,t \quad \therefore \quad dx = -dt$

As x varies from $-\,a$ to 0, t varies from a to 0

$$\begin{aligned} I &= \int_a^0 f(-t)\,(-dt) = -\int_a^0 f(-t)\,dt \\ &= \int_0^a f(-t)\,dt \ldots \left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right) \\ &= \int_0^a f(-x)\,dx \ldots \left(\int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right) \end{aligned}$$

Equation (i) becomes

$$\begin{aligned} \int_{-a}^{a} f(x)\,dx &= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \\ &= \int_0^a [f(-x) + f(x)]\,dx \end{aligned}$$

If $f(x)$ is odd function then $f(-x) = -f(x)$, hence

$$\int_{-a}^{a} f(x)\,dx = 0$$

If $f(x)$ is even function then $f(-x) = f(x)$, hence

$$\int_{-a}^{a} f(x)\,dx = 2 \cdot \int_0^a f(x)\,dx$$

Hence :

$$\int_{-a}^{a} f(x)\,dx = 2 \cdot \int_0^a f(x)\,dx \text{ , if } f(x) \text{ even function}$$

$$= 0 \text{ , if } f(x) \text{ is odd function}$$

Ex. :

1. $\int_{-\pi/4}^{\pi/4} x^3 \cdot \sin^4 x \cdot dx$

Let $f(x) = x^3 \cdot \sin^4 x$

$f(-x) = (-x)^3 \cdot [\sin(-x)]^4 = -x^3 \cdot [-\sin x]^4 = -x^3 \cdot \sin^4 x$

$= -f(x)$

$f(x)$ is odd function.

$\therefore \int_{-\pi/4}^{\pi/4} x^3 \cdot \sin^4 x \cdot dx = 0$

2. $\int_{-1}^{1} \frac{x^2}{1+x^2} \cdot dx$

Let $f(x) = \frac{x^2}{1+x^2}$

$f(-x) = \frac{(-x)^2}{1+(-x)^2}$

$= \frac{x^2}{1+x^2}$

$= f(x)$

$f(x)$ is even function.

$$\int_{-1}^{1} \frac{x^2}{1+x^2} \cdot dx = 2\int_0^1 \frac{x^2}{1+x^2} \cdot dx$$

$$= 2\int_0^1 \frac{1+x^2-1}{1+x^2} \cdot dx$$

$$= 2\int_0^1 \left[1 - \frac{1}{1+x^2}\right] \cdot dx$$

$$= 2\left[x - \tan^{-1}x\right]_0^1$$

$$= 2\left\{(1 - \tan^{-1}x) - (0 - \tan^{-1}x)\right\}$$

$$= 2\left\{1 - \frac{\pi}{4} - 0\right\}$$

$$= 2\left(1 - \frac{\pi}{4}\right) = \left(\frac{4-\pi}{2}\right)$$

$$\therefore \int_{-1}^{1} \frac{x^2}{1+x^2} \cdot dx = \frac{4-\pi}{2}$$

SOLVED EXAMPLES

Ex. 1 : $\int_1^3 \frac{1}{\sqrt{2+x}+\sqrt{x}} \cdot dx$

Solution : $= \int_1^3 \left(\frac{1}{\sqrt{2+x}+\sqrt{x}}\right)\left(\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}\right) \cdot dx$

$$= \int_1^3 \left(\frac{\sqrt{2+x}-\sqrt{x}}{2+x-x}\right) \cdot dx$$

$$= \frac{1}{2} \cdot \int_1^3 (\sqrt{2+x}-\sqrt{x}) \cdot dx$$

$$= \frac{1}{2} \cdot \left[\frac{(2+x)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^3$$

$$= \frac{1}{3} \cdot \left[(2+x)^{\frac{3}{2}} - (x)^{\frac{3}{2}}\right]_1^3$$

$$= \frac{1}{3}\left\{\left[(2+3)^{\frac{3}{2}} - (3)^{\frac{3}{2}}\right] - \left[(2+1)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]\right\}$$

$$= \frac{1}{3}\left\{5^{\frac{3}{2}} - 3^{\frac{3}{2}} - 3^{\frac{3}{2}} + 1^{\frac{3}{2}}\right\}$$

$$= \frac{1}{3}\left\{5^{\frac{3}{2}} - 2(3)^{\frac{3}{2}} + 1\right\}$$

$$\therefore \int_1^3 \frac{1}{\sqrt{2+x}+\sqrt{x}} \, dx = \frac{1}{3}\left[5^{\frac{3}{2}} - 2(3)^{\frac{3}{2}} + 1\right]$$

Ex. 2 : $\int_0^{\pi/2} \sqrt{1 - \cos 4x} \cdot dx$

Solution : Let $I = \int_0^{\pi/2} \sqrt{1 - \cos 4x} \cdot dx$

$$I = \int_0^{\pi/2} \sqrt{2 \sin^2 2x} \cdot dx$$

$$\left(\because 1 - \cos A = 2 \sin^2 \frac{A}{2}\right)$$

$$= \sqrt{2} \cdot \int_0^{\pi/2} \sin 2x \cdot dx$$

$$= \sqrt{2} \cdot \left[\frac{-\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{\sqrt{2}}{2} \cdot \left[\cos 2\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{\sqrt{2}}{2} \cdot [\cos \pi - \cos 0]$$

