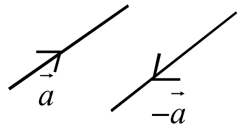
VECTOR ALGEBRA

Vectors

Quantities having both magnitude and direction.

Eg: Velocity, accelaration force, weight, momentum etc.



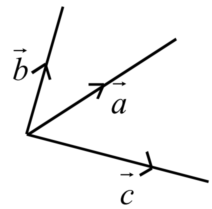
Types of Vector

Zero vector - vector having zero magnitude.

Unit vector - vector having magnitude 1.

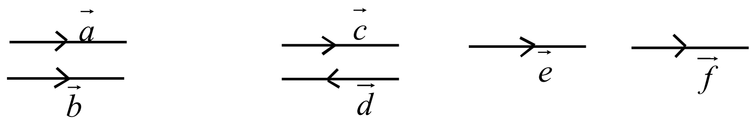
$\vec{i}, \vec{j}, \vec{k}$ are unit vectors along OX, OY, OZ axis,

Co-initial vectors - Two or more vectors having same initial point.



$\vec{a}, \vec{b}$ & $\vec{c}$ are co-initial vectors.

Collinear vectors - The vectors having same or parallel line of action.

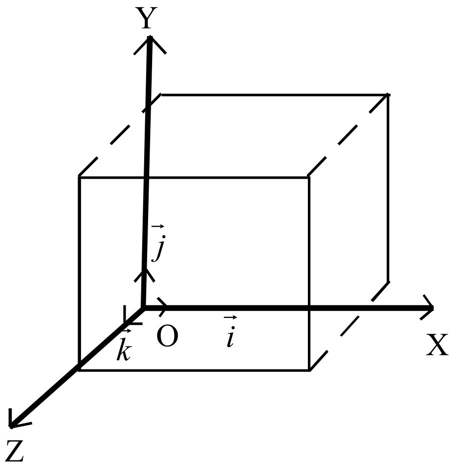


here $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$, $\vec{f}$ are collinear vectors.

Equal vectors - two vectors having same magnitude and direction are called equal vectors.

Negative of a vector - If $\vec{a}$ is a vector then -$\vec{a}$ is called negative of vector $\vec{a}$. Which has same magnitude and opposite direction as that of $\vec{a}$.

Components of vectors-



Consider the rectangular co-ordinate system in 3 dimensional geometry.

O is the origin and 0X, OY, OZ are +ve X axis, +ve Y axis and +ve z axis.

Let $\vec{i}$, $\vec{j}$, $\vec{k}$ be the unit vectors along OX OY and OZ axis respectively. Any vector in space can be expressed in terms of these unit vectors as $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

If P(x, y, z) is a point in the space then $\overline{OP}$ is called position vector of P and $\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$

Magnitude (modulus) of a vector $a\vec{i} + b\vec{j} + c\vec{k}$ is $\sqrt{a^2 + b^2 + c^2}$.

Addition : Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Then $\vec{a} + \vec{b} = (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3) + b_3)\vec{k}$

ie, if $\vec{a} = 3\vec{i} - 2\vec{j} + 4\vec{k}$ &

$\vec{b} = 2\vec{i} + \vec{j} + 5\vec{k}$

Then $\vec{a} + \vec{b} = 5\vec{i} - \vec{j} + 9\vec{k}$

Vector joining two points

If P(x_1 y_1 z_1) Q(x_2, y_2, z_3) are any two points then,

$\overrightarrow{PQ} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$

Section formula : Position vector of the point P which divided the line segment joining $A(\vec{a})$ & $B(\vec{b})$ in the ratio m:n is given by $\vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

Qn.: Find the magnitude of the vector $\vec{a} = 2\vec{i} - 7\vec{j} - 3\vec{k}$

Ans: $|\vec{a}| = \sqrt{2^2 + 7^2 + 3^2} = \sqrt{4 + 49 + 9} = \sqrt{62}$

- Unit vector along a vector $\vec{a}$ is given by $\dfrac{\vec{a}}{|\vec{a}|}$

Qn: Find the unit vector along $\vec{a} = 3\vec{i} + 2\vec{j} + 4\vec{k}$

Ans: $\vec{a} = 3\vec{i} + 2\vec{j} + 4\vec{k}$

$|\vec{a}| = \sqrt{3^2 + 2^2 + 4^2} = \sqrt{9 + 4 + 16} = \sqrt{29}$

$\therefore$ unit vector along $\vec{a} = \dfrac{3\vec{i} + 2\vec{j} + 4\vec{k}}{\sqrt{29}}$

Qn: Find the unit vector in the direction of $\vec{a} + \vec{b}$ where $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = -\vec{i} - \vec{j} - \vec{k}$

