CHAPTER 7

Techniques of Integration

OVERVIEW We have seen how integrals arise in modeling real phenomena and in measuring objects in the world around us, and we know in theory how integrals are evaluated with antiderivatives. The more sophisticated our models become, however, the more involved our integrals become. We need to know how to change these more involved integrals into forms we can work with. The goal of this chapter is to show how to change unfamiliar integrals into integrals we can recognize, find in a table, or evaluate with a computer.

7.1 Basic Integration Formulas

As we saw in Section 4.1, we evaluate an indefinite integral by finding an antiderivative of the integrand and adding an arbitrary constant. Table 7.1 (on the following page) shows the basic forms of the integrals we have evaluated so far. There is a more extensive table at the back of the book; we will discuss it in Section 7.5.

Algebraic Procedures

We often have to rewrite an integral to match it to a standard formula.

EXAMPLE 1 *A simplifying substitution*

Evaluate $\displaystyle\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx.$

Solution

$$\begin{aligned}
\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx &= \int \frac{du}{\sqrt{u}} && \begin{array}{l} u = x^2-9x+1 \\ du = (2x-9)\,dx \end{array} \\
&= \int u^{-1/2}\,du \\
&= \frac{u^{(-1/2)+1}}{(-1/2)+1} + C && \text{Table 7.1, Formula 4, with } n = -1/2 \\
&= 2u^{1/2} + C \\
&= 2\sqrt{x^2-9x+1} + C
\end{aligned}$$

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Table 7.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\, du = ku + C \qquad \text{(any number } k\text{)}$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\, du = \dfrac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \dfrac{du}{u} = \ln |u| + C$
6. $\int \sin u\, du = -\cos u + C$
7. $\int \cos u\, du = \sin u + C$
8. $\int \sec^2 u\, du = \tan u + C$
9. $\int \csc^2 u\, du = -\cot u + C$
10. $\int \sec u \tan u\, du = \sec u + C$
11. $\int \csc u \cot u\, du = -\csc u + C$
12. $\int \tan u\, du = -\ln |\cos u| + C$
    $= \ln |\sec u| + C$
13. $\int \cot u\, du = \ln |\sin u| + C$
    $= -\ln |\csc u| + C$
14. $\int e^u du = e^u + C$
15. $\int a^u du = \dfrac{a^u}{\ln a} + C \qquad (a > 0, \quad a \neq 1)$
16. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
17. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a} \tan^{-1}\left(\dfrac{u}{a}\right) + C$
18. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a} \sec^{-1}\left|\dfrac{u}{a}\right| + C$

EXAMPLE 2 *Completing the square*

Evaluate $\displaystyle\int \frac{dx}{\sqrt{8x - x^2}}$.

Solution We complete the square to write the radicand as

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 4, \quad u = (x - 4) \\ & && du = dx \\ &= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 7.1, Formula 16} \\ &= \sin^{-1}\left(\frac{x - 4}{4}\right) + C. \end{aligned}$$

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EXAMPLE 3 *Expanding a power and using a trigonometric identity*

Evaluate $\int (\sec x + \tan x)^2\, dx$.

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are old friends; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned}\int (\sec x + \tan x)^2\, dx &= \int (\sec^2 x + 2\sec x \tan x + \sec^2 x - 1)\, dx \\ &= 2\int \sec^2 x\, dx + 2\int \sec x \tan x\, dx - \int 1\, dx \\ &= 2\tan x + 2\sec x - x + C.\end{aligned}$$

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EXAMPLE 4 *Eliminating a square root*

Evaluate $\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx$.

Solution We use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence,

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx &= \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx \\ &= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx && \sqrt{u^2} = |u| \\ &= \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx && \text{On } [0, \pi/4],\ \cos 2x \ge 0 \text{ so } |\cos 2x| = \cos 2x. \\ &= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} \\ &= \sqrt{2}\left[\frac{1}{2} - 0\right] = \frac{\sqrt{2}}{2}.\end{aligned}$$

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EXAMPLE 5 *Reducing an improper fraction*

Evaluate $\int \frac{3x^2 - 7x}{3x + 2}\, dx$.

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

$$\begin{array}{r|l} & x \quad - 3 \\ \hline 3x + 2 & 3x^2 - 7x \\ & 3x^2 + 2x \\ \hline & \quad - 9x \\ & \quad - 9x - 6 \\ \hline & \qquad + 6 \end{array}$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2 \ln |3x + 2| + C.$$ ❑

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We will see what to do about that in Section 7.3.

EXAMPLE 6 *Separating a fraction*

Evaluate $\displaystyle\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx.$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = 3 \int \frac{x\, dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals we substitute

$$u = 1 - x^2, \qquad du = -2x\, dx, \qquad \text{and} \qquad x\, dx = -\frac{1}{2}\, du.$$

$$3 \int \frac{x\, dx}{\sqrt{1 - x^2}} = 3 \int \frac{(-1/2)\, du}{\sqrt{u}} = -\frac{3}{2} \int u^{-1/2}\, du$$

$$= -\frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1.$$

The second of the new integrals is a standard form,

$$2 \int \frac{dx}{\sqrt{1 - x^2}} = 2 \sin^{-1} x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = -3\sqrt{1 - x^2} + 2 \sin^{-1} x + C.$$ ❑

EXAMPLE 7 *Multiplying by a form of* 1

Evaluate $\displaystyle\int \sec x\, dx.$

Solution

$$\int \sec x\, dx = \int (\sec x)(1)\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{du}{u} \qquad \begin{aligned} u &= \tan x + \sec x \\ du &= (\sec^2 x + \sec x \tan x)\, dx \end{aligned}$$

$$= \ln |u| + C = \ln |\sec x + \tan x| + C$$

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Table 7.2 The secant and cosecant integrals

1.	$\int \sec u \, du = \ln \lvert \sec u + \tan u \rvert + C$
2.	$\int \csc u \, du = -\ln \lvert \csc u + \cot u \rvert + C$

With cosecants and cotangents in place of secants and tangents, the method of Example 7 leads to a companion formula for the integral of the cosecant (see Exercise 95).

Procedures for Matching Integrals to Basic Formulas

Procedure	Example
Making a simplifying substitution	$\frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\, dx = \frac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x - x^2} = \sqrt{16 - (x - 4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2 \sec x \tan x + \tan^2 x$ $= \sec^2 x + 2 \sec x \tan x + (\sec^2 x - 1)$ $= 2 \sec^2 x + 2 \sec x \tan x - 1$
Eliminating a square root	$\sqrt{1 + \cos 4x} = \sqrt{2 \cos^2 2x} = \sqrt{2}\, \lvert \cos 2x \rvert$
Reducing an improper fraction	$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}$
Separating a fraction	$\frac{3x + 2}{\sqrt{1 - x^2}} = \frac{3x}{\sqrt{1 - x^2}} + \frac{2}{\sqrt{1 - x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$ $= \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$

Exercises 7.1

Basic Substitutions

Evaluate the integrals in Exercises 1–36 by using substitutions that reduce them to standard forms.

1. $\int \frac{16x\,dx}{\sqrt{8x^2+1}}$

2. $\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$

3. $\int 3\sqrt{\sin v}\cos v\,dv$

4. $\int \cot^3 y\csc^2 y\,dy$

5. $\int_0^1 \frac{16x\,dx}{8x^2+2}$

6. $\int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z}\,dz$

7. $\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

8. $\int \frac{dx}{x-\sqrt{x}}$

9. $\int \cot(3-7x)\,dx$

10. $\int \csc(\pi x-1)\,dx$

11. $\int e^\theta \csc(e^\theta+1)\,d\theta$

12. $\int \frac{\cot(3+\ln x)}{x}\,dx$

13. $\int \sec\frac{t}{3}\,dt$

14. $\int x\sec(x^2-5)\,dx$

15. $\int \csc(s-\pi)\,ds$

16. $\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta$

17. $\int_0^{\sqrt{\ln 2}} 2xe^{x^2}\,dx$

18. $\int_{\pi/2}^{\pi} \sin(y)e^{\cos y}\,dy$

19. $\int e^{\tan v}\sec^2 v\,dv$

20. $\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}}$

21. $\int 3^{x+1}\,dx$

22. $\int \frac{2^{\ln x}}{x}\,dx$

23. $\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}$

24. $\int 10^{2\theta}\,d\theta$

25. $\int \frac{9\,du}{1+9u^2}$

26. $\int \frac{4\,dx}{1+(2x+1)^2}$

27. $\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}}$

28. $\int_0^1 \frac{dt}{\sqrt{4-t^2}}$

29. $\int \frac{2s\,ds}{\sqrt{1-s^4}}$

30. $\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$

31. $\int \frac{6\,dx}{x\sqrt{25x^2-1}}$

32. $\int \frac{dr}{r\sqrt{r^2-9}}$

33. $\int \frac{dx}{e^x+e^{-x}}$

34. $\int \frac{dy}{\sqrt{e^{2y}-1}}$

35. $\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}$

36. $\int \frac{\ln x\,dx}{x+4x\ln^2 x}$

Completing the Square

Evaluate the integrals in Exercises 37–42 by completing the square and using substitutions to reduce them to standard forms.

37. $\int_1^2 \frac{8\,dx}{x^2-2x+2}$

38. $\int_2^4 \frac{2\,dx}{x^2-6x+10}$

39. $\int \frac{dt}{\sqrt{-t^2+4t-3}}$

40. $\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$

41. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

42. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Trigonometric Identities

Evaluate the integrals in Exercises 43–46 by using trigonometric identities and substitutions to reduce them to standard forms.

43. $\int (\sec x+\cot x)^2 dx$

44. $\int (\csc x-\tan x)^2 dx$

45. $\int \csc x\sin 3x\,dx$

46. $\int (\sin 3x\cos 2x-\cos 3x\sin 2x)\,dx$

Improper Fractions

Evaluate each integral in Exercises 47–52 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

47. $\int \frac{x}{x+1}dx$

48. $\int \frac{x^2}{x^2+1}dx$

49. $\int_{\sqrt{2}}^3 \frac{2x^3}{x^2-1}dx$

50. $\int_{-1}^3 \frac{4x^2-7}{2x+3}dx$

51. $\int \frac{4t^3-t^2+16t}{t^2+4}dt$

52. $\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}d\theta$

Separating Fractions

Evaluate each integral in Exercises 53–56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

53. $\int \frac{1-x}{\sqrt{1-x^2}}dx$

54. $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}dx$

55. $\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}dx$

56. $\int_0^{1/2} \frac{2-8x}{1+4x^2}dx$

Multiplying by a Form of 1

Evaluate each integral in Exercises 57–62 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

57. $\int \frac{1}{1+\sin x} dx$

58. $\int \frac{1}{1+\cos x} dx$

59. $\int \frac{1}{\sec\theta+\tan\theta} d\theta$

60. $\int \frac{1}{\csc\theta+\cot\theta} d\theta$

61. $\int \frac{1}{1-\sec x} dx$

62. $\int \frac{1}{1-\csc x} dx$

Eliminating Square Roots

Evaluate each integral in Exercises 63–70 by eliminating the square root.

63. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx$

64. $\int_0^{\pi} \sqrt{1-\cos 2x}\, dx$

65. $\int_{\pi/2}^{\pi} \sqrt{1+\cos 2t}\, dt$

66. $\int_{-\pi}^{0} \sqrt{1+\cos t}\, dt$

67. $\int_{-\pi}^{0} \sqrt{1-\cos^2\theta}\, d\theta$

68. $\int_{\pi/2}^{\pi} \sqrt{1-\sin^2\theta}\, d\theta$

69. $\int_{-\pi/4}^{\pi/4} \sqrt{1+\tan^2 y}\, dy$

70. $\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy$

Assorted Integrations

Evaluate the integrals in Exercises 71–82 using any technique you think is appropriate.

71. $\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx$

72. $\int_0^{\pi/4} (\sec x + 4\cos x)^2\, dx$

73. $\int \cos\theta\, \csc(\sin\theta)\, d\theta$

74. $\int \left(1+\frac{1}{x}\right) \cot(x+\ln x)\, dx$

75. $\int (\csc x - \sec x)(\sin x + \cos x)\, dx$

76. $\int (\csc x + \sec x)(\tan x + \cot x)\, dx$

77. $\int \frac{6\, dy}{\sqrt{y}(1+y)}$

78. $\int \frac{dx}{x\sqrt{4x^2-1}}$

79. $\int \frac{7\, dx}{(x-1)\sqrt{x^2-2x-48}}$

80. $\int \frac{dx}{(2x+1)\sqrt{4x^2+4x}}$

81. $\int \sec^2 t \tan(\tan t)\, dt$

82. $\int \frac{\tan\theta\, d\theta}{2\sec\theta + 1}$

Trigonometric Powers

83. a) Evaluate $\int \cos^3\theta\, d\theta$. (*Hint:* $\cos^2\theta = 1 - \sin^2\theta$.)
b) Evaluate $\int \cos^5\theta\, d\theta$.
c) Without actually evaluating the integral, explain how you would evaluate $\int \cos^9\theta\, d\theta$.

84. a) Evaluate $\int \sin^3\theta\, d\theta$. (*Hint:* $\sin^2\theta = 1 - \cos^2\theta$.)
b) Evaluate $\int \sin^5\theta\, d\theta$.
c) Evaluate $\int \sin^7\theta\, d\theta$.
d) Without actually evaluating the integral, explain how you would evaluate $\int \sin^{13}\theta\, d\theta$.

85. a) Express $\int \tan^3\theta\, d\theta$ in terms of $\int \tan\theta\, d\theta$. Then evaluate $\int \tan^3\theta\, d\theta$. (*Hint:* $\tan^2\theta = \sec^2\theta - 1$.)
b) Express $\int \tan^5\theta\, d\theta$ in terms of $\int \tan^3\theta\, d\theta$.
c) Express $\int \tan^7\theta\, d\theta$ in terms of $\int \tan^5\theta\, d\theta$.
d) Express $\int \tan^{2k+1}\theta\, d\theta$, where k is a positive integer, in terms of $\int \tan^{2k-1}\theta\, d\theta$.

86. a) Express $\int \cot^3\theta\, d\theta$ in terms of $\int \cot\theta\, d\theta$. Then evaluate $\int \cot^3\theta\, d\theta$. (*Hint:* $\cot^2\theta = \csc^2\theta - 1$.)
b) Express $\int \cot^5\theta\, d\theta$ in terms of $\int \cot^3\theta\, d\theta$.
c) Express $\int \cot^7\theta\, d\theta$ in terms of $\int \cot^5\theta\, d\theta$.
d) Express $\int \cot^{2k+1}\theta\, d\theta$, where k is a positive integer, in terms of $\int \cot^{2k-1}\theta\, d\theta$.

Theory and Examples

87. Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

88. Find the area of the "triangular" region that is bounded from above and below by the curves $y = \csc x$ and $y = \sin x$, $\pi/6 \le x \le \pi/2$, and on the left by the line $x = \pi/6$.

89. Find the volume of the solid generated by revolving the region in Exercise 87 about the x-axis.

90. Find the volume of the solid generated by revolving the region in Exercise 88 about the x-axis.

91. Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

92. Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

93. Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

94. Find the centroid of the region that is bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

95. *The integral of* $\csc x$. Repeat the derivation in Example 7, using cofunctions, to show that

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C.$$

96. Show that the integral

$$\int \left((x^2-1)(x+1)\right)^{-2/3} dx$$

can be evaluated with any of the following substitutions.

a) $u = 1/(x+1)$
b) $u = ((x-1)/(x+1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1
c) $u = \tan^{-1} x$
d) $u = \tan^{-1}\sqrt{x}$
e) $u = \tan^{-1}((x-1)/2)$
f) $u = \cos^{-1} x$
g) $u = \cosh^{-1} x$

What is the value of the integral? (From "Problems and Solutions," *College Mathematics Journal*, Vol. 21, No. 5, Nov. 1990, pp. 425–426.)

7.2

Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx \tag{1}$$

in which f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x\,dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x\,dx,$$

in which each part of the integrand appears again after repeated differentiation or integration.

In this section, we describe integration by parts and show how to apply it.

The Formula

The formula for integration by parts comes from the Product Rule,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

In its differential form, the rule becomes

$$d(uv) = u\,dv + v\,du,$$

which is then written as

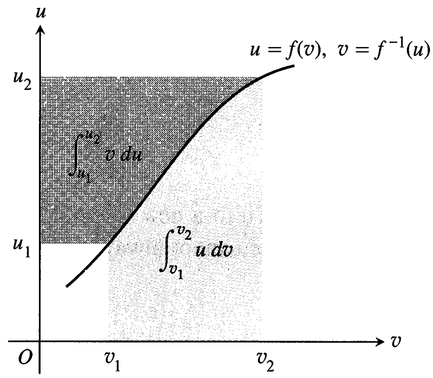
$$u\,dv = d(uv) - v\,du$$

and integrated to give the following formula.

The Integration-by-Parts Formula

$$\int u\,dv = uv - \int v\,du. \tag{2}$$

The integration-by-parts formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. This is the reason for the importance of the formula. When faced with an integral we cannot handle, we can replace it by one with which we might have more success.



7.1 The area of the blue region, $\int_{v_1}^{v_2} u\,dv$, equals the area of the large rectangle, $u_2 v_2$, minus the areas of the small rectangle, $u_1 v_1$, and the gray region,

$$\int_{u_1}^{u_2} v\,du.$$

In symbols,

$$\int_{v_1}^{v_2} u\,dv = (u_2 v_2 - u_1 v_1) - \int_{u_1}^{u_2} v\,du.$$

The equivalent formula for definite integrals is

$$\int_{v_1}^{v_2} u\,dv = (u_2 v_2 - u_1 v_1) - \int_{u_1}^{u_2} v\,du. \qquad (3)$$

Figure 7.1 shows how the different parts of the formula may be interpreted as areas.

EXAMPLE 1 Find $\int x \cos x\,dx$.

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + C.$$

❑

Let us examine the choices available for u and dv in Example 1.

When and How to Use Integration by Parts

When: If substitution doesn't work, try integration by parts.

How: Start with an integral of the form

$$\int f(x)g(x)\,dx.$$

Match this with an integral of the form

$$\int u\,dv$$

by choosing dv to be part of the integrand including dx and possibly $f(x)$ or $g(x)$.

Guideline for choosing u and dv: The formula

$$\int u\,dv = uv - \int v\,du$$

gives a new integral on the right side of the equation. If the new integral is more complex than the original one, try a different choice for u and dv.

EXAMPLE 2 *Example 1 revisited*

To apply integration by parts to

$$\int x \cos x\,dx = \int u\,dv$$

we have four possible choices:

1. Let $u = 1$ and $dv = x \cos x\,dx$.
2. Let $u = x$ and $dv = \cos x\,dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x\,dx$.

