CHAPTER TWENTY FIVE

Trigonometrical Ratios, Identities and Equations

INTRODUCTION

An angle is the amount of revolution which a line *OP* revolving about the point *O* has undergone in passing from its initial position *OA* into its final position *OB*.

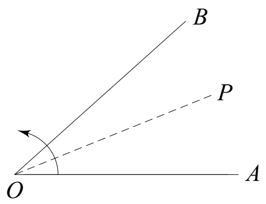

Fig. 25.1

If the rotation is in the clockwise sense, the angle measured is negative and it is positive if the rotation is in the anti-clockwise sense.

The two commonly used systems of measuring an angle are

1. *Sexagesimal system* in which

   1 right angle = 90 degrees (90°)

   1 degree = 60 minutes (60′)

   1 minute = 60 seconds (60″)

2. *Circular systems* in which the unit of measurement is the angle subtended at the centre of a circle by an arc whose length is equal to the radius and is called a *radian*.

Relation between degree and radian

$$\pi \text{ radian} = 180 \text{ degrees}$$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

$$= 57° \ 17' \ 45'' \text{ (approximately)}$$

The *trigonometrical ratios* of an angle are numerical quantities. Each one of them represents the ratio of the length of one side to another of a right angled triangle.

DOMAIN AND RANGE OF TRIGONOMETRICAL FUNCTIONS

	Domain	Range
sin *A*	$\mathbf{R}$	$[-1, 1]$
cos *A*	$\mathbf{R}$	$[-1, 1]$
tan *A*	$\mathbf{R} - \{(2n+1)\,\pi/2, n \in \mathbf{I}\}$	$\mathbf{R} = (-\infty, \infty)$
cosec *A*	$R - \{n\pi, n \in \mathbf{I}\}$	$(-\infty, -1] \cup [1, \infty)$
sec *A*	$\mathbf{R} - \{(2n+1)\,\pi/2, n \in \mathbf{I}\}$	$(-\infty, -1] \cup [1, \infty)$
cot *A*	$\mathbf{R} - \{n\pi, n \in \mathbf{I}\}$	$\mathbf{R} = (-\infty, \infty)$

We find, $|\sin A| \le 1$, $|\cos A| \le 1$

$\sec A \ge 1$ or $\sec A \le -1$

and $\operatorname{cosec} A \ge 1$ or $\operatorname{cosec} A \le -1$

SOME BASIC FORMULAE

1. $\sin^2 A + \cos^2 A = 1$ or $\cos^2 A = 1 - \sin^2 A$ or $\sin^2 A = 1 - \cos^2 A$.
2. $1 + \tan^2 A = \sec^2 A$ or $\sec^2 A - \tan^2 A = 1$.
3. $1 + \cot^2 A = \operatorname{cosec}^2 A$ or $\operatorname{cosec}^2 A - \cot^2 A = 1$.
4. $\sin A \operatorname{cosec} A = \tan A \cot A = \cos A \sec A = 1$.

A system of rectangular coordinate axes divides a plane into four quadrants. An angle θ lies in one and only one of these quadrants. The values of the trigonometric ratios in the various quadrants are shown in Fig. 25.2.

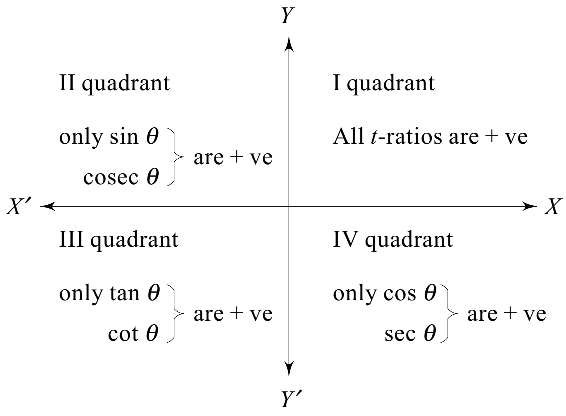

Fig. 25.2

Sine, cosine and tangent of some angles less than 90°

	0°	**15°**	**18°**	**30°**
sin	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$
cos	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}}{2}$
tan	0	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\frac{1}{\sqrt{3}}$
	36°	**45°**	**60°**	**90°**
sin	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	$\sqrt{5-2\sqrt{5}}$	1	$\sqrt{3}$	not defined

ALLIED OR RELATED ANGLES

The angles $\frac{n\pi}{2}+\theta$ and $\frac{n\pi}{2}-\theta$, where n is any integer, are known as *allied* or *related angles*. The trigonometric functions of these angles can be expressed as trigonometric functions of θ, with either a plus or a minus sign. The following working rules can be used in determining these functions:

1. Assuming that $0 < \theta < 90°$, note the quadrant in which the given angle lies. The result has a plus or minus sign according as the given function is positive or negative in that quadrant.
2. If n is even, the result contains the same trigonometric function as the given function. But if n is odd, the result contains the corresponding *co-function*, i.e., sine becomes cosine, tangent becomes cotangent, secant becomes cosecant and *vice versa*.

Illustration 1

(a) To determine $\sin(540° - \theta)$, we note that $540° - \theta = 6 \times 90° - \theta$ is a second quadrant angle if $0° < \theta < 90°$. In this quadrant, sine is positive and, since the given angle contains an even multiple of $\pi/2$, the sine function is retained. Hence, $\sin(540° - \theta) = \sin\theta$.

(b) To determine $\cos(630° - \theta)$, we note $630° - \theta = 7 \times 90° - \theta$ is a third quadrant angle if $0° < \theta < 90°$. In this quadrant, cosine is negative and, since the given angle contains an odd multiple of $\pi/2$, cosine is replaced by sine. Hence, $\cos(630° - \theta) = -\sin\theta$.

COMPOUND ANGLES

An angle made up of the algebraic sum of two or more angles is called a compound angle.

Some formulae and results regarding compound angles:

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$.
2. $\sin(A-B) = \sin A\cos B - \cos A\sin B$.
3. $\cos(A+B) = \cos A\cos B - \sin A\sin B$.
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$.
5. $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$.
6. $\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$.
7. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$.
8. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.
9. $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$.
10. $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \frac{1-\tan^2 A}{1+\tan^2 A}$.
11. $1+\cos 2A = 2\cos^2 A$, $\quad 1-\cos 2A = 2\sin^2 A$.
12. $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$.
13. (a) $\frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$ (b) $\frac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$

    (c) $\frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2}$.

14. (a) $\sin 3x = 3 \sin x - 4 \sin^3 x$
    (b) $\cos 3x = 4 \cos^3 x - 3 \cos x$
    (c) $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$.
15. $\tan (A + B + C)$
    $= \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}$.
16. (a) $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$
    (b) $\sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$
    (c) $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$
    (d) $\cos C - \cos D = 2 \sin \dfrac{C+D}{2} \sin \dfrac{D-C}{2}$.
    (*Note*)
17. (a) $\tan A + \tan B = \dfrac{\sin (A+B)}{\cos A \cos B}$
    (b) $\tan A - \tan B = \dfrac{\sin (A-B)}{\cos A \cos B}$.
18. (a) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$
    (b) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
    (c) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
    (d) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$
    (*Note*)
19. $\sin nA = \cos^n A \,({}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - \dots)$
20. $\cos nA = \cos^n A \,(1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A \dots)$
21. $\tan nA = \dfrac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A \dots}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A \dots}$
22. $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots + \sin (\alpha + (n - 1)\beta)$
    $= \dfrac{\sin(\alpha + (n-1)\beta/2)}{\sin(\beta/2)} \sin (n\beta/2)$
23. $\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \dots + \cos (\alpha + (n - 1)\beta)$
    $= \dfrac{\cos(\alpha + (n-1)\beta/2)}{\sin(\beta/2)} \sin (n\beta/2)$
24. $\tan (A_1 + A_2 + \dots + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \dots}{1 - S_2 + S_4 - S_6 + \dots}$

where $S_1 = \tan A_1 + \tan A_2 + \dots + \tan A_n$
(sum of the tangents taken one at a time)
$S_2 = \tan A_1 \tan A_2 + \tan A_2 \tan A_3 + \dots$
(sum of the tangents taken two at a time, there are ${}^nC_2$ such terms)
$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \dots$
(sum of the tangents taken three at a time, there are ${}^nC_3$ such terms and so on.

SOME IMPORTANT RESULTS

1. $\sin \theta \sin (60° - \theta) \sin (60° + \theta) = (1/4) \sin 3\theta$.
2. $\cos \theta \cos (60° - \theta) \cos (60° + \theta) = (1/4) \cos 3\theta$.
3. $\tan \theta \tan (60° - \theta) \tan (60° + \theta) = \tan 3\theta$
4. $\sin 9° = (1/4)\left[\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\right] = \cos 81°$
5. $\cos 9° = (1/4)\left[\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}\right] = \sin 81°$
6. $\cos 36° - \cos 72° = 1/2$
7. $\cos 36° \cos 72° = 1/4$
8. $\sin 22\frac{1}{2}° = \left(\frac{1}{2}\right)\left[\sqrt{2-\sqrt{2}}\right]$
9. $\cos 22\frac{1}{2}° = \left(\frac{1}{2}\right)\left[\sqrt{2+\sqrt{2}}\right]$
10. $\tan 22\frac{1}{2}° = \sqrt{2} - 1$
11. $\cot 22\frac{1}{2}° = \sqrt{2} + 1$
12. $-\sqrt{a^2 + b^2} \le a \sin x + b \cos x \le \sqrt{a^2 + b^2}$ for all $x \in \mathbf{R}$.

IDENTITIES

A trigonometric equation is an identity if it is true for all values of the angle or angles involved. A given identity may be established by (i) reducing either side to the other one, or (ii) reducing each side to the same expression, or (iii) any convenient modifications of the methods given in (i) and (ii).

