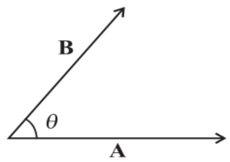
Chapter 5
Work ,Energy and Power

The Scalar Product or Dot Product

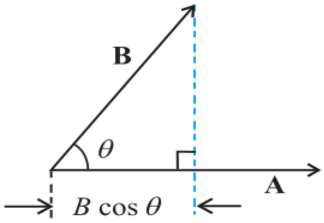
The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}.\vec{B}$
(read A dot B) is defined as

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

where θ is the angle between the two vectors

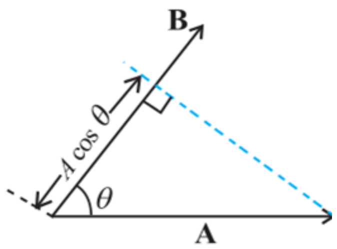


Since A, B and cos θ are scalars, the dot product of A and B is a scalar quantity. Each vector, A and B, has a direction but their scalar product does not have a direction.



$$\vec{A} \cdot \vec{B} = A(B\cos\theta\ )$$

$\vec{A} \cdot \vec{B} =$ magnitude of Ax projection of B onto A



$$\vec{A} \cdot \vec{B} = (A\cos\theta\ )B$$

$\vec{A} \cdot \vec{B} =$ magnitude of Bx projection of Aonto B

Properties of scalar product

- The scalar product follows the commutative law

$$\vec{A} \cdot \vec{B} = \vec{B}.\vec{A}$$

- Scalar product obeys the distributive law

$$\vec{A} \cdot (\vec{B} + \vec{C}\,) = \vec{A} \cdot \vec{B} + \vec{A}\,.\vec{C}$$

$$\vec{A}\,.\,(\lambda\,\vec{B}) = \lambda\,(\,\vec{A} \cdot \vec{B})$$ where λ is a real number.

- For unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ we have

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1$$

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\imath}} = 0$$

- For two vectors $\overline{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

$$\overline{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = \mathbf{A_xB_x + A_yB_y + A_zB_z}$$

- $\vec{A}\cdot\vec{A} = A_xA_x + A_yA_y + A_z A_z$

  $\vec{A}\cdot\vec{A} = {A_x}^2 + {A_y}^2 + {A_z}^2 = A^2$

- $\vec{A}\cdot\vec{A} = \mathbf{A\ A\cos 0} = A^2$

- If $\vec{A}$ and $\vec{B}$ **are perpendicular**

  $$\vec{A}\cdot\vec{B} = A\,B\cos 90 = 0$$

Example

Find the angle between force $\vec{F} = (3\hat{i} + 4\hat{j} - 5\hat{k})$ unit and displacement $\vec{d} = (5\hat{i} + 4\hat{j} + 3\hat{k})$ unit. Also find the projection of F on d.

$$\vec{F}.\vec{d} = Fd\cos\theta$$

$$\cos\theta = \frac{\vec{F}.\vec{d}}{F\,d} \quad \text{----------(1)}$$

$$\vec{F}.\vec{d} = F_x d_x + F_y d_y + F_z d_z$$

$$= (3x5) + (4x4) + (-5\,x3)$$

$$\vec{\mathbf{F}}.\vec{\mathbf{d}} = \mathbf{16\ unit}$$

$$\mathbf{F} = \sqrt{{F_x}^2 + {F_y}^2 + {F_z}^2} = \sqrt{3^2 + 4^2 + (-5)^2}$$

$$= \sqrt{9 + 16 + 25}$$

$$\mathbf{F} = \sqrt{50}\ \mathbf{unit}$$

$$\mathbf{d} = \sqrt{{d_x}^2 + {d_y}^2 + {d_z}^2} = \sqrt{5^2 + 4^2 + 3^2}$$

$$= \sqrt{25 + 16 + 9}$$

$$\mathbf{d} = \sqrt{50}\mathbf{unit}$$

Substituting the values in eq(1)

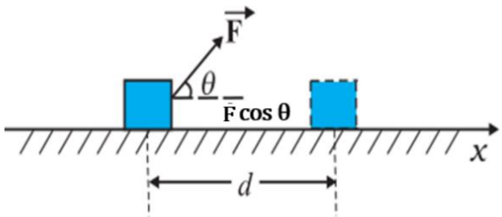
$$\cos\theta = \frac{16}{\sqrt{50}\sqrt{50}} = \frac{16}{50} = 0.32$$

$$\boldsymbol{\theta} = \mathbf{cos}^{-1}\, 0.32$$

The projection of F on d = F cosθ= $\sqrt{50}$ x 0.32 **=2.26**

Work

Consider a constant force F acting on an object of mass m. The object undergoes a displacement d in the positive x-direction



The work done by the force is defined to be the product of component of the force in the direction of the displacement and the magnitude of this displacement.

$$W = (F \cos \theta)d$$

$$W = F\, d \cos \theta$$

$$W = \vec{F} \cdot \vec{d}$$

Work can be zero, positive or negative.