Let's examine these one at a time.

Choice 1 won't do because we don't know how to integrate $dv = x \cos x\,dx$ to get v.

Choice 2 works well, as we saw in Example 1.

Choice 3 leads to

$$u = x \cos x, \qquad dv = dx,$$
$$du = (\cos x - x \sin x)\,dx, \qquad v = x,$$

and the new integral

$$\int v\,du = \int (x \cos x - x^2 \sin x)\,dx.$$

This is worse than the integral we started with.

Choice 4 leads to

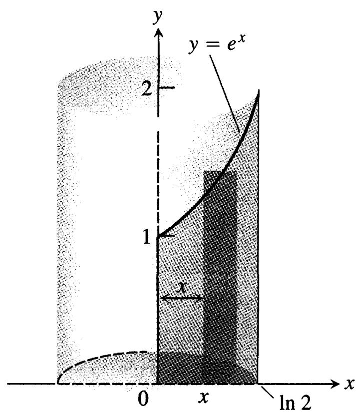
$$u = \cos x, \qquad dv = x\,dx,$$
$$du = -\sin x\,dx, \qquad v = x^2/2,$$

so the new integral is

$$\int v\,du = -\int \frac{x^2}{2}\sin x\,dx.$$

This, too, is worse.

Summary. Keep in mind that the object is to go from $\int u\,dv$ to a new integral that is simpler. Integration by parts does not always work, so we cannot always achieve the goal. ❑



7.2 The solid in Example 3.

EXAMPLE 3 Find the volume of the solid generated by revolving about the y-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = e^x$ and the line $x = \ln 2$ (Fig. 7.2).

Solution Using the method of cylindrical shells, we find

$$V = \int_a^b 2\pi x\, f(x)\,dx \qquad \text{The shell volume formula}$$

$$= 2\pi \int_0^{\ln 2} x\, e^x\,dx.$$

To evaluate the integral, we use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = e^x\,dx$$

$$du = dx, \qquad v = e^x. \qquad \text{Simplest antiderivative of } e^x$$

Then

$$\int x\,e^x\,dx = x\,e^x - \int e^x\,dx,$$

so

$$\int_0^{\ln 2} x\,e^x dx = x\,e^x\Big]_0^{\ln 2} - \int_0^{\ln 2} e^x\,dx$$

$$= \left[\ln 2\, e^{\ln 2} - 0\right] - \left[e^x\right]_0^{\ln 2}$$

$$= 2\ln 2 - [2 - 1]$$

$$= 2\ln 2 - 1.$$

The solid's volume is therefore

$$V = 2\pi \int_0^{\ln 2} x\,e^x\,dx$$

$$= 2\pi(2\ln 2 - 1).$$ ❑

Integration by parts can be useful even when the integrand has only a single factor. For example, we can use this method to find $\int \ln x\,dx$ (next example) or $\int \cos^{-1} x\,dx$ (Exercise 47).

EXAMPLE 4 Find $\int \ln x\,dx$.

Solution Since $\int \ln x\, dx$ can be written as $\int \ln x \cdot 1\ dx$, we use the formula $\int u\, dv = uv - \int v\, du$ with

$$u = \ln x \quad \text{Simplifies when differentiated} \qquad dv = dx \quad \text{Easy to integrate}$$

$$du = \frac{1}{x}dx \qquad\qquad v = x. \quad \text{Simplest antiderivative}$$

Then

$$\int \ln x\, dx = x \ln x - \int x \cdot \frac{1}{x}dx = x \ln x - \int dx = x \ln x - x + C.$$

❑

Repeated Use

Sometimes we have to use integration by parts more than once to obtain an answer.

EXAMPLE 5 Find $\displaystyle\int x^2 e^x\, dx.$

Solution We use the formula $\displaystyle\int u\, dv = uv - \int v\, du$ with

$$u = x^2, \qquad dv = e^x\, dx, \qquad v = e^x, \qquad du = 2x\, dx.$$

This gives

$$\int x^2 e^x\, dx = x^2 e^x - 2\int x e^x\, dx.$$

It takes a second integration by parts to find the integral on the right. As in Example 3, its value is $xe^x - e^x + C'$. Hence

$$\int x^2 e^x\, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

❑

Solving for the Unknown Integral

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 6 Find $\displaystyle\int e^x \cos x\, dx.$

Solution We first use the formula $\displaystyle\int u\, dv = uv - \int v\, du$ with

$$u = e^x, \qquad dv = \cos x\, dx, \qquad v = \sin x, \qquad du = e^x dx.$$

Then

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx. \tag{4}$$

The second integral is like the first, except it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\, dx, \qquad v = -\cos x, \qquad du = e^x\, dx.$$

Then

$$\int e^x \cos x\,dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\,dx)\right)$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

The unknown integral now appears on both sides of the equation. Combining the two expressions gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C'.$$

The choice of $u = e^x$ and $dv = \sin x\,dx$ in the second integration may have seemed arbitrary but it wasn't. In theory, we could have chosen $u = \sin x$ and $dv = e^x\,dx$. Doing so, however, would have turned Eq. (4) into

$$\int e^x \cos x\,dx = e^x \sin x - \left(e^x \sin x - \int e^x \cos x\,dx\right)$$
$$= \int e^x \cos x\,dx.$$

The resulting identity is correct, but useless. *Moral:* Once you have decided on what to differentiate and integrate in circumstances like these, stick with them. Formulas for the integrals of $e^{ax}\cos bx$ and the closely related $e^{ax}\sin bx$ can be found in the integral table at the end of this book. ❑

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE 7 Find $\int x^2 e^x\,dx$ by tabular integration.

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list

$f(x)$ and its derivatives		**$g(x)$ and its integrals**
x^2	(+)	e^x
$2x$	(−)	e^x
2	(+)	e^x
0		e^x.

We add the products of the functions connected by the arrows, with the middle sign changed, to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

❑

EXAMPLE 8 Find $\int x^3 \sin x \, dx$ by tabular integration.

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list

For more about tabular integration, see the Additional Exercises at the end of this chapter.

$f(x)$ and its derivatives		**$g(x)$ and its integrals**
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x.$

Again we add the products of the functions connected by the arrows, with every other sign changed, to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

❑

Exercises 7.2

Integration by Parts

Evaluate the integrals in Exercises 1–24.

1. $\int x \sin \frac{x}{2} \, dx$

2. $\int \theta \cos \pi\theta \, d\theta$

3. $\int t^2 \cos t \, dt$

4. $\int x^2 \sin x \, dx$

5. $\int_1^2 x \ln x \, dx$

6. $\int_1^e x^3 \ln x \, dx$

7. $\int \tan^{-1} y \, dy$

8. $\int \sin^{-1} y \, dy$

9. $\int x \sec^2 x \, dx$

10. $\int 4x \sec^2 2x \, dx$

11. $\int x^3 e^x \, dx$

12. $\int p^4 e^{-p} \, dp$

13. $\int (x^2 - 5x)e^x \, dx$

14. $\int (r^2 + r + 1)e^r \, dr$

15. $\int x^5 e^x \, dx$

16. $\int t^2 e^{4t} \, dt$

17. $\int_0^{\pi/2} \theta^2 \sin 2\theta \, d\theta$

18. $\int_0^{\pi/2} x^3 \cos 2x \, dx$

19. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t \, dt$

20. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) \, dx$

21. $\int e^\theta \sin \theta \, d\theta$

22. $\int e^{-y} \cos y \, dy$

23. $\int e^{2x} \cos 3x \, dx$

24. $\int e^{-2x} \sin 2x \, dx$

Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}} \, ds$

26. $\int_0^1 x\sqrt{1-x} \, dx$

27. $\int_0^{\pi/3} x \tan^2 x \, dx$

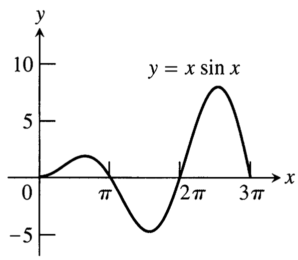
28. $\int \ln(x + x^2) \, dx$

29. $\int \sin(\ln x) \, dx$

30. $\int z (\ln z)^2 \, dz$

Theory and Examples

31. Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis for (a) $0 \leq x \leq \pi$, (b) $\pi \leq x \leq 2\pi$, (c) $2\pi \leq x \leq 3\pi$. (d) What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \leq x \leq (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.

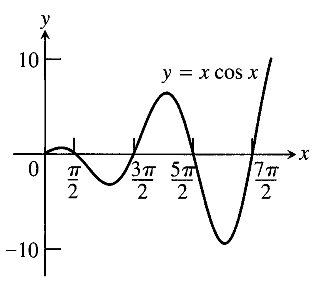


32. Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the figure below) for

a) $\pi/2 \leq x \leq 3\pi/2$,
b) $3\pi/2 \leq x \leq 5\pi/2$,
c) $5\pi/2 \leq x \leq 7\pi/2$.
d) What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \leq x \leq \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



33. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

34. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$ (a) about the y-axis, (b) about the line $x = 1$.

35. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \leq x \leq \pi/2$, about (a) the y-axis, (b) the line $x = \pi/2$.

36. Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \leq x \leq \pi$, about (a) the y-axis, (b) the line $x = \pi$. (See Exercise 31 for a graph.)

37. a) Find the centroid of a thin plate of constant density covering the region in the first quadrant enclosed by the curve $y = x^2 e^x$, the x-axis, and the line $x = 1$.

b) CALCULATOR Find the coordinates of the centroid to 2 decimal places. Show the center of mass in a rough sketch of the plate.

38. a) Find the centroid of a thin plate of constant density covering the region enclosed by the curve $y = \ln x$, the x-axis, and the line $x = e$.

b) CALCULATOR Find the coordinates of the centroid to 2 decimal places. Show the centroid in a rough sketch of the plate.

39. Find the moment about the y-axis of a thin plate of density $\delta = 1 + x$ covering the region bounded by the x-axis and the curve $y = \sin x$, $0 \leq x \leq \pi$.

40. Although we usually drop the constant of integration in determining v as $\int dv$ in integration by parts, choosing the constant to be different from zero can occasionally be helpful. As a case in point, evaluate

$$\int x \tan^{-1} x \, dx,$$

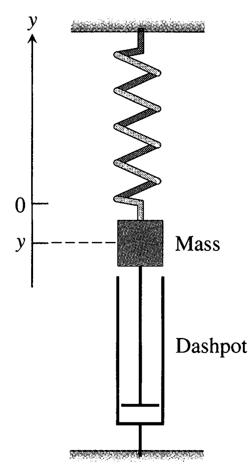
with $u = \tan^{-1} x$ and $v = (x^2/2) + C$, and find a value of C that simplifies the resulting formula.

41. A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \quad t \geq 0.$$

a) Find the average value of y over the interval $0 \leq t \leq 2\pi$.

b) GRAPHER Graph y over the interval $0 \leq t \leq 2\pi$. Copy the graph and mark the average value of y as a point on the y-axis.



42. In a mass-spring-dashpot system like the one in Exercise 41, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \quad t \geq 0.$$

a) Find the average value of y over the interval $0 \le t \le 2\pi$.

b) GRAPHER Graph y over the interval $0 \le t \le 2\pi$. Copy the graph and mark the average value of y as a point on the y-axis.

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\int f^{-1}(x)\,dx = \int y f'(y)\,dy \qquad \begin{matrix} y = f^{-1}(x), & x = f(y) \\ dx = f'(y)\,dy \end{matrix}$$

$$= yf(y) - \int f(y)\,dy \qquad \text{Integration by parts with } u = y,\ dv = f'(y)\,dy$$

$$= xf^{-1}(x) - \int f(y)\,dy$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\int \ln x\,dx = \int ye^y\,dy \qquad \begin{matrix} y = \ln x, & x = e^y \\ dx = e^y\,dy \end{matrix}$$

$$= ye^y - e^y + C$$

$$= x\ln x - x + C.$$

For the integral of $\cos^{-1} x$ we get

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \int \cos y\,dy \qquad y = \cos^{-1} x$$

$$= x\cos^{-1} x - \sin y + C$$

$$= x\cos^{-1} x - \sin(\cos^{-1} x) + C.$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \tag{5}$$

to evaluate the integrals in Exercises 43–46. Express your answers in terms of x.

43. $\int \sin^{-1} x\,dx$ **44.** $\int \tan^{-1} x\,dx$

45. $\int \sec^{-1} x\,dx$ **46.** $\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \tag{6}$$

Exercises 47 and 48 compare the results of using Eqs. (5) and (6).

47. Equations (5) and (6) give different formulas for the integral of $\cos^{-1} x$:

a) $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (5)

b) $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1 - x^2} + C$ Eq. (6)

Can both integrations be correct? Explain.

48. Equations (5) and (6) lead to different formulas for the integral of $\tan^{-1} x$:

a) $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sec(\tan^{-1} x) + C$ Eq. (5)

b) $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{1 + x^2} + C$ Eq. (6)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 49 and 50 with (a) Eq. (5) and (b) Eq. (6). In each case, check your work by differentiating your answer with respect to x.

49. $\int \sinh^{-1} x\,dx$ **50.** $\int \tanh^{-1} x\,dx$

7.3 Partial Fractions

A theorem from advanced algebra (mentioned later in more detail) says that every rational function, no matter how complicated, can be rewritten as a sum of simpler fractions that we can integrate with techniques we already know. For instance,

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3}, \tag{1}$$

so we can integrate the rational function on the left by integrating the fractions on the right instead.

The method for rewriting rational functions this way is called the **method of partial fractions.** In this particular case, it consists of finding constants A and B

such that

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}. \tag{2}$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x+1)$ and $B/(x-3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until proper values for them have been found.

To find A and B, we first clear Eq. (2) of fractions, obtaining

$$5x - 3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

EXAMPLE 1 *Two distinct linear factors in the denominator*

Find

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx.$$

Solution From the preceding discussion,

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$
$$= 2\ln|x+1| + 3\ln|x-3| + C.$$
❑

EXAMPLE 2 *A repeated linear factor in the denominator*

Express

$$\frac{6x+7}{(x+2)^2}$$

as a sum of partial fractions.

Solution Since the denominator has a repeated linear factor, $(x+2)^2$, we must express the fraction in the form

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}. \tag{3}$$

Clearing Eq. (3) of fractions gives

$$6x + 7 = A(x+2) + B = Ax + (2A + B).$$

Matching coefficients of like terms gives $A = 6$ and

$$7 = 2A + B = 12 + B, \qquad \text{or} \qquad B = -5.$$

Hence,

$$\frac{6x+7}{(x+2)^2} = \frac{6}{x+2} - \frac{5}{(x+2)^2}.$$
❑

How to Evaluate Undetermined Coefficients

1. Clear the given equation of fractions.
2. Equate the coefficients of like terms (powers of x).
3. Solve the resulting equations for the coefficients.

EXAMPLE 3 *An improper fraction*

Express

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}$$

as a sum of partial fractions.

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction. Then we write the proper fraction as a sum of partial fractions. Long division gives

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{\smash{\big)}\, 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x } \\ 5x - 3 \end{array}$$

Hence,

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3} \qquad \text{Result of the division}$$

$$= 2x + \frac{2}{x + 1} + \frac{3}{x - 3}. \qquad \text{Proper fraction expanded as in Example 1}$$

❑

EXAMPLE 4 *An irreducible quadratic factor in the denominator*

Express

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2}$$

as a sum of partial fractions.

A quadratic polynomial is **irreducible** if it cannot be written as the product of two linear factors with real coefficients.

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \tag{4}$$

Notice the numerator over $x^2 + 1$: For quadratic factors, we use first degree numerators, not constant numerators. Clearing the equation of fractions gives

$$-2x + 4 = (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1)$$

$$= (A + C)x^3 + (-2A + B - C + D)x^2$$

$$+ (A - 2B + C)x + (B - C + D).$$

Equating coefficients of like terms gives

Coefficients of x^3: $\quad 0 = A + C$

Coefficients of x^2: $\quad 0 = -2A + B - C + D$

Coefficients of x^1: $\quad -2 = A - 2B + C$

Coefficients of x^0: $\quad 4 = B - C + D$

We solve these equations simultaneously to find the values of A, B, C, and D:

$$-4 = -2A, \quad A = 2 \qquad \text{Subtract fourth equation from second.}$$

$$C = -A = -2 \qquad \text{From the first equation}$$

$$B = 1 \qquad A = 2 \text{ and } C = -2 \text{ in third equation.}$$

$$D = 4 - B + C = 1. \qquad \text{From the fourth equation}$$

We substitute these values into Eq. (4), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$ ❑

EXAMPLE 5 Evaluate $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx.$

Solution We expand the integrand by partial fractions, as in Example 4, and integrate the terms of the expansion:

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \qquad \text{Example 4}$$

$$= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx$$

$$= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C.$$

❑

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

1. *The degree of $f(x)$ must be less than the degree of $g(x)$.* (If it isn't, divide and work with the remainder term.)
2. *We must know the factors of $g(x)$.* (In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.)

A theorem from advanced algebra says that when these two conditions are met, we may write $f(x)/g(x)$ as the sum of partial fractions by taking these steps.