CONDITIONAL IDENTITIES

When the angles A, B and C satisfy a given relation, many interesting identities can be established connecting the trigonometric functions of these angles. In proving these identities, we require the properties of complementary and supplementary angles. For example, if $A + B + C = \pi$, then

1. $\sin (B + C) = \sin A, \quad \cos B = -\cos (C + A)$
2. $\cos (A + B) = -\cos C, \quad \sin C = \sin (A + B)$
3. $\tan (C + A) = -\tan B, \quad \cot A = -\cot (B + C)$.
4. $\cos \dfrac{A+B}{2} = \sin \dfrac{C}{2}, \quad \cos \dfrac{C}{2} = \sin \dfrac{A+B}{2}$.
5. $\sin \dfrac{C+A}{2} = \cos \dfrac{B}{2}, \quad \sin \dfrac{A}{2} = \cos \dfrac{B+C}{2}$.

6. $\tan\dfrac{B+C}{2} = \cot\dfrac{A}{2}$, $\tan\dfrac{B}{2} = \cot\dfrac{C+A}{2}$.

Some Important Identities If $A + B + C = \pi$, then

1. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
2. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.
3. $\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1$.
4. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.
5. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.
6. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$.
7. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$.
8. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.
9. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

GENERAL SOLUTIONS OF TRIGONOMETRICAL EQUATIONS

The following formulae are used in solving trigonometrical equations:

1. If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n \alpha$ $(n \in \mathbf{I})$
2. If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$ $(n \in \mathbf{I})$
3. If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$ $(n \in \mathbf{I})$
4. If $\sin\theta = \sin\alpha$, $\cos\theta = \cos\alpha$, then $\theta = 2n\pi + \alpha$ $(n \in \mathbf{I})$

Illustration 2

If $\sin\theta = 1/2$, then $\sin\theta = \sin(\pi/6)$ so that $\theta = n\pi + (-1)^n \pi/6$ $(n \in \mathbf{I})$
In solving trigonometrical equations, the general values of the angle should be given, unless the solution is required in any specified interval or range.

Principal Value

Since all the trigonometrical ratios are periodic functions, an equation of the form $\sin\theta = k$, $\cos\theta = k$ or $\tan\theta = k$ etc. can have an infinite number of angles satisfying it. The set of all such values is called general value of θ.
Numerically least value in this set is called **Principal Value.**

Illustration 3

If $\sin\theta = 1/\sqrt{2}$, the general value of θ is $n\pi + (-1)^n \pi/4$ and the principal value is $\pi/4$.

Method for Finding Principal Value

(i) First note the quadrant in which the angle lies.
(ii) For 1st and 2nd quadrants consider anticlockwise direction and for 3rd and 4th clockwise direction.
(iii) Find the angles in the Ist rotation.
(iv) Select the numerically least angle among these two values. The angle thus found will be the principal value.

Illustration 4

If $\tan\theta = -1$, θ lies in 2nd or 4th quadrant. For 2nd quadrant we will select anticlockwise and for 4th quadrant we will select clockwise direction. So we get two values $3\pi/4$ and $-\pi/4$, of which $-\pi/4$ is numerically least angle. Hence principal value is $-\pi/4$.

Illustration 5

For $\sin\theta = -1/2$, the principal value of θ is $-\pi/6$ and for $\cos\theta = 1/2$, the principal value of θ is $\pi/3$. [Note $\cos\theta$ is positive in Ist and 4th quadrant, so we get two values $-\pi/3$ and $\pi/3$ of θ satisfying the relation. In such case. We select the angle with positive sign as principal value].

SOLVED EXAMPLES
Concept-based
Straight Objective Type Questions

Example 1: If $0 < \theta < \pi/2$, and $f(\theta) = (1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta)$, then $f(\pi/4)$ is equal to

(a) 0 (b) 1
(c) 2 (d) $2 + \sqrt{2}$

Ans. (c)

Solution: $f(\theta) = \left(1 + \dfrac{\cos\theta}{\sin\theta} - \dfrac{1}{\sin\theta}\right)\left(1 + \dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta}\right)$

$$= \left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right)\left(\frac{\sin\theta + \cos\theta + 1}{\cos\theta}\right)$$

$$= \frac{(\sin\theta+\cos\theta)^2-1}{\sin\theta\cos\theta} = \frac{1+2\sin\theta+\cos\theta-1}{\sin\theta\cos\theta} = 2$$

$\therefore \quad f(\pi/4) = 2$

Example 2: If $2\cos\theta = \sqrt{x} + \frac{1}{\sqrt{x}}$ for some $x > 0$, then which one of the following is **not** equal to $\cos(5\theta)$

(a) $\frac{1}{2}\left(x^{5/2} + \frac{1}{x^{5/2}}\right)$ (b) $\frac{1}{2}\left(x^{3/2} + \frac{1}{x^{3/2}}\right)$

(c) 1 (d) – 1

Ans. (d)

Solution: For $x > 0$, $\sqrt{x} + \frac{1}{\sqrt{x}} \geq 2$ and

$\sqrt{x} + \frac{1}{\sqrt{x}} = 2 \Leftrightarrow \sqrt{x} = 1$ or $x = 1$.

We have $2\cos\theta \leq 2$ and $\sqrt{x} + \frac{1}{\sqrt{x}} \geq 2$.

Thus, $2\cos\theta = \sqrt{x} + \frac{1}{\sqrt{x}} \Leftrightarrow x = 1, \cos\theta = 1$

$\therefore \quad \theta = 2n\pi$, where $n \in \mathbf{Z}$.

$\Rightarrow \quad \cos(5\theta) = \cos(10n\pi) = 1 \neq -1$

Also, for $x = 1$, $\frac{1}{2}\left(x^{5/2} + \frac{1}{x^{5/2}}\right) = \frac{1}{2}\left(x^{3/2} + \frac{1}{x^{3/2}}\right) = 1$

Example 3: For $0 < \alpha, \beta < \pi/2$, let $E = (\tan\alpha + \tan\beta)(1 - \cot\alpha\cot\beta) + (\cot\alpha + \cot\beta)(1 - \tan\alpha\tan\beta)$ then E is

(a) independent of α only

(b) independent of β only

(c) independent of α and β both

(d) equal to $2(\tan\alpha + \tan\beta)$

Ans. (c)

Solution: $E = (\tan\alpha + \tan\beta)\left(1 - \frac{1}{\tan\alpha\tan\beta}\right)$

$$+ \frac{(\tan\alpha+\tan\beta)}{\tan\alpha\tan\beta}(1-\tan\alpha\tan\beta)$$

$$= (\tan\alpha + \tan\beta)(1 - \tan\alpha\tan\beta)p$$

where

$$p = -\frac{1}{\tan\alpha\tan\beta} + \frac{1}{\tan\alpha\tan\beta} = 0$$

$\therefore \quad E = 0$

Example 4: If $\cot A = \tan((n-1)A)$, where $n \in \mathbf{N}$, then a value of A can be

(a) $\pi/2n$ (b) π/n

(c) $2\pi/n$ (d) $\pi/4n$

Ans. (a)

Solution: $\tan(\pi/2 - A) = \tan((n-1)A)$

One possible solution is

$\pi/2 - A = (n-1)A \Rightarrow A = \pi/2n$

Example 5: $\frac{1}{2}(\sqrt{3}\cos 23° - \sin 23°)$ is not equal to

(a) cos 53° (b) sin 53°

(c) sin 37° (d) sin 143°

Ans. (b)

Solution: $\frac{1}{2}(\sqrt{3}\cos 23° - \sin 23°)$

$= \cos 30°\cos 23° - \sin 30°\sin 23°$

$= \cos(30° + 23°) = \cos 53°$

$= \cos(90° - 37°) = \sin 37° = \sin(180° - 37°)$

$= \sin(143°)$

Example 6: Suppose $\alpha, \beta \in \mathbf{R}$, and let $E = \sin^2(\alpha+\beta) + \sin^2\alpha + \sin^2\beta - 2\cos\alpha\cos\beta\cos(\alpha+\beta)$ then E is equal to:

(a) 0 (b) 1

(c) 2 (d) $2\cos\alpha\cos\beta$

Ans. (c)

Solution:

$$E = 1 + \sin^2\alpha + \sin^2\beta - \cos^2(\alpha+\beta) - 2\cos\alpha\cos\beta\cos(\alpha+\beta)$$

$$= 1 + \sin^2\alpha + \sin^2\beta - \cos(\alpha+\beta)[\cos(\alpha+\beta) - 2\cos\alpha\cos\beta]$$

$$= 1 + \sin^2\alpha + \sin^2\beta + \cos(\alpha+\beta)\cos(\alpha-\beta)$$

$$= 1 + \sin^2\alpha + \sin^2\beta + \cos^2\alpha - \sin^2\beta$$

$$= 2$$

Example 7: If α, β, γ are three distinct real numbers such that $0 < \alpha, \beta, \gamma < \pi/2$, then $\tan(\alpha-\beta) + \tan(\beta-\gamma) + \tan(\gamma-\alpha)$ is equal to:

(a) 0 (b) – 1

(c) 1 (d) none of these

Ans. (d)

Solution: Let $A = \alpha - \beta$, $B = \beta - \gamma$, $C = \gamma - \alpha$, then $A + B + C = 0$.

We have $\tan(A + B) = \tan(-C)$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C$

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A\tan B\tan C$

Example 8: If $\tan A - \tan B = \frac{1}{2}$ and $\cot A - \cot B = \frac{1}{3}$, then $\cot(A - B)$ is equal to:

(a) 1 (b) –1

(c) 5 (d) –5

Ans. (b)

Solution: $\frac{1}{2} = \frac{1}{\cot A} - \frac{1}{\cot B} = -\frac{\cot A - \cot B}{\cot A\cot B}$

$\Rightarrow \cot A\cot B = -2/3$.