$$= -\frac{\sqrt{2}}{2} \cdot (-1 - 1) = \sqrt{2}$$

$$\therefore \quad \int_0^{\pi/2} \sqrt{1 - \cos 4x} \cdot dx = \sqrt{2}$$

Ex. 3 : $\int_0^{\pi/2} \cos^3 x \cdot dx$

Solution : Let $I = \int_0^{\pi/2} \cos^3 x \cdot dx$

$$= \int_0^{\pi/2} \frac{1}{4} [\cos 3x + 3 \cos x] \cdot dx$$

$$= \frac{1}{4}\left[\sin 3x \cdot \frac{1}{3} + 3 \sin x\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(\frac{1}{3} \sin 3 \frac{\pi}{2} + 3 \sin \frac{\pi}{2}\right) - \left(\frac{1}{3} \sin 3 (0) + 3 \sin (0)\right)\right]$$

$$= \frac{1}{4}\left[\frac{1}{3} \sin \frac{3\pi}{2} + 3 \sin \frac{\pi}{2} - \frac{1}{3} \sin 0 + 3 \sin 0\right]$$

$$= \frac{1}{4}\left[\frac{1}{3}(-1) + 3(1) - 0\right]$$

$$= \frac{1}{4}\left[-\frac{1}{3} + 3\right] = \frac{1}{4}\left[\frac{8}{3}\right] = \frac{2}{3}$$

$$\therefore \quad \int_0^{\pi/2} \cos^3 x \cdot dx = \frac{2}{3}$$

Ex. 4 : $\int_0^{\pi/4} \frac{\sec^2 x}{2 \tan^2 x + 5 \tan x + 1} \cdot dx$

Solution : Let $I = \int_0^{\pi/4} \frac{\sec^2 x}{2 \tan^2 x + 5 \tan x + 1} \cdot dx$

put $\tan x = t \quad \therefore \sec^2 x \cdot dx = 1 \cdot dt$

As x varies from 0 to $\frac{\pi}{4}$

t varies from 0 to 1

$$= \int_0^1 \frac{1}{2t^2 + 4t + 1} \cdot dt$$

$$= \frac{1}{2} \cdot \int_0^1 \frac{1}{t^2 + 2t + \frac{1}{2}} \cdot dt$$

$$= \frac{1}{2} \cdot \int_0^1 \frac{1}{t^2 + 2t + 1 - 1 + \frac{1}{2}} \cdot dt$$

$$= \frac{1}{2} \cdot \int_0^1 \frac{1}{(t+1)^2 - \left(\frac{1}{\sqrt{2}}\right)^2} \cdot dt$$

$$= \frac{1}{2} \cdot \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)}\left[\log\left[\frac{(t+1) - \frac{1}{\sqrt{2}}}{(t+1) + \frac{1}{\sqrt{2}}}\right]\right]_0^1$$

$$= \frac{\sqrt{2}}{4} \log\left[\left(\frac{\sqrt{2}\,t + \sqrt{2} - 1}{\sqrt{2}\,t + \sqrt{2} + 1}\right)\right]_0^1$$

$$= \frac{\sqrt{2}}{4}\left[\log\left(\frac{\sqrt{2}\,(1) + \sqrt{2} - 1}{\sqrt{2}\,(1) + \sqrt{2} + 1}\right) - \log\left(\frac{\sqrt{2}\,(0) + \sqrt{2} - 1}{\sqrt{2}\,(0) + \sqrt{2} + 1}\right)\right]$$

$$= \frac{\sqrt{2}}{4}\left[\log\left(\frac{2\sqrt{2} - 1}{2\sqrt{2} + 1}\right) - \log\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right)\right]$$

$$= \frac{\sqrt{2}}{4} \log\left[\left(\frac{2\sqrt{2} - 1}{2\sqrt{2} + 1}\right) \div \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right)\right]$$

$$= \frac{\sqrt{2}}{4} \log\left[\frac{3 + \sqrt{2}}{3 - \sqrt{2}}\right]$$

Ex. 5 : $\int_1^2 \frac{\log x}{x^2} \cdot dx$

Solution : Let $I = \int_1^2 (\log x)\left(\frac{1}{x^2}\right) \cdot dx$

$$= \left[(\log x) \cdot \int \frac{1}{x^2} \cdot dx\right]_1^2 - \int_1^2 \frac{d}{dx} \log x \cdot \int \frac{1}{x^2} \cdot dx \cdot dx$$

$$= \left[(\log x) \cdot \left(-\frac{1}{x}\right)\right]_1^2 - \int_1^2 \frac{1}{x} \cdot \left(-\frac{1}{x}\right) \cdot dx$$

$$= \left[-\frac{1}{x} \log x\right]_1^2 + \int_1^2 \frac{1}{x^2} \cdot dx$$

$$= \left[-\frac{1}{x} \log x\right]_1^2 + \left[-\frac{1}{x}\right]_1^2$$

$$= \left[\left(-\frac{1}{2} \log 2\right) - \left(-\frac{1}{1} \log 1\right)\right] + \left[\left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)\right]$$

$$= -\frac{1}{2} \log 2 - 0 - \frac{1}{2} + 1 = \frac{1}{2} - \frac{1}{2} \log 2 \qquad \because \log 1 = 0$$

$$\therefore \int_1^2 \frac{\log x}{x^2} \cdot dx = \frac{1}{2}\left(1 - \log 2\right)$$