Cases discussed so far

Proper fraction	*Decomposition*
$\dfrac{\text{numerator}}{(x+p)(x+q)}$	$= \dfrac{A}{(x+p)} + \dfrac{B}{(x+q)}$
$\dfrac{\text{numerator}}{(x+p)^2}$	$= \dfrac{A}{(x+p)} + \dfrac{B}{(x+p)^2}$
$\dfrac{\text{numerator}}{(x^2+p)(x+q)^2}$	$= \dfrac{Ax+B}{x^2+p} + \dfrac{C}{x+q} + \dfrac{D}{(x+q)^2}$

The Method of Partial Fractions ($f(x)/g(x)$ *Proper*)

Step 1 Let $x - r$ be a linear factor of $g(x)$. Suppose $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then assign the sum of m partial fractions to this factor, as follows:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

Step 2 Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$. Suppose $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then to this factor assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

Step 3 Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

Step 4 Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

⁂ The Heaviside "Cover-up" Method for Linear Factors

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$, and

$$g(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

EXAMPLE 6 Find A, B, and C in the partial-fraction expansion

$$\frac{x^2 + 1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}. \tag{5}$$

Solution If we multiply both sides of Eq. (5) by $(x - 1)$ to get

$$\frac{x^2 + 1}{(x-2)(x-3)} = A + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x-3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2 + 1}{(1-2)(1-3)} = A + 0 + 0,$$
$$A = 1.$$

Thus, the value of A is the number we would have obtained if we had covered the factor $(x - 1)$ in the denominator of the original fraction

$$\frac{x^2 + 1}{(x-1)(x-2)(x-3)} \tag{6}$$

and evaluated the rest at $x = 1$:

$$A = \frac{(1)^2 + 1}{\boxed{(x-1)}\,(1-2)(1-3)} = \frac{2}{(-1)(-2)} = 1.$$

$\Uparrow$
Cover

Similarly, we find the value of B in Eq. (5) by covering the factor $(x-2)$ in (6) and evaluating the rest at $x=2$:

$$B = \frac{(2)^2+1}{(2-1)\boxed{(x-2)}(2-3)} = \frac{5}{(1)(-1)} = -5.$$

⇧ Cover

Finally, C is found by covering the $(x-3)$ in (6) and evaluating the rest at $x=3$:

$$C = \frac{(3)^2+1}{(3-1)(3-2)\boxed{(x-3)}} = \frac{10}{(2)(1)} = 5.$$

⇧ Cover

❑

The steps in the cover-up method are these:

Step 1: Write the quotient with $g(x)$ factored:

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)}. \tag{7}$$

Step 2: Cover the factors $(x-r_i)$ of $g(x)$ in (7) one at a time, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1-r_2)\cdots(r_1-r_n)},$$

$$A_2 = \frac{f(r_2)}{(r_2-r_1)(r_2-r_3)\cdots(r_2-r_n)},$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n-r_1)(r_n-r_2)\cdots(r_n-r_{n-1})}.$$

Step 3: Write the partial-fraction expansion of $f(x)/g(x)$ as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x-r_1)} + \frac{A_2}{(x-r_2)} + \cdots + \frac{A_n}{(x-r_n)}.$$

EXAMPLE 7 Evaluate

$$\int \frac{x+4}{x^3+3x^2-10x}\,dx.$$

Solution The degree of $f(x)=x+4$ is less than the degree of $g(x)=x^3+3x^2-10x$, and, with $g(x)$ factored,

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x-2)(x+5)}.$$

The roots of $g(x)$ are $r_1=0$, $r_2=2$, and $r_3=-5$. We find

$$A_1 = \frac{0+4}{\boxed{x}(0-2)(0+5)} = \frac{4}{(-2)(5)} = -\frac{2}{5},$$

⇧ Cover

$$A_2 = \frac{2+4}{2\ \boxed{(x-2)}\ (2+5)} = \frac{6}{(2)(7)} = \frac{3}{7},$$

⇧ Cover

$$A_3 = \frac{-5+4}{(-5)(-5-2)\ \boxed{(x+5)}} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$

⇧ Cover

Therefore,

$$\frac{x+4}{x(x-2)(x+5)} = -\frac{2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)},$$

and

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C.$$ ❑

Other Ways to Determine the Constants

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x.

EXAMPLE 8 *Differentiation*

Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Solution We first clear of fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}.$$ ❑

In some problems, assigning small values to x such as $x = 0, \pm 1, \pm 2$, to get equations in A, B, and C provides a fast alternative to other methods.

EXAMPLE 9 *Assigning numerical values to x*

Find A, B, and C in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Solution Clear of fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned} x = 1: \quad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\ 2 &= 2A \\ A &= 1 \\ x = 2: \quad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0) \\ 5 &= -B \\ B &= -5 \\ x = 3: \quad (3)^2 + 1 &= A(0) + B(0) + C(2)(1) \\ 10 &= 2C \\ C &= 5. \end{aligned}$$

Conclusion:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$ ❑

Exercises 7.3

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x-13}{(x-3)(x-2)}$
2. $\dfrac{5x-7}{x^2-3x+2}$
3. $\dfrac{x+4}{(x+1)^2}$
4. $\dfrac{2x+2}{x^2-2x+1}$
5. $\dfrac{z+1}{z^2(z-1)}$
6. $\dfrac{z}{z^3-z^2-6z}$
7. $\dfrac{t^2+8}{t^2-5t+6}$
8. $\dfrac{t^4+9}{t^4+9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrands as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1-x^2}$
10. $\displaystyle\int \frac{dx}{x^2+2x}$
11. $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx$
12. $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$
13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$
14. $\displaystyle\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy$
15. $\displaystyle\int \frac{dt}{t^3+t^2-2t}$
16. $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrands as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2+2x+1}$
18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$
19. $\displaystyle\int \frac{dx}{(x^2-1)^2}$
20. $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

Irreducible Quadratic Factors

In Exercises 21–28, express the integrands as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)}$
22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$
23. $\displaystyle\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$
24. $\displaystyle\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx$
25. $\displaystyle\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds$
26. $\displaystyle\int \frac{s^4+81}{s(s^2+9)^2}\,ds$
27. $\displaystyle\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta$
28. $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta$

Improper Fractions

In Exercises 29–34, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

29. $\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\, dx$

30. $\int \frac{x^4}{x^2 - 1}\, dx$

31. $\int \frac{9x^2 - 3x + 1}{x^3 - x^2}\, dx$

32. $\int \frac{16x^3}{4x^2 - 4x + 1}\, dx$

33. $\int \frac{y^4 + y^2 - 1}{y^3 + y}\, dy$

34. $\int \frac{2y^4}{y^3 - y^2 + y - 1}\, dy$

Evaluating Integrals

Evaluate the integrals in Exercises 35–40.

35. $\int \frac{e^t\, dt}{e^{2t} + 3e^t + 2}$

36. $\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\, dt$

37. $\int \frac{\cos y\, dy}{\sin^2 y + \sin y - 6}$

38. $\int \frac{\sin \theta\, d\theta}{\cos^2 \theta + \cos \theta - 2}$

39. $\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2 + 1)(x - 2)^2}\, dx$

40. $\int \frac{(x+1)^2 \tan^{-1}(3x) + 9x^3 + x}{(9x^2 + 1)(x + 1)^2}\, dx$

Initial Value Problems

Solve the initial value problems in Exercises 41–44 for x as a function of t.

41. $(t^2 - 3t + 2)\frac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

42. $(3t^4 + 4t^2 + 1)\frac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

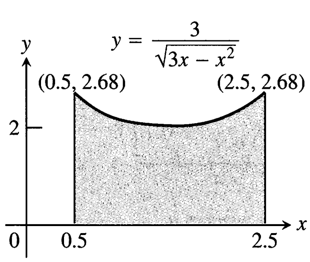
43. $(t^2 + 2t)\frac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

44. $(t + 1)\frac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = \pi/4$

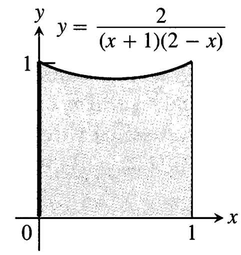
Applications and Examples

In Exercises 45 and 46, find the volume of the solid generated by revolving the shaded region about the indicated axis.

45. The x-axis

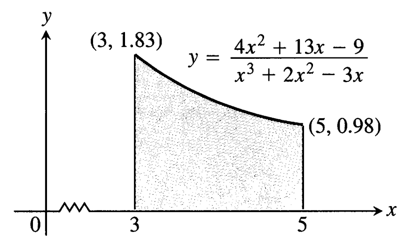


46. The y-axis



47. CALCULATOR Find, to 2 decimal places, the x-coordinate of the centroid of the region in the first quadrant bounded by the x-axis, the curve $y = \tan^{-1} x$, and the line $x = \sqrt{3}$.

48. CALCULATOR Find the x-coordinate of the centroid of this region to 2 decimal places.



49. *Social diffusion.* Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t, and the rate of diffusion, dx/dt, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where N is the number of people in the population.

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

a) Find x as a function of t.

b) When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

50. *Second order chemical reactions.* Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the amount of substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k,$$

where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t (a) if $a = b$, and (b) if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.

51. *An integral connecting π to the approximation 22/7*

a) Evaluate $\displaystyle\int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,dx$.

b) CALCULATOR How good is the approximation $\pi \approx 22/7$? Find out by expressing $(\pi - 22/7)$ as a percentage of π.

c) GRAPHER Graph the function $y = \dfrac{x^4(x-1)^4}{x^2+1}$ for $0 \leq x \leq 1$. Experiment with the range on the y-axis set between 0 and 1, then between 0 and 0.5, and then decreasing the range until the graph can be seen. What do you conclude about the area under the curve?

52. Find the second degree polynomial $P(x)$ such that $P(0) = 1$, $P'(0) = 0$, and

$$\int \frac{P(x)}{x^3(x-1)^2}\,dx$$

is a rational function.

7.4 Trigonometric Substitutions

Trigonometric substitutions enable us to replace the binomials $a^2 + x^2$, $a^2 - x^2$, and $x^2 - a^2$ by single squared terms and thereby transform a number of integrals containing square roots into integrals we can evaluate directly.

Three Basic Substitutions

The most common substitutions are $x = a \tan\theta$, $x = a \sin\theta$, and $x = a \sec\theta$. They come from the reference right triangles in Fig. 7.3.

With $x = a \tan\theta$,

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta. \quad (1)$$

With $x = a \sin\theta$,

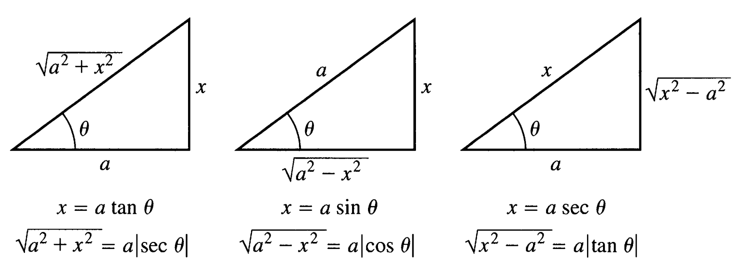
$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta. \quad (2)$$

With $x = a \sec\theta$,

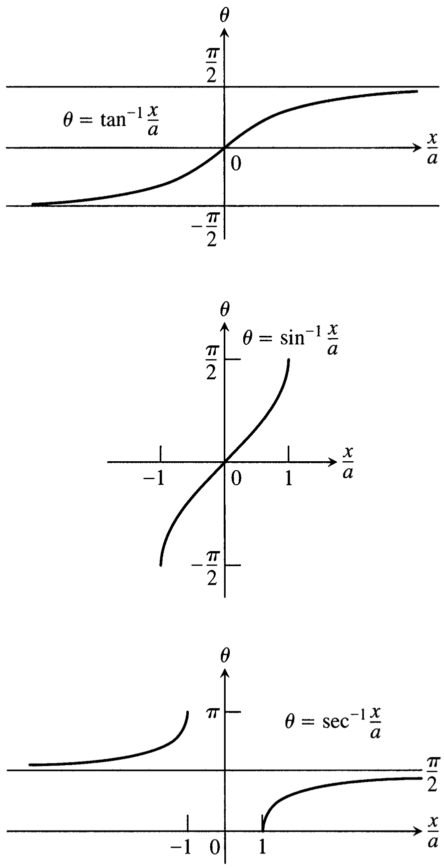
$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta. \quad (3)$$

Trigonometric Substitutions

1. $x = a\tan\theta$ replaces $a^2 + x^2$ by $a^2\sec^2\theta$.
2. $x = a\sin\theta$ replaces $a^2 - x^2$ by $a^2\cos^2\theta$.
3. $x = a\sec\theta$ replaces $x^2 - a^2$ by $a^2\tan^2\theta$.



7.3 Reference triangles for trigonometric substitutions that change binomials into single squared terms.



7.4 The arc tangent, arc sine, and arc secant of *x*/*a*, graphed as functions of *x*/*a*.

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a\tan\theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a\sin\theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a\sec\theta$.

As we know from Section 6.8, the functions in these substitutions have inverses only for selected values of θ (Fig. 7.4). For reversibility,

$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\ \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

To simplify calculations with the substitution $x = a\sec\theta$, we will restrict its use to integrals in which $x/a \ge 1$. This will place θ in $[0, \pi/2)$ and make $\tan\theta \ge 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, free of absolute values, provided $a > 0$.

EXAMPLE 1 Evaluate $\displaystyle\int \frac{dx}{\sqrt{4 + x^2}}$.

Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$

Then

$$\int \frac{dx}{\sqrt{4 + x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \sqrt{\sec^2\theta} = |\sec\theta|$$

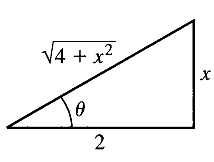
$$= \int \sec\theta\, d\theta \qquad \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4 + x^2}}{2} + \frac{x}{2}\right| + C \qquad \text{From Fig. 7.5}$$

$$= \ln\left|\sqrt{4 + x^2} + x\right| + C'. \qquad \text{Taking } C' = C - \ln 2$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Fig. 7.5) and read the ratios from the triangle. ❑



7.5 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4 + x^2}}{2}.$$

EXAMPLE 2 Evaluate $\displaystyle\int \frac{x^2\, dx}{\sqrt{9 - x^2}}$.

Solution To replace $9 - x^2$ by a single squared term, we set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$9 - x^2 = 9(1 - \sin^2\theta) = 9\cos^2\theta.$$

Then

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\,d\theta \qquad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta$$

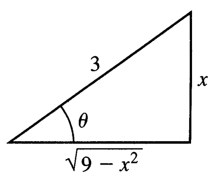
$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C \qquad \text{Fig. 7.6}$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

❑



7.6 Reference triangle for $x = 3\sin\theta$ (Example 2):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

EXAMPLE 3 Evaluate $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}}, \quad x > \frac{2}{5}.$

Solution We first rewrite the radical as

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$$

$$= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

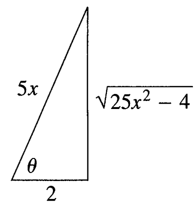
to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$

$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta,$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta. \qquad \tan\theta > 0 \text{ for } 0 < \theta < \pi/2$$



7.7 If $x = (2/5)\sec\theta$, $0 \le \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$ and we can read the values of the other trigonometric functions of θ from this right triangle.

With these substitutions, we have

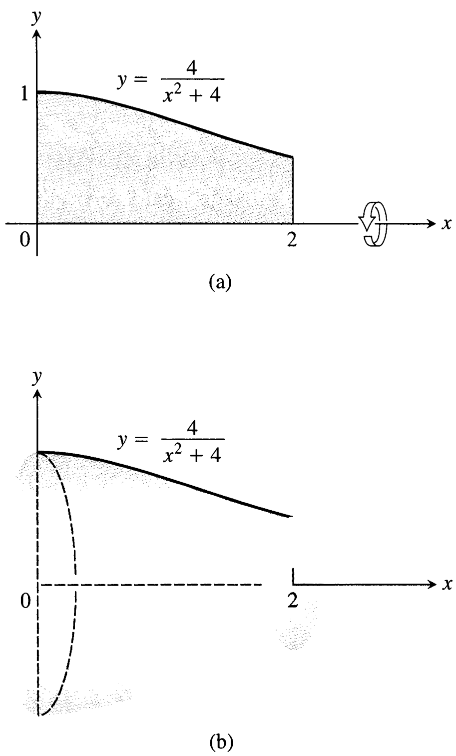
$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{5\sqrt{x^2-(4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C \qquad \text{Fig. 7.7}$$

❑

A trigonometric substitution can sometimes help us to evaluate an integral containing an integer power of a quadratic binomial, as in the next example.



7.8 The region (a) and solid (b) in Example 4.

EXAMPLE 4 Find the volume of the solid generated by revolving about the x-axis the region bounded by the curve $y = 4/(x^2+4)$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution We sketch the region (Fig. 7.8) and use the disk method (Section 5.3):

$$V = \int_0^2 \pi[R(x)]^2\, dx = 16\pi \int_0^2 \frac{dx}{(x^2+4)^2}. \qquad R(x) = \frac{4}{x^2+4}$$

To evaluate the integral, we set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad \theta = \tan^{-1}\frac{x}{2},$$

$$x^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$$

(Fig. 7.9). With these substitutions,

$$V = 16\pi \int_0^2 \frac{dx}{(x^2+4)^2}$$

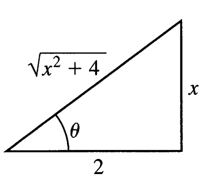
$$= 16\pi \int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{(4\sec^2\theta)^2} \qquad \begin{array}{l}\theta = 0 \\ \text{when } x = 0; \\ \theta = \pi/4 \\ \text{when } x = 2\end{array}$$

$$= 16\pi \int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta} = \pi \int_0^{\pi/4} 2\cos^2\theta\, d\theta$$

$$= \pi \int_0^{\pi/4} (1+\cos 2\theta)\, d\theta = \pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} \qquad \begin{array}{l}2\cos^2\theta = \\ 1+\cos 2\theta\end{array}$$

$$= \pi\left[\frac{\pi}{4} + \frac{1}{2}\right] \approx 4.04.$$

❑



7.9 Reference triangle for $x = 2\tan\theta$ (Example 4).

Exercises 7.4

Basic Trigonometric Substitutions

Evaluate the integrals in Exercises 1–28.

1. $\int \frac{dy}{\sqrt{9+y^2}}$

2. $\int \frac{3\,dy}{\sqrt{1+9y^2}}$

3. $\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\int_0^2 \frac{dx}{8+2x^2}$

5. $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\int \sqrt{25-t^2}\,dt$

8. $\int \sqrt{1-9t^2}\,dt$

9. $\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

10. $\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

11. $\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

12. $\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

13. $\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

14. $\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

15. $\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

16. $\int \frac{dx}{x^2\sqrt{x^2+1}}$

17. $\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

18. $\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

19. $\int_0^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

20. $\int_0^1 \frac{dx}{(4-x^2)^{3/2}}$

21. $\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

22. $\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

23. $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

24. $\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

25. $\int \frac{8\,dx}{(4x^2+1)^2}$

26. $\int \frac{6\,dt}{(9t^2+1)^2}$

27. $\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

28. $\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 29–36, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

29. $\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

30. $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

31. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

32. $\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

33. $\int \frac{dx}{x\sqrt{x^2-1}}$

34. $\int \frac{dx}{1+x^2}$

35. $\int \frac{x\,dx}{\sqrt{x^2-1}}$

36. $\int \frac{dx}{\sqrt{1-x^2}}$

Initial Value Problems

Solve the initial value problems in Exercises 37–40 for y as a function of x.

37. $x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \geq 2, \quad y(2) = 0$

38. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

39. $(x^2+4)\,\frac{dy}{dx} = 3, \quad y(2) = 0$

40. $(x^2+1)^2\,\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

Applications

41. Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9-x^2}/3$.

42. Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = 2/(1+x^2)$, and the line $x = 1$.

The Substitution $z = \tan(x/2)$

The substitution

$$z = \tan\frac{x}{2} \tag{4}$$

reduces the problem of integrating a rational expression in sin x and cos x to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions. Thus the substitution (4) is a powerful tool. It is cumbersome, however, and is used only when simpler methods fail.

