अध्याय 7(B)

निश्चित समाकलन

[DEFINITE INTEGRAL]

दीर्घ उत्तरीय प्रश्न-II

प्रश्न 1. समाकलन $\int_2^3 x^5 dx$ का मान ज्ञात कीजिए। (म.प्र. 2022)

हल :

$$I = \int_2^3 x^5 dx$$

$$\int x^n dx = \frac{x^n + 1}{n+1} = \left[\frac{x^6}{6}\right]_2^3$$

$$= \frac{1}{6}[x^6]_2^3 = \frac{1}{6}[3^6 - 2^6] = \frac{1}{6}[729 - 64] = \frac{665}{6}.$$ उत्तर

प्रश्न 2. सिद्ध कीजिए कि

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{12}.$$ (NCERT)

हल : माना

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx \quad ...(1)$$

अत:

$$I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}+\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}dx,$$

[प्रगुण $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$ से]

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad ...(2)$$

समी. (1) और (2) को जोड़ने पर,

$$I+I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow \quad 2I=\int_{\pi/6}^{\pi/3}1.dx$$

$$=[x]_{\pi/6}^{\pi/3}=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

$$\therefore \quad I=\frac{\pi}{12}.$$

यही सिद्ध करना था।

प्रश्न 3. मान ज्ञात करो—

$$\int_{-1}^{1}\sin^5 x\cos^4 x dx.$$

(म.प्र. 2023)

हल :

$$f(x)=\sin^5 x\cos^4 x$$

$$f(-x)=-\sin^5 x\cos^4 x$$

$$\therefore \quad f(x)=-f(-x)$$

$\therefore \quad \int_{-1}^{1}\sin^5 x\cos^4 x=0,$ (क्योंकि $f(x)$ एक विषम फलन है।) **उत्तर**

प्रश्न 4. सिद्ध कीजिए— $\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx=\frac{\pi}{4}.$ **(म.प्र. 2022)**

हल :

$$I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad ...(1)$$

$$\Rightarrow \quad I=\int_0^{\pi/2}\frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}dx$$

$$\Rightarrow \qquad I=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad ...(2)$$

समी. (1) व (2) को जोड़ने पर,

$$2I=\int_0^{\pi/2}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow \qquad 2I=\int_0^{\pi/2}dx=[x]_0^{\pi/2}=\frac{\pi}{2}-0$$

$$\Rightarrow \qquad I=\frac{\pi}{4}.$$

यही सिद्ध करना था।

प्रश्न 5. $\int_0^{\pi/2}\frac{\sin^4 x}{\sin^4 x+\cos^4 x}dx$ **का मान ज्ञात कीजिए।** **(म.प्र. 2020)**

हल : मान लीजिए कि $I=\int_0^{\pi/2}\frac{\sin^4 x}{\sin^4 x+\cos^4 x}dx$ ...(1)

$$I=\int_0^{\pi/2}\frac{\sin^4\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx$$

$$I=\int_0^{\pi/2}\frac{\cos^4 x}{\cos^4 x+\sin^4 x}dx \qquad ...(2)$$

समी. (1) और (2) को जोड़ने पर,

$$I+I=\int_0^{\pi/2}\frac{\sin^4 x}{\sin^4 x+\cos^4 x}dx+\int_0^{\pi/2}\frac{\cos^4 x}{\cos^4 x+\sin^4 x}dx$$

$$2I=\int_0^{\pi/2}\frac{\sin^4 x+\cos^4 x}{\sin^4 x+\cos^4 x}dx$$

$$2I=\int_0^{\pi/2}1\,dx$$

$$2I=[x]_0^{\pi/2}$$

$$2I=\frac{\pi}{2}-0$$

$$\therefore \qquad I=\frac{\pi}{4}.$$

उत्तर

प्रश्न 6. मान ज्ञात करो—

$$\int_{-1}^{2}|x^3-x|\,dx.$$

(म.प्र. 2023)