Here $\vec{a} + \vec{b} = \vec{i} + \vec{k}$

$|\vec{a} + \vec{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

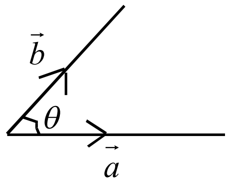
hence unit vector in the direction of $\vec{a} + \vec{b} = \dfrac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \dfrac{\vec{i} + \vec{k}}{\sqrt{2}}$

Product of two Vectors

1) Scalar (dot) Product of two vectors

If $\vec{a}$ and $\vec{b}$ are two vectors, then $\vec{a} \cdot \vec{b} = |\vec{a}|.|\vec{b}|.\text{Cos}\,\theta$

Where θ is the angle between $\vec{a}$ & $\vec{b}$



Results

1) $\vec{a} \cdot \vec{b}$ is a real number

2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

3) $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

4) $\vec{a}.\vec{a} = |\vec{a}|^2$

5) $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$ & $\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0$

6) Angle between two vectors $\vec{a}$ & $\vec{b}$ is $\text{Cos}\,\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$

7) Projection of $\vec{a}$ in the direction of $\vec{b}$ is $\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

8) If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ & $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ then

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Ex. 1 Find $\vec{a}.\vec{b}$ where $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$ *and* $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$

Sol. $\vec{a}.\vec{b} = (1x3) + (^-2x^-2) + (3x1) = 3+4+3 = 10$

Ex. 2 Show that the vectors $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} - 3\vec{j} - 5\vec{k}$ and $3\vec{i} - 4\vec{j} - 4\vec{k}$ form the vertices of a right angled triangle.

Sol: Let the vertices be $A(2\vec{i} - \vec{j} + \vec{k})$, $B(\vec{i} - 3\vec{j} - 5\vec{k})$ $C(3\vec{i} - 4\vec{j} - 4\vec{k})$

Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (1-2)\vec{i} + (^-3+1)\vec{j} + (^-5-1)\vec{k} = ^-\vec{i} - 2\vec{j} - 6\vec{k}$

$\overrightarrow{BC} = (3-1)\vec{i} + (^-4+3)\vec{j} + (^-4+5)\vec{k} = 2\vec{i} - \vec{j} + \vec{k}$

$\overrightarrow{AC} = (3-2)\vec{i} + (^-4+1)\vec{j} + (^-4-1)\vec{k} = \vec{i} - 3\vec{j} - 5\vec{k}$

Now $\overrightarrow{BC}.\overrightarrow{AC} = (2x1) + (^-1x-3) + (1x^-5) = 2+3-5 = 0$

ie, $\overrightarrow{BC} \perp \overrightarrow{AC}$ $\therefore \Delta le\ ABC$ *is right angled* Δle

Ex. 3 Find the projection of the vector $\vec{i} + 3\vec{j} + 7\vec{k}$ on the vector $7\vec{i} - \vec{j} + 8\vec{k}$

Sol. Let $\vec{a} = \vec{i} + 3\vec{j} + 7\vec{k}$ $\quad \vec{b} = 7\vec{i} - \vec{j} + 8\vec{k}$

Projection of $\vec{a}$ *on* $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|} = \dfrac{(1x7) + (3x-1) + (7x8)}{\sqrt{7^2 + (^-1)^2 + 8^2}} = \dfrac{7-3+56}{\sqrt{49+1+64}} = \dfrac{60}{\sqrt{114}}$

Ex. 4 For and two vectors $\vec{a}$ & $\vec{b}$. $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$ (triangle enequality)

$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}).(\vec{a} - \vec{b})$

$= \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b}.$

$= |\vec{a}|^2 + 2(\vec{a}.\vec{b}) + |\vec{b}|^2$

$\leq |a|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2$

$= \left(|\vec{a}| + |\vec{b}|\right)^2$

$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

Cauchy Schwartz inequality

$|\vec{a}.\vec{b}| \leq |\vec{a}|.|\vec{b}|$

Proof

$$\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = |Cos\theta| \leq 1$$

$$|\vec{a}.\vec{b}| \leq |\vec{a}||\vec{b}|, \ |\vec{a}| \neq 0, \ |\vec{b}| \neq 0$$

Vector (or Cross) Product of two vectors

The Vector product of two vectors $\vec{a}$ & $\vec{b}$ denoted by $\vec{a}x\vec{b}$ defined as $\vec{a}x\vec{b} = |\vec{a}||\vec{b}|\, Sin\theta\ \vec{n}$

Where θ is the angle between $\vec{a}x\vec{b}$ and $\vec{n}$ is a unit vector perpendicular to $\vec{a}$ *and* $\vec{b}$.

So that $\vec{a}, \vec{b}, \vec{n}$ form a right handed system.