Figure 7.10 shows how tan $(x/2)$ expresses a rational function of sin x and cos x. To see the effect of the substitution, we calculate

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1$$

$$= \frac{2}{1+\tan^2(x/2)} - 1 = \frac{2}{1+z^2} - 1$$

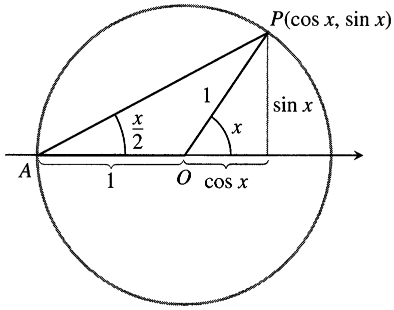
$$\cos x = \frac{1-z^2}{1+z^2}, \tag{5}$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin(x/2)}{\cos(x/2)} \cdot \cos^2\left(\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2} \cdot \frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1+\tan^2(x/2)}$$

$$\sin x = \frac{2z}{1+z^2}. \tag{6}$$



7.10 From this figure, we can read the relation

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

Finally, $x = 2 \tan^{-1} z$, so

$$dx = \frac{2\,dz}{1 + z^2}. \tag{7}$$

EXAMPLE

a)
$$\begin{aligned}\int \frac{1}{1+\cos x}\,dx &= \int \frac{1+z^2}{2}\,\frac{2\,dz}{1+z^2} \\ &= \int dz = z + C \\ &= \tan\left(\frac{x}{2}\right) + C\end{aligned}$$

b)
$$\begin{aligned}\int \frac{1}{2+\sin x}\,dx &= \int \frac{1+z^2}{2+2z+2z^2}\,\frac{2\,dz}{1+z^2} \\ &= \int \frac{dz}{z^2+z+1} = \int \frac{dz}{(z+(1/2))^2+3/4} \\ &= \int \frac{du}{u^2+a^2} \\ &= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \\ &= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}} + C \\ &= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1+2\tan(x/2)}{\sqrt{3}} + C\end{aligned}$$ ❑

Use the substitutions in Eqs. (4)–(7) to evaluate the integrals in Exercises 43–50. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

43. $\displaystyle\int \frac{dx}{1-\sin x}$

44. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

45. $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x}$

46. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{1-\cos x}$

47. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta}$

48. $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta+\sin\theta}$

49. $\displaystyle\int \frac{dt}{\sin t-\cos t}$

50. $\displaystyle\int \frac{\cos t\,dt}{1-\cos t}$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 51 and 52.

51. $\displaystyle\int \sec\theta\,d\theta$

52. $\displaystyle\int \csc\theta\,d\theta$

7.5 Integral Tables and CAS

As you know, the basic techniques of integration are substitution and integration by parts. We apply these techniques to transform unfamiliar integrals into integrals whose forms we recognize or can find in a table. But where do the integrals in the tables come from? They come from applying substitutions and integration by parts. We could derive them all from scratch if we had to, but having the table saves us the trouble of repeating laborious calculations. When an integral matches an integral in the table or can be changed into one of the tabulated integrals with some appropriate combination of algebra, trigonometry, substitution, and calculus, we have a ready-made solution for the problem at hand. The examples and exercises of this section show how the formulas in integral tables are derived and used. The emphasis is on use. The integration formulas at the back of this book are stated in terms of constants a, b, c, m, n, and so on. These constants can usually assume any real value and need not be integers. Occasional limitations on their values are stated with the formulas. Formula 5 requires $n \neq -1$, for example, and Formula 11 requires $n \neq -2$.

The formulas also assume that the constants do not take on values that require dividing by zero or taking even roots of negative numbers. For example, Formula 8 assumes $a \neq 0$, and Formula 13(a) cannot be used unless b is negative.

Many indefinite integrals can also be evaluated with a Computer Algebra System (CAS). These systems are generally faster than tables and usually do not require you to rewrite integrals in special recognizable forms first. We discuss computer algebra systems in the last third of the section.

Integration with Tables

EXAMPLE 1 Find $\displaystyle\int x(2x+5)^{-1}\,dx.$

Solution We use Formula 8 (not 7, which requires $n \neq -1$):

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x+5)^{-1}\,dx = \frac{x}{2} - \frac{5}{4}\ln|2x+5| + C.$$

❑

EXAMPLE 2 Find $\displaystyle\int \frac{dx}{x\sqrt{2x+4}}.$

Solution We use Formula 13(b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C, \qquad \text{if } b > 0.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x+4}} = \frac{1}{\sqrt{4}}\ln\left|\frac{\sqrt{2x+4}-\sqrt{4}}{\sqrt{2x+4}+\sqrt{4}}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{2x+4}-2}{\sqrt{2x+4}+2}\right| + C.$$

Formula 13(a), which requires $b < 0$, would not have been appropriate here. It *is* appropriate, however, in the next example. ❑

EXAMPLE 3 Find $\displaystyle\int \frac{dx}{x\sqrt{2x-4}}.$

Solution We use Formula 13(a):

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}} \tan^{-1} \sqrt{\frac{2x-4}{4}} + C = \tan^{-1} \sqrt{\frac{x-2}{2}} + C.$$

❑

EXAMPLE 4 Find $\displaystyle\int \frac{dx}{x^2\sqrt{2x-4}}$.

Solution We begin with Formula 15:

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{ax+b}} + C.$$

With $a = 2$ and $b = -4$, we have

$$\int \frac{dx}{x^2\sqrt{2x-4}} = -\frac{\sqrt{2x-4}}{-4x} + \frac{2}{2 \cdot 4} \int \frac{dx}{x\sqrt{2x-4}} + C.$$

We then use Formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x-4}} = \frac{\sqrt{2x-4}}{4x} + \frac{1}{4} \tan^{-1} \sqrt{\frac{x-2}{2}} + C.$$

❑

EXAMPLE 5 Find $\displaystyle\int x \sin^{-1} x\, dx$.

Solution We use Formula 99:

$$\int x^n \sin^{-1} ax\, dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1}\, dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x\, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2\, dx}{\sqrt{1-x^2}}.$$

The integral on the right is found in the table as Formula 33:

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\, dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2} x\sqrt{a^2-x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2} \sin^{-1} x - \frac{1}{2} x\sqrt{1-x^2} + C.$$

The combined result is

$$\begin{aligned}
\int x \sin^{-1} x\, dx &= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2}\left(\frac{1}{2} \sin^{-1} x - \frac{1}{2} x\sqrt{1-x^2}\right) + C' \\
&= \left(\frac{x^2}{2} - \frac{1}{4}\right) \sin^{-1} x + \frac{1}{4} x\sqrt{1-x^2} + C'.
\end{aligned}$$

❑

Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying formulas like

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \tag{1}$$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx \tag{2}$$

$$\int \sin^n x \cos^m x\,dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2} x \cos^m x\,dx \qquad (n \neq -m). \tag{3}$$

Formulas like these are called **reduction formulas** because they replace an integral containing some power of a function with an integral of the same form with the power reduced. By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly.

EXAMPLE 6 Find $\displaystyle\int \tan^5 x\,dx.$

Solution We apply Eq. (1) with $n = 5$ to get

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx.$$

We then apply Eq. (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \int \tan x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

The combined result is

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C'.$$

❑

As their form suggests, reduction formulas are derived by integration by parts.

EXAMPLE 7 *Deriving a reduction formula*

Show that for any positive integer n,

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

Solution We use the integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

with

$$u = (\ln x)^n, \qquad du = n(\ln x)^{n-1}\frac{dx}{x}, \qquad dv = dx, \qquad v = x,$$

to obtain

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx.$$

❑

Sometimes two reduction formulas come into play.

EXAMPLE 8 Find $\displaystyle\int \sin^2 x \cos^3 x \, dx.$

Solution 1 We apply Eq. (3) with $n = 2$ and $m = 3$ to get

$$\begin{aligned}\int \sin^2 x \cos^3 x \, dx &= -\frac{\sin x \cos^4 x}{2+3} + \frac{1}{2+3} \int \sin^0 x \cos^3 x \, dx \\ &= -\frac{\sin x \cos^4 x}{5} + \frac{1}{5} \int \cos^3 x \, dx.\end{aligned}$$

We can evaluate the remaining integral with Formula 61 (another reduction formula):

$$\int \cos^n ax \, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} ax \, dx.$$

With $n = 3$ and $a = 1$, we have

$$\begin{aligned}\int \cos^3 x \, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \int \cos x \, dx \\ &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \sin x + C.\end{aligned}$$

The combined result is

$$\begin{aligned}\int \sin^2 x \cos^3 x \, dx &= -\frac{\sin x \cos^4 x}{5} + \frac{1}{5} \left(\frac{\cos^2 x \sin x}{3} + \frac{2}{3} \sin x + C \right) \\ &= -\frac{\sin x \cos^4 x}{5} + \frac{\cos^2 x \sin x}{15} + \frac{2}{15} \sin x + C'.\end{aligned}$$

Solution 2 Equation (3) corresponds to Formula 68 in the table, but there is another formula we might use, namely Formula 69. With $a = 1$, Formula 69 gives

$$\int \sin^n x \cos^m x \, dx = \frac{\sin^{n+1} x \cos^{m-1} x}{m+n} + \frac{m-1}{m+n} \int \sin^n x \cos^{m-2} x \, dx.$$

In our case, $n = 2$ and $m = 3$, so that

$$\begin{aligned}\int \sin^2 x \cos^3 x \, dx &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5} \int \sin^2 x \cos x \, dx \\ &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5} \left(\frac{\sin^3 x}{3} \right) + C \\ &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{15} \sin^3 x + C.\end{aligned}$$

As you can see, it is faster to use Formula 69, but we often cannot tell beforehand how things will work out. Do not spend a lot of time looking for the "best" formula. Just find one that will work and forge ahead.

Notice also that Formulas 68 (Solution 1) and 69 (Solution 2) lead to different-looking answers. That is often the case with trigonometric integrals and is no cause for concern. The results are equivalent, and we may use whichever one we please. ❑

Nonelementary Integrals

The development of computers and calculators that find antiderivatives by symbolic manipulation has led to a renewed interest in determining which antiderivatives can be expressed as finite combinations of elementary functions (the functions we have been studying) and which cannot. Integrals of functions that do not have elementary antiderivatives are called **nonelementary** integrals. They require infinite series (Chapter 8) or numerical methods for their evaluation. Examples of the latter include the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

and integrals such as

$$\int \sin x^2\,dx \qquad \text{and} \qquad \int \sqrt{1 + x^4}\,dx$$

that arise in engineering and physics. These and a number of others, such as

$$\int \frac{e^x}{x}\,dx, \qquad \int e^{(e^x)}\,dx, \qquad \int \frac{1}{\ln x}\,dx, \qquad \int \ln(\ln x)\,dx, \qquad \int \frac{\sin x}{x}\,dx,$$

$$\int \sqrt{1 - k^2 \sin^2 x}\,dx, \qquad 0 < k < 1,$$

look so easy they tempt us to try them just to see how they turn out. It can be proved, however, that there is no way to express these integrals as finite combinations of elementary functions. The same applies to integrals that can be changed into these by substitution. The integrands all have antiderivatives—they are, after all, continuous—but none of the antiderivatives is elementary.

None of the integrals you are asked to evaluate in the present chapter falls into this category, but you may encounter nonelementary integrals from time to time in your other work.

A General Procedure for Indefinite Integration

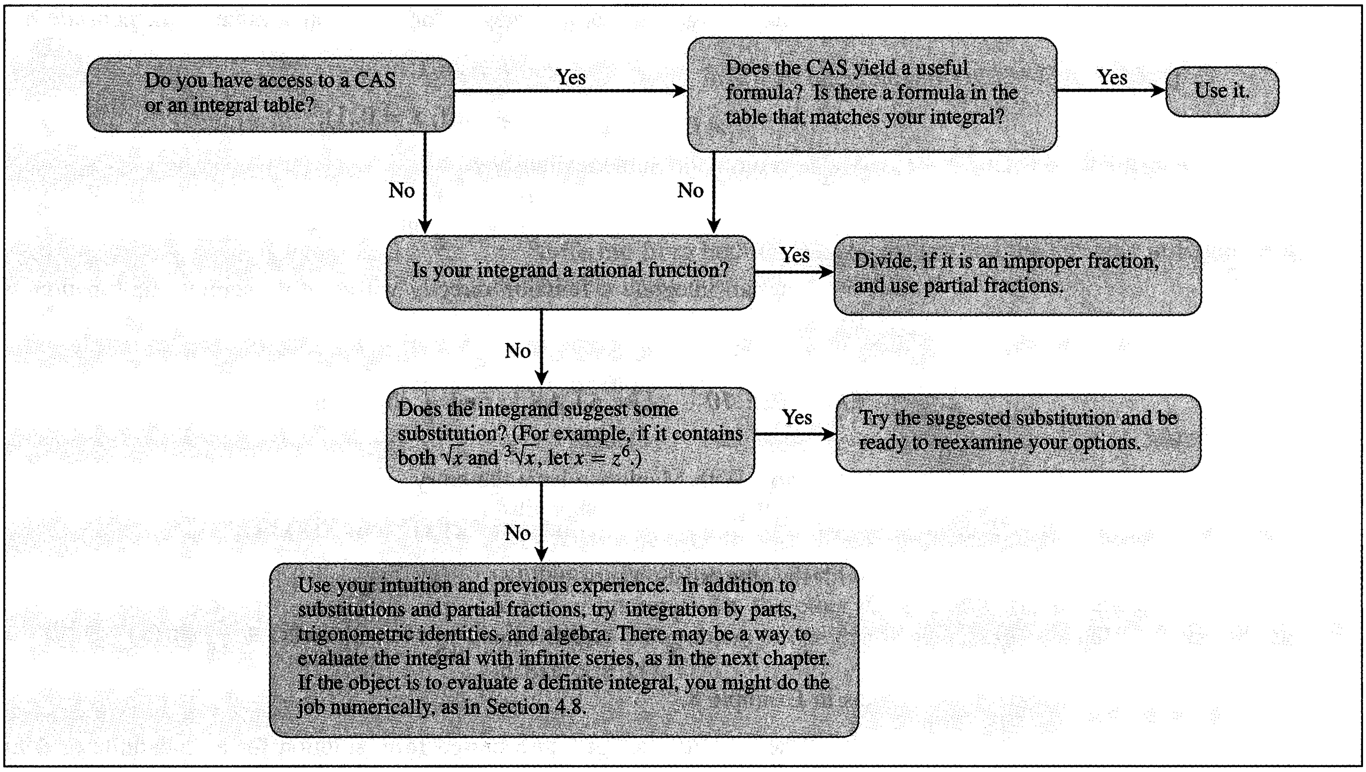
While there is no surefire way to evaluate all indefinite integrals, the procedure in Flowchart 7.1 may help.

Integration with a Computer Algebra System (CAS)

A powerful capability of Computer Algebra Systems is their facility to integrate symbolically. This is performed with the **integrate command** specified by the particular system (e.g., **int** in Maple, **Integrate** in Mathematica).

EXAMPLE 9 Suppose you want to evaluate the indefinite integral of the function

$$f(x) = x^2\sqrt{a^2 + x^2}.$$



Flowchart 7.1 Procedure for indefinite integration

Using Maple you first define the function:

```
> f := x^2 * sqrt (a^2 + x^2);
```

Then you use the integrate command on f, identifying the variable of integration:

```
> int(f, x);
```

Maple returns the answer

$$\frac{1}{4}x(a^2+x^2)^{3/2} - \frac{1}{8}a^2x\sqrt{a^2+x^2} - \frac{1}{8}a^4\ln\left(x+\sqrt{a^2+x^2}\right).$$

If you want to see if the answer can be simplified, enter

```
> simplify(");
```

Maple returns

$$\frac{1}{8}a^2x\sqrt{a^2+x^2} + \frac{1}{4}x^3\sqrt{a^2+x^2} - \frac{1}{8}a^4\ln\left(x+\sqrt{a^2+x^2}\right).$$

If you want the definite integral for $0 \le x \le \pi/2$, you can use the format

```
> int(f, x = 0..Pi/2);
```

Maple (Version 3.0) will return the expression

$$\frac{1}{64}(4a^2+\pi^2)^{3/2}\pi - \frac{1}{8}a^4\ln\left(\frac{1}{2}\pi+\frac{1}{2}\sqrt{4a^2+\pi^2}\right) - \frac{1}{32}a^2\sqrt{4a^2+\pi^2}\,\pi$$
$$+\frac{1}{8}a^4\ln\left(\sqrt{a^2}\right).$$

You can also find the definite integral for a particular value of the constant a:

```
> a := 1;
> int(f, x = 0..1);
```

Maple returns the numerical answer

$$\frac{3}{8}\sqrt{2} - \frac{1}{8}\ln(1+\sqrt{2})$$ ❑

You can integrate a function directly without first naming the function as in Example 9.

EXAMPLE 10 Use a CAS to find $\int \sin^2 x \cos^3 x\, dx$.

Solution With Maple we have the entry

```
> int((sin^2)(x) * (cos^3)(x),x);
```

with the immediate return

$$-\frac{1}{5}\sin(x)\cos(x)^4 + \frac{1}{15}\cos(x)^2\sin(x) + \frac{2}{15}\sin(x)$$

as in Example 8. ❑

When a CAS cannot find a closed form solution for an indefinite or definite integral it just returns the integral expression you asked for.

EXAMPLE 11 Use a CAS to find $\int (\cos^{-1} ax)^2\, dx$.

Solution Using Maple we enter

```
> int((arccos(a*x))^2, x);
```

and Maple returns the expression

$$\int \arccos(ax)^2\, dx$$

indicating it does not have a closed form solution. In the next chapter you will see how series expansion may help to evaluate such an integral. ❑

Computer Algebra Systems vary in how they process integrations. We used Maple in Examples 9–11. Mathematica would have returned somewhat different results:

1. In Example 9, given

   ```
   In[1]: = Integrate [x^2 * Sqrt [a^2 + x^2], x]
   ```

   Mathematica returns

   $$\text{Out}[1] = \text{Sqrt}\,[a^2 + x^2]\left(\frac{a^2 x}{8} + \frac{x^3}{4}\right) - \frac{a^4\,\text{Log}\,[x + \text{Sqrt}\,[a^2 + x^2]]}{8}$$

   without having to simplify an intermediate result. The answer is close to Formula 22 in the integral tables.

2. The Mathematica answer to the integral

$$\text{In [2]: = Integrate [Sin [x]\^{}2 * Cos [x]\^{}3, x]}$$

in Example 10 is

$$\text{Out [2]} = \frac{30\,\text{Sin}\,[x] - 5\,\text{Sin}\,[3x] - 3\,\text{Sin}\,[5x]}{240}$$

differing from both the Maple answer and the answers in Example 8.