Now,

$$\cot(A - B) = \frac{\cot B - \cot A}{1 + \cot A\cot B} = \frac{-1/3}{1 - 2/3} = -1$$

Example 9: If $\tan\beta = \dfrac{3\sin\alpha\cos\alpha}{1-3\sin^2\alpha}$, then $\dfrac{\tan(\alpha-\beta)}{\tan\alpha}$ is equal to:

(a) 3 (b) 2
(c) – 2 (d) none of these

Ans. (c)

Solution: $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$$= \dfrac{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{3\sin\alpha\cos\alpha}{1-3\sin^2\alpha}}{1+\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{3\sin\alpha\cos\alpha}{1-3\sin^2\alpha}}$$

$= -2\tan\alpha$

Example 10: Let $f(\theta) = \tan\left(\dfrac{\pi}{3}+\theta\right)\tan\left(\dfrac{\pi}{3}-\theta\right)$, $-\dfrac{\pi}{3} < \theta < \dfrac{\pi}{3}$, then $f\left(\dfrac{\pi}{12}\right)$ is equal to:

(a) $2+\sqrt{3}$ (b) $2-\sqrt{3}$
(c) $\sqrt{3}+1$ (d) $\sqrt{3}-1$

Ans. (a)

Solution: $f(\theta) = \dfrac{\sin\left(\dfrac{\pi}{3}+\theta\right)\sin\left(\dfrac{\pi}{3}-\theta\right)}{\cos\left(\dfrac{\pi}{3}+\theta\right)\cos\left(\dfrac{\pi}{3}-\theta\right)}$

$$= \dfrac{\sin^2\left(\dfrac{\pi}{3}\right)-\sin^2\theta}{\cos^2\left(\dfrac{\pi}{3}\right)-\sin^2\theta}$$

$$= \dfrac{3-4\sin^2\theta}{1-4\sin^2\theta} = \dfrac{2\cos(2\theta)+1}{2\cos(2\theta)-1}$$

$$\therefore \quad f\left(\dfrac{\pi}{12}\right) = \dfrac{2\cos\left(\dfrac{\pi}{6}\right)+1}{2\cos\left(\dfrac{\pi}{6}\right)-1} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$$

$= 2+\sqrt{3}$

Example 11: If $\operatorname{cosec}\theta = \dfrac{p+q}{p-q}$ where $p > q > 0$, then $\left|\cot\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)\right|$ is equal to:

(a) $\sqrt{\dfrac{p}{q}}$ (b) $\sqrt{\dfrac{q}{p}}$
(c) $\sqrt{pq}$ (d) pq

Ans. (a)

Solution: $\sin\theta = \dfrac{p-q}{p+q} \Rightarrow \cos\left(\dfrac{\pi}{2}+\theta\right) = \dfrac{p-q}{p+q}$

$$\Rightarrow \quad \dfrac{1-\tan^2\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)}{1+\tan^2\left(\dfrac{\theta}{4}+\dfrac{\theta}{2}\right)} = \dfrac{1-q/p}{1+q/p}$$

$$\Rightarrow \quad \tan^2\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right) = \dfrac{q}{p}$$

$$\Rightarrow \quad \left|\cot\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)\right| = \sqrt{\dfrac{p}{q}}$$

Example 12: If $2\cos\theta + \sin\theta = 1$, $(\theta \ne (4k+1)\pi/2$, $k \in \mathbf{I})$ then $7\cos\theta + 6\sin\theta$ is equal to:

(a) 1/2 (b) 2
(c) 11/2 (d) 46/5

Ans. (b)

Solution: $2\cos\theta + \sin\theta = 1 \Rightarrow 4\cos^2\theta = (1-\sin\theta)^2$

$\Rightarrow \quad 4(1-\sin^2\theta) = (1-\sin\theta)^2$

$\Rightarrow \quad 4(1+\sin\theta) = 1-\sin\theta$

$\Rightarrow \quad \sin\theta = -3/5$

As $2\cos\theta + \sin\theta = 1$, we get $\cos\theta = 4/5$.

Thus, $7\cos\theta + 6\sin\theta = 7\left(\dfrac{4}{5}\right)+6\left(\dfrac{-3}{5}\right) = 2$

Example 13: The number of solutions of the equation $\tan x + \sec x = 2\cos x$, $x \in [0, 2\pi]$ is

(a) 1 (b) 2
(c) 3 (d) 0

Ans.(b)

Solution: $\tan x + \sec x = 2\cos x$

$\Rightarrow \quad 1 + \sin x = 2\cos^2 x = 2(1-\sin^2 x)$

$\Rightarrow \quad (1+\sin x)(1-2+2\sin x) = 0$

$\Rightarrow \quad \sin x = -1$ or $\sin x = 1/2$

But $\sin x = -1 \Rightarrow \cos x = 0$. This is not possible.

Thus, $\sin x = 1/2 \Rightarrow x = \pi/6, 5\pi/6, \in [0, 2\pi]$

Example 14: Let $A = \sin^4 x + \cos^4 x$, $x \in \mathbf{R}$, then

(a) $\dfrac{1}{2} \le A \le 1$ (b) $\dfrac{1}{\sqrt{2}} \le A \le 1$
(c) $\dfrac{1}{4} \le A \le 1$ (d) $\dfrac{1}{2\sqrt{2}} \le A \le 1$

Ans. (a)

Solution: $A = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$= 1 - \dfrac{1}{2}\sin^2 2x$

Now, $\quad 0 \le \sin^2 2x \le 1$

$$\Rightarrow \quad \dfrac{1}{2} = 1-\dfrac{1}{2} \le 1-\dfrac{1}{2}\sin^2 2x = A \le 1$$

Example 15: If $0 < x < \pi$, and $\sin x + \cos x = 1/3$, then a possible value of $\tan x$ is

(a) $\frac{1}{8}\left(-9+\sqrt{17}\right)$ (b) $\frac{1}{8}\left(9-\sqrt{17}\right)$

(c) $-\frac{1}{5}\left(9+\sqrt{5}\right)$ (d) $\frac{1}{5}\left(9-\sqrt{5}\right)$

Ans. (a)

Solution: $3(\tan x + 1) = \sec x$

$\Rightarrow$ $9(1 + \tan^2 x + 2\tan x) = 1 + \tan^2 x$

$\Rightarrow$ $4\tan^2 x + 9\tan x + 4 = 0$

$\Rightarrow$ $\tan x = \frac{-9 \pm \sqrt{81-64}}{8} = \frac{-9 \pm \sqrt{17}}{8}$

Example 16: Let $f(\theta) = (1 + \cos\theta)(1 + \cos 3\theta)(1 + \cos 5\theta)(1 + \cos 7\theta)$ then $f\left(-\frac{\pi}{8}\right)$ is equal to:

(a) – 1/8 (b) 1/4

(c) – 1/4 (d) 1/8

Ans. (d)

Solution: As $f(-\theta) = f(\theta)$, we evaluate $f(\pi/8)$.

For $\theta = \pi/8$, $3\theta = \pi/2 - \pi/8$, $5\theta = \pi/2 + \pi/8$ and $7\theta = \pi - \pi/8$, so that

$$f\left(\frac{\pi}{8}\right) = \left(1+\cos\frac{\pi}{8}\right)\left(1+\sin\frac{\pi}{8}\right)\left(1-\sin\frac{\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$$

$$= \cos^2\left(\frac{\pi}{8}\right)\sin^2\left(\frac{\pi}{8}\right)$$

$$= \frac{1}{4}\left[2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right]^2$$

$$= \frac{1}{4}\sin^2\left(\frac{\pi}{4}\right) = \frac{1}{8}$$

Example 17: If $\theta = \pi/8$, and $f(\theta) = \cos^6\theta + \cos^6 3\theta + \cos^6 5\theta + \cos^6 7\theta$ then $f(\theta)$ is equal to:

(a) 3/4 (b) 1

(c) 5/4 (d) 3/2

Ans. (c)

Solution: As in example 16, using

$3\theta = \pi/2 - \pi/8$, $5\theta = \pi/2 + \pi/8$, $7\theta = \pi - \pi/8$, we get

$$f(\theta) = 2(\cos^6\theta + \sin^6\theta)$$

$$= 2[(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)]$$

$$= 2\left[1-\frac{3}{4}(2\cos\theta\sin\theta)^2\right] = 2\left[1-\frac{3}{4}\left(\frac{1}{\sqrt{2}}\right)^2\right]$$

$$= 5/4$$

Example 18: If α, β, γ are three real numbers, then

$\cos^2(\beta-\gamma) + \cos^2(\gamma-\alpha) + \cos^2(\alpha-\beta) - 2\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)$ is equal to

(a) 0 (b) 1

(c) –1 (d) none of these

Ans. (b)

Solution: Put $A = \beta - \gamma$, $B = \gamma - \alpha$, $C = \alpha - \beta$, so that $A + B + C = 0$. Now

$\cos^2 A + \cos^2 B + \cos^2 C - 2\cos A\cos B\cos C$

$= 1 + \cos^2 A - \sin^2 B + \cos C[\cos C - 2\cos A\cos B]$

$= 1 + \cos(A+B)\cos(A-B) + \cos C[\cos(-A-B) - 2\cos A\cos B]$

$= 1 + \cos(-C)\cos(A-B) - \cos C[\sin A\sin B + \cos A\cos B]$

$= 1 + \cos C\cos(A-B) - \cos C\cos(A-B)$

$= 1$

Example 19: Number of values of $x \in [0, 2\pi]$ and satisfying the equation

$\cos x\cos 2x\cos 3x = 1/4$ is:

(a) 6 (b) 8

(c) 20 (d) 24

Ans. (b)

Solution: $2(2\cos 3x\cos x)\cos 2x = 1$

$\Rightarrow$ $2(\cos 4x + \cos 2x)\cos 2x = 1$

$\Rightarrow$ $2\cos 4x\cos 2x = 1 - 2\cos^2 2x = -\cos 4x$

$\Rightarrow$ $\cos 4x(2\cos 2x + 1) = 0$

$\Rightarrow$ $\cos 4x = 0$ or $\cos 2x = -1/2$

Now $\cos 4x = 0 \Rightarrow 4x = (2n+1)\pi/2$, $n \in \mathbf{I}$

$\Rightarrow$ $x = (2n+1)\pi/8$, $n \in \mathbf{I}$.

and $0 \le (2n+1)\pi/4 \le 2\pi \Rightarrow 0 \le 2n+1 \le 8$

$\Rightarrow$ $n = 0, 1, 2, 3$.