Ex. 6 : $\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} \cdot dx$

Solution : Let $I = \int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} \cdot dx$

$$= \int_0^{\pi/2} \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{2 \cos^2\left(\frac{x}{2}\right) + 2 \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)} \cdot dx$$

$$= \int_0^{\pi/2} \frac{\left[\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]}{2\left[\cos\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]} \cdot dx$$

$$= \int_0^{\pi/2} \left[\frac{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right] \cdot dx = \int_0^{\pi/2} \left[1 - \tan\left(\frac{x}{2}\right)\right] \cdot dx$$

$$= \frac{1}{2} \cdot \left[x - \log \left(\sec \frac{x}{2} \right) \cdot \frac{1}{\frac{1}{2}} \right]_0^{\pi/2}$$

$$= \frac{1}{2} \cdot \left[\frac{\pi}{2} - 2 \cdot \log \left(\sec \frac{\pi}{4} \right) - (0 - 2 \log \sec 0) \right]$$

$$= \frac{1}{2} \cdot \left[\frac{\pi}{2} - 2 \log \sqrt{2} - 0 + 2(0) \right] \qquad = \frac{1}{2} \cdot \left[\frac{\pi}{2} - 2 \log \sqrt{2} \right] \qquad = \frac{\pi}{4} - \log \sqrt{2}$$

$$\therefore \quad \int_0^{\pi/2} \frac{\sec^2 x}{1 + \cos x + \sin x} \cdot dx = \frac{\pi}{4} - \log \sqrt{2}$$

Ex. 7 : $\displaystyle\int_0^{1/2} \frac{1}{(1 - 2x^2)\sqrt{1 - x^2}} \cdot dx$

Solution : Let $\displaystyle I = \int_0^{1/2} \frac{1}{(1 - 2x^2)\sqrt{1 - x^2}} \cdot dx$

put $x = \sin\theta \quad \therefore \quad 1 \cdot dx = \cos\theta \cdot d\theta$

As x varies from 0 to $\frac{1}{2}$, θ varies from 0 to $\frac{\pi}{6}$

$$= \int_0^{\pi/6} \frac{\cos\theta}{(1 - 2\sin^2\theta)\sqrt{1 - \sin^2\theta}} \cdot d\theta = \int_0^{\pi/6} \frac{\cos\theta}{(\cos 2\theta)\sqrt{\cos^2\theta}} \cdot d\theta$$

$$= \int_0^{\pi/6} \frac{1}{\cos 2\theta} \cdot d\theta$$

$$= \int_0^{\pi/6} \sec 2\theta \cdot d\theta$$

$$= \left[\log(\sec 2\theta + \tan 2\theta) \cdot \frac{1}{2} \right]_0^{\pi/6}$$

$$= \frac{1}{2} \cdot \left[\log \left(\sec 2\left(\frac{\pi}{6}\right) + \tan 2\left(\frac{\pi}{6}\right) \right) - \log(\sec 0 + \tan 0) \right]$$

$$= \frac{1}{2} \cdot \left[\log \left(\sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right) - \log(1 + 0) \right] \qquad \because \log 1 = 0$$

$$= \frac{1}{2} \cdot \left[\log(2 + \sqrt{3}) - 0 \right]$$

$$= \frac{1}{2} \log(2 + \sqrt{3})$$

$$\therefore \quad \int_0^{1/2} \frac{1}{(1 - 2x^2)\sqrt{1 - x^2}} \cdot dx = \frac{1}{2} \log(2 + \sqrt{3})$$

Ex. 8 : $\int_0^2 \frac{2^x}{2^x(1+4^x)} \cdot dx$

Solution : Let $I = \int_0^2 \frac{2^x}{2^x(1+4^x)} \cdot dx$

put $2^x = t \quad \therefore \quad 2^x \cdot \log 2 \cdot dx = 1 \cdot dt$

As x varies from 0 to 2, t varies from 1 to 4

$$= \int_1^4 \frac{\frac{1}{\log 2}}{t(1+t^2)} \cdot dt$$

$$= \frac{1}{\log 2} \cdot \int_1^4 \frac{1}{t(1+t^2)} \cdot dt$$

$$= \frac{1}{\log 2} \cdot \int_1^4 \frac{1+t^2-t^2}{t(1+t^2)} \cdot dt$$

may be solved by method of partial fraction

$$= \frac{1}{\log 2} \cdot \int_1^4 \left[\frac{1+t^2}{t(1+t^2)} - \frac{t^2}{t(1+t^2)}\right] \cdot dt$$

$$= \frac{1}{\log 2} \cdot \int_1^4 \left[\frac{1}{t} - \frac{t}{1+t^2}\right] \cdot dt$$

$$= \frac{1}{\log 2} \cdot \left[\int_1^4 \frac{1}{t} \cdot dt - \frac{1}{2}\int_1^4 \frac{2t}{1+t^2} \cdot dt\right]$$

$$= \frac{1}{\log 2} \cdot \left[\log(t) - \frac{1}{2}\log(1+t^2)\right]_1^4$$

$$= \frac{1}{\log 2} \cdot \left[\left(\log 4 - \frac{1}{2}\log 17\right) - \left(\log 1 - \frac{1}{2}\log 2\right)\right]$$

$$= \frac{1}{\log 2} \cdot \left[\log 4 - \frac{1}{2}\log 17 + \frac{1}{2}\log 2\right]$$