हल : $\int_{-1}^{2}|x^3-x|\,dx.=\int_{-1}^{0}(x^3-x)dx-\int_0^1(x^3-x)dx+\int_1^2(x^3-x)dx$

$$=\left[\left(\frac{x^4}{4}-\frac{x^2}{2}\right)\right]_{-1}^{0}-\left[\left(\frac{x^4}{4}-\frac{x^2}{2}\right)\right]_0^1+\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^2$$

$$= -\frac{1}{4} - \left[\frac{-1}{2}\right] - \frac{1}{4} + \frac{1}{2} + (4-2) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} + 2 - \frac{1}{4} + \frac{1}{2}$$

$$= \frac{11}{4}.$$

उत्त

प्रश्न 7. सिद्ध कीजिए—

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\,dx = \frac{\pi}{4}.$$

हल : माना
$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\,dx \quad ...(1)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{1+\sqrt{\cot x}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\sqrt{\tan x}}}{1+\frac{1}{\sqrt{\tan x}}}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\tan x}} \times \frac{\sqrt{\tan x}}{\sqrt{\tan x}+1}\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt{\tan x}}\,dx \quad ...(2)$$

समी. (1) और (2) को जोड़ने पर,

$$I + I = \int_0^{\frac{\pi}{2}} \left[\frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} + \frac{1}{1+\sqrt{\tan x}}\right] dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}+1}{1+\sqrt{\tan x}}\,dx \Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\,dx = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow \quad 2I = \left[\frac{\pi}{2} - 0\right] \Rightarrow I = \frac{\pi}{4}.$$

यही सिद्ध करना था।

प्रश्न 8. सिद्ध कीजिए कि

$$\int_0^{\frac{\pi}{4}} \log_e(1+\tan x)\,dx = \frac{\pi}{8}\log_e 2.$$

हल : माना
$$I = \int_0^{\frac{\pi}{4}} \log_e(1+\tan x)\,dx$$

$$= \int_0^{\frac{\pi}{4}} \log_e \left[1 + \tan\left(\frac{\pi}{4} - x\right)\right] dx$$

$$= \int_0^{\frac{\pi}{4}} \log_e \left[1 + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}.\tan x}\right] dx = \int_0^{\frac{\pi}{4}} \log_e \left[1 + \frac{1 - \tan x}{1 + \tan x}\right] dx$$

$$= \int_0^{\frac{\pi}{4}} \log_e \left[\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right] dx = \int_0^{\frac{\pi}{4}} \log_e \left[\frac{2}{1 + \tan x}\right] dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2\, dx - \int_0^{\frac{\pi}{4}} \log(1 + \tan x)\, dx$$

$\therefore$ $$I = \log_e 2 \int_0^{\frac{\pi}{4}} dx - I$$

$\Rightarrow$ $$2I = \log_e 2 [x]_0^{\frac{\pi}{4}} = \log_e 2\left(\frac{\pi}{4}\right)$$

$\therefore$ $$I = \frac{\pi}{8}\log_e 2.$$ यही सिद्ध करना था।

प्रश्न 9. $\int_0^{\pi} \frac{x}{1 + \sin x} dx$ का मान ज्ञात कीजिए।

हल : माना $$I = \int_0^{\pi} \frac{x}{1 + \sin x} dx \quad ...(1)$$

$\Rightarrow$ $$I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin(\pi - x)} dx$$

$\Rightarrow$ $$I = \int_0^{\pi} \frac{\pi - x}{1 + \sin x} dx \quad ...(2)$$

समी. (1) और (2) को जोड़ने पर,

$$2I = \int_0^{\pi} \frac{x + \pi - x}{1 + \sin x} dx = \pi \int_0^{\pi} \frac{dx}{1 + \sin x}$$

$$= \pi \int_0^{\pi} \frac{1 - \sin x}{(1 + \sin x)(1 - \sin x)} dx = \pi \int_0^{\pi} \frac{1 - \sin x}{1 - \sin^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx = \pi \int_0^{\pi} \frac{1}{\cos^2 x} dx - \pi \int_0^{\pi} \frac{\sin x}{\cos^2 x} dx$$

$$= \pi \int_0^{\pi} \sec^2 x\, dx - \pi \int_0^{\pi} \sec x \tan x\, dx = \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi[0 - (-1) - 0 + 1] = 2\pi$$

$\therefore$ $$I = \pi.$$ उत्तर

प्रश्न 10. $\int_0^{\pi} \frac{dx}{5+4\cos x}$ का मान ज्ञात कीजिए।

हल : माना $$I = \int_0^{\pi} \frac{dx}{5+4\cos x}$$

$$= \int_0^{\pi} \frac{dx}{5\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+4\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int_0^{\pi} \frac{dx}{9\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{9+\tan^2\frac{x}{2}} dx$$