3. Mathematica does give a result for the integration

$$\text{In [3]: = Integrate [ArcCos [a * x]\^{}2, x]}$$

in Example 11:

$$\text{Out [3]} = -2x - \frac{2\,\text{Sqrt}\,[1 - a^2x^2]\ \text{ArcCos}\,[ax]}{a} + x\text{ArcCos}\,[ax]^2$$

Although a CAS is very powerful and can aid us in solving difficult problems, each CAS has its own limitations. There are even situations where a CAS may further complicate a problem (in the sense of producing an answer that is extremely difficult to use or interpret). On the other hand, a little mathematical thinking on your part may reduce the problem to one that is quite easy to handle. We provide an example in Exercise 111.

Exercises 7.5

Using Integral Tables

Use the table of integrals at the back of the book to evaluate the integrals in Exercises 1–38.

1. $\displaystyle\int \frac{dx}{x\sqrt{x-3}}$

2. $\displaystyle\int \frac{dx}{x\sqrt{x+4}}$

3. $\displaystyle\int \frac{x\,dx}{\sqrt{x-2}}$

4. $\displaystyle\int \frac{x\,dx}{(2x+3)^{3/2}}$

5. $\displaystyle\int x\sqrt{2x-3}\,dx$

6. $\displaystyle\int x(7x+5)^{3/2}\,dx$

7. $\displaystyle\int \frac{\sqrt{9-4x}}{x^2}\,dx$

8. $\displaystyle\int \frac{dx}{x^2\sqrt{4x-9}}$

9. $\displaystyle\int x\sqrt{4x-x^2}\,dx$

10. $\displaystyle\int \frac{\sqrt{x-x^2}}{x}\,dx$

11. $\displaystyle\int \frac{dx}{x\sqrt{7+x^2}}$

12. $\displaystyle\int \frac{dx}{x\sqrt{7-x^2}}$

13. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$

14. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

15. $\displaystyle\int \sqrt{25-p^2}\,dp$

16. $\displaystyle\int q^2\sqrt{25-q^2}\,dq$

17. $\displaystyle\int \frac{r^2}{\sqrt{4-r^2}}\,dr$

18. $\displaystyle\int \frac{ds}{\sqrt{s^2-2}}$

19. $\displaystyle\int \frac{d\theta}{5+4\sin 2\theta}$

20. $\displaystyle\int \frac{d\theta}{4+5\sin 2\theta}$

21. $\displaystyle\int e^{2t}\cos 3t\,dt$

22. $\displaystyle\int e^{-3t}\sin 4t\,dt$

23. $\displaystyle\int x\cos^{-1}x\,dx$

24. $\displaystyle\int x\sin^{-1}x\,dx$

25. $\displaystyle\int \frac{ds}{(9-s^2)^2}$

26. $\displaystyle\int \frac{d\theta}{(2-\theta^2)^2}$

27. $\displaystyle\int \frac{\sqrt{4x+9}}{x^2}\,dx$

28. $\displaystyle\int \frac{\sqrt{9x-4}}{x^2}\,dx$

29. $\displaystyle\int \frac{\sqrt{3t-4}}{t}\,dt$

30. $\displaystyle\int \frac{\sqrt{3t+9}}{t}\,dt$

31. $\displaystyle\int x^2\tan^{-1}x\,dx$

32. $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx$

33. $\displaystyle\int \sin 3x\cos 2x\,dx$

34. $\displaystyle\int \sin 2x\cos 3x\,dx$

35. $\int 8 \sin 4t \sin \frac{t}{2}\, dt$

36. $\int \sin \frac{t}{3} \sin \frac{t}{6}\, dt$

37. $\int \cos \frac{\theta}{3} \cos \frac{\theta}{4}\, d\theta$

38. $\int \cos \frac{\theta}{2} \cos 7\theta\, d\theta$

Substitution and Integral Tables

In Exercises 39–52, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.

39. $\int \frac{x^3 + x + 1}{(x^2 + 1)^2}\, dx$

40. $\int \frac{x^2 + 6x}{(x^2 + 3)^2}\, dx$

41. $\int \sin^{-1} \sqrt{x}\, dx$

42. $\int \frac{\cos^{-1} \sqrt{x}}{\sqrt{x}}\, dx$

43. $\int \frac{\sqrt{x}}{\sqrt{1 - x}}\, dx$

44. $\int \frac{\sqrt{2 - x}}{\sqrt{x}}\, dx$

45. $\int \cot t \sqrt{1 - \sin^2 t}\, dt, \quad 0 < t < \pi/2$

46. $\int \frac{dt}{\tan t \sqrt{4 - \sin^2 t}}$

47. $\int \frac{dy}{y\sqrt{3 + (\ln y)^2}}$

48. $\int \frac{\cos \theta\, d\theta}{\sqrt{5 + \sin^2 \theta}}$

49. $\int \frac{3\, dr}{\sqrt{9r^2 - 1}}$

50. $\int \frac{3\, dy}{\sqrt{1 + 9y^2}}$

51. $\int \cos^{-1} \sqrt{x}\, dx$

52. $\int \tan^{-1} \sqrt{y}\, dy$

Using Reduction Formulas

Use reduction formulas to evaluate the integrals in Exercises 53–72.

53. $\int \sin^5 2x\, dx$

54. $\int \sin^5 \frac{\theta}{2}\, d\theta$

55. $\int 8 \cos^4 2\pi t\, dt$

56. $\int 3 \cos^5 3y\, dy$

57. $\int \sin^2 2\theta \cos^3 2\theta\, d\theta$

58. $\int 9 \sin^3 \theta \cos^{3/2} \theta\, d\theta$

59. $\int 2 \sin^2 t \sec^4 t\, dt$

60. $\int \csc^2 y \cos^5 y\, dy$

61. $\int 4 \tan^3 2x\, dx$

62. $\int \tan^4 \left(\frac{x}{2}\right) dx$

63. $\int 8 \cot^4 t\, dt$

64. $\int 4 \cot^3 2t\, dt$

65. $\int 2 \sec^3 \pi x\, dx$

66. $\int \frac{1}{2} \csc^3 \frac{x}{2}\, dx$

67. $\int 3 \sec^4 3x\, dx$

68. $\int \csc^4 \frac{\theta}{3}\, d\theta$

69. $\int \csc^5 x\, dx$

70. $\int \sec^5 x\, dx$

71. $\int 16x^3 (\ln x)^2\, dx$

72. $\int (\ln x)^3\, dx$

Powers of x Times Exponentials

Evaluate the integrals in Exercises 73–80 using table Formulas 103–106. These integrals can also be evaluated using tabular integration (Section 7.2).

73. $\int x e^{3x}\, dx$

74. $\int x e^{-2x}\, dx$

75. $\int x^3 e^{x/2}\, dx$

76. $\int x^2 e^{\pi x}\, dx$

77. $\int x^2 2^x\, dx$

78. $\int x^2 2^{-x}\, dx$

79. $\int x \pi^x\, dx$

80. $\int x 2^{\sqrt{2} x}\, dx$

Substitutions with Reduction Formulas

Evaluate the integrals in Exercises 81–86 by making a substitution (possibly trigonometric) and then applying a reduction formula.

81. $\int e^t \sec^3 (e^t - 1)\, dt$

82. $\int \frac{\csc^3 \sqrt{\theta}}{\sqrt{\theta}}\, d\theta$

83. $\int_0^1 2\sqrt{x^2 + 1}\, dx$

84. $\int_0^{\sqrt{3}/2} \frac{dy}{(1 - y^2)^{5/2}}$

85. $\int_1^2 \frac{(r^2 - 1)^{3/2}}{r}\, dr$

86. $\int_0^{1/\sqrt{3}} \frac{dt}{(t^2 + 1)^{7/2}}$

Hyperbolic Functions

Use the integral tables to evaluate the integrals in Exercises 87–92.

87. $\int \frac{1}{8} \sinh^5 3x\, dx$

88. $\int \frac{\cosh^4 \sqrt{x}}{\sqrt{x}}\, dx$

89. $\int x^2 \cosh 3x\, dx$

90. $\int x \sinh 5x\, dx$

91. $\int \operatorname{sech}^7 x \tanh x\, dx$

92. $\int \operatorname{csch}^3 2x \coth 2x\, dx$

Theory and Examples

Exercises 93–100 refer to formulas in the table of integrals at the back of the book.

93. Derive Formula 9 by using the substitution $u = ax + b$ to evaluate

$$\int \frac{x}{(ax + b)^2}\, dx.$$

94. Derive Formula 17 by using a trigonometric substitution to evaluate

$$\int \frac{dx}{(a^2 + x^2)^2}.$$

95. Derive Formula 29 by using a trigonometric substitution to evaluate

$$\int \sqrt{a^2 - x^2}\, dx.$$

96. Derive Formula 46 by using a trigonometric substitution to evaluate

$$\int \frac{dx}{x^2\sqrt{x^2 - a^2}}.$$

97. Derive Formula 80 by evaluating

$$\int x^n \sin ax\, dx$$

by integration by parts.

98. Derive Formula 110 by evaluating

$$\int x^n (\ln ax)^m\, dx$$

by integration by parts.

99. Derive Formula 99 by evaluating

$$\int x^n \sin^{-1} ax\, dx$$

by integration by parts.

100. Derive Formula 101 by evaluating

$$\int x^n \tan^{-1} ax\, dx$$

by integration by parts.

101. Find the area of the surface generated by revolving the curve $y = \sqrt{x^2 + 2}$, $0 \le x \le \sqrt{2}$, about the x-axis.

102. Find the length of the curve $y = x^2$, $0 \le x \le \sqrt{3}/2$.

103. Find the centroid of the region cut from the first quadrant by the curve $y = 1/\sqrt{x+1}$ and the line $x = 3$.

104. A thin plate of constant density $\delta = 1$ occupies the region enclosed by the curve $y = 36/(2x + 3)$ and the line $x = 3$ in the first quadrant. Find the moment of the plate about the y-axis.

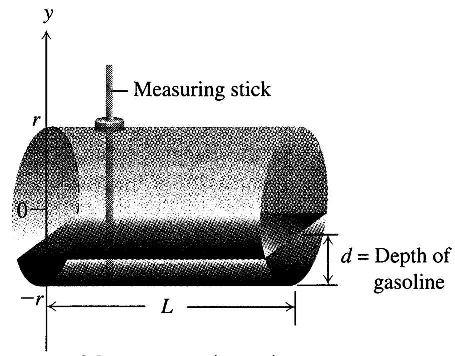
105. CALCULATOR Use the integral table and a calculator to find to 2 decimal places the area of the surface generated by revolving the curve $y = x^2$, $-1 \le x \le 1$, about the x-axis.

106. The head of your firm's accounting department has asked you to find a formula she can use in a computer program to calculate the year-end inventory of gasoline in the company's tanks. A typical tank is shaped like a right circular cylinder of radius r and length L, mounted horizontally, as shown here. The data come to the accounting office as depth measurements taken with a vertical measuring stick marked in centimeters.

a) Show, in the notation of the figure here, that the volume of gasoline that fills the tank to a depth d is

$$V = 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy.$$

b) Evaluate the integral.



Measurements in centimeters

107. What is the largest value

$$\int_a^b \sqrt{x - x^2}\, dx$$

can have for any a and b? Give reasons for your answer.

108. What is the largest value

$$\int_a^b x\sqrt{2x - x^2}\, dx$$

can have for any a and b? Give reasons for your answer.

CAS Explorations and Projects

In Exercises 109 and 110, use a CAS to perform the integrations.

109. Evaluate the integrals

a) $\int x \ln x\, dx$ **b)** $\int x^2 \ln x\, dx$

c) $\int x^3 \ln x\, dx.$

d) What pattern do you see? Predict the formula for $\int x^4 \ln x\, dx$ and then see if you are correct by evaluating it with a CAS.

e) What is the formula for $\int x^n \ln x\, dx$, $n \ge 1$? Check your answer using a CAS.

110. Evaluate the integrals

a) $\int \frac{\ln x}{x^2}\, dx$ **b)** $\int \frac{\ln x}{x^3}\, dx$

c) $\int \frac{\ln x}{x^4}\, dx.$

d) What pattern do you see? Predict the formula for

$$\int \frac{\ln x}{x^5}\, dx$$

and then see if you are correct by evaluating it with a CAS.

e) What is the formula for

$$\int \frac{\ln x}{x^n}\, dx, n \ge 2?$$

Check your answer using a CAS.

111. a) Use a CAS to evaluate

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx$$

where n is an arbitrary positive integer. Does your CAS find the result?

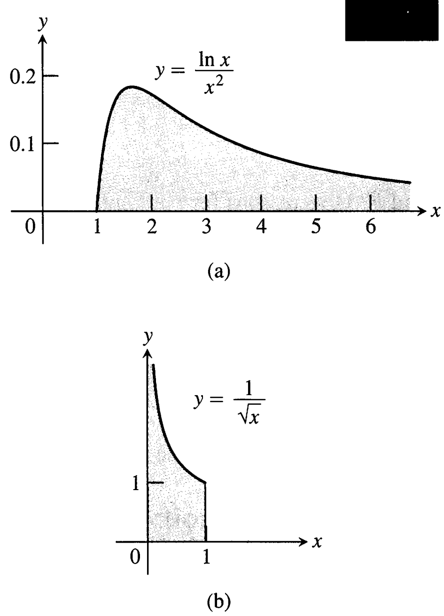
b) In succession, find the integral when $n = 1, 2, 3, 5, 7$. Comment on the complexity of the results.

c) Now substitute $x = (\pi/2) - u$ and add the new and old integrals. What is the value of

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx?$$

This exercise illustrates how a little mathematical ingenuity solves a problem not immediately amenable to solution by a CAS.

7.6 Improper Integrals



7.11 Are the areas under these infinite curves finite? See Examples 1 and 2.

Up to now, we have required our definite integrals to have two properties. First, that the domain of integration, from a to b, be finite. Second, that the range of the integrand be finite on this domain. In practice, however, we frequently encounter problems that fail to meet one or both of these conditions. As an example of an infinite domain, we might want to consider the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ (Fig. 7.11a). As an example of an infinite range, we might want to consider the area under the curve $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ (Fig. 7.11b). We treat both examples in the same reasonable way. We ask, "What is the integral when the domain is slightly less?" and examine the answer as the domain increases to the limit. We do the finite case and then see what happens as we approach infinity.

EXAMPLE 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the infinite curve (Fig. 7.12). The area from 1 to b is

$$\int_1^b \frac{\ln x}{x^2} dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right) dx$$

Integration by parts with $u = \ln x,\ dv = dx/x^2,\ du = dx/x,\ v = -1/x$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

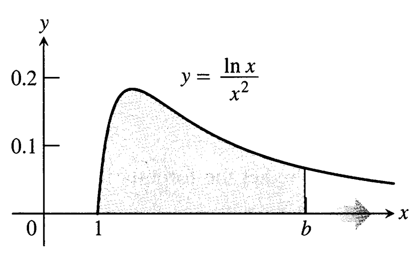
$$\lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] = -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1$$

$$= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1.$$

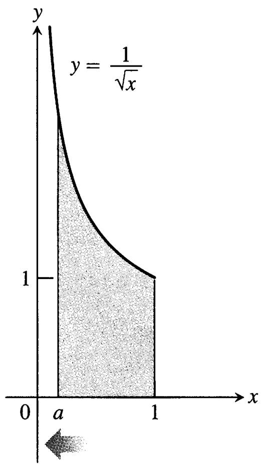
l'Hôpital's rule

In integral notation, the area under the infinite curve from 1 to ∞ is

$$\int_1^\infty \frac{\ln x}{x^2}\, dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\, dx = 1. \qquad \square$$



7.12 The area under this curve is $\lim_{b\to\infty}\int_1^b ((\ln x)/x^2)\, dx$ (Example 1).



7.13 The area under this curve is

$$\lim_{a \to 0^+} \int_a^1 (1/\sqrt{x})\, dx$$

(Example 2).

EXAMPLE 2 Is the area under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ finite? If so, what is it?

Solution We find the area under the curve from a to 1 and examine the limit as $a \to 0^+$. If the limit is finite, we take it to be the area under the infinite curve (Fig. 7.13). The area from a to 1 is

$$\int_a^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}.$$

The limit as $a \to 0^+$ is

$$\lim_{a \to 0^+} (2 - 2\sqrt{a}) = 2 - 0 = 2.$$

In integral notation, the area under the infinite curve from 0 to 1 is

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\, dx = 2.$$

❑

Improper Integrals

The integrals for the areas in Examples 1 and 2 are improper integrals.

Definition

Integrals with infinite limits of integration and integrals of functions that become infinite at a point within the interval of integration are **improper integrals.** When the limits involved exist, we evaluate such integrals with the following definitions:

1. If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx. \tag{1}$$

2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx. \tag{2}$$

3. If f is continuous on $(a, b]$, then

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx. \tag{3}$$

4. If f is continuous on $[a, b)$, then

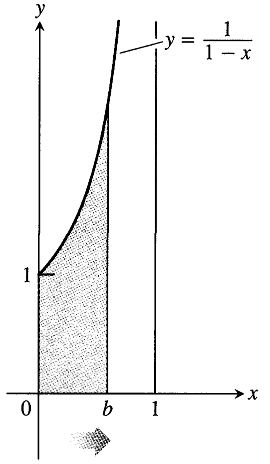
$$\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx. \tag{4}$$

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist the improper integral **diverges.**

Example 1 illustrates Part 1 of the definition:

$$\int_1^\infty \frac{\ln x}{x^2}\, dx = \lim_{b \to \infty} \int_1^b \frac{\ln x}{x^2}\, dx = 1$$

Infinite upper limit of integration



7.14 If the limit exists $\int_0^1 (1/(1-x))\,dx = \lim_{b\to 1^-} \int_0^b (1/(1-x))\,dx$ (Example 3).

Example 2 illustrates Part 3 of the definition:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = 2$$

Integrand becomes infinite at lower limit of integration

In each case, the integral converges. The integral in the next example diverges.