$\because \log 1 = 0$

$$= \frac{1}{\log 2} \cdot \left[\log \frac{4\sqrt{2}}{\sqrt{17}}\right]$$

$$\therefore \int_0^2 \frac{2^x}{2^x(1+4^x)} \cdot dx = \frac{1}{(\log 2)} \cdot \left[\log \frac{4\sqrt{2}}{\sqrt{17}}\right]$$

$$= \log_2\left(\frac{4\sqrt{2}}{\sqrt{17}}\right)$$

Ex. 9 : $\int_{-1}^{1} |5x-3| \cdot dx$

Solution : Let $I = \int_{-1}^{1} |5x-3| \cdot dx$

$|5x-3| = -(5x-3)$ for $(5x-3) < 0$ i.e. $x < \frac{3}{5}$

$= (5x-3)$ for $(5x-3) > 0$ i.e. $x > \frac{3}{5}$

$$= \int_{-1}^{3/5} |5x-3| \cdot dx + \int_{3/5}^{1} |5x-3| \cdot dx \quad = \int_{-1}^{3/5} -(5x-3) \cdot dx + \int_{3/5}^{1} (5x-3) \cdot dx$$

$$= \left[-\left(5\frac{x^2}{2} - 3x\right)\right]_{-1}^{3/5} + \left[\left(5\frac{x^2}{2} - 3x\right)\right]_{3/5}^{1} \quad = \left[3x - \frac{5}{2}x^2\right]_{-1}^{3/5} + \left[\frac{5}{2}x^2 - 3x\right]_{3/5}^{1}$$

$$= \left[\left(3\left(\frac{3}{5}\right) - \frac{5}{2}\left(\frac{3}{5}\right)^2\right) - \left(3(-1) - \frac{5}{2}(-1)^2\right)\right] + \left[\left(\frac{5}{2}(1)^2 - 3(1)\right) - \left(\frac{5}{2}\left(\frac{3}{5}\right)^2 - 3\left(\frac{3}{5}\right)\right)\right]$$

$$= \left[\left(\frac{9}{5} - \frac{9}{10}\right) - \left(-3 - \frac{5}{2}\right)\right] + \left[\left(\frac{5}{2} - 3\right) - \left(\frac{9}{10} - \frac{9}{5}\right)\right]$$

$$= \frac{9}{5} - \frac{9}{10} + 3 + \frac{5}{2} + \frac{5}{2} - 3 - \frac{9}{10} + \frac{9}{5} = 2\left(\frac{9}{5} - \frac{9}{10} + \frac{5}{2}\right) = 2\left(\frac{18 - 9 + 25}{5}\right) \quad = \frac{34}{5}$$

$$\therefore \quad \int_{-1}^{1} |\, 5x - 3 \,| \cdot dx = \frac{34}{5}$$

Ex. 10 : $\displaystyle\int_0^{\pi/2} \frac{1}{1 + \sqrt[3]{\tan x}} \cdot dx$

Solution : Let $\displaystyle I = \int_0^{\pi/2} \frac{1}{1 + \sqrt[3]{\tan x}} \cdot dx$

$$= \int_0^{\pi/2} \left[\frac{1}{1 + \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\cos x}}}\right] \cdot dx$$

$$= \int_0^{\pi/2} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}} \cdot dx \qquad \ldots \text{(i)}$$

By property $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

$$I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt[3]{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt[3]{\sin\left(\frac{\pi}{2} - x\right)}} \cdot dx$$

$$= \int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}} \cdot dx \qquad \ldots \text{(ii)}$$

adding (i) and (ii)

$$I + I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}} \cdot dx + \int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}} \cdot dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}} \cdot dx$$

$$2I = \int_0^{\pi/2} 1 \cdot dx$$

$$I = \frac{1}{2}\Big[\, x \,\Big]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{4}$$

$$\therefore \quad \int_0^{\pi/2} \frac{1}{1 + \sqrt[3]{\tan x}} \cdot dx = \frac{\pi}{4}$$

with the help of the above solved/ illustrative example verify whether the following examples evaluates their definite integrate to be equal to / as $\frac{\pi}{4}$

$\displaystyle\int_0^{\pi/2} \frac{1}{1 + \cot^3 x} \cdot dx$; $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} \cdot dx$; $\displaystyle\int_0^{\pi/2} \frac{\sec x}{\sec x + \operatorname{cosec} x} \cdot dx$;

$\displaystyle\int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} \cdot dx$; $\displaystyle\int_0^{\pi/2} \frac{\operatorname{cosec}^{\frac{5}{2}} x}{\operatorname{cosec}^{\frac{5}{2}} x + \sec^{\frac{5}{2}} x} \cdot dx$

Ex. 11 : $\int_3^8 \frac{(11-x)^2}{x^2+(1-x)^2} \cdot dx$

Solution : Let $I = \int_3^8 \frac{(11-x)^2}{x^2+(1-x)^2} \cdot dx \quad \ldots (i)$

By property $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

$$I = \int_3^8 \frac{[11-(8+3-x)]^2}{[8+3-x]^2+[11-(8+3-x)]^2} \cdot dx \quad = \int_3^8 \frac{[11-(11-x)]^2}{(11-x)^2+[11-(11-x)]^2} \cdot dx$$

$$= \int_3^8 \frac{x^2}{(11-x)^2+x^2} \cdot dx \quad \ldots (ii)$$

adding (i) and (ii)

$$I + I = \int_3^8 \frac{(11-x)^2}{x^2+(1+x)^2} \cdot dx + \int_3^8 \frac{x^2}{(11-x)^2+x^2} \cdot dx$$

$$2I = \int_3^8 \frac{(11-x)^2+x^2}{x^2+(11-x)^2} \cdot dx$$

$$I = \frac{1}{2}\int_3^8 1 \cdot dx$$

$$I = \frac{1}{2}\Big[x\Big]_3^8 = \frac{1}{2}[8-3] = \frac{5}{2}$$

$$\therefore \int_3^8 \frac{(11-x)^2}{x^2+(1+x)^2} \cdot dx = \frac{5}{2}$$

Note that : In general $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} \cdot dx = \frac{1}{2}(b-a)$

verify the generalisation for the following examples :