माना $\tan\frac{\pi}{2} = t$,

$\therefore$ $$\frac{1}{2}\sec^2\frac{\pi}{2} dx = dt$$

जब $x = 0$, तब $t = \tan 0 = 0$

तथा जब $x = \pi$, तब $t = \tan\frac{\pi}{2} = \infty$

$\therefore$ $$I = 2\int_0^{\infty} \frac{dt}{9+t^2} = \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{\infty} = \frac{2}{3}[\tan^{-1}\infty - \tan^{-1}0]$$

$$= \frac{2}{3}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{3}.$$ उत्तर

प्रश्न 11. मान ज्ञात करो—

$$\int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x+\cos^5 x} dx.$$ **(म.प्र. 2023)**

हल : माना $$I = \int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x+\cos^5 x} dx \quad ...(1)$$

$\Rightarrow$ $$I = \int_0^{\pi/2} \frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)} dx$$

$\Rightarrow$ $$I = \int_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x+\sin^5 x} dx \quad ...(2)$$

समी. (1) और (2) को जोड़ने पर,

$$2I = \int_0^{\pi/2} \frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x} dx$$

$$2I=\int_0^{\pi/2}dx$$

$$2I=[x]_0^{\pi/2}$$

$$2I=\frac{\pi}{2}$$

$$I=\frac{\pi}{4}.$$ उत्तर

प्रश्न 12. सिद्ध कीजिए—

$$\int_0^{\pi/2}\log_e \sin x\,dx=-\frac{\pi}{2}\log_e 2.$$

हल : माना $I=\int_0^{\pi/2}\log_e \sin x\,dx$ ...(1)

$\Rightarrow$ $I=\int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}-x\right)dx$

$\Rightarrow$ $I=\int_0^{\pi/2}\log\cos x\,dx$ ...(2)

समी. (1) और (2) को जोड़ने पर,

$$2I=\int_0^{\pi/2}(\log\sin x+\log\cos x)\,dx$$

$$=\int_0^{\pi/2}\log\sin x\cos x\,dx=\int_0^{\pi/2}\log\frac{2\sin x\cos x}{2}dx$$

$$=\int_0^{\pi/2}(\log\sin 2x-\log 2)\,dx$$

$$=\int_0^{\pi/2}\log\sin 2x\,dx-\int_0^{\pi/2}\log 2\,dx$$

$$=\int_0^{\pi}\log\sin t\frac{dt}{2}-\log 2\int_0^{\pi/2}dx,$$ $\left[2x=t\Rightarrow 2dx=dt\right.$

जब $x=0$ तब $t=0$, जब $x=\frac{\pi}{2}$ तब $\left.t=\pi\right]$

$$=2.\frac{1}{2}\int_0^{\pi/2}\log\sin t\,dt-\log 2[x]_0^{\pi/2}=\int_0^{\pi/2}\log\sin t\,dt-\log 2.\frac{\pi}{2}$$

$$=I-\frac{\pi}{2}\log 2$$

$\Rightarrow$ $I=-\frac{\pi}{2}\log 2.$ यही सिद्ध करना था।

प्रश्न 13. सिद्ध कीजिए कि—

$$\int_0^{\frac{\pi}{2}}(2\log\sin x-\log\sin 2x)dx=-\frac{\pi}{2}\log 2.$$ **(NCERT)**

हल : माना $$I=\int_0^{\frac{\pi}{2}}(2\log\sin x-\log\sin 2x)dx$$
$$=\int_0^{\frac{\pi}{2}}[2\log\sin x-\log(2\sin x\cos x)]dx$$
$$=\int_0^{\frac{\pi}{2}}[2\log\sin x-\log 2-\log\sin x\ -\log\cos x]dx$$
$$=\int_0^{\frac{\pi}{2}}[\log\sin x-\log 2-\log\cos x]dx$$
$$=\int_0^{\frac{\pi}{2}}\log\sin x dx-\int_0^{\frac{\pi}{2}}\log 2dx-\int_0^{\frac{\pi}{2}}\log\cos x dx$$
$$=\int_0^{\frac{\pi}{2}}\log\sin x dx-\log 2\int_0^{\frac{\pi}{2}}1dx-\int_0^{\frac{\pi}{2}}\log\cos\left(\frac{\pi}{2}-x\right)dx$$
$$=\int_0^{\frac{\pi}{2}}\log\sin x dx-\log 2[x]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\log\sin x dx$$
$$=-\log 2\left[\frac{\pi}{2}-0\right]$$
$\Rightarrow$ $$I=-\frac{\pi}{2}\log 2.$$ यही सिद्ध करना था।