Also, $\cos(2x) = -1/2 = \cos(2\pi/3)$

$\Rightarrow$ $2x = 2n\pi \pm 2\pi/3$, $n \in \mathbf{I}$

$\Rightarrow$ $x = n\pi \pm \pi/3$, $n \in \mathbf{I}$.

As $0 \le x \le 2\pi$, $x = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$.

Thus, there are 8 values of x.

Example 20: Number of values of $x \in [0, 4\pi]$ and satisfying $\sqrt{2}\sec x + \tan x = 1$ is

(a) 0 (b) 1

(c) 2 (d) 4

Ans. (c)

Solution: We can write the given equation as

$\sqrt{2} = \cos x - \sin x$

$\Rightarrow$ $\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = 1$

$\Rightarrow$ $\cos\left(x+\frac{\pi}{4}\right) = 1$

$\Rightarrow$ $x + \frac{\pi}{4} = 2n\pi$, $n \in \mathbf{I}$.

$\Rightarrow$ $x = \frac{7\pi}{4}, \frac{15\pi}{4}$

LEVEL 1

Straight Objective Type Questions

Example 21: $\cos(540° - \theta) - \sin(630° - \theta)$ is equal to

(a) 0 (b) $2\cos\theta$
(c) $2\sin\theta$ (d) $\sin\theta - \cos\theta$

Ans. (a)

Solution: $\cos(540° - \theta) = \cos(6(\pi/2) - \theta) = -\cos\theta$

$\sin(630° - \theta) = \sin(7(\pi/2) - \theta) = -\cos\theta$

Example 22: $2\sec^2\alpha - \sec^4\alpha - 2\operatorname{cosec}^2\alpha + \operatorname{cosec}^4\alpha = 15/4$ if $\tan\alpha$ is equal to

(a) $1/\sqrt{2}$ (b) 1/2
(c) $1/2\sqrt{2}$ (d) 1/4

Ans. (a)

Solution: L.H.S. $= 2(1 + \tan^2\alpha - 1 - \cot^2\alpha) - [(1 + \tan^2\alpha)^2 - (1 + \cot^2\alpha)^2]$

$= \cot^4\alpha - \tan^4\alpha = 15/4$

$\Rightarrow \quad \tan^4\alpha = 1/4 \Rightarrow \tan\alpha = \pm 1/\sqrt{2}$.

Example 23: If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15°, then the values of $2 + q - p$ is

(a) 1 (b) 2
(c) 3 (d) 0

Ans. (c)

Solution: $\tan 30° + \tan 15° = -p$

and $\tan 30° \tan 15° = q$

Now $\quad 1 = \tan 45° = \dfrac{\tan 30° + \tan 15°}{1 - \tan 30° \tan 15°}$

$\Rightarrow \quad 1 = \dfrac{-p}{1-q} \Rightarrow 1 - q + p = 0$

$\Rightarrow \quad 2 + q - p = 2 + 1 = 3.$

Example 24: If $\tan 25° = x$, then $\dfrac{\tan 155° - \tan 115°}{1 + \tan 155° \tan 115°}$ is equal to

(a) $\dfrac{1-x^2}{2x}$ (b) $\dfrac{1+x^2}{2x}$

(c) $\dfrac{1+x^2}{1-x^2}$ (d) $\dfrac{1-x^2}{1+x^2}$

Ans. (a)

Solution: $\dfrac{\tan 155° - \tan 115°}{1 + \tan 155° \tan 115°}$

$$= \frac{\tan(180° - 25°) - \tan(90° + 25°)}{1 + \tan(180° - 25°)\tan(90° + 25°)}$$

$$= \frac{-\tan 25° + \cot 25°}{1 + \tan 25° \cot 25°} = \frac{1}{2}\left(-x + \frac{1}{x}\right) = \frac{1-x^2}{2x}.$$

Example 25: The value of the determinant

$$\begin{vmatrix} \sin^2 13° & \sin^2 77° & \tan 135° \\ \sin^2 77° & \tan 135° & \sin^2 13° \\ \tan 135° & \sin^2 13° & \sin^2 77° \end{vmatrix}$$ is equal to

(a) – 1 (b) 0
(c) 1 (d) 2

Ans. (b)

Solution: The given determinant is equal to

$$\begin{vmatrix} \sin^2 13° & \cos^2 13° & -1 \\ \cos^2 13° & -1 & \sin^2 13° \\ -1 & \sin^2 13° & \cos^2 13° \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \cos^2 13° & -1 \\ 0 & -1 & \sin^2 13° \\ 0 & \sin^2 13° & \cos^2 13° \end{vmatrix} = 0$$

(Applying $C_1 \to C_1 + C_2 + C_3$)

Example 26: If $A = 130°$ and $x = \sin A + \cos A$, then

(a) $x > 0$ (b) $x < 0$
(c) $x = 0$ (d) $x \geq 0$

Ans. (a)

Solution: $x = \sin 130° + \cos 130° = \sin 50° - \sin 40° > 0$

($\because$ $\sin x$ is increasing for $0 < x < \pi/2$)

Example 27: If $\tan^2 36° + k(\sin 18° + \cos 36°) = 5$, then the value of k is

(a) 2 (b) $2\sqrt{5}$
(c) 4 (d) $4\sqrt{5}$

Ans. (c)

Solution: From the tables, we have

$$\text{L.H.S.} = 5 - 2\sqrt{5} + k\left(\frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}+1}{4}\right)$$

$$= 5 - 2\sqrt{5} + \frac{\sqrt{5}}{2}k = 5 \text{ (given)} \quad \Rightarrow \quad k = 4$$

Example 28: If $\frac{\sin x}{a} = \frac{\cos x}{b} = \frac{\tan x}{c} = k$, then $bc + \frac{1}{ck} + \frac{ak}{1+bk}$ is equal to

(a) $k\left(a+\frac{1}{a}\right)$ (b) $\frac{1}{k}\left(a+\frac{1}{a}\right)$

(c) $\frac{1}{k^2}$ (d) $\frac{a}{k}$

Ans. (b)

Solution: The given expression is equal to

$$\frac{\cos x \cdot \tan x}{k^2} + \frac{1}{\tan x} + \frac{\sin x}{1+\cos x}$$

$$= \frac{\sin x}{k^2} + \frac{\cos x(1+\cos x)+\sin^2 x}{\sin x(1+\cos x)}$$

$$= \frac{a}{k} + \frac{1}{\sin x} = \frac{a}{k} + \frac{1}{ak} = \frac{1}{k}\left(a+\frac{1}{a}\right).$$

Example 29: $\sin^2 \alpha + \cos^2 (\alpha + \beta) + 2 \sin \alpha \sin \beta \cos (\alpha + \beta)$ is independent of

(a) α (b) β

(c) both α and β (d) none

Ans. (a)

Solution: The given expression is equal to

$\sin^2 \alpha + \cos (\alpha + \beta) [\cos (\alpha + \beta) + 2 \sin \alpha \sin \beta]$

$= \sin^2 \alpha + \cos (\alpha + \beta) [\cos \alpha \cos \beta + \sin \alpha \sin \beta]$

$= \sin^2 \alpha + \cos (\alpha + \beta) \cos (\alpha - \beta)$

$= \sin^2 \alpha + \cos^2 \alpha - \sin^2 \beta = 1 - \sin^2 \beta = \cos^2 \beta$

which is independent of α only.

Example 30: $\tan 203° + \tan 22° + \tan 203° \tan 22° =$

(a) – 1 (b) 0

(c) 1 (d) 2

Ans. (c)

Solution: $\tan (203° + 22°) = \frac{\tan 203° + \tan 22°}{1 - \tan 203° \tan 22°}$

$\Rightarrow$ $1 = \tan (180° + 45°) = \frac{\tan 203° + \tan 22°}{1 - \tan 203° \tan 22°}$

$\Rightarrow$ $\tan 203° + \tan 22° + \tan 203° \tan 22° = 1$.

Example 31: If $\sin 32° = k$ and $\cos x = 1 - 2k^2$; α, β are the values of x between 0° and 360° with $\alpha < \beta$, then

(a) $\alpha + \beta = 180°$ (b) $\beta - \alpha = 200°$

(c) $\beta = 4\alpha + 40°$ (d) $\beta = 5\alpha - 20°$

Ans. (c)

Solution: $\cos x = 1 - 2k^2 = 1 - 2 \sin^2 32° = \cos 64°$

$\Rightarrow$ $x = 64°$ or $296°$

$\therefore$ $\alpha = 64°$ and $\beta = 296°$

which satisfy (c).

Example 32: The minimum value of $27 \tan^2 \theta + 3 \cot^2 \theta$ is

(a) 9 (b) 18

(c) 27 (d) 30

Ans. (b)

Solution: A.M. ≥ G.M.

$$\Rightarrow \quad \frac{27\tan^2 \theta + 3\cot^2 \theta}{2} \geq \sqrt{27\tan^2 \theta \cdot 3\cot^2 \theta}$$

$\Rightarrow$ $27 \tan^2 \theta + 3 \cot^2 \theta \geq 18$.