Results:

1) $\vec{a}x\vec{b}$ is a vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$

2) $\vec{i}x\vec{j} = \vec{k}, \ \vec{j}x\vec{k} = \vec{i}$ & $\vec{k}x\vec{i} = \vec{j}$

   $\vec{j}x\vec{i} = \vec{k}, \ \vec{k}x\vec{j} = ^{-}\vec{i}$ & $\vec{i}x\vec{k} = ^{-}\vec{j}$

3) Unit vector perpendicular to $\vec{a}$ & $\vec{b}$ is $\dfrac{\vec{a}x\vec{b}}{|\vec{a}x\vec{b}|}$

4) If $\vec{a}$ and $\vec{b}$ represents adjacent sides of a parallelogram $|\vec{a}x\vec{b}|$ gives its area.

5) If $\vec{a}$ and $\vec{b}$ represents sides of a triangle, its area ½$|\vec{a}x\vec{b}|$

6) If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ *and* $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\vec{a}x\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Ex.1 Find $\vec{a}x\vec{b}$ if $\vec{a} = 2\vec{i} + \vec{j} + 3\vec{k}$ *and* $\vec{b} = 3\vec{i} + 5\vec{j} - 2\vec{k}$

Sol: $$\vec{a}x\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix} = \vec{i}(^{-}2-5) - \vec{j}(^{-}4-9) + \vec{k}(10-3)$$

$= ^{-}17\vec{i} + 13\vec{j} - 7\vec{k}$

Ex.2 Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where

$\vec{a}=\vec{i}+\vec{j}+\vec{k}, \quad \vec{b}=\vec{i}+2\vec{j}+3\vec{k}$

Sol: $\vec{a}+\vec{b}=2\vec{i}+3\vec{j}+4\vec{k} \quad and \quad \vec{a}-\vec{b}={}^{-}\vec{j}-2\vec{k}$

u.v. perpendiculat to $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is $\dfrac{(\vec{a}+\vec{b})x(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})x(\vec{a}-\vec{b})|}$

$$\vec{a}+\vec{b} \ x \ \vec{a}-\vec{b}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}={}^{-}2\vec{i}+4\vec{j}-2\vec{k}$$

$$\text{Req. u.v.} = \frac{{}^{-}2\vec{i}+4\vec{j}-2\vec{k}}{\sqrt{4+16+4}}=\frac{2\left({}^{-}\vec{i}+2\vec{j}-\vec{k}\right)}{2\sqrt{6}}=\frac{-\vec{i}+2\vec{j}-\vec{k}}{\sqrt{6}}$$

Ex. 3 Find the area of the parallelogram whose adjacent sides are $\vec{a}=\vec{i}-\vec{j}+3\vec{k} \quad and \quad 2\vec{i}-7\vec{j}+\vec{k}$

Sol. Area of a parallelogram is $|\vec{a}x\vec{b}|$

$$|\vec{a}x\vec{b}|=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}=\vec{i}\left({}^{-}1+21\right)-\vec{j}(1-6)+\vec{k}\left({}^{-}7+2)\right)$$

$=\vec{i}(20)+5\vec{j}-5\vec{k}$

Req. Area = $\sqrt{400+25+25}=\sqrt{500}=10\sqrt{5} \ sq.unit$

- If $\vec{a}$ and $\vec{b}$ are two collinear vectors $\vec{a}x\vec{b}$ =0
- If A B C are collinear points then, $\overrightarrow{AB}x\overrightarrow{BC}=0$
- If $\vec{a},\vec{b},\vec{c}$ are three coplanar vectors then, $(\vec{a}x\vec{b}).\vec{c}=0$ (Since $\vec{a}x\vec{b}$ & $\vec{c}$ are $\perp r$)

UNIT TEST

1) If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ then angle between $\vec{a} \& \vec{b} = ?$

$\left(0,\ \pi/4,\ \pi/3,\ \pi/2\right)$ (1)

2) Given ABC are the points (2, 1, -1), (3, 2, -1) & (3, 1, 0) find the angle between the vectors $\overrightarrow{AB}$ & $\overrightarrow{AC}$ (3)

3) Find the valur of λ so that the following vectors are $\perp r$ (2)

$\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ and $3 + 2\vec{j} - \lambda\vec{k}$

4) Find a vector orthogonal to both $\vec{j} + \vec{j} + 5\vec{k}$ and $2\vec{i} - \vec{k}$ (2)

5) If D, E, F are midpoints of sides of a triangle ABC. Prove that area of triangle DEF = $\frac{1}{4}$ area of triangle ABC. (3)

(Hints : Position vector of midpoint of AB is $\frac{\vec{a}+\vec{b}}{2}$)

6) It θ is the angle between two vectors $\vec{a} \& \vec{b}$ then $\frac{\vec{a} x \vec{b}}{\vec{a}.\vec{b}}$ *is* $(cot\theta, -\cot\theta, \tan\theta, -\tan\theta)$ (1)

7) Determine area of the parallelogram whose adjacent sides are $\vec{a}+\vec{b}$ *and* $\vec{a}-\vec{b}$. (3)

Where $\vec{a} = \vec{i} - \vec{j} - \vec{k}$ & $\vec{b} = 3\vec{c} + 4\vec{j} - 5\vec{k}$