Zero Work

The work can be zero,if

(i)the displacement is zero .

When you push hard against a rigid brick wall, the force you exert on the wall does no work.

A weightlifter holding a 150 kg mass steadily on his shoulder for 30 s does no work on the load during this time.

(ii) the force is zero.

A block moving on a smooth horizontal table is not acted upon by a horizontal force (since there is no friction), but may undergo a large displacement.

(iii) the force and displacement are mutually perpendicular

Here $\theta = 90^{\circ}$, $\cos(90) = 0$.

For the block moving on a smooth horizontal table, the gravitational force mg does no work since it acts at right angles to the displacement.

Positive Work

If θ is between 0° and 90°, cos θ is positive and work positive.

Eg: Workdone by Gravitational force on a freely falling body is positive

Negative work

If θ is between 90° and 180°, cos θ is negative and work negative.

Eg: the frictional force opposes displacement and $\theta = 180^{\circ}$.

Then the work done by friction is negative ($\cos 180^{\circ} = -1$).

Units of Work and Energy

- Work and Energy are scalar quantities.
- Work and energy have the same dimensions, $[ML^2 T^{-2}]$.
- The SI unit is $\mathbf{kgm^2s^{-2}}$ or **joule (J)**, named after the famous British physicist James Prescott Joule.

Alternative Units of Work/Energy in J

erg	10^{-7} J
electron volt (eV)	1.6×10^{-19} J
calorie (cal)	4.186 J
kilowatt hour (kWh)	3.6×10^{6} J

Example

Example 6.3 A cyclist comes to a skidding stop in 10 m. During this process, the force on the cycle due to the road is 200 N and is directly opposed to the motion. (a) How much work does the road do on the cycle ? (b) How much work does the cycle do on the road ?

Answer Work done on the cycle by the road is the work done by the stopping (frictional) force on the cycle due to the road.

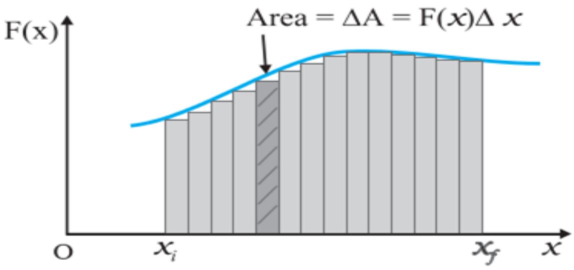
(a) The stopping force and the displacement make an angle of 180° (π rad) with each other. Thus, work done by the road,

$$W_r = Fd\cos\theta$$
$$= 200 \times 10 \times \cos\pi$$
$$= -2000 \text{ J}$$

It is this negative work that brings the cycle to a halt in accordance with WE theorem.

(b) From Newton's Third Law an equal and opposite force acts on the road due to the cycle. Its magnitude is 200 N. However, the road undergoes no displacement. Thus, work done by cycle on the road is zero.

Work done by a Variable Force



If the displacement Δx is small, we can take the force F (x) as approximately constant and the work done is then

$$\Delta W = F(x)\,\Delta x$$
$$W = \int_{x_1}^{x_2} F_{(x)}\,\Delta x$$

In the limit Δx tends to zero

$$\mathbf{W = \int_{x_1}^{x_2} F_{(x)}\, dx}$$

Kinetic Energy

The kinetic energy is the energy possessed by a body by virtue of its motion.

If an object of mass m has velocity v, its kinetic energy K is

$$\mathbf{K = \frac{1}{2} m\bar{v} \cdot \bar{v} = \frac{1}{2} mv^2}$$

Kinetic energy is a scalar quantity.

Example

In a ballistics demonstration a police officer fires a bullet of mass 50.0 g with speed 200 m s-1 on soft plywood of thickness 2.00 cm. The bullet emerges with only 10% of its initial kinetic energy. What is the emergent speed of the bullet ?