Example 33: If $3 \sin \beta = \sin (2\alpha + \beta)$, then $\tan (\alpha + \beta) - 2 \tan \alpha$ is

(a) independent of α

(b) independent of β

(c) independent of both α and β

(d) independent of none of them

Ans. (c)

Solution: $\sin (2\alpha + \beta) = 3 \sin \beta$

$$\Rightarrow \quad \frac{\sin(2\alpha+\beta)+\sin\beta}{\sin(2\alpha+\beta)-\sin\beta} = \frac{3+1}{3-1}$$

$$\Rightarrow \quad \frac{2\sin(\alpha+\beta)\cos\alpha}{2\cos(\alpha+\beta)\sin\alpha} = 2 \Rightarrow \tan(\alpha+\beta) - 2\tan\alpha = 0$$

Example 34: If $A = \sin^2 \theta + \cos^4 \theta$, then for all values of θ

(a) $1 \leq A \leq 2$ (b) $3/4 \leq A \leq 1$

(c) $13/16 \leq A \leq 1$ (d) $3/4 \leq A \leq 13/16$

Ans. (b)

Solution: $A = \sin^2 \theta + (1 - \sin^2 \theta)^2$

$= 1 + \sin^2 \theta\ (\sin^2 \theta - 1)$

$= 1 - \sin^2 \theta \cos^2 \theta \leq 1$

Also $A = 1 - (1/4) \sin^2 2\theta \geq 1 - (1/4) = (3/4)$. Hence $3/4 \leq A \leq 1$

Example 35: If $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ where $a, b > 0$. Then the difference between the maximum and minimum values of u^2 is given by

(a) $(a + b)^2$ (b) $2\sqrt{a^2 + b^2}$

(c) $2(a^2 + b^2)$ (d) $(a - b)^2$

Ans. (d)

Solution: We have

$$u = \sqrt{\frac{a^2}{2}(1+\cos 2\theta) + \frac{b^2}{2}(1-\cos 2\theta)}$$

$$+ \sqrt{\frac{a^2}{2}(1-\cos 2\theta) + \frac{b^2}{2}(1+\cos 2\theta)}$$

$$= \sqrt{\frac{a^2+b^2}{2} + \frac{a^2-b^2}{2}\cos 2\theta}\ .$$

$$+\sqrt{\frac{a^2+b^2}{2}-\frac{a^2-b^2}{2}\cos 2\theta}$$

Squaring we get

$$u^2 = a^2+b^2+2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2-\left(\frac{a^2-b^2}{2}\right)^2\cos^2 2\theta}$$

Thus max $(u^2) = a^2+b^2+a^2+b^2 = 2(a^2+b^2)$

and min $(u^2) = a^2+b^2+2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2-\left(\frac{a^2-b^2}{2}\right)^2}$

$$= a^2+b^2+2ab = (a+b)^2$$

So the required difference $= 2(a^2+b^2)-(a+b)^2 = (a-b)^2$.

Example 36: $\begin{vmatrix} \sin^2 x & \cos^2 x & 1 \\ \cos^2 x & \sin^2 x & 1 \\ -10 & 12 & 2 \end{vmatrix}$ is equal to

(a) 0
(b) $12\cos^2 x - 10\sin^2 x$
(c) $12\sin^2 x - 10\cos^2 x - 2$
(d) $10\sin^2 x$

Ans. (a)

Solution: Apply $C_1 \to C_1 + C_2 - C_3$, then the given determinant is equal to

$$\begin{vmatrix} 0 & \cos^2 x & 1 \\ 0 & \sin^2 x & 1 \\ 0 & 12 & 2 \end{vmatrix} = 0.$$

Example 37: If $\frac{\cos A}{3} = \frac{\cos B}{4} = \frac{1}{5}, -\pi/2 < A, B < 0$, then value of $2\sin A + 4\sin B$ is equal to

(a) 4 (b) 2
(c) – 4 (d) 0

Ans. (c)

Solution: $\cos A = 3/5 \Rightarrow \sin A = -(4/5)$

$\cos B = (4/5) \Rightarrow \sin B = -3/5.$

So that $2\sin A + 4\sin B$

$= -(8/5) - (12/5) = -4.$

Example 38: If $\sin A = 3\sin(A+2B)$, angle B is acute and A is obtuse; then

(a) $\tan B = 1/\sqrt{2}$ (b) $\tan B > 1/\sqrt{2}$
(c) $\tan B < 1/\sqrt{2}$ (d) $0 < \tan B < 1/\sqrt{2}$

Ans. (d)

Solution: $\frac{\sin(A+2B)}{\sin A} = \frac{1}{3}$,

By componendo and dividendo, $\tan(A+B) = -2\tan B$

$\Rightarrow \quad \frac{\tan A + \tan B}{1-\tan A \tan B} = -2\tan B$

$\Rightarrow \quad \tan A = \frac{3\tan B}{2\tan^2 B - 1} < 0$ (as A is obtuse)

$\Rightarrow \quad 2\tan^2 B - 1 < 0$ (as $\tan B > 0$)

$\Rightarrow \quad 0 < \tan B < 1/\sqrt{2}$

Example 39: The value of $\cos^2\left(\frac{3\pi}{5}\right)+\cos^2\left(\frac{4\pi}{5}\right)$ is equal to

(a) 3/4 (b) 5/4
(c) 5/2 (d) 4/5

Ans. (a)

Solution: $\cos^2\left(\frac{3\pi}{5}\right)+\cos^2\left(\frac{4\pi}{5}\right)$

$= \cos^2(108°) + \cos^2(144°)$
$= (\cos(90° + 18°))^2 + (\cos(180° - 36°))^2$
$= \sin^2 18° + \cos^2 36°$
$= \left(\frac{\sqrt{5}-1}{4}\right)^2 + \left(\frac{\sqrt{5}+1}{4}\right)^2 = \frac{2\times 6}{16} = \frac{3}{4}.$

Example 40: The value of
$\cos 12° + \cos 84° + \cos 156° + \cos 132°$ is

(a) 1/8 (b) – 1/2
(c) 1 (d) 1/2

Ans. (b)

Solution: $(\cos 12° + \cos 132°) + (\cos 84° + \cos 156°)$

$= 2\cos\frac{12°+132°}{2}\cos\frac{12°-132°}{2} + 2\cos\frac{84°+156°}{2}\cos\frac{84°-156°}{2}.$

$= 2\cos 72°\cos 60° + 2\cos 120°\cos 36°$

$= 2\times\frac{\sqrt{5}-1}{4}\times\frac{1}{2} + 2\times\left(-\frac{1}{2}\right)\times\frac{\sqrt{5}+1}{4} = -\frac{1}{2}.$

Example 41: Let $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$, then Δ lies in the interval.

(a) [2, 4] (b) [3, 4]
(c) [1, 4] (d) none of these

Ans. (a)

Solution: $\Delta = \begin{vmatrix} 0 & 0 & 2 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$ $[R_1 \to R_1 + R_3]$

$= 2(1 + \sin^2\theta)$

$\Rightarrow \quad 2 \le \Delta \le 4$ as $0 \le \sin^2\theta \le 1$

Example 42: If $\sin(120° - \alpha) = \sin(120° - \beta)$ and $0 < \alpha, \beta < \pi$, then all values of α, β are given by

(a) $\alpha + \beta = \pi/3$ (b) $\alpha = \beta$ or $\alpha + \beta = \pi/3$
(c) $\alpha = \beta$ (d) $\alpha + \beta = 0$

Ans. (b)

◎ **Solution:** $\sin(120° - \alpha) = \sin(120° - \beta)$

$\Rightarrow$ either $120° - \alpha = 120° - \beta \Rightarrow \alpha = \beta$

or $120° - \alpha = 180° - (120° - \beta)$

$\Rightarrow$ $\alpha + \beta = 60° = \pi/3$

◉ **Example 43:** If $\sin\theta + \text{cosec}\,\theta = 2$, then $\sin^n\theta + \text{cosec}^n\theta$ is equal to

(a) 2 (b) 2^n
(c) 4^n (d) none of these

Ans. (a)

◎ **Solution:** We can write $\sin^2\theta + 1 = 2\sin\theta$

$\Rightarrow$ $\sin^2\theta - 2\sin\theta + 1 = 0$ $\Rightarrow$ $(\sin\theta - 1)^2 = 0$

$\Rightarrow$ $\sin\theta = 1$ $\Rightarrow$ $\text{cosec}\,\theta = 1$

and thus $\sin^n\theta + \text{cosec}^n\theta = 2$.

◉ **Example 44:** If $a = \cos\phi\cos\psi + \sin\phi\sin\psi\cos\delta$

$b = \cos\phi\sin\psi - \sin\phi\cos\psi\cos\delta$

and $c = \sin\phi\sin\delta$. Then $a^2 + b^2 + c^2 =$

(a) -1 (b) 0
(c) 1 (d) none of these

Ans. (c)

◎ **Solution:** $a^2 + b^2 + c^2$

$$= \cos^2\phi\cos^2\psi + \sin^2\phi\sin^2\psi\cos^2\delta + \cos^2\phi\sin^2\psi + \sin^2\phi\cos^2\psi\cos^2\delta + \sin^2\phi\sin^2\delta$$

$$= \cos^2\phi + \sin^2\phi\cos^2\delta + \sin^2\phi\sin^2\delta$$

$$= \cos^2\phi + \sin^2\phi = 1.$$

◉ **Example 45:** In a triangle ABC, BP is drawn perpendicular to BC to meet CA in P, such that $CA = AP$, then $\dfrac{BP}{AB} =$

(a) $2\sin A$ (b) $2\sin B$
(c) $2\sin C$ (d) none of these

Ans. (c)