$\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}} \cdot dx$; $\int_2^7 \frac{x^3}{(9-x)^3+x^3} \cdot dx$;

$\int_4^9 \frac{x^{\frac{1}{4}}}{(13-x)^{\frac{1}{4}}+x^{\frac{1}{4}}} \cdot dx$ $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}} \cdot dx$

$\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\operatorname{cosec} x}} \cdot dx$

Ex. 12 : $\int_0^{\pi} x \cdot \sin^2 x \cdot dx$

Solution :

Consider, $\quad I = \int_0^{\pi} x \cdot \sin^2 x \cdot dx \ldots \text{(i)}$

$$I = \int_0^{\pi} (\pi - x) \cdot [\sin(\pi - x)]^2 x \cdot dx$$

$$I = \int_0^{\pi} (\pi - x) \cdot \sin^2 x \cdot dx$$

$$I = \int_0^{\pi} \pi \cdot \sin^2 x \cdot dx - \int_0^{\pi} x \cdot \sin^2 x \cdot dx$$

$$I = \pi \cdot \int_0^{\pi} \frac{1}{2}(1 - \cos 2x) \cdot dx - I \ldots \text{by (i)}$$

$$I + I = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \cdot dx$$

$$2I = \frac{\pi}{2}\left[x - \sin 2x \cdot \frac{1}{2}\right]_0^{\pi}$$

$$I = \frac{\pi}{4}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{\pi}{4}[\pi] \qquad \because \sin 0 = 0;\ \sin 2\pi = 0$$

$$= \frac{\pi^2}{4}$$

$$\therefore \int_0^{\pi} x^2 \cdot \sin^2 x \cdot dx = \frac{\pi^2}{4}$$

Ex. 13 : Evaluate the integral $\int_0^{\pi} \cos^2 x \cdot dx$ using the result/ property.

Solution :

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx$$

Let, $I = \int_0^{\pi} \cos^2 x \cdot dx$

$$= \int_0^{2\left(\frac{\pi}{2}\right)} \cos^2 x \cdot dx$$

$$= \int_0^{\pi/2} \cos^2 x \cdot dx + \int_0^{\pi/2} \left[\cos\left(2\frac{\pi}{2} - x\right)\right]^2 \cdot dx$$

$$= \int_0^{\pi/2} \cos^2 x \cdot dx + \int_0^{\pi/2} \cos^2 x \cdot dx$$

$\because \cos(\pi - x) = -\cos x$

$$= 2 \cdot \int_0^{\pi/2} \cos^2 x \cdot dx$$

$$= \int_0^{\pi/2} (1 + \cos 2x) \cdot dx$$

$$= \left[x + \sin 2x \cdot \frac{1}{2}\right]_0^{\pi/2}$$

$$= \left[\left(\frac{\pi}{2} + \frac{1}{2}\sin 2\frac{\pi}{2}\right) - \left(0 + \frac{1}{2}\sin 2(0)\right)\right]$$

$$= \frac{\pi}{2} + 0 \qquad \because \sin 0 = 0;\ \sin \pi = 0$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{\pi} \cos^2 x \cdot dx = \frac{\pi}{2}$$

Ex. 14 : $\displaystyle\int_{-\pi}^{\pi} \frac{x(1+\sin x)}{1+\cos^2 x}\cdot dx$

Solution : Let $\displaystyle I = \int_{-\pi}^{\pi} \frac{x(1+\sin x)}{1+\cos^2 x}\cdot dx$

$$= \left[\left(\int_{-\pi}^{\pi} \frac{x}{1+\cos^2 x}\cdot dx\right) + \left(\int_{-\pi}^{\pi} \frac{x\cdot\sin x}{1+\cos^2 x}\cdot dx\right)\right]$$

The function $\dfrac{x}{1+\cos^2 x}$ is odd function and the function $\dfrac{x\cdot\sin x}{1+\cos^2 x}$ is even function.

$\displaystyle\int_{-a}^{a} f(x)\,dx = 2\cdot\int_{0}^{a} f(x)\,dx$, if $f(x)$ even function