Answer The initial kinetic energy of the bullet is $mv^2/2 = 1000$ J. It has a final kinetic energy of $0.1 \times 1000 = 100$ J. If v_f is the emergent speed of the bullet,

$$\frac{1}{2} m v_f^2 = 100 \text{ J}$$

$$v_f = \sqrt{\frac{2 \times 100 \text{ J}}{0.05 \text{ kg}}}$$

$$= 63.2 \text{ m s}^{-1}$$

The Work-Energy Theorem

The work-energy theorem can be stated as :**The change in kinetic energy of a particle is equal to the work done on it by the net force.**

Proof

For uniformly accelerated motion

$$v^2 - u^2 = 2as$$

Multiplying both sides by $\frac{1}{2}m$, we have

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mas = Fs$$

$$K_f - K_i = W$$

Change in KE = Work

Potential Energy

Potential energy is the 'stored energy' by virtue of the position or configuration of a body.

- A body at a height h above the surface of earth possesses potential energy due to its position.
- A Stretched or compressed spring possesses potential energy due to its state of strain.

Gravitational potential energy of a body of mass m at a height h above the surface of earth is mgh.

Gravitational Potential Energy , $V = mgh$

Show that gravitational potential energy of the object at height h, is equal to the kinetic energy of the object on reaching the ground, when the object is released.

PE at a height h, **$V = mgh$----------(1)**

When the object is released from a height it gains KE

$$K = \frac{1}{2} mv^2$$

$$v^2 = u^2 + 2as$$

$$u=0, a=g, s=h$$

$$v^2 = 2gh$$

$$K = \frac{1}{2} m \times 2gh$$

$K = mgh$----------(2)

From eq(1) and (2)

Kinetic energy= Potential energy

Conservative Force

A force is said to be conservative, if it can be derived from a scalar quantity.

$$F = \frac{-dV}{dx} \text{ where V is a scalar}$$

Eg: Gravitational force, Spring force.

- The work done by a conservative force depends only upon initial and final positions of the body
- The work done by a conservative force in a cyclic process is zero

Note: Frictional force , air resistance are non conservative forces.

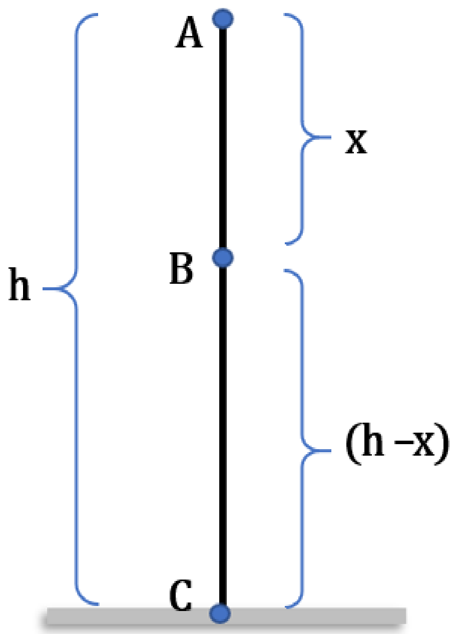
The Conservation of Mechanical Energy

The principle of conservation of total mechanical energy can be stated as,

The total mechanical energy of a system is conserved if the forces, doing work on it, are conservative.

Conservation of Mechanical Energy for a Freely Falling Body

Consider a body of mass m falling freely from a height h



At Point A

PE = mgh

KE = 0 (since v=0)

TE = PE + KE

= mgh + 0

TE = mgh-----------(1)

At Point B

PE = mg (h-x)

KE = ½ mv^2

$v^2 = u^2 + 2as$

u=0, a=g , s= x

$v^2 = 2gx$

KE = ½ m x 2gx

KE= mgx

TE = PE + KE

TE = mg (h-x) + mgx

TE = mgh--------------(2)

At Po int C

PE = 0 (Since h=0)

KE = ½ mv^2

$v^2 = u^2 + 2as$

u=0, a=g , s= h

$v^2 = 2gh$

KE = ½ m x 2gh

KE= mgh

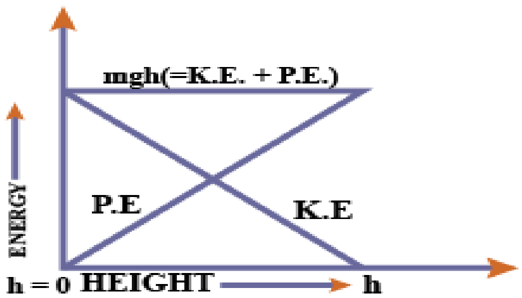
TE = PE + KE

TE = 0 + mgh

TE = mgh--------------(3)

From eqns (1), (2) and (3), it is clear that the total mechanical energy is conserved during the free fall.