◎ **Solution:** We have $\dfrac{BP}{AB} = \dfrac{BP}{AC}$

(AB, AC are the radii of the circle on CP as diametre)

$$= 2\frac{BP}{CP} = 2\sin C.$$

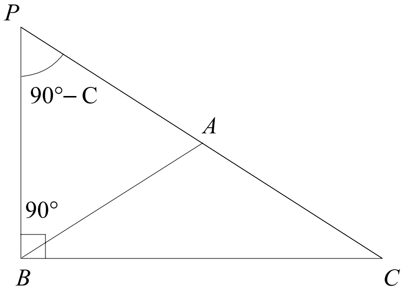

Fig. 25.3

◉ **Example 46:** If $\cos\theta = \cos\alpha\cos\beta$, then $\tan\dfrac{\theta+\alpha}{2}\tan\dfrac{\theta-\alpha}{2}$ is equal to

(a) $\tan^2(\alpha/2)$ (b) $\tan^2(\beta/2)$
(c) $\tan^2(\theta/2)$ (d) $\cot^2(\beta/2)$

Ans. (b)

◎ **Solution:** $\tan\dfrac{\theta+\alpha}{2}\tan\dfrac{\theta-\alpha}{2}$

$$= \frac{\tan^2(\theta/2) - \tan^2(\alpha/2)}{1 - \tan^2(\theta/2)\tan^2(\alpha/2)}$$

$$= \frac{\dfrac{1-\cos\theta}{1+\cos\theta} - \dfrac{1-\cos\alpha}{1+\cos\alpha}}{1 - \dfrac{1-\cos\theta}{1+\cos\theta}\cdot\dfrac{1-\cos\alpha}{1+\cos\alpha}}$$

$$= \frac{2(\cos\alpha - \cos\theta)}{2(\cos\alpha + \cos\theta)} = \frac{\cos\alpha(1-\cos\beta)}{\cos\alpha(1+\cos\beta)} = \tan^2\frac{\beta}{2}.$$

◉ **Example 47:** The equation $a\sin x + b\cos x = c$, where $|c| > \sqrt{a^2+b^2}$ has

(a) a unique solution
(b) infinite number of solutions
(c) no solution
(d) none of these

Ans. (c)

◎ **Solution:** Let $a = r\cos\alpha$, $b = r\sin\alpha$ so that $r = \sqrt{a^2+b^2}$.

The given equation can be written as

$$r\sin(x+\alpha) = c \Rightarrow \sin(x+\alpha) = \frac{c}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad |\sin(x+\alpha)| = \frac{|c|}{\sqrt{a^2+b^2}} > 1 \text{ as } |c| > \sqrt{a^2+b^2}$$

which is not possible for any value of x.

◉ **Example 48:** If $\cot\alpha$ equals the integral solution of the inequality $4x^2 - 16x + 15 < 0$ and $\sin\beta$ equals to the slope of the bisector of the first quadrant, then $\sin(\alpha+\beta)\sin(\alpha-\beta)$ is equal to

(a) $-3/5$ (b) $-4/5$
(c) $2/\sqrt{5}$ (d) 3

Ans. (b)

◎ **Solution:** We have $4x^2 - 16x + 15 < 0$

$\Rightarrow$ $3/2 < x < 5/2$

$\Rightarrow \cot\alpha = 2$, the integral solution of the given inequality and $\sin\beta = \tan 45° = 1$

$\therefore$ $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$

$$= \frac{1}{1+\cot^2\alpha} - 1 = \frac{1}{1+4} - 1 = -\frac{4}{5}.$$

Example 49: The value of $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{7\pi}{7}$ is

(a) 1 (b) – 1
(c) 1/2 (d) – 3/2

Ans. (d)

Solution: $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{7\pi}{7}$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[2\sin\frac{\pi}{7}\cos\frac{2\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{4\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{6\pi}{7}\right] - 1$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[\sin\frac{3\pi}{7} - \sin\frac{\pi}{7} + \sin\frac{5\pi}{7} - \sin\frac{3\pi}{7} + \sin\frac{7\pi}{7} - \sin\frac{5\pi}{7}\right] - 1$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left(-\sin\frac{\pi}{7}\right) - 1 = -\frac{1}{2} - 1 = -\frac{3}{2}.$$

Example 50: The greatest value of $f(x) = 2\sin x + \sin 2x$ on $[0, 3\pi/2]$, is given by

(a) 9/2 (b) 5/2
(c) $3\sqrt{3}/2$ (d) 3/2

Ans. (c)

Solution: $f'(x) = 2\cos x + 2\cos 2x$
and $f''(x) = -2\sin x - 4\sin 2x$
For extreme value $f'(x) = 0$

$\Rightarrow$ $\cos x + 2\cos^2 x - 1 = 0$
$\Rightarrow$ $\cos x = -1$ or $1/2$
$\Rightarrow$ $x = \pi$ or $\pi/3$ as $x \in [0, 3\pi/2]$

Now $f(\pi) = 0$ and $f(\pi/3) = \frac{2\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

Also $f(0) = 0$ and $f(3\pi/2) = -2$ so the greatest value of $f(x)$ is $3\sqrt{3}/2$

Example 51: If $x = a\cos^3\theta\sin^2\theta$, $y = a\sin^3\theta\cos^2\theta$ and $\frac{(x^2+y^2)^p}{(xy)^q}$ ($p, q, \in \mathbf{N}$) is independent of θ, then

(a) $4p = 5q$ (b) $4q = 5p$
(c) $p + q = 9$ (d) $pq = 20$

Ans. (a)

Solution: $$\frac{(x^2+y^2)^p}{(xy)^q} = \frac{\left[a\sin^2\theta\cos^2\theta\right]^{2p}}{\left[a^2\sin^5\theta\cos^5\theta\right]^q}$$

$$= \frac{a^{2p}(\sin\theta\cos\theta)^{4p}}{a^{2q}(\sin\theta\cos\theta)^{5q}}$$

which is independent of θ if $4p = 5q$.

Example 52: If $\cos\alpha + \cos\beta = a$, $\sin\alpha + \sin\beta = b$ and $\alpha - \beta = 2\theta$, then $\frac{\cos 3\theta}{\cos\theta} =$

(a) $a^2 + b^2 - 2$ (b) $a^2 + b^2 - 3$
(c) $3 - a^2 - b^2$ (d) $(a^2 + b^2)/4$

Ans. (b)

Solution: From the given relations we have

$$a^2 + b^2 = \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta$$
$$= 2 + 2\cos(\alpha - \beta)$$
$$= 2 + 2\cos 2\theta = 4\cos^2\theta$$

Now $$\frac{\cos 3\theta}{\cos\theta} = \frac{4\cos^3\theta - 3\cos\theta}{\cos\theta}$$
$$= 4\cos^2\theta - 3 = a^2 + b^2 - 3.$$

Example 53: If $\cos A = 3/4$, then the value of $16\cos^2(A/2) - 32\sin(A/2)\sin(5A/2)$ is

(a) – 4 (b) – 3
(c) 3 (d) 4

Ans. (c)

Solution: The given expression is equal to

$$8(1 + \cos A) - 16(\cos 2A - \cos 3A)$$
$$= 8(1 + \cos A) - 16[2\cos^2 A - 1 - \cos A(4\cos^2 A - 3)]$$
$$= 8\left(1 + \frac{3}{4}\right) - 16\left[2\times\frac{9}{16} - 1 - \frac{3}{4}\left(4\times\frac{9}{16} - 3\right)\right]$$
$$= 14 - (18 - 16 - 27 + 36) = 3.$$

Example 54: If $D = \begin{vmatrix} 1 & \cos\theta & 1 \\ -\sin\theta & 1 & -\cos\theta \\ -1 & \sin\theta & 1 \end{vmatrix}$ then D lies in the interval

(a) [0, 4] (b) [2, 4]
(c) $[2-\sqrt{2}, 2+\sqrt{2}]$ (d) [–2, 2]

Ans. (c)

Solution: Expanding D, we get

$$D = 1 + \sin\theta\cos\theta - \cos\theta(-\sin\theta - \cos\theta) + (-\sin^2\theta + 1)$$
$$= 2 + \sin 2\theta + \cos 2\theta = 2 + \sqrt{2}\cos(2\theta - \pi/4)$$

As $-1 \le \cos(2\theta - \pi/4) \le 1$, $2 - \sqrt{2} \le D \le 2 + \sqrt{2}$

Example 55: A value of θ lying between $\theta = 0$ and $\theta = \pi/2$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0 \text{ is}$$

(a) $3\pi/24$ (b) $5\pi/24$
(c) $11\pi/24$ (d) $\pi/24$

Ans. (c)

Solution: Applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$ to the given determinant we get

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

$\Rightarrow$ $1 + 4\sin 4\theta + \cos^2\theta + \sin^2\theta = 0$

$\Rightarrow$ $\sin 4\theta = -1/2 \Rightarrow 4\theta = \pi + \pi/6$ or $2\pi - \pi/6$ $[\because 0 < 4\theta < 2\pi]$

$\Rightarrow$ $\theta = 7\pi/24$ or $11\pi/24$.

Example 56: Let $\theta \in (\pi/4, \pi/2)$, which of the following statements is true?

(a) $(\cos\theta)^{\cos\theta} < (\sin\theta)^{\cos\theta} < (\cos\theta)^{\sin\theta}$
(b) $(\cos\theta)^{\sin\theta} < (\cos\theta)^{\cos\theta} < (\sin\theta)^{\cos\theta}$
(c) $(\sin\theta)^{\cos\theta} < (\cos\theta)^{\cos\theta} < (\cos\theta)^{\sin\theta}$
(d) $(\cos\theta)^{\cos\theta} < (\cos\theta)^{\sin\theta} < (\sin\theta)^{\cos\theta}$

Ans. (b)

Solution: For $\theta \in (\pi/4, \pi/2)$, $0 < \cos\theta < 1/\sqrt{2} < \sin\theta < 1$

$\Rightarrow$ $(\cos\theta)^{\cos\theta} < (\sin\theta)^{\cos\theta}$ and $(\cos\theta)^{\cos\theta} > (\cos\theta)^{\sin\theta}$
(as $0 < \cos\theta < 1$ and $\cos\theta < \sin\theta$)

Showing that (b) is correct.