प्रश्न 14. $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$ का मान ज्ञात कीजिए। **(म.प्र. 2023; NCERT)**

हल : माना $$I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx \quad ...(1)$$
$\Rightarrow$ $$I=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx, \quad \left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$
$\Rightarrow$ $$I=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx \quad ...(2)$$
समी. (1) और (2) को जोड़ने पर,
$$I+I=\int_0^{\pi}\left[\frac{x\sin x}{1+\cos^2 x}+\frac{(\pi-x)\sin x}{1+\cos^2 x}\right]dx$$
$\Rightarrow$ $$2I=\int_0^{\pi}\frac{x\sin x+\pi\sin x-x\sin x}{1+\cos^2 x}dx=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx$$
माना $\cos x=t$, तब $-\sin x dx=dt$

जब $x=0$, तब $t=1$ तथा जब $x=\pi$, तब $t=-1$

$\therefore$ $$2I=-\int_1^{-1}\frac{\pi\,dt}{1+t^2}=\pi\int_{-1}^{1}\frac{dt}{1+t^2}=\pi\left[\tan^{-1}t\right]_{-1}^{1}$$
$$=\pi\left[\tan^{-1}1-\tan^{-1}(-1)\right]=\pi\left[\frac{\pi}{4}+\frac{\pi}{4}\right]=\pi\times\frac{2\pi}{4}$$
$\Rightarrow$ $$I=\frac{\pi^2}{2}.$$ उत्तर

प्रश्न 15. $\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$ का मान ज्ञात कीजिए। (NCERT; CBSE 2018)

हल : माना $$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9[1 - (1 - \sin 2x)]} dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9[1 - (\cos^2 x + \sin^2 x - 2\sin x \cos x)]} dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9[1 - (\sin x - \cos x)^2]} dx$$

[$\sin x - \cos x = t$ रखने पर, $(\cos x + \sin x)dx = dt$

$x = 0, t = -1; x = \frac{\pi}{4}, t = 0$]

$$= \int_{-1}^{0} \frac{dt}{16 + 9(1 - t^2)} = \int_{-1}^{0} \frac{dt}{16 + 9 - 9t^2} = \int_{-1}^{0} \frac{dt}{25 - 9t^2}$$

$$= \int_{-1}^{0} \frac{dt}{9\left(\frac{25}{9} - t^2\right)} = \frac{1}{9} \int_{-1}^{0} \frac{dt}{\left(\frac{5}{3}\right)^2 - t^2}$$

$$= \frac{1}{9} \cdot \frac{1}{2 \cdot \frac{5}{3}} \left[\log \left(\frac{\frac{5}{3} + t}{\frac{5}{3} - t} \right) \right]_{-1}^{0}$$

$$= \frac{1}{30} \left[\log \frac{5/3}{5/3} - \log \frac{2/3}{8/3} \right]$$

$$= \frac{1}{30} \left[\log 1 - \log \frac{1}{4} \right]$$

$\therefore$ $$I = -\frac{1}{30} \log \frac{1}{4} = \frac{1}{30} \log 4.$$ उत्तर

प्रश्न 16. $\int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx$ का मान ज्ञात कीजिए। (म. प्र. 2019)

हल : माना $$I = \int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx$$

$\tan^{-1} x = t$ रखने पर,

$$\frac{d}{dx} \tan^{-1} x = \frac{dt}{dx} \Rightarrow \left(\frac{1}{1 + x^2} \right) dx = dt$$

जब $x = 0$, तब $t = \tan^{-1} 0 = 0$

तथा जब $x = 1$, तब $t = \tan^{-1} 1 = \frac{\pi}{4}$

$$\therefore \qquad I = \int_0^{\pi/4} t\, dt = \left[\frac{t^2}{2}\right]_0^{\pi/4} = \frac{1}{2} \times \frac{\pi^2}{16}$$

$$I = \frac{\pi^2}{32}.$$