EXAMPLE 3 *A divergent improper integral*

Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but becomes infinite as $x \to 1^-$ (Fig. 7.14). We evaluate the integral as

$$\lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx = \lim_{b\to 1^-} \Big[-\ln|1-x|\Big]_0^b$$

$$= \lim_{b\to 1^-} [-\ln(1-b) + 0] = \infty.$$

The limit is infinite, so the integral diverges. ❑

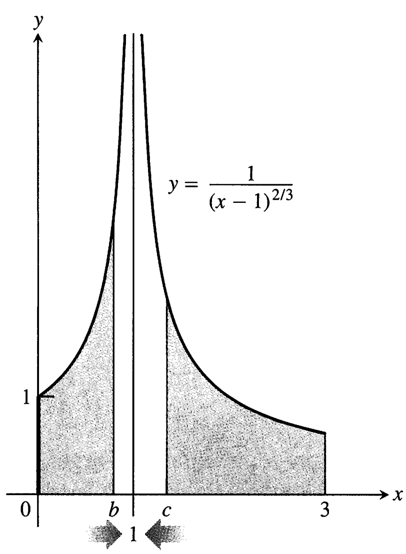
The list in the preceding definition extends in a natural way to integrals with two infinite limits of integration. We will treat these later in the section. The list also extends to integrals of functions that become infinite at an interior point d of the interval of integration. In this case, we define the integral from a to b to be the sum of the integrals from a to d and d to b.

Definition

If f becomes infinite at an interior point d of $[a,b]$, then

$$\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx. \tag{5}$$

The integral from a to b **converges** if the integrals from a to d and d to b both converge. Otherwise, the integral from a to b **diverges.**



7.15 Example 4 investigates the convergence of

$\int_0^3 (1/(x-1)^{2/3})\,dx.$

EXAMPLE 4 *Infinite at an interior point*

Investigate the convergence of

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

Solution The integrand $f(x) = 1/(x-1)^{2/3}$ becomes infinite at $x = 1$ but is continuous on $[0, 1)$ and $(1, 3]$ (Fig. 7.15). The convergence of the integral over $[0, 3]$ depends on the integrals from 0 to 1 and 1 to 3. On $[0, 1]$ we have

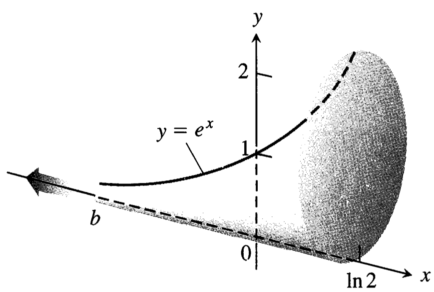
$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{b\to 1^-} [3(b-1)^{1/3} - 3(0-1)^{1/3}] = 3.$$

On [1, 3] we have

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}$$
$$= \lim_{c\to 1^+} [3(3-1)^{1/3} - 3(c-1)^{1/3}] = 3\sqrt[3]{2}.$$

Both limits are finite, so the integral of f from 0 to 3 converges and its value is $3 + 3\sqrt[3]{2}$. ❑



7.16 The calculation in Example 5 shows that this infinite horn has a finite volume.

EXAMPLE 5 The cross sections of the solid horn in Fig. 7.16 perpendicular to the x-axis are circular disks with diameters reaching from the x-axis to the curve $y = e^x$, $-\infty < x \le \ln 2$. Find the volume of the horn.

Solution The area of a typical cross section is

$$A(x) = \pi(\text{radius})^2 = \pi\left(\frac{1}{2}y\right)^2 = \frac{\pi}{4}e^{2x}.$$

We define the volume of the horn to be the limit as $b \to -\infty$ of the volume of the portion from b to ln 2. As in Section 5.2 (the method of slicing), the volume of this portion is

$$V = \int_b^{\ln 2} A(x)\,dx = \int_b^{\ln 2} \frac{\pi}{4}e^{2x}\,dx = \frac{\pi}{8}e^{2x}\bigg]_b^{\ln 2}$$
$$= \frac{\pi}{8}(e^{\ln 4} - e^{2b}) = \frac{\pi}{8}(4 - e^{2b}).$$

As $b \to -\infty$, $e^{2b} \to 0$ and $V \to (\pi/8)(4-0) = \pi/2$. The volume of the horn is $\pi/2$. ❑

EXAMPLE 6 Evaluate $\displaystyle\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx.$

Solution

$$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx = \lim_{b\to\infty} \int_2^b \frac{x+3}{(x-1)(x^2+1)}\,dx$$
$$= \lim_{b\to\infty} \int_2^b \left(\frac{2}{x-1} - \frac{2x+1}{x^2+1}\right) dx \qquad \text{Partial fractions}$$
$$= \lim_{b\to\infty} \left[2\ln(x-1) - \ln(x^2+1) - \tan^{-1} x\right]_2^b$$
$$= \lim_{b\to\infty} \left[\ln\frac{(x-1)^2}{x^2+1} - \tan^{-1} x\right]_2^b \qquad \text{Combine the logarithms.}$$
$$= \lim_{b\to\infty} \left[\ln\left(\frac{(b-1)^2}{b^2+1}\right) - \tan^{-1} b\right] - \ln\left(\frac{1}{5}\right) + \tan^{-1} 2$$
$$= 0 - \frac{\pi}{2} + \ln 5 + \tan^{-1} 2 \approx 1.1458$$

Notice that we combined the logarithms in the antiderivative *before* we calculated the limit as $b \to \infty$. Had we not done so, we would have encountered the indeterminate form

$$\lim_{b \to \infty} (2 \ln (b-1) - \ln (b^2+1)) = \infty - \infty.$$

The way to evaluate the indeterminate form, of course, is to combine the logarithms, so we would have arrived at the same answer in the end. But our original route was shorter. ❑

Integrals from $-\infty$ to ∞

In the mathematics underlying studies of light, electricity, and sound we encounter integrals with two infinite limits of integration. The next definition addresses the convergence of such integrals.

Definition

If f is continuous on $(-\infty, \infty)$ and if $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ both converge, we say that $\int_{-\infty}^{\infty} f(x)\,dx$ **converges** and define its value to be

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx. \qquad (6)$$

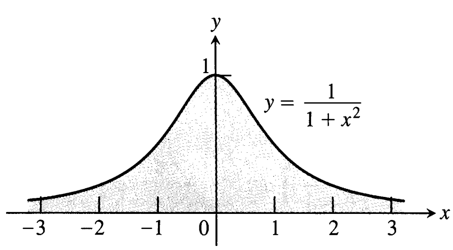
If either or both of the integrals on the right-hand side of this equation diverge, the integral of f from $-\infty$ to ∞ **diverges.**

It can be shown that the choice of a in Eq. (6) is unimportant. We can evaluate or determine the convergence of $\int_{-\infty}^{\infty} f(x)\,dx$ with any convenient choice.

The integral of f from $-\infty$ to ∞ need not equal $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$, which may exist even if $\int_{-\infty}^{\infty} f(x)\,dx$ does not converge (Exercise 75).

EXAMPLE 7

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} \qquad \text{Eq. (6) with } a=0 \\ &= \lim_{b\to-\infty} \left[\tan^{-1} x\right]_b^0 + \lim_{c\to\infty} \left[\tan^{-1} x\right]_0^c \\ &= \lim_{b\to-\infty} \left[\tan^{-1} 0 - \tan^{-1} b\right] + \lim_{c\to\infty} \left[\tan^{-1} c - \tan^{-1} 0\right] \\ &= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi.\end{aligned}$$

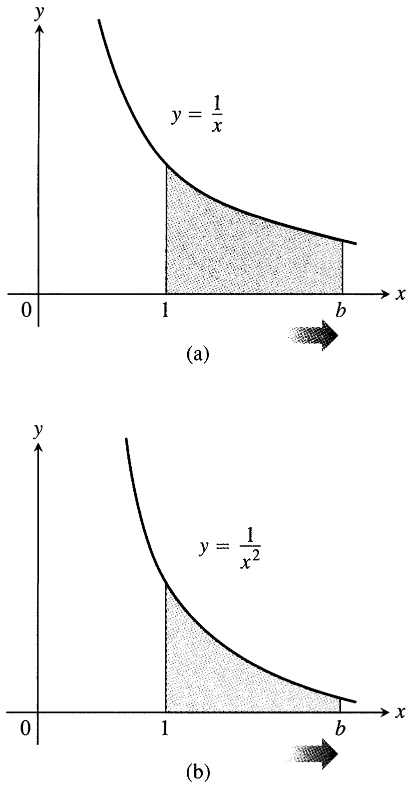


7.17 The area under this "doubly" infinite curve is finite (Example 7).

We interpret the integral as the area of the infinite region between the curve $y = 1/(1+x^2)$ and the x-axis (Fig. 7.17). ❑

The Integral $\int_1^{\infty} dx/x^p$

The convergence of the integral $\int_1^{\infty} dx/x^p$ depends on p. The next example illustrates this with $p=1$ and $p=2$.



7.18 One of these limits is finite; the other is not (Example 8).

EXAMPLE 8 Investigate the convergence of

$$\int_1^{\infty} \frac{dx}{x} \quad \text{and} \quad \int_1^{\infty} \frac{dx}{x^2}.$$

Solution The functions involved are continuous on $[1, \infty)$ and their graphs both approach the x-axis as $x \to \infty$ (Fig. 7.18), so it is reasonable to think that the areas under these infinite curves might be finite. In the first case,

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x} = \lim_{b\to\infty} (\ln b - \ln 1) = \infty,$$

so the integral diverges. In the second case,

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x^2} = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1,$$

so the integral converges and its value is 1. ❑

Generally, $\int_1^{\infty} dx/x^p$ converges if $p > 1$ and diverges if $p \leq 1$ (Exercise 67).

Tests for Convergence and Divergence

When an improper integral cannot be evaluated directly (often the case in practice) we turn to the two-step procedure of first establishing the fact of convergence and then approximating the integral numerically. The principal tests for convergence are the direct comparison and limit comparison tests.

EXAMPLE 9 Investigate the convergence of $\int_1^{\infty} e^{-x^2} dx$.

Solution By definition,

$$\int_1^{\infty} e^{-x^2} dx = \lim_{b\to\infty} \int_1^{b} e^{-x^2} dx.$$

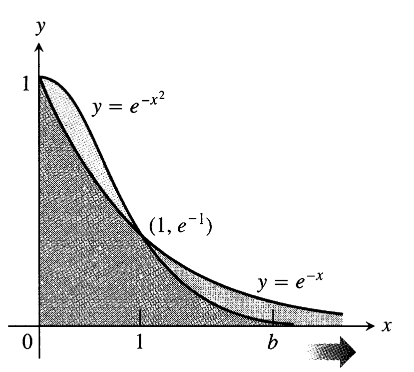
We cannot evaluate the latter integral directly because it is nonelementary. But we *can* show that its limit as $b \to \infty$ is finite. We know that $\int_1^b e^{-x^2} dx$ is an increasing function of b. Therefore either it becomes infinite as $b \to \infty$ or it has a finite limit as $b \to \infty$. It does not become infinite: For every value of $x \geq 1$ we have $e^{-x^2} \leq e^{-x}$ (Fig. 7.19), so that

$$\int_1^{b} e^{-x^2} dx \leq \int_1^{b} e^{-x} dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788.$$

Hence

$$\int_1^{\infty} e^{-x^2} dx = \lim_{b\to\infty} \int_1^{b} e^{-x^2} dx$$

converges to some definite finite value. We do not know exactly what the value is except that it is something less than 0.37. ❑



7.19 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 9).

The comparison of e^{-x^2} and e^{-x} in Example 9 is a special case of the following test.

Theorem 1

Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ and suppose that $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\displaystyle\int_a^{\infty} f(x)\,dx$ converges if $\displaystyle\int_a^{\infty} g(x)\,dx$ converges.

2. $\displaystyle\int_a^{\infty} g(x)\,dx$ diverges if $\displaystyle\int_a^{\infty} f(x)\,dx$ diverges.

EXAMPLE 10

a) $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ converges because $0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$ on $[1, \infty)$ and $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ converges.

b) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because $\dfrac{1}{\sqrt{x^2 - 0.1}} \ge \dfrac{1}{x}$ on $[1, \infty)$ and $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges. ❑

Theorem 2

Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \qquad (0 < L < \infty),$$

then $\int_a^{\infty} f(x)\,dx$ and $\int_a^{\infty} g(x)\,dx$ both converge or both diverge.

In the language of Section 6.7, Theorem 2 says that if two positive functions grow at the same rate as $x \to \infty$, then their integrals from a to ∞ behave alike: They both converge or both diverge. This does not mean that their integrals have the same value, however, as the next example shows.

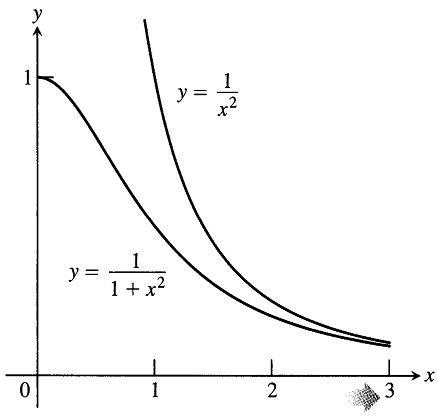
EXAMPLE 11 Compare

$$\int_1^{\infty} \frac{dx}{x^2} \qquad \text{and} \qquad \int_1^{\infty} \frac{dx}{1+x^2}$$

with the Limit Comparison Test.

Solution With $f(x) = 1/x^2$ and $g(x) = 1/(1+x^2)$, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{1/x^2}{1/(1+x^2)} \\ &= \lim_{x\to\infty} \frac{1+x^2}{x^2} = \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,\end{aligned}$$



7.20 The functions in Example 11.

a positive finite limit (Fig. 7.20). Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

The integrals converge to different values, however.

$$\int_1^\infty \frac{dx}{x^2} = 1, \qquad \text{Example 8}$$

and

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

❑

EXAMPLE 12

$$\int_1^\infty \frac{3}{e^x+5}\,dx \quad \text{converges because} \quad \int_1^\infty \frac{1}{e^x}\,dx \text{ converges}$$

and

$$\lim_{x\to\infty} \frac{1/e^x}{3/(e^x+5)} = \lim_{x\to\infty} \frac{e^x+5}{3e^x}$$

$$= \lim_{x\to\infty} \left(\frac{1}{3} + \frac{5}{3e^x}\right) = \frac{1}{3} + 0 = \frac{1}{3},$$

a positive finite limit. As far as the convergence of the improper integral is concerned, $3/(e^x+5)$ behaves like $1/e^x$. ❑

Computer Algebra Systems

Computer Algebra Systems can evaluate many convergent improper integrals.

EXAMPLE 13 Evaluate the integral $\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx$ from Example 6.

Solution Using Maple, enter

```
> f: = (x + 3)/((x − 1)*(x^2 + 1));
```

Then use the integration command

```
> int (f, x=2..infinity);
```

Maple returns the answer

$$-\frac{1}{2}\pi + \ln(5) + \arctan(2).$$

To obtain a numerical result use the evaluation command **evalf** and specify the number of digits, as follows:

```
> evalf(", 6);
```

The ditto symbol (") instructs the computer to evaluate the last expression on the screen, in this case $-\frac{1}{2}\pi + \ln(5) + \text{arc tan}(2)$. Maple returns 1.14579.

Using Mathematica, entering

In [1]: = Integrate [(x + 3)/((x − 1)(x^2 + 1)), {x, 2, Infinity}]

returns

$$\text{Out [1]} = \frac{-\text{Pi}}{2} + \text{ArcTan [2]} + \text{Log [5]}.$$

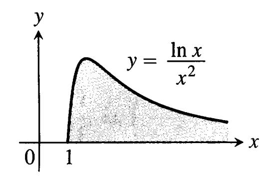
To obtain a numerical result with six digits, use the command "N[%, 6]" which also yields 1.14579. ❑

Types of Improper Integrals Discussed in This Section

INFINITE LIMITS OF INTEGRATION

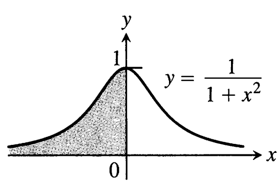
1. Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\,dx$$



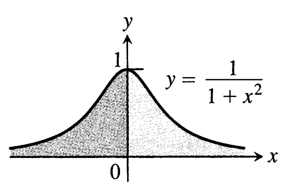
2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^{0} \frac{dx}{1+x^2}$$



3. Both limits

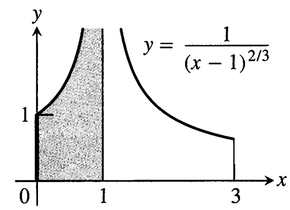
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^{0} \frac{dx}{1+x^2} + \lim_{c\to\infty} \int_0^{c} \frac{dx}{1+x^2}$$



INTEGRAND BECOMES INFINITE

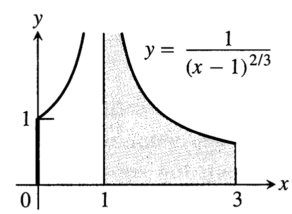
4. Upper endpoint

$$\int_0^{1} \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^{b} \frac{dx}{(x-1)^{2/3}}$$



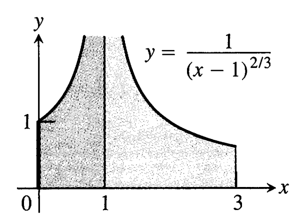
5. Lower endpoint

$$\int_1^{3} \frac{dx}{(x-1)^{2/3}} = \lim_{d\to 1^+} \int_d^{3} \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^{3} \frac{dx}{(x-1)^{2/3}} = \int_0^{1} \frac{dx}{(x-1)^{2/3}} + \int_1^{3} \frac{dx}{(x-1)^{2/3}}$$



Exercises 7.6

Evaluating Improper Integrals

Evaluate the integrals in Exercises 1–34 without using tables.

1. $\int_0^{\infty} \frac{dx}{x^2+1}$

2. $\int_1^{\infty} \frac{dx}{x^{1.001}}$

3. $\int_0^1 \frac{dx}{\sqrt{x}}$

4. $\int_0^4 \frac{dx}{\sqrt{4-x}}$

5. $\int_{-1}^1 \frac{dx}{x^{2/3}}$

6. $\int_{-8}^1 \frac{dx}{x^{1/3}}$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

8. $\int_0^1 \frac{dr}{r^{0.999}}$

9. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\int_{-\infty}^2 \frac{2\,dx}{x^2+4}$

11. $\int_2^{\infty} \frac{2}{v^2-v}\,dv$

12. $\int_2^{\infty} \frac{2\,dt}{t^2-1}$

13. $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$

14. $\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$

18. $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$

20. $\int_0^{\infty} \frac{16\tan^{-1} x}{1+x^2}\,dx$

21. $\int_{-\infty}^0 \theta e^{\theta}\,d\theta$

22. $\int_0^{\infty} 2e^{-\theta}\sin\theta\,d\theta$

23. $\int_{-\infty}^0 e^{-|x|}\,dx$

24. $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

25. $\int_0^1 x\ln x\,dx$

26. $\int_0^1 (-\ln x)\,dx$

27. $\int_0^2 \frac{ds}{\sqrt{4-s^2}}$

28. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}}$

29. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$

30. $\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$

31. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

32. $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

33. $\int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6}$

34. $\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\int_0^{\pi/2} \tan\theta\,d\theta$

36. $\int_0^{\pi/2} \cot\theta\,d\theta$

37. $\int_0^{\pi} \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$

38. $\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi-2\theta)^{1/3}}$

39. $\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

41. $\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$

42. $\int_0^1 \frac{dt}{t-\sin t}$ $\left(\text{Hint: } t \ge \sin t \text{ for } t \ge 0\right)$

43. $\int_0^2 \frac{dx}{1-x^2}$

44. $\int_0^2 \frac{dx}{1-x}$

45. $\int_{-1}^1 \ln|x|\,dx$

46. $\int_{-1}^1 -x\ln|x|\,dx$

47. $\int_1^{\infty} \frac{dx}{x^3+1}$

48. $\int_4^{\infty} \frac{dx}{\sqrt{x}-1}$

49. $\int_2^{\infty} \frac{dv}{\sqrt{v-1}}$

50. $\int_0^{\infty} \frac{d\theta}{1+e^{\theta}}$

51. $\int_0^{\infty} \frac{dx}{\sqrt{x^6+1}}$

52. $\int_2^{\infty} \frac{dx}{\sqrt{x^2-1}}$

53. $\int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\,dx$

54. $\int_2^{\infty} \frac{x\,dx}{\sqrt{x^4-1}}$

55. $\int_{\pi}^{\infty} \frac{2+\cos x}{x}\,dx$

56. $\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$

57. $\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$

58. $\int_2^\infty \frac{1}{\ln x}\,dx$

59. $\int_1^\infty \frac{e^x}{x}\,dx$

60. $\int_{e^e}^\infty \ln(\ln x)\,dx$

61. $\int_1^\infty \frac{1}{\sqrt{e^x - x}}\,dx$

62. $\int_1^\infty \frac{1}{e^x - 2^x}\,dx$

63. $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$

64. $\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}}$

Theory and Examples

65. *Estimating the value of a convergent improper integral whose domain is infinite*

a) Show that

$$\int_3^\infty e^{-3x}\,dx = \frac{1}{3}e^{-9} < 0.000042,$$

and hence that $\int_3^\infty e^{-x^2}\,dx < 0.000042$. Explain why this means that $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error of magnitude greater than 0.000042.

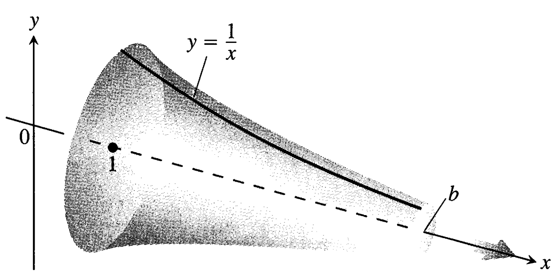
b) NUMERICAL INTEGRATOR Evaluate $\int_0^3 e^{-x^2}\,dx$ numerically.

66. *The infinite paint can or Gabriel's horn.* As Example 8 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \le x$, about the x-axis, diverges also. By comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_1^b 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > 2\pi \int_1^b \frac{1}{x}\,dx.$$



However, the integral

$$\int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges. (a) Calculate it. (b) This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

67. **a)** Show that

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}$$

if $p > 1$ but that the integral is infinite if $p < 1$. Example 8 shows what happens if $p = 1$.

b) Show that

$$\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}$$

if $p < 1$ but that the integral diverges if $p \ge 1$.

68. Find the values of p for which each integral converges:

a) $\int_1^2 \frac{dx}{x(\ln x)^p}$, **b)** $\int_2^\infty \frac{dx}{x(\ln x)^p}$.

Exercises 69–72 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the x-axis.

69. Find the area of the region.

70. Find the centroid of the region.

71. Find the volume of the solid generated by revolving the region about the y-axis.

72. Find the volume of the solid generated by revolving the region about the x-axis.

73. Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

74. The region in Exercise 73 is revolved about the x-axis to generate a solid.

a) Find the volume of the solid.

b) Show that the inner and outer surfaces of the solid have infinite area.

75. *$\int_{-\infty}^\infty f(x)\,dx$ may not equal $\lim_{b\to\infty}\int_{-b}^b f(x)\,dx$.* Show that

$$\int_0^\infty \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty}\int_{-b}^b \frac{2x\,dx}{x^2+1} = 0.$$

76. Here is an argument that ln 3 equals $\infty - \infty$. Where does the argument go wrong? Give reasons for your answer.

$$\begin{aligned}
\ln 3 &= \ln 1 + \ln 3 = \ln 1 - \ln \frac{1}{3} \\
&= \lim_{b\to\infty} \ln\left(\frac{b-2}{b}\right) - \ln \frac{1}{3} \\
&= \lim_{b\to\infty} \left[\ln \frac{x-2}{x}\right]_3^b \\
&= \lim_{b\to\infty} \left[\ln (x-2) - \ln x\right]_3^b \\
&= \lim_{b\to\infty} \int_3^b \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \\
&= \int_3^\infty \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \\
&= \int_3^\infty \frac{1}{x-2} dx - \int_3^\infty \frac{1}{x} dx \\
&= \lim_{b\to\infty} \Big[\ln (x-2)\Big]_3^b - \lim_{b\to\infty} \Big[\ln x\Big]_3^b \\
&= \infty - \infty.
\end{aligned}$$

77. Show that if $f(x)$ is integrable on every interval of real numbers and a and b are real numbers with $a < b$, then

a) $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ both converge if and only if $\int_{-\infty}^b f(x)\,dx$ and $\int_b^\infty f(x)\,dx$ both converge.

b) $\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx$ when the integrals involved converge.

78. a) Show that if f is even and the necessary integrals exist, then

$$\int_{-\infty}^\infty f(x)\,dx = 2\int_0^\infty f(x)\,dx.$$

b) Show that if f is odd and the necessary integrals exist, then

$$\int_{-\infty}^\infty f(x)\,dx = 0.$$

Use direct evaluation, the comparison tests, and the results in Exercise 78, as appropriate, to determine the convergence or divergence of the integrals in Exercises 79–86. If more than one method applies, use whatever method you prefer.

79. $\displaystyle\int_{-\infty}^\infty \frac{dx}{\sqrt{x^2+1}}$

80. $\displaystyle\int_{-\infty}^\infty \frac{dx}{\sqrt{x^6+1}}$

81. $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

82. $\displaystyle\int_{-\infty}^\infty \frac{e^{-x}\,dx}{x^2+1}$

83. $\displaystyle\int_{-\infty}^\infty e^{-|x|}\,dx$

84. $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x+1)^2}$

85. $\displaystyle\int_{-\infty}^\infty \frac{|\sin x| + |\cos x|}{|x|+1}\,dx$

(*Hint:* $|\sin\theta| + |\cos\theta| \ge \sin^2\theta + \cos^2\theta$.)

86. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)(x^2+2)}$

CAS Explorations and Projects

In Exercises 87–90, use a CAS to explore the integrals for various values of p (include noninteger values). For what values of p does the integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of p.

87. $\displaystyle\int_0^e x^p \ln x\,dx$

88. $\displaystyle\int_e^\infty x^p \ln x\,dx$

89. $\displaystyle\int_0^\infty x^p \ln x\,dx$

90. $\displaystyle\int_{-\infty}^\infty x^p \ln |x|\,dx$

91. The integral

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt,$$

called the **sine-integral function,** has important applications in optics.

a) Plot the integrand $(\sin t)/t$ for $t > 0$. Is the Si function everywhere increasing or decreasing? Do you think $\text{Si}(x) = 0$ for $x > 0$? Check your answers by graphing the function $\text{Si}(x)$ for $0 \le x \le 25$.

b) Explore the convergence of

$$\int_0^\infty \frac{\sin t}{t}\,dt.$$

If it converges, what is its value?

92. The function

$$\text{erf}(x) = \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt,$$

called the **error function,** has important applications in probability and statistics.

a) Plot the error function for $0 \le x \le 25$.

b) Explore the convergence of

$$\int_0^\infty \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 13.3, Exercise 37.

Chapter 7 Questions to Guide Your Review

1. What basic integration formulas do you know?
2. What procedures do you know for matching integrals to basic formulas?
3. What is the formula for integration by parts? Where does it come from? Why might you want to use it?
4. When applying the formula for integration by parts, how do you choose the u and dv? How can you apply integration by parts to an integral of the form $\int f(x)\,dx$?
5. What is tabular integration? Give an example.
6. What is the goal of the method of partial fractions?
7. When the degree of a polynomial $f(x)$ is less than the degree of a polynomial $g(x)$, how do you write $f(x)/g(x)$ as a sum of partial fractions if $g(x)$
   - **a)** is a product of distinct linear factors?
   - **b)** consists of a repeated linear factor?
   - **c)** contains an irreducible quadratic factor?

   What do you do if the degree of f is *not* less than the degree of g?
8. What substitutions are sometimes used to change quadratic binomials into single squared terms? Why might you want to make such a change?
9. What restrictions can you place on the variables involved in the three basic trigonometric substitutions to make sure the substitutions are reversible (have inverses)?
10. What is a reduction formula? How are reduction formulas typically derived? How are reduction formulas used? Give an example.
11. How are integral tables typically used? What do you do if a particular integral you want to evaluate is not listed in the table?
12. What is an improper integral? How are the values of various types of improper integrals defined? Give examples.
13. What tests are available for determining the convergence and divergence of improper integrals that cannot be evaluated directly? Give examples of their use.

Chapter 7 Practice Exercises

Integration Using Substitutions

Evaluate the integrals in Exercises 1–82. To transform each integral into a recognizable basic form, it may be necessary to use one or more of the techniques of algebraic substitution, completing the square, separating fractions, long division, or trigonometric substitution.

1. $\int x\sqrt{4x^2-9}\,dx$

2. $\int 6x\sqrt{3x^2+5}\,dx$

3. $\int x(2x+1)^{1/2}\,dx$

4. $\int x(1-x)^{-1/2}\,dx$

5. $\int \frac{x\,dx}{\sqrt{8x^2+1}}$

6. $\int \frac{x\,dx}{\sqrt{9-4x^2}}$

7. $\int \frac{y\,dy}{25+y^2}$

8. $\int \frac{y^3\,dy}{4+y^4}$

9. $\int \frac{t^3\,dt}{\sqrt{9-4t^4}}$

10. $\int \frac{2t\,dt}{t^4+1}$

11. $\int z^{2/3}(z^{5/3}+1)^{2/3}\,dz$

12. $\int z^{-1/5}(1+z^{4/5})^{-1/2}\,dz$

13. $\int \frac{\sin 2\theta\,d\theta}{(1-\cos 2\theta)^2}$

14. $\int \frac{\cos\theta\,d\theta}{(1+\sin\theta)^{1/2}}$

15. $\int \frac{\sin t}{3+4\cos t}\,dt$

16. $\int \frac{\cos 2t}{1+\sin 2t}\,dt$

17. $\int \sin 2x\, e^{\cos 2x}\,dx$

18. $\int \sec x\tan x\, e^{\sec x}\,dx$

19. $\int e^{\theta}\sin(e^{\theta})\cos^2(e^{\theta})\,d\theta$

20. $\int e^{\theta}\sec^2(e^{\theta})\,d\theta$

21. $\int 2^{x-1}\,dx$

22. $\int 5^{x\sqrt{2}}\,dx$

23. $\int \frac{dv}{v\ln v}$

24. $\int \frac{dv}{v(2+\ln v)}$

25. $\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$

26. $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

27. $\int \frac{2\,dx}{\sqrt{1-4x^2}}$

28. $\int \frac{dx}{\sqrt{49-x^2}}$

29. $\int \frac{dt}{\sqrt{16-9t^2}}$

30. $\int \frac{dt}{\sqrt{9-4t^2}}$

31. $\int \frac{dt}{9+t^2}$

32. $\int \frac{dt}{1+25t^2}$

33. $\int \frac{4\,dx}{5x\sqrt{25x^2-16}}$

34. $\int \frac{6\,dx}{x\sqrt{4x^2-9}}$

35. $\int \frac{dx}{\sqrt{4x-x^2}}$

36. $\int \frac{dx}{\sqrt{4x-x^2-3}}$

37. $\int \frac{dy}{y^2-4y+8}$

38. $\int \frac{dt}{t^2+4t+5}$

39. $\int \frac{dx}{(x-1)\sqrt{x^2-2x}}$

40. $\int \frac{dv}{(v+1)\sqrt{v^2+2v}}$

41. $\int \sin^2 x\,dx$

42. $\int \cos^2 3x\,dx$

43. $\int \sin^3 \frac{\theta}{2}\,d\theta$

44. $\int \sin^3\theta\cos^2\theta\,d\theta$

45. $\int \tan^3 2t\,dt$

46. $\int 6\sec^4 t\,dt$

47. $\int \frac{dx}{2\sin x\cos x}$

48. $\int \frac{2\,dx}{\cos^2 x-\sin^2 x}$

49. $\int_{\pi/4}^{\pi/2} \sqrt{\csc^2 y-1}\,dy$

50. $\int_{\pi/4}^{3\pi/4} \sqrt{\cot^2 t+1}\,dt$

51. $\int_0^{\pi} \sqrt{1-\cos^2 2x}\,dx$

52. $\int_0^{2\pi} \sqrt{1-\sin^2 \frac{x}{2}}\,dx$

53. $\int_{-\pi/2}^{\pi/2} \sqrt{1-\cos 2t}\,dt$

54. $\int_{\pi}^{2\pi} \sqrt{1+\cos 2t}\,dt$

55. $\int \frac{x^2}{x^2+4}\,dx$

56. $\int \frac{x^3}{9+x^2}\,dx$

57. $\int \frac{4x^2+3}{2x-1}\,dx$

58. $\int \frac{2x}{x-4}\,dx$

59. $\int \frac{2y-1}{y^2+4}\,dy$

60. $\int \frac{y+4}{y^2+1}\,dy$

61. $\int \frac{t+2}{\sqrt{4-t^2}}\,dt$

62. $\int \frac{2t^2+\sqrt{1-t^2}}{t\sqrt{1-t^2}}\,dt$

63. $\int \frac{\tan x\,dx}{\tan x+\sec x}$

64. $\int \frac{\cot x}{\cot x+\csc x}\,dx$

65. $\int \sec(5-3x)\,dx$

66. $\int x\csc(x^2+3)\,dx$

67. $\int \cot\left(\frac{x}{4}\right)dx$

68. $\int \tan(2x-7)\,dx$

69. $\int x\sqrt{1-x}\,dx$

70. $\int 3x\sqrt{2x+1}\,dx$

71. $\int \sqrt{z^2+1}\,dz$

72. $\int (16+z^2)^{-3/2}\,dz$

73. $\int \frac{dy}{\sqrt{25+y^2}}$

74. $\int \frac{dy}{\sqrt{25+9y^2}}$

75. $\int \frac{dx}{x^2\sqrt{1-x^2}}$

76. $\int \frac{x^3\,dx}{\sqrt{1-x^2}}$

77. $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

78. $\int \sqrt{4-x^2}\,dx$

79. $\int \frac{dx}{\sqrt{x^2-9}}$

80. $\int \frac{12\,dx}{(x^2-1)^{3/2}}$

81. $\int \frac{\sqrt{w^2-1}}{w}\,dw$

82. $\int \frac{\sqrt{z^2-16}}{z}\,dz$

Integration by Parts

Evaluate the integrals in Exercises 83–90 using integration by parts.

83. $\int \ln(x+1)\,dx$

84. $\int x^2\ln x\,dx$

85. $\int \tan^{-1} 3x\,dx$

86. $\int \cos^{-1}\left(\frac{x}{2}\right)dx$

87. $\int (x+1)^2 e^x\,dx$

88. $\int x^2\sin(1-x)\,dx$

89. $\int e^x\cos 2x\,dx$

90. $\int e^{-2x}\sin 3x\,dx$

Partial Fractions

Evaluate the integrals in Exercises 91–110. It may be necessary to use a substitution first.

91. $\int \frac{x\,dx}{x^2-3x+2}$

92. $\int \frac{x\,dx}{x^2+4x+3}$

93. $\int \frac{dx}{x(x+1)^2}$

94. $\int \frac{x+1}{x^2(x-1)}\,dx$

95. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta+\cos\theta-2}$

96. $\int \frac{\cos\theta\,d\theta}{\sin^2\theta+\sin\theta-6}$

97. $\int \frac{3x^2+4x+4}{x^3+x}\,dx$

98. $\int \frac{4x\,dx}{x^3+4x}$

99. $\int \frac{v+3}{2v^3-8v}\,dv$

100. $\int \frac{(3v-7)\,dv}{(v-1)(v-2)(v-3)}$

101. $\int \frac{dt}{t^4+4t^2+3}$

102. $\int \frac{t\,dt}{t^4-t^2-2}$

103. $\int \frac{x^3+x^2}{x^2+x-2}\,dx$

104. $\int \frac{x^3+1}{x^3-x}\,dx$

105. $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx$

106. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx$

107. $\int \frac{dx}{x\left(3\sqrt{x+1}\right)}$

108. $\int \frac{dx}{x\left(1+\sqrt[3]{x}\right)}$

109. $\int \frac{ds}{e^s - 1}$

110. $\int \frac{ds}{\sqrt{e^s + 1}}$

Improper Integrals

Evaluate the improper integrals in Exercises 111–120.

111. $\int_0^3 \frac{dx}{\sqrt{9 - x^2}}$

112. $\int_0^1 \ln x \, dx$

113. $\int_{-1}^1 \frac{dy}{y^{2/3}}$

114. $\int_{-2}^0 \frac{d\theta}{(\theta + 1)^{3/5}}$

115. $\int_3^\infty \frac{2\,du}{u^2 - 2u}$

116. $\int_1^\infty \frac{3v - 1}{4v^3 - v^2} dv$

117. $\int_0^\infty x^2 e^{-x} \, dx$

118. $\int_{-\infty}^0 x e^{3x} \, dx$

119. $\int_{-\infty}^\infty \frac{dx}{4x^2 + 9}$

120. $\int_{-\infty}^\infty \frac{4\,dx}{x^2 + 16}$

Convergence or Divergence

Which of the improper integrals in Exercises 121–126 converge and which diverge?

121. $\int_6^\infty \frac{d\theta}{\sqrt{\theta^2 + 1}}$

122. $\int_0^\infty e^{-u} \cos u \, du$

123. $\int_1^\infty \frac{\ln z}{z} dz$

124. $\int_1^\infty \frac{e^{-t}}{\sqrt{t}} dt$

125. $\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}}$

126. $\int_{-\infty}^\infty \frac{dx}{x^2(1 + e^x)}$

Trigonometric Substitutions

Evaluate the integrals in Exercises 127–130 (a) without using a trigonometric substitution, (b) using a trigonometric substitution.

127. $\int \frac{y\,dy}{\sqrt{16 - y^2}}$

128. $\int \frac{x\,dx}{\sqrt{4 + x^2}}$

129. $\int \frac{x\,dx}{4 - x^2}$

130. $\int \frac{t\,dt}{\sqrt{4t^2 - 1}}$

Quadratic Terms

Evaluate the integrals in Exercises 131–134.

131. $\int \frac{x\,dx}{9 - x^2}$

132. $\int \frac{dx}{x(9 - x^2)}$

133. $\int \frac{dx}{9 - x^2}$

134. $\int \frac{dx}{\sqrt{9 - x^2}}$

Assorted Integrations

Evaluate the integrals in Exercises 135–202. The integrals are listed in random order.

135. $\int \frac{x\,dx}{1 + \sqrt{x}}$

136. $\int \frac{x^3 + 2}{4 - x^2} dx$

137. $\int \frac{dx}{x(x^2 + 1)^2}$

138. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

139. $\int \frac{dx}{\sqrt{-2x - x^2}}$

140. $\int \frac{(t - 1)\,dt}{\sqrt{t^2 - 2t}}$

141. $\int \frac{du}{\sqrt{1 + u^2}}$

142. $\int e^t \cos e^t \, dt$

143. $\int \frac{2 - \cos x + \sin x}{\sin^2 x} dx$

144. $\int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta$

145. $\int \frac{9\,dv}{81 - v^4}$

146. $\int \frac{\cos x\,dx}{1 + \sin^2 x}$

147. $\int \theta \cos (2\theta + 1)\, d\theta$

148. $\int_2^\infty \frac{dx}{(x - 1)^2}$

149. $\int \frac{x^3\,dx}{x^2 - 2x + 1}$

150. $\int \frac{d\theta}{\sqrt{1 + \sqrt{\theta}}}$

151. $\int \frac{2 \sin \sqrt{x}\,dx}{\sqrt{x} \sec \sqrt{x}}$

152. $\int \frac{x^5\,dx}{x^4 - 16}$

153. $\int \frac{dy}{\sin y \cos y}$

154. $\int \frac{d\theta}{\theta^2 - 2\theta + 4}$

155. $\int \frac{\tan x}{\cos^2 x} dx$

156. $\int \frac{dr}{(r + 1)\sqrt{r^2 + 2r}}$

157. $\int \frac{(r + 2)\,dr}{\sqrt{-r^2 - 4r}}$

158. $\int \frac{y\,dy}{4 + y^4}$

159. $\int \frac{\sin 2\theta\, d\theta}{(1 + \cos 2\theta)^2}$

160. $\int \frac{dx}{(x^2 - 1)^2}$

161. $\int_{\pi/4}^{\pi/2} \sqrt{1 + \cos 4x}\, dx$

162. $\int (15)^{2x+1} dx$

163. $\int \frac{x\,dx}{\sqrt{2 - x}}$

164. $\int \frac{\sqrt{1 - v^2}}{v^2} dv$

165. $\int \frac{dy}{y^2 - 2y + 2}$

166. $\int \ln \sqrt{x - 1}\, dx$

167. $\int \theta^2 \tan (\theta^3)\, d\theta$

168. $\int \frac{x\,dx}{\sqrt{8 - 2x^2 - x^4}}$

169. $\int \frac{z + 1}{z^2(z^2 + 4)} dz$

170. $\int x^3 e^{(x^2)} dx$

171. $\int \frac{t\,dt}{\sqrt{9 - 4t^2}}$

172. $\int_0^{\pi/10} \sqrt{1 + \cos 5\theta}\, d\theta$

173. $\int \frac{\cot \theta\, d\theta}{1 + \sin^2 \theta}$

174. $\int \frac{\tan^{-1} x}{x^2} dx$

175. $\int \frac{\tan \sqrt{y}}{2\sqrt{y}} dy$

176. $\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

177. $\int \frac{\theta^2 d\theta}{4 - \theta^2}$

178. $\int \frac{1 - \cos 2x}{1 + \cos 2x} dx$

179. $\int \frac{\cos (\sin^{-1} x)}{\sqrt{1 - x^2}} dx$

180. $\int \frac{\cos x\,dx}{\sin^3 x - \sin x}$

181. $\int \sin \frac{x}{2} \cos \frac{x}{2}\, dx$

182. $\int \frac{x^2 - x + 2}{(x^2+2)^2}\, dx$

183. $\int \frac{e^t\, dt}{1+e^t}$

184. $\int \tan^3 t\, dt$

185. $\int_1^{\infty} \frac{\ln y}{y^3} dy$

186. $\int \frac{3 + \sec^2 x + \sin x}{\tan x}\, dx$

187. $\int \frac{\cot v\, dv}{\ln \sin v}$

188. $\int \frac{dx}{(2x-1)\sqrt{x^2 - x}}$

189. $\int e^{\ln \sqrt{x}}\, dx$

190. $\int e^{\theta}\sqrt{3 + 4e^{\theta}}\, d\theta$

191. $\int \frac{\sin 5t\, dt}{1 + (\cos 5t)^2}$

192. $\int \frac{dv}{\sqrt{e^{2v} - 1}}$

193. $\int (27)^{3\theta+1} d\theta$

194. $\int x^5 \sin x\, dx$

195. $\int \frac{dr}{1 + \sqrt{r}}$

196. $\int \frac{4x^3 - 20x}{x^4 - 10x^2 + 9}\, dx$

197. $\int \frac{8\, dy}{y^3(y+2)}$

198. $\int \frac{(t+1)\, dt}{(t^2 + 2t)^{2/3}}$

199. $\int \frac{8\, dm}{m\sqrt{49m^2 - 4}}$

200. $\int \frac{dt}{t(1 + \ln t)\sqrt{(\ln t)(2 + \ln t)}}$

201. $\int_0^1 3(x-1)^2 \left(\int_0^x \sqrt{1 + (t-1)^4}\, dt \right) dx$

202. $\int_2^{\infty} \frac{4v^3 + v - 1}{v^2(v-1)(v^2+1)} dv$

203. Suppose for a certain function f it is known that

$$f'(x) = \frac{\cos x}{x}, \quad f(\pi/2) = a, \quad \text{and} \quad f(3\pi/2) = b.$$

Use integration by parts to evaluate

$$\int_{\pi/2}^{3\pi/2} f(x)\, dx.$$

204. Find a positive number a satisfying

$$\int_0^a \frac{dx}{1+x^2} = \int_a^{\infty} \frac{dx}{1+x^2}.$$

CHAPTER 7 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Challenging Integrals

Evaluate the integrals in Exercises 1–10.

1. $\int (\sin^{-1} x)^2\, dx$

2. $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

3. $\int x \sin^{-1} x\, dx$

4. $\int \sin^{-1} \sqrt{y}\, dy$

5. $\int \frac{d\theta}{1 - \tan^2 \theta}$

6. $\int \ln(\sqrt{x} + \sqrt{1+x})\, dx$

7. $\int \frac{dt}{t - \sqrt{1 - t^2}}$

8. $\int \frac{(2e^{2x} - e^x)\, dx}{\sqrt{3e^{2x} - 6e^x - 1}}$

9. $\int \frac{dx}{x^4 + 4}$

10. $\int \frac{dx}{x^6 - 1}$

Limits

Evaluate the limits in Exercises 11 and 12.

11. $\lim_{x\to\infty} \int_{-x}^{x} \sin t\, dt$

12. $\lim_{x\to 0^+} x \int_x^1 \frac{\cos t}{t^2}\, dt$

Evaluate the limits in Exercises 13 and 14 by identifying them with definite integrals and evaluating the integrals.

13. $\lim_{n\to\infty} \sum_{k=1}^{n} \ln \sqrt[n]{1 + \frac{k}{n}}$

14. $\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2 - k^2}}$

Theory and Applications

15. Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt, \quad 0 \le x \le \pi/4.$$

16. Find the length of the curve $y = \ln(1 - x^2)$, $0 \le x \le 1/2$.

17. The region in the first quadrant that is enclosed by the x-axis and the curve $y = 3x\sqrt{1-x}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

18. The region in the first quadrant that is enclosed by the x-axis, the curve $y = 5/(x\sqrt{5-x})$, and the lines $x = 1$ and $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

19. The region in the first quadrant enclosed by the coordinate axes, the curve $y = e^x$, and the line $x = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

20. The region in the first quadrant that is bounded above by the curve $y = e^x - 1$, below by the x-axis, and on the right by the line $x = \ln 2$ is revolved about the line $x = \ln 2$ to generate a solid. Find the volume of the solid.

21. Let R be the "triangular" region in the first quadrant that is bounded above by the line $y = 1$, below by the curve $y = \ln x$, and on the left by the line $x = 1$. Find the volume of the solid generated by revolving R about

a) the x-axis
b) the line $y = 1$.

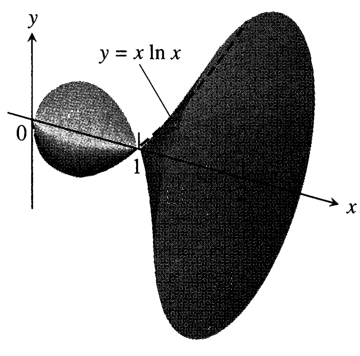
22. (*Continuation of Exercise 21.*) Find the volume of the solid generated by revolving the shaded region about (a) the y-axis, (b) the line $x = 1$.

23. The region between the curve

$$y = f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \le 2 \end{cases}$$

is revolved about the x-axis to generate the solid shown here.

a) Show that f is continuous at $x = 0$.
b) Find the volume of the solid.



24. The infinite region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant is revolved about the x-axis to generate a solid. Find the volume of the solid.

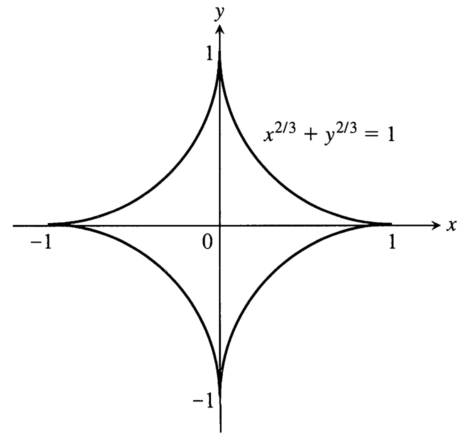
25. Find the centroid of the region in the first quadrant that is bounded below by the x-axis, above by the curve $y = \ln x$, and on the right by the line $x = e$.

26. Find the centroid of the region in the plane enclosed by the curves $y = \pm(1 - x^2)^{-1/2}$ and the lines $x = 0$ and $x = 1$.

27. Find the length of the curve $y = \ln x$ from $x = 1$ to $x = e$.

28. Find the area of the surface generated by revolving the curve in Exercise 27 about the y-axis.

29. *The length of an astroid.* The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (Fig. 7.21). Find the length of this particular astroid.



7.21 The astroid in Exercises 29 and 30.

30. *The surface generated by an astroid.* Find the area of the surface generated by revolving the curve in Fig. 7.21 about the x-axis.

31. Find a curve through the origin whose length is

$$\int_0^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

32. Without evaluating either integral, explain why

$$2\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}.$$

(Source: Peter A. Lindstrom, *Mathematics Magazine,* Vol. 45, No. 1, January 1972, p. 47.)

33. a) GRAPHER Graph the function $f(x) = e^{(x - e^x)}$, $-5 \le x \le 3$.

b) Show that $\int_{-\infty}^{\infty} f(x)\, dx$ converges and find its value.

34. Find $\lim_{n \to \infty} \int_0^1 \frac{n\, y^{n-1}}{1 + y} dy.$

35. Derive the integral formula

$$\int x \left(\sqrt{x^2 - a^2}\right)^n dx = \frac{\left(\sqrt{x^2 - a^2}\right)^{n+2}}{n + 2} + C, \quad n \ne -2.$$

36. Prove that

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi\sqrt{2}}{8}.$$

(*Hint:* Observe that for $0 < x < 1$, we have $4 - x^2 > 4 - x^2 - x^3 > 4 - 2x^2$, with the left-hand side becoming an equality for $x = 0$ and the right-hand side becoming an equality for $x = 1$.)

37. For what value or values of a does

$$\int_1^{\infty} \left(\frac{ax}{x^2 + 1} - \frac{1}{2x}\right) dx$$

converge? Evaluate the corresponding integral(s).

38. For each $x > 0$, let $G(x) = \int_0^\infty e^{-xt}\,dt$. Prove that $xG(x) = 1$ for each $x > 0$.

39. *Infinite area and finite volume.* What values of p have the following property: The area of the region between the curve $y = x^{-p}$, $1 \le x < \infty$, and the x-axis is infinite but the volume of the solid generated by revolving the region about the x-axis is finite.

40. *Infinite area and finite volume.* What values of p have the following property: The area of the region in the first quadrant enclosed by the curve $y = x^{-p}$, the y-axis, the line $x = 1$, and the interval $[0, 1]$ on the x-axis is infinite but the volume of the solid generated by revolving the region about one of the coordinate axes is finite.

Tabular Integration

The technique of tabular integration also applies to integrals of the form $\int f(x)\,g(x)\,dx$ when neither function can be differentiated repeatedly to become zero. For example, to evaluate

$$\int e^{2x} \cos x\,dx$$

we begin as before with a table listing successive derivatives of e^{2x} and integrals of $\cos x$:

e^{2x} and its derivatives		$\cos x$ and its integrals	
e^{2x}	$+$	$\cos x$	
$2e^{2x}$	$-$	$\sin x$	
$4e^{2x}$	$+$	$-\cos x$	⟵ *Stop here:* Row is same as first row except for multiplicative constants (4 on the left, -1 on the right)

We stop differentiating and integrating as soon as we reach a row that is the same as the first row except for multiplicative constants. We interpret the table as saying

$$\int e^{2x}\cos x\,dx$$

$$= +(e^{2x}\sin x) - (2e^{2x}(-\cos x)) + \int (4e^{2x})(-\cos x)\,dx.$$

We take signed products from the diagonal arrows and a signed integral for the last horizontal arrow. Transposing the integral on the right-hand side over to the left-hand side now gives

$$5\int e^{2x}\cos x\,dx = e^{2x}\sin x + 2e^{2x}\cos x$$

or

$$\int e^{2x}\cos x\,dx = \frac{e^{2x}\sin x + 2e^{2x}\cos x}{5} + C,$$

after dividing by 5 and adding the constant of integration.

Use tabular integration to evaluate the integrals in Exercises 41–48.

41. $\int e^{2x}\cos 3x\,dx$

42. $\int e^{3x}\sin 4x\,dx$

43. $\int \sin 3x \sin x\,dx$

44. $\int \cos 5x \sin 4x\,dx$

45. $\int e^{ax}\sin bx\,dx$

46. $\int e^{ax}\cos bx\,dx$

47. $\int \ln(ax)\,dx$

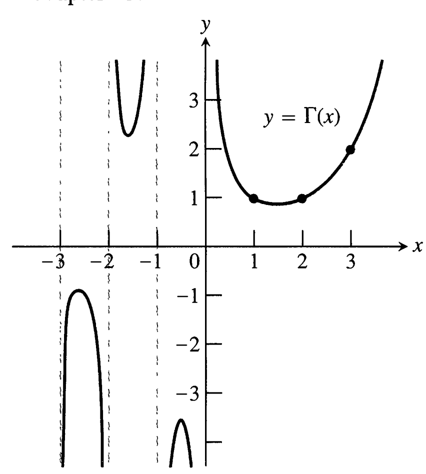
48. $\int x^2 \ln(ax)\,dx$

The Gamma Function and Stirling's Formula

Euler's gamma function $\Gamma(x)$ ("gamma of x"; Γ is a Greek capital g) uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad x > 0.$$

For each positive x, the number $\Gamma(x)$ is the integral of $t^{x-1}e^{-t}$ with respect to t from 0 to ∞. Figure 7.22 shows the graph of Γ near the origin. You will see how to calculate $\Gamma(1/2)$ if you do Additional Exercise 31 in Chapter 13.



7.22 $\Gamma(x)$ is a continuous function of x whose value at each positive integer $n + 1$ is $n!$. The defining integral formula for Γ is valid only for $x > 0$, but we can extend Γ to negative noninteger values of x with the formula $\Gamma(x) = (\Gamma(x + 1))/x$, which is the subject of Exercise 49.

49. *If n is a nonnegative integer, $\Gamma(n+1) = n!$*

a) Show that $\Gamma(1) = 1$.

b) Then apply integration by parts to the integral for $\Gamma(x+1)$ to show that $\Gamma(x+1) = x\Gamma(x)$. This gives

$$\Gamma(2) = 1\Gamma(1) = 1$$

$$\Gamma(3) = 2\Gamma(2) = 2$$

$$\Gamma(4) = 3\Gamma(3) = 6$$

$$\vdots$$

$$\Gamma(n+1) = n\Gamma(n) = n! \qquad (1)$$

c) Use mathematical induction to verify Eq. (1) for every nonnegative integer n.

50. *Stirling's formula.* Scottish mathematician James Stirling (1692–1770) showed that

$$\lim_{x\to\infty} \left(\frac{e}{x}\right)^x \sqrt{\frac{x}{2\pi}}\,\Gamma(x) = 1,$$

so for large x,

$$\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}(1+\epsilon(x)), \quad \epsilon(x) \to 0 \text{ as } x \to \infty. \qquad (2)$$

Dropping $\epsilon(x)$ leads to the approximation

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}. \qquad \textbf{(Stirling's formula)} \qquad (3)$$

a) *Stirling's approximation for $n!$.* Use Eq. (3) and the fact that $n! = n\Gamma(n)$ to show that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}. \qquad \textbf{(Stirling's approximation)} \qquad (4)$$

As you will see if you do Exercise 68 in Section 8.2, Eq. (4) leads to the approximation

$$\sqrt[n]{n!} \approx \frac{n}{e}. \qquad (5)$$

b) CALCULATOR Compare your calculator's value for $n!$ with the value given by Stirling's approximation for $n = 10, 20, 30, \ldots$, as far as your calculator can go.

c) CALCULATOR A refinement of Eq. (2) gives

$$\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}e^{1/(12x)}(1+\epsilon(x)),$$

or

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}e^{1/(12x)}$$

which tells us that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}\,e^{1/(12n)}. \qquad (6)$$

Compare the values given for 10! by your calculator, Stirling's approximation, and Eq. (6).