$= 0$, if $f(x)$ is odd function

$$\therefore I = 0 + 2\cdot\int_{0}^{\pi} \frac{x\cdot\sin x}{1+\cos^2 x}\cdot dx$$

$$\therefore I = 2\cdot\int_{0}^{\pi} \frac{x\cdot\sin x}{1+\cos^2 x}\cdot dx \qquad \text{. . . (i)}$$

$$= 2\cdot\int_{0}^{\pi} \frac{(\pi-x)\cdot\sin(\pi-x)}{1+[\cos(\pi-x)]^2}\cdot dx$$

$$= 2\cdot\int_{0}^{\pi} \frac{(\pi-x)\cdot\sin x}{1+(-\cos x)^2}\cdot dx$$

$$= 2\pi\cdot\int_{0}^{\pi} \frac{\pi\cdot\sin x - x\cdot\sin x}{1+\cos^2 x}\cdot dx$$

$$= 2\pi\cdot\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} - 2\cdot\int_{0}^{\pi} \frac{x\cdot\sin x}{1+\cos^2 x}\cdot dx$$

$$I = 2\pi\cdot\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} - I \qquad \text{. . . by eq.(i)}$$

$$I + I = 2\pi\cdot\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} \qquad \text{. . . (ii)}$$

put $\cos x = t$ $\therefore$ $-\sin x\cdot dx = +\,dt$

As varies from 0 to π, t varies from 1 to -1

$$2I = 2\pi\cdot\int_{-1}^{1} \frac{-1}{1+t^2}\cdot dt$$

$$I = \pi\cdot 2\int_{0}^{1} \frac{1}{1+t^2}\cdot dt \qquad \left(\text{where } \frac{1}{1+t^2} \text{ is even function.}\right)$$

$$\text{I} = 2\pi \cdot \left[\tan^{-1} t\right]_0^1 \cdot$$

$$= 2\pi\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= 2\pi\left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{2}$$

$$\therefore \int_{-\pi}^{\pi} \frac{x(1+\sin x)}{1+\cos^2 x} \cdot dx = \frac{\pi^2}{2}$$

Ex. 15 : $\int_0^3 x[x] \cdot dx$, where $[x]$ denote greatest integrate function not greater than x.

Solution : Let $\text{I} = \int_0^3 x[x] \cdot dx$

$$\text{I} = \int_0^1 x[x] \cdot dx + \int_1^2 x[x] \cdot dx + \int_2^3 x[x] \cdot dx$$

$$= \int_0^1 x(0) \cdot dx + \int_1^2 x(1) \cdot dx + \int_2^3 x(2) \cdot dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_1^2 + \left[x^2\right]_2^3$$

$$= 0 + \left(\frac{4}{2} - \frac{1}{2}\right) + (9-4)$$

$$= \frac{3}{2} + 5 = \frac{13}{2}$$

$$\therefore \int_0^3 x[x] \cdot dx = \frac{13}{2}$$

EXERCISE 4.2

I. Evaluate :

(1) $\int_1^9 \frac{x+1}{\sqrt{x}} \cdot dx$

(2) $\int_2^3 \frac{1}{x^2+5x+6} \cdot dx$

(3) $\int_0^{\pi/4} \cot^2 \cdot dx$

(4) $\int_{-\pi/4}^{\pi/4} \frac{1}{1-\sin x} \cdot dx$

(5) $\int_3^5 \frac{1}{\sqrt{2x+3} - \sqrt{2x-3}} \cdot dx$

(6) $\int_0^1 \frac{x^2-2}{x^2+1} \cdot dx$

(7) $\int_0^{\pi/4} \sin 4x \sin 3x \cdot dx$

(8) $\int_0^{\pi/4} \sqrt{1+\sin 2x} \cdot dx$

(9) $\int_0^{\pi/4} \sin^4 x \cdot dx$

(10) $\int_{-4}^{2} \frac{1}{x^2+4x+13} \cdot dx$

(11) $\int_0^4 \frac{1}{\sqrt{4x-x^2}} \cdot dx$

(12) $\int_0^1 \frac{1}{\sqrt{3+2x-x^2}} \cdot dx$

(13) $\int_0^{\pi/2} x \cdot \sin x \cdot dx$

(14) $\int_0^1 x \cdot \tan^{-1} x \cdot dx$

(15) $\int_0^{\infty} x \cdot e^{-x} \cdot dx$

II. Evaluate :

(1) $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} \cdot dx$

(2) $\int_0^{\pi/4} \frac{\sec^2 x}{3\tan^2 x + 4\tan x + 1} \cdot dx$

(3) $\int_0^{4\pi} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \cdot dx$

(4) $\int_0^{2\pi} \sqrt{\cos x} \cdot \sin^3 x \cdot dx$

(5) $\int_0^{\pi/2} \frac{1}{5 + 4\cos x} \cdot dx$

(6) $\int_0^{\pi/4} \frac{\cos x}{4 - \sin^2 x} \cdot dx$

(7) $\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)} \cdot dx$

(8) $\int_{-1}^{1} \frac{1}{a^2 e^x + b^2 e^{-x}} \cdot dx$

(9) $\int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x} \cdot dx$

(10) $\int_0^{\pi/4} \sec^4 x \cdot dx$

(11) $\int_0^{1} \sqrt{\frac{1-x}{1+x}} \cdot dx$

(12) $\int_0^{\pi} \sin^3 x\,(1 + 2\cos x)\,(1+\cos x)^2 \cdot dx$

(13) $\int_0^{\pi/2} \sin 2x \cdot \tan^{-1}(\sin x) \cdot dx$

(14) $\int_{\frac{1}{\sqrt{2}}}^{1} \frac{(e^{\cos^{-1} x})(\sin^{-1} x)}{\sqrt{1-x^2}} \cdot dx$

(15) $\int_2^{3} \frac{\cos(\log x)}{x} \cdot dx$

III. Evaluate :

(1) $\int_0^{a} \frac{1}{x + \sqrt{a^2 - x^2}} \cdot dx$

(2) $\int_0^{\pi/2} \log \tan x \cdot dx$

(3) $\int_0^{1} \log\left(\frac{1}{x} - 1\right) \cdot dx$

(4) $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cdot \cos x} \cdot dx$

(5) $\int_0^{3} x^2 (3-x)^{\frac{5}{2}} \cdot dx$

(6) $\int_{-3}^{3} \frac{x^3}{9 - x^2} \cdot dx$

(7) $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2+\sin x}{2-\sin x}\right) \cdot dx$