Graphical variation of KE and PE with height from ground



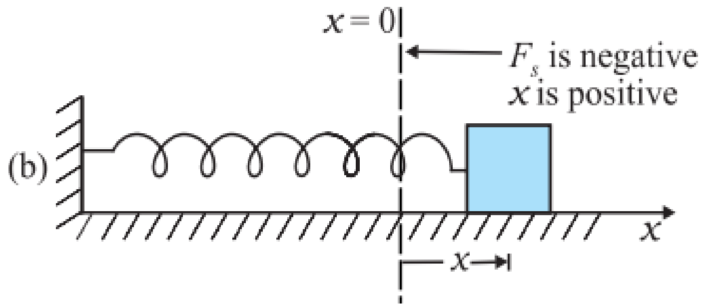
Hooke's Law

Hooke's law states that ,for an ideal spring, the spring force F is proportional displacement x of the block from the equilibrium position.

$$\mathbf{F = -kx}$$

The displacement could be either positive or negative.
The constant k is called the spring constant. Its unit is Nm^{-1}
The spring is said to be stiff if k is large and soft if k is small.

The Potential Energy of a Spring



Consider a block of mass m attached to a spring and resting on a smooth horizontal surface. The other end of the spring is attached to a rigid wall. Let the spring be pulled through a distance x.

Then the spring force $\mathbf{F = -kx}$

The work done by the spring force is

$$W = \int_0^x F\,dx$$

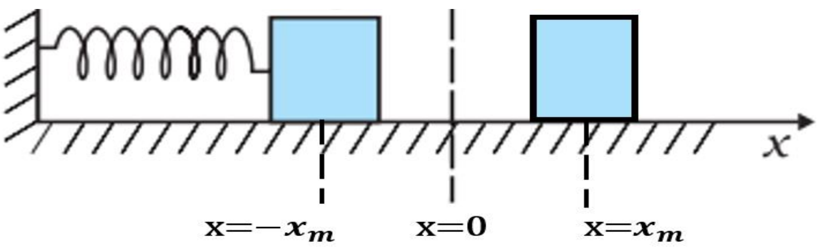
$$W = -\int_0^x kx\,dx$$

$$W = -\frac{1}{2}kx^2$$

This work is stored as potential energy of spring

$$\mathbf{PE = \frac{1}{2}kx^2}$$

Conservation of Mechanical Energy of an Oscillating Spring



Consider a spring oscillating between $-x_m$ and x_m .
At any point x between $-x_m$ *and* x_m , the total mechanical energy of the spring

$$TE = PE + KE$$

$$\frac{1}{2}k\mathbf{x_m}^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

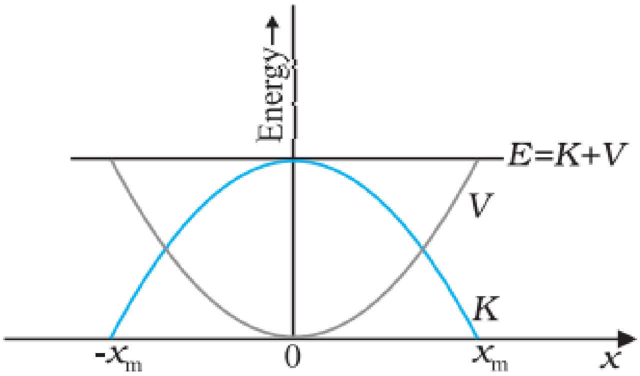
At equilibrium position x=0,

$$\frac{1}{2}k\mathbf{x_m}^2 = \frac{1}{2}mv^2$$

$$\mathbf{v = \sqrt{\frac{k}{m}}x_m}$$

- **At equilibrium position PE is zero and KE is max.**
- **At extreme ends, the PE is maximum and KE is zero.**
- The kinetic energy gets converted to potential energy and vice versa, however, the total mechanical energy remains constant.

Graphical variation of kinetic Energy and potential of a spring



Power

Power is defined as the time rate at which work is done or energy is transferred.