Example 57: If $a\cos A - b\sin A = c$, then $a\sin A + b\cos A$ is equal to

(a) $\pm\sqrt{a^2+b^2-c^2}$ (b) $\pm\sqrt{b^2+c^2-a^2}$
(c) $\pm\sqrt{c^2+a^2-b^2}$ (d) $\pm\sqrt{a^2+b^2+c^2}$

Ans. (a)

Solution: Put $a\sin A + b\cos A = x$, then

$$c^2 + x^2 = (a\cos A - b\sin A)^2 + (a\sin A + b\cos A)^2 = a^2 + b^2$$

$\Rightarrow$ $x = \pm\sqrt{a^2+b^2-c^2}$

Example 58: The general solution of the trigonometrical equation $\sin x + \cos x = 1$ is

(a) $x = 2n\pi$, $n \in \mathbf{I}$
(b) $x = 2n\pi + \pi/2$, $n \in \mathbf{I}$
(c) $x = n\pi + (-1)^n \pi/4 - \pi/4$, $n \in \mathbf{I}$
(d) none of these

Ans. (c)

Solution: $\sin x + \cos x = 1 \Rightarrow \sin(x + \pi/4) = 1/\sqrt{2}$ $= \sin\pi/4$

$\Rightarrow$ $x + \pi/4 = n\pi + (-1)^n \pi/4$

$\Rightarrow$ $x = n\pi + (-1)^n \pi/4 - \pi/4$

which includes $x = 2n\pi$ and $x = 2n\pi + \pi/2$.

Example 59: The values of x between 0 and 2π which satisfy the equation $\sin x\sqrt{8\cos^2 x} = 1$ are in A.P. The common difference of the A.P. is

(a) $\pi/8$ (b) $\pi/4$
(c) $3\pi/8$ (d) $5\pi/8$

Ans. (b)

Solution: From the given equation we have $2\sin x\,|\cos x| = 1/\sqrt{2} \Rightarrow \sin 2x = 1/\sqrt{2}$ if $\cos x > 0$ and $\sin 2x = -1/2$ if $\cos x < 0$

$\therefore$ when $\cos x > 0$, $\sin 2x = 1/\sqrt{2} \Rightarrow x = \pi/8, 3\pi/8$

when $\cos x < 0$, $\sin 2x = -1/\sqrt{2} \Rightarrow x = 5\pi/8, 7\pi/8$. So the required values of x are

$$\pi/8, 3\pi/8, 5\pi/8, 7\pi/8$$

which form an A.P. with common difference $\pi/4$.

Example 60: $15[\tan 2\theta + \sin 2\theta] + 8 = 0$ if

(a) $\tan\theta = 1/2$ (b) $\sin\theta = 1/4$
(c) $\tan\theta = 2$ (d) $\cos\theta = 1/5$

Ans. (c)

Solution: We have $\tan 2\theta + \sin 2\theta = -8/15$

$$\Rightarrow \quad \frac{2\tan\theta}{1-\tan^2\theta} + \frac{2\tan\theta}{1+\tan^2\theta} = \frac{-8}{15}$$

$\Rightarrow$ $15 \times 4\tan\theta + 8(1 - \tan^4\theta) = 0$

$\Rightarrow$ $8\tan^4\theta + 60\tan\theta - 8 = 0$

which is satisfied if $\tan\theta = 2$

Example 61: The numbers of solutions of the pair of equations

$$2\sin^2\theta - \cos 2\theta = 0$$
$$2\cos^2\theta - 3\sin\theta = 0$$

in the interval $[0, 2\pi]$ is

(a) zero (b) one
(c) two (d) four

Ans. (c)

Solution: $2\sin^2\theta - \cos 2\theta = 0$

$\Rightarrow$ $1 - \cos 2\theta - \cos 2\theta = 0 \Rightarrow \cos 2\theta = 1/2$

$\Rightarrow \quad 2\cos^2\theta - 1 = 1/2 \Rightarrow 2\cos^2\theta = 3/2$

So that from $2\cos^2\theta - 3\sin\theta = 0$, we have

$\sin\theta = 1/2 \Rightarrow \theta = \pi/6, 5\pi/6$ as $\theta \in [0, 2\pi]$.

Example 62: The solution set of the equation

$$\tan(\pi\tan x) = \cot(\pi\cot x) \text{ is}$$

(a) $\{0\}$ (b) $\{\pi/4\}$
(c) ϕ (d) none of these

Ans. (c)

Solution: $\tan(\pi\tan x) = \tan(\pi/2 - \pi\cot x)$

$\Rightarrow \quad \pi\tan x = \pi/2 - \pi\cot x \Rightarrow \tan x + \cot x = 1/2$

$\Rightarrow \quad 2\tan^2 x - \tan x + 2 = 0$

$\Rightarrow \quad \tan x = \dfrac{1 \pm \sqrt{1-16}}{4}$

which does not give real values of $\tan x$.

Example 63: The least positive root of the function $\sin x - \pi/2 + 1 = 0$ lies in the interval

(a) $[0, \pi/2]$ (b) $[\pi/2, \pi]$
(c) $[\pi/2, 3\pi/2]$ (d) none of these

Ans. (a)

Solution: $\sin x = \dfrac{\pi}{2} - 1 = \dfrac{22}{14} - 1 = \dfrac{8}{14}$

$= \dfrac{4}{7}$ so the least positive root lies in $[0, \pi/2]$

Example 64: The number of solutions of the equation

$$\cos(\pi\sqrt{x-4})\cos(\pi\sqrt{x}) = 1 \text{ is}$$

(a) 1 (b) 2
(c) more than two (d) 0

Ans. (a)

Solution: Given equation is possible if $\cos(\pi\sqrt{x-4}) = 1$ and $\cos(\pi\sqrt{x}) = 1$.

Since $\quad x - 4 \geq 0 \Rightarrow x \geq 4$ and $x \geq 0$

So $\quad x = 4$ is the only solution.

Example 65: The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is

(a) 2 (b) 4
(c) 6 (d) 1

Ans. (b)

Solution: We have $(\sin x + 3)(2\sin x - 1) = 0$

$\Rightarrow \quad \sin x = -3$ which is not possible

or $\quad \sin x = 1/2 \Rightarrow x = \pi/6, 5\pi/6, 13\pi/6, 17\pi/6$ as $x \in [0, 3\pi]$.

Example 66: If A, B, C are the angles of a triangle and $\cos B + \cos C = 4\sin^2(A/2)$, then $\tan B/2 \tan C/2$ is equal to

(a) 1/2 (b) 1/3
(c) 2/3 (d) 3/2

Ans. (b)

Solution: $\cos B + \cos C = 4\sin^2(A/2)$

$\Rightarrow \quad 2\cos\dfrac{B+C}{2}\cos\dfrac{B-C}{2} = 4\sin\dfrac{A}{2}\sin\left(\dfrac{\pi}{2} - \dfrac{B+C}{2}\right)$

$\Rightarrow \quad 2\sin\dfrac{A}{2}\cos\dfrac{B-C}{2} = 4\sin\dfrac{A}{2}\cos\dfrac{B+C}{2}$

$\Rightarrow \quad \cos\dfrac{B}{2}\cos\dfrac{C}{2} + \sin\dfrac{B}{2}\sin\dfrac{C}{2} = 2\left(\cos\dfrac{B}{2}\cos\dfrac{C}{2} - \sin\dfrac{B}{2}\sin\dfrac{C}{2}\right)$

$\Rightarrow \quad 3\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \cos\dfrac{B}{2}\cos\dfrac{C}{2}$

$\Rightarrow \quad \tan\dfrac{B}{2}\tan\dfrac{C}{2} = \dfrac{1}{3}.$

Example 67: If $15\sin^4 x + 10\cos^4 x = 6$, Then $\tan^2 x =$

(a) 1/5 (b) 2/5
(c) 2/3 (d) 1/3

Ans. (c)

Solution: $15\sin^4 x + 10\cos^4 x = 6(\sin^2 x + \cos^2 x)^2$

$\Rightarrow \quad 9\sin^4 x + 4\cos^4 x - 12\sin^2 x\cos^2 x = 0$

$\Rightarrow \quad (3\sin^2 x - 2\cos^2 x)^2 = 0$

$\Rightarrow \quad \tan^2 x = 2/3.$

Example 68: Sum of the root of the equation $2\sin^2\theta + \sin^2 2\theta = 2;\ 0 \leq \theta \leq \pi/2$ is

(a) $\pi/2$ (b) $3\pi/4$
(c) $7\pi/2$ (d) $5\pi/12$

Ans. (b)

Solution: $4\sin^2\theta\cos^2\theta = 2(1 - \sin^2\theta)$

$\Rightarrow \quad (2\sin^2\theta - 1)\cos^2\theta = 0$

$\Rightarrow \quad \sin^2\theta = 1/2 \quad \text{or} \quad \cos^2\theta = 0$

$\Rightarrow \quad \theta = \pi/4 \quad \text{or} \quad \theta = \pi/2$

Example 69: If $\tan x/2 = \operatorname{cosec} x - \sin x$, then $\sec^2(x/2) =$

(a) $\sqrt{5} + 1$ (b) $\sqrt{5} - 1$
(c) $\sqrt{5} - 2$ (d) $\sqrt{5} + 2$.

Ans. (b)

Solution: $\tan(x/2) = \dfrac{1 + \tan^2(x/2)}{2\tan(x/2)} - \dfrac{2\tan(x/2)}{1 + \tan^2(x/2)}$

$\Rightarrow \quad 2\tan^2(x/2)\,(1 + \tan^2(x/2)) = [1 + \tan^2(x/2)]^2 - 4\tan^2(x/2)$

$\Rightarrow \quad 2\tan^4(x/2) + 2\tan^2(x/2) = 1 + \tan^4(x/2) - 2\tan^2(x/2)$

$\Rightarrow \quad \tan^4(x/2) + 4\tan^2(x/2) - 1 = 0$

$\Rightarrow \quad \tan^2(x/2) = \sqrt{5} - 2 \quad \Rightarrow \sec^2(x/2) = \sqrt{5} - 1$

Example 70: Let A and B denote the statements

$A : \cos\alpha + \cos\beta + \cos\gamma = 0$

$B : \sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -3/2$

Then

(a) both A and B are true
(b) both A and B are false
(c) A is true B is false
(d) A is false B is true.