(8) $\int_{-\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} \cdot dx$

(9) $\int_{-\pi/4}^{\pi/4} x^3 \cdot \sin^4 x \cdot dx$

(10) $\int_0^{1} \frac{\log(x+1)}{x^2+1} \cdot dx$

(11) $\int_{-1}^{1} \frac{x^3 + 2}{\sqrt{x^2 + 4}} \cdot dx$

(12) $\int_{-a}^{a} \frac{x + x^3}{16 - x^2} \cdot dx$

(13) $\int_0^{1} t^2 \sqrt{1-t} \cdot dx$

(14) $\int_0^{\pi} x \cdot \sin x \cdot \cos^2 x \cdot dx$

(15) $\int_0^{1} \frac{\log x}{\sqrt{1-x^2}} \cdot dx$

Note that :

To evaluate the integrals of the type $\int_0^{\pi/2} \sin^n x \cdot dx$ and $\int_0^{\pi/2} \cos^n x \cdot dx$, the results used are known as **'reduction formulae'** which are stated as follows :

$$\int_0^{\pi/2} \sin^n x \cdot dx = \frac{(n-1)}{n}\cdot\frac{(n-3)}{(n-2)}\cdot\frac{(n-5)}{(n-4)} \cdots \frac{4}{5}\,\frac{2}{3}, \qquad \text{if } n \text{ is odd.}$$

$$= \frac{(n-1)}{n}\cdot\frac{(n-3)}{(n-2)}\cdot\frac{(n-5)}{(n-4)} \cdots \frac{3}{4}\,\frac{1}{2}\cdot\frac{\pi}{2}, \qquad \text{if } n \text{ is even.}$$

$$\int_0^{\pi/2} \cos^n x \cdot dx = \int_0^{\pi/2} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^n \cdot dx \qquad \text{. . . by property}$$

$$= \int_0^{\pi/2} [\,\sin x\,]^n \cdot dx$$

$$= \int_0^{\pi/2} \sin^n x \cdot dx$$

$$\int_0^{\pi/2} \sin^7 x \cdot dx = \frac{(7-1)}{7}\cdot\frac{(7-3)}{(7-2)}\cdot\frac{(7-5)}{(7-4)}$$

$$= \frac{(7-1)\cdot(7-3)\cdot(7-5)}{7\cdot(7-2)\cdot(7-4)}$$

$$= \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3} = \frac{16}{35}$$

$$\int_0^{\pi/2} \cos^8 x \cdot dx = \frac{(8-1)}{8}\cdot\frac{(8-3)}{(8-2)}\cdot\frac{(8-5)}{(8-4)}\cdot\frac{(8-7)}{(8-6)}\cdot\frac{\pi}{2}$$

$$= \frac{(8-1)\cdot(8-3)\cdot(8-5)\cdot(8-7)}{8\cdot(8-2)\cdot(8-4)\cdot(8-6)}\cdot\frac{\pi}{2}$$

$$= \frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \frac{35\pi}{256}$$

Let us Remember

- $\sum_{r=0}^{n-1} (x_{r+1} - x_r) \cdot f(t_r) = \sum_{r=0}^{n-1} g(x_{r+1}) - g(x_r) = g(b) - g(a)$

  Thus taking limit as $n \to \infty$

  $$g(b) - g(a) = \lim_{n\to\infty} \sum (x_{r+1} - x_r) \cdot f(t_r) = \lim_{n\to\infty} S_n = \int_a^b f(x)\,dx$$

- **Fundamental theorem of integral calculus :** $\int_a^b f(x)\,dx = g(b) - g(a)$

  Property I : $\int_a^a f(x)\,dx = 0$

  Property II : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

  Property III : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

  Property IV : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$ i.e. $c \in [a, b]$

  Property V : $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

  Property VI : $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$

  Property VII : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx$

  Property VIII : $\int_{-a}^a f(x)\,dx = 2 \cdot \int_0^a f(x)\,dx$, if $f(x)$ even function

  $= 0$, if $f(x)$ is odd function

  $f(x)$ even function if $f(-x) = f(x)$ and $f(x)$ odd function if $f(-x) = -f(x)$

- **'Reduction formulae'** which are stated as follows :

  $$\int_0^{\pi/2} \sin^n x \cdot dx = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{4}{5}\,\frac{2}{3}, \quad \text{if } n \text{ is odd.}$$

  $$= \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4}\,\frac{1}{2} \cdot \frac{\pi}{2}, \quad \text{if } n \text{ is even.}$$

  $$\int_0^{\pi/2} \cos^n x \cdot dx = \int_0^{\pi/2} \left[\cos\left(\frac{\pi}{2} - 0\right)\right]^n \cdot dx = \int_0^{\pi/2} [\sin x]^n \cdot dx = \int_0^{\pi/2} \sin^n x \cdot dx$$

MISCELLANEOUS EXERCISE 4

(I) Choose the correct option from the given alternatives :

(1) $\int_{2}^{3} \frac{dx}{x(x^3 - 1)} =$

(A) $\frac{1}{3}\log\left(\frac{208}{189}\right)$ (B) $\frac{1}{3}\log\left(\frac{189}{208}\right)$ (C) $\log\left(\frac{208}{189}\right)$ (D) $\log\left(\frac{189}{208}\right)$