The average power of a force is defined as the ratio of the work, W, to the total time t taken.

$$\mathbf{P_{av} = \frac{W}{t}}$$

The instantaneous power

The instantaneous power is defined as the limiting value of the average power as time interval approaches zero.

$$\mathbf{P = \frac{dW}{dt}}$$

The work done, dW = F. dr.

$$P = F\,.\frac{dr}{dt}$$

$$\mathbf{P = F\,.\,v}$$

where **v** is the instantaneous velocity when the force is F.

- Power, like work and energy, is a scalar quantity.
- Its dimensions are ML^2T^{-3}.
- SI unit of power is called a watt (W). 1W = 1 J/s
- The unit of power is named after James Watt.

- Another unit of power is the horse-power (hp)

  1 hp = 746 W

  This unit is still used to describe the output of automobiles, motorbikes, etc

kilowatt hour

Electrical energy is measured in kilowatt hour (kWh).

$$1\text{kWh} = 3.6 \times 10^6 \text{ J}$$

Note:

A 100 watt bulb which is on for 10 hours uses 1 kilowatt hour (kWh) of energy.

$$\begin{aligned}\text{Energy} &= \text{Power x Time} \\ &= 100 \text{ (watt)} \times 10 \text{ (hour)} \\ &= 1000 \text{ watt hour} = \\ &= 1 \text{ kilowatt hour (kWh)} \\ &= 10^3 \text{ (W)} \times 3600 \text{ (s)} \\ &= 3.6 \times 10^6 \text{ J}\end{aligned}$$

Problem

An elevator can carry a maximum load of 1800 kg (elevator + passengers) is moving up with a constant speed of 2 m s–1. The frictional force opposing the motion is 4000 N. Determine the minimum power delivered by the motor to the elevator in watts as well as in horse power.

The downward force on the elevator is F = m g + Frictional Force

$$\begin{aligned} &= (1800 \times 10) + 4000 \\ &= 22000 \text{ N}\end{aligned}$$

$$\begin{aligned}\text{Power, } P &= F.v \\ &= 22000 \times 2 \\ &= 44000 \text{ W}\end{aligned}$$

In horse power, power = 44000/746

=59 hp

Collisions

In all collisions the total linear momentum is conserved; the initial momentum of the system is equal to the final momentum of the system. There are two types of collisions Elastic and Inelastic.

Elastic Collisions

The collisions in which both linear momentum and kinetic energy are conserved are called elastic collisions.

Eg: Collision between sub atomic particles

Inelastic Collisions

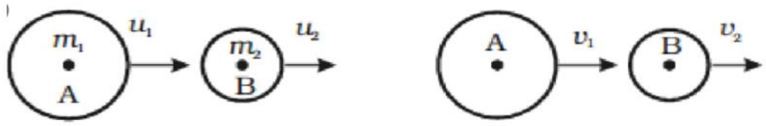
The collisions in which linear momentum is conserved, but kinetic energy is not conserved are called inelastic collisions. . Part of the initial kinetic energy is transformed into other forms of energy such as heat,sound etc..

Eg: Collision between macroscopic objects

A collision in which the two particles move together after the collision is a perfectly inelastic collision.

Elastic Collisions in One Dimension

If the initial velocities and final velocities of both the bodies are along the same straight line, then it is called a **one-dimensional collision, or head-on collision.**



Before Collision **After Collision**

Consider two masses m_1 and m_2 making elastic collision in one dimension.

By the conservation of momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \text{ --------------(1)}$$

$$m_1u_1 - m_1v_1 = m_2v_2 - m_2u_2$$

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \text{----------------(2)}$$

By the conservation of kinetic energy

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \text{-----------(3)}$$

$$\frac{1}{2}m_1u_1^2 - \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_2v_2^2 - \frac{1}{2}m_2u_2^2$$

$$\frac{1}{2}m_1 (u_1^2 - v_1^2) = \frac{1}{2}m_2(v_2^2 - u_2^2)$$

$$m_1 (u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \quad \text{-------------(4)}$$

$$\text{Eqn } \frac{(4)}{(2)} \quad \text{------------} \quad \frac{m_1 (u_1^2-v_1^2)}{m_1(u_1-v_1)} = \frac{m_2(v_2^2-u_2^2)}{m_2(v_2-u_2)}$$

$$\frac{(u_1+v_1)(u_1-v_1)}{(u_1-v_1)} = \frac{(v_2+u_2)(v_2-u_2)}{(v_2-u_2)}$$

$$u_1 + v_1 = v_2 + u_2 \quad \text{-------------}(5)$$

$$\mathbf{u_1 - u_2 = -(v_1 - v_2)}\text{--------}(6)$$

i.e., relative velocity before collision is numerically equal to relative velocity after collision.