Ans. (a)

Solution: $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -3/2$

$$\Rightarrow \quad 2[\cos\beta\cos\gamma + \cos\gamma\cos\alpha + \cos\alpha\cos\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha + \sin\alpha\sin\beta] + \left(\sin^2\alpha + \cos^2\alpha\right) + \left(\sin^2\beta + \cos^2\beta\right) + \left(\sin^2\gamma + \cos^2\gamma\right) = 0$$

$$\Rightarrow \quad (\sin\alpha + \sin\beta + \sin\gamma)^2 + (\cos\alpha + \cos\beta + \cos\gamma)^2 = 0$$

$$\Rightarrow \quad \cos\alpha + \cos\beta + \cos\gamma = 0$$

and $\quad \sin\alpha + \sin\beta + \sin\gamma = 0$

so both A and B are true.

Example 71: $\cos^2 u + \cos^2(u + x) - 2\cos u\cos x\cos(u + x) = 1/2$ if

(a) $x = \pi/4$ (b) $u = \pi/4$
(c) $x = \pi/2$ (d) $u = \pi/2$

Ans. (a)

Solution: L.H.S $= \cos^2 u + \cos^2(u + x) - [\cos(u + x) + \cos(u - x)]\cos(u + x)$

$$= \cos^2 u - \cos(u - x)\cos(u + x)$$

$$= \cos^2 u - (\cos^2 u - \sin^2 x) = \sin^2 x.$$

so $\quad \sin^2 x = 1/2 \quad \Rightarrow \quad \sin x = \pm 1/\sqrt{2}$

Example 72: For a regular polygon, let r, R be the radii of the inscribed and circumscribed circles. There is no regular polygon with

(a) $\frac{r}{R} = \frac{2}{3}$ (b) $\frac{r}{R} = \frac{\sqrt{3}}{2}$
(c) $\frac{r}{R} = \frac{1}{2}$ (d) $\frac{r}{R} = \frac{1}{\sqrt{2}}$.

Ans. (a)

Solution: We have $\frac{r}{R} = \cos\frac{\pi}{n}$

When $\frac{r}{R} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{n} = \cos\frac{\pi}{6}$. $\Rightarrow n = 6$

when $\frac{r}{R} = \frac{1}{2}$, $\cos\frac{\pi}{n} = \cos\frac{\pi}{3} \quad \Rightarrow \quad n = 3$

and when $\frac{r}{R} = \frac{1}{\sqrt{2}}$, $\cos\frac{\pi}{n} = \cos\frac{\pi}{4} \quad \Rightarrow \quad n = 4$

But when $\frac{r}{R} = \frac{2}{3}$, $\quad \cos\frac{\pi}{n} = \frac{2}{3}$

Which does not give a positive integral value of n.

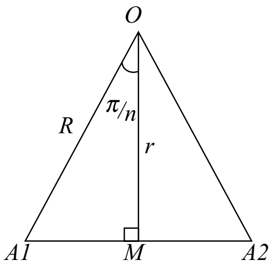

Fig. 25.4

Example 73: If $\cos(\alpha + \beta) = 4/5$ and $\sin(\alpha - \beta) = 5/13$ where $0 \le \alpha, \beta \le \pi/4$, then $\tan 2\alpha =$

(a) 19/12 (b) 20/7
(c) 25/16 (d) 56/33

Ans. (d)

Solution: $0 \le \alpha, \beta \le \pi/4$

$\Rightarrow \quad 0 \le \alpha + \beta \le \pi/2 \quad \Rightarrow \quad -\pi/4 \le \alpha - \beta \le \pi/4$

Now $\quad \cos(\alpha + \beta) = 4/5 \quad \Rightarrow \quad \tan(\alpha + \beta) = 3/4$

and $\quad \sin(\alpha - \beta) = 5/13 \quad \Rightarrow \quad \tan(\alpha - \beta) = 5/12$

we have $\quad \tan 2\alpha = \tan[(\alpha + \beta) + (\alpha - \beta)]$

$$= \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)}$$

$$= \frac{(3/4) + (5/12)}{1 - (3/4)(5/12)} = \frac{14/12}{33/48} = \frac{56}{33}.$$

Example 74: The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are

(a) $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{12}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$

(b) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{35\pi}{36}$

(c) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$

(d) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$

Ans. (d)

Solution: $\sin\theta + \sin 4\theta + \sin 7\theta = 0$

$\Rightarrow \quad 2\sin 4\theta\cos 3\theta + \sin 4\theta = 0$

$\Rightarrow \quad \sin 4\theta(2\cos 3\theta + 1) = 0$

$\Rightarrow \quad \sin 4\theta = 0 \quad$ or $\quad \cos 3\theta = -1/2 = \cos(2\pi/3)$

$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ or $3\theta = 2\pi/3, \quad 2\pi \pm 2\pi/3$

$\Rightarrow \quad \theta = \frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{4\pi}{9}, \frac{3\pi}{4}, \frac{8\pi}{9}.$

Example 75: In a ΔPQR, if $3 \sin P + 4\cos Q = 6$ and $4 \sin Q + 3 \cos P = 1$, Then the angle R is equal to

(a) $\pi/4$ (b) $3\pi/4$

(c) $5\pi/6$ (d) $\pi/6$.

Ans. (d)

Solution: Squaring and adding the given relations we get

$16 + 9 + 24 \sin (P + Q) = 37$

$\Rightarrow \quad \sin (P + Q) = 1/2 \Rightarrow \sin R = 1/2$

$\Rightarrow \quad R = \pi/6 \quad$ or $\quad 5\pi/6$

If R = $5\pi/6$, then $P < \pi/6 \quad \Rightarrow 3\sin P < 3/2$.

$\Rightarrow 3 \sin P + 4 \cos Q < 3/2 + 4 < 6$

So $R \neq 5\pi/6$

Assertion-Reason Type Questions

Example 76: Statement-1:
$1/2 \le \sin^2 \theta + 3 \sin \theta \cos \theta + 5\cos^2 \theta \le 11/2$

Statement-2: $-\sqrt{a^2 + b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2 + b^2}$.

Ans. (a)

Solution: Take $a = r \cos\alpha$ and $b = r \sin\alpha$

$\Rightarrow \quad r = \sqrt{a^2 + b^2}$ and $a \sin\theta + b \cos\theta = r \sin(\theta + \alpha)$

Now $\quad -r \le r \sin (\theta + \alpha) \le r.$

$\Rightarrow \quad -\sqrt{a^2 + b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2 + b^2}$

$\Rightarrow$ Statement-2 is true.

Now $\sin^2\theta + 3 \sin\theta \cos\theta + 5 \cos^2\theta$

$= 1 + (3/2)\sin 2\theta + 4\cos^2 \theta$

$= 1 + (3/2)\sin 2\theta + 2(1 + \cos 2\theta)$

$= 3 + (3/2)\sin 2\theta + 2 \cos 2\theta = A(\text{say})$

From statement-2

$3 - \sqrt{(9/4) + 4} \le A \le 3 + \sqrt{9/4 + 4}$

$\Rightarrow \quad 1/2 \le A \le 11/2$

and the statement-1 is also true.

Example 77: Statement-1: The number of values of θ in the interval $(-\pi/2, \pi/2)$ such that $\theta \neq n\pi/5$ for $n = 0, \pm 1, \pm 2$ and $\tan \theta = \cot 5\theta$ is 6.

Statement-2: No value of θ in statement-1 satisfies $\sin 2\theta = \cos 4\theta$.

Ans. (c)

Solution: $\tan \theta = \cot 5\theta$

$\Rightarrow \quad 2 \sin\theta \sin 5\theta = 2 \cos\theta \cos 5\theta$

$\Rightarrow \quad \cos 4\theta - \cos 6\theta = \cos 6\theta + \cos 4\theta$

$\Rightarrow \quad \cos 6\theta = 0$

$\Rightarrow \quad 6\theta = \pm\frac{5\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{\pi}{2}$

as $-3\pi < 6\theta < 3\pi$

Gives 6 values of θ, $\pm\frac{5\pi}{12}, \pm\frac{3\pi}{12}, \pm\frac{\pi}{12}$

out of which $\frac{5\pi}{12}, \frac{-\pi}{4}$ and $\frac{\pi}{12}$

satisfy $\sin 2\theta = \cos 4\theta$.

so statement -1: is true and statement-2 is false.

Example 78: Statement-1: $\sin 52° + \sin 78° + \sin 50° = 4 \cos 26° \cos 39° \cos 25°$

Statement-2: If $A + B + C = \pi$, then
$\sin A + \sin B + \sin C = 4 \cos (A/2) \cos (B/2) \cos (C/2)$

Ans. (a)

Solution: Statement-2 is True from conditional identities
$\Rightarrow$ statement-1, is also True.

Example 79: Statement-1: If A, B, C are the angles of a triangle such that angle A is obtuse, then $\tan B \tan C < 1$

Statement-2: In a triangle ABC

$$\tan A = \frac{\tan B + \tan C}{1 - \tan B \tan C}$$

Ans. (c)

Solution: Statement-2 is false because
$A = \tan (\pi - (B + C))$

$$\tan A = -\tan (B + C) = \frac{\tan B + \tan C}{\tan B \tan C - 1}.$$

In statement-1 If A is obtuse, $\tan A < 0 \quad \Rightarrow \quad \tan B \tan