(2) $\int_{0}^{\pi/2} \frac{\sin^2 x \cdot dx}{(1 + \cos x)^2} =$

(A) $\frac{4 - \pi}{2}$ (B) $\frac{\pi - 4}{2}$ (C) $4 - \frac{\pi}{2}$ (D) $\frac{4 + \pi}{2}$

(3) $\int_{0}^{\log 5} \frac{e^x \sqrt{e^x - 1}}{e^x + 3} \cdot dx =$

(A) $3 + 2\pi$ (B) $4 - \pi$ (C) $2 + \pi$ (D) $4 + \pi$

(4) $\int_{0}^{\pi/2} \sin^6 x \cos^2 x \cdot dx =$

(A) $\frac{7\pi}{256}$ (B) $\frac{3\pi}{256}$ (C) $\frac{5\pi}{256}$ (D) $\frac{-5\pi}{256}$

(5) If $\int_{0}^{1} \frac{dx}{\sqrt{1 + x} - \sqrt{x}} = \frac{k}{3}$, then k is equal to

(A) $\sqrt{2}(2\sqrt{2} - 2)$ (B) $\frac{\sqrt{2}}{3}(2 - 2\sqrt{2})$ (C) $\frac{2\sqrt{2} - 2}{3}$ (D) $4\sqrt{2}$

(6) $\int_{1}^{2} \frac{1}{x^2} e^{\frac{1}{x}} \cdot dx =$

(A) $\sqrt{e} + 1$ (B) $\sqrt{e} - 1$ (C) $\sqrt{e}(\sqrt{e} - 1)$ (D) $\frac{\sqrt{e} - 1}{e}$

(7) If $\int_{2}^{e} \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] \cdot dx = a + \frac{b}{\log 2}$, then

(A) $a = e, b = -2$ (B) $a = e, b = 2$ (C) $a = -e, b = 2$ (D) $a = -e, b = -2$

(8) Let $I_1 = \int_{e}^{e^2} \frac{dx}{\log x}$ and $I_2 = \int_{1}^{2} \frac{e^x}{x} \cdot dx$, then

(A) $I_1 = \frac{1}{3} I_2$ (B) $I_1 + I_2 = 0$ (C) $I_1 = 2I_2$ (D) $I_1 = I_2$

(9) $\int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} \cdot dx =$

(A) 9 (B) $\frac{9}{2}$ (C) 0 (D) 1

(10) The value of $\int_{-\pi/4}^{\pi/4} \log\left(\frac{2+\sin\theta}{2-\sin\theta}\right) \cdot d\theta$ is

(A) 0 (B) 1 (C) 2 (D) π

(II) Evaluate the following :

(1) $\int_0^{\pi/2} \frac{\cos x}{3 \cdot \cos x + \sin x} \cdot dx$

(2) $\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\left[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right]^3} \cdot d\theta$

(3) $\int_0^1 \frac{1}{1+\sqrt{x}} \cdot dx$

(4) $\int_0^{\pi/4} \frac{\tan^3 x}{1+\cos 2x} \cdot dx$

(5) $\int_0^1 t^5 \cdot \sqrt{1-t^2} \cdot dt$

(6) $\int_0^1 (\cos^{-1} x)^2 \cdot dx$

(7) $\int_{-1}^1 \frac{1+x^3}{9-x^2} \cdot dx$

(8) $\int_0^{\pi} x \cdot \sin x \cdot \cos^4 x \cdot dx$

(9) $\int_0^{\pi} \frac{x}{1+\sin^2 x} \cdot dx$

(10) $\int_1^{\infty} \frac{1}{\sqrt{x}\,(1+x)} \cdot dx$

(III) Evaluate :

(1) $\int_0^1 \left(\frac{1}{1+x^2}\right) \sin^{-1}\left(\frac{2x}{1+x^2}\right) \cdot dx$

(2) $\int_0^{\pi/2} \frac{1}{6-\cos x} \cdot dx$

(3) $\int_0^a \frac{1}{a^2+ax-x^2} \cdot dx$

(4) $\int_{\pi/5}^{3\pi/10} \frac{\sin x}{\sin x + \cos x} \cdot dx$

(5) $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) \cdot dx$

(6) $\int_0^{\pi/4} \frac{\cos 2x}{1+\cos 2x + \sin 2x} \cdot dx$

(7) $\int_0^{\pi/2} \left(2 \cdot \log \sin x - \log \sin 2x\right) \cdot dx$

(8) $\int_0^{\pi} \left(\sin^{-1} x + \cos^{-1} x\right)^3 \cdot \sin^3 x \cdot dx$

(9) $\int_0^4 \left[\sqrt{x^2+2x+3}\,\right]^{-1} \cdot dx$

(10) $\int_{-2}^3 \left|x-2\right| \cdot dx$

(IV) Evaluate the following :

(1) If $\int_{0}^{a} \sqrt{x} \cdot dx = 2a \cdot \int_{0}^{\pi/2} \sin^3 x \cdot dx$ then find the value of $\int_{a}^{a+1} x \cdot dx$.

(2) If $\int_{0}^{k} \frac{1}{2 + 8x^2} \cdot dx = \frac{\pi}{16}$. Find k.

(3) If $f(x) = a + bx + cx^2$, show that $\int_{0}^{1} f(x) = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$.

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