From eqn(5), $v_2 = u_1 + v_1 - u_2$

Substituting in eqn (1)

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2(u_1 + v_1 - u_2)$$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2u_1 + m_2v_1 - m_2u_2$$

$$m_1u_1 + m_2u_2 - m_2u_1 + m_2u_2 = m_1v_1 + m_2v_1$$

$$(m_1 - m_2)u_1 + 2m_2u_2 = (m_1 + m_2)v_1$$

$$\mathbf{v_1 = \frac{(m_1-m_2)u_1}{m_1+m_2} + \frac{2m_2u_2}{m_1+m_2}} \text{--------} (7)$$

Similarly, $$\mathbf{v_2 = \frac{(m_2-m_1)u_2}{m_1+m_2} + \frac{2m_1u_1}{m_1+m_2}} \text{-------} (8)$$

Case 1 -If two masses are equal, $\mathbf{m_1 = m_2 = \boldsymbol{m}}$

Substituting in eqns (7) and (8)

$$v_1 = \frac{2mu_2}{2m} = u_2$$

$$v_2 = \frac{2mu_1}{2m} = u_1$$

ie.,the bodies will exchange their velocities

Case 2- If one mass dominates, $\mathbf{m_2 >> m_1}$ **and** $\mathbf{u_2} = 0$

$$m_1 + m_2 = m_2 \qquad and \qquad m_1 - m_2 = -m_2$$

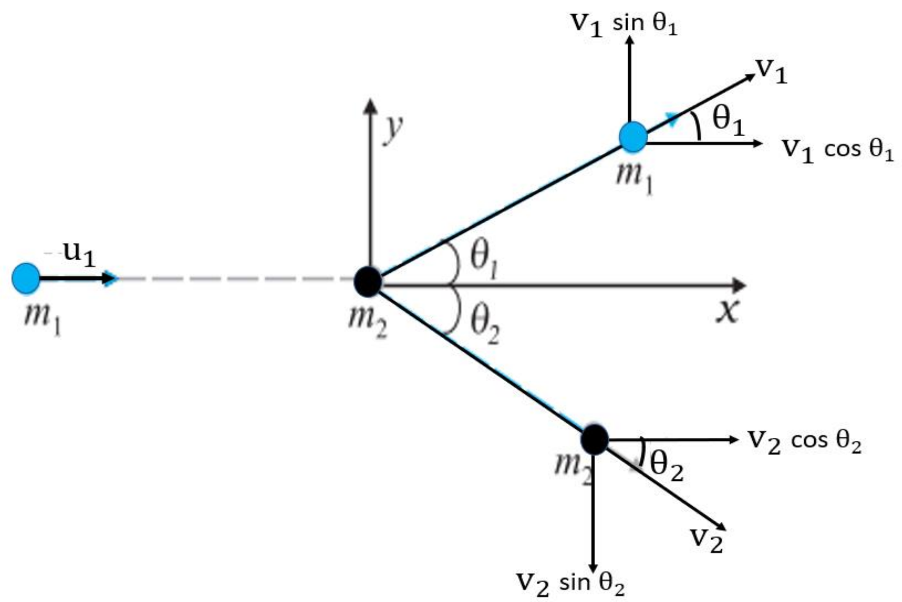
$$v_1 = \frac{(m_1-m_2)u_1}{m_1+m_2} = -\frac{m_2u_1}{m_2} = -u_1$$

$$v_2 = \frac{2m_1u_1}{m_1+m_2} = \frac{2 \times 0 \times u_1}{m_2} = 0$$

(since m_1 is very small , it can be neglected)

The heavier mass comes to rest while the lighter mass reverses its velocity.

Elastic Collisions in Two Dimensions



Consider the elastic collision of a moving mass m_1 with the stationary mass m_2.
Since momentum is a vector ,it has 2 equations in x and y directions.

Equation for conservation of momentum in x direction

$$\mathbf{m_1 u_1 = m_1 v_1 cos\theta_1 + m_2 v_2 cos\theta_2}$$

Equation for conservation of momentum in y direction

$$0 = \mathbf{m_1 v_1 sin\theta_1 - m_2 v_2 sin\,\theta_2}$$

Equation for conservation of kinetic energy,(KE is a scalar quantity)

$$\mathbf{\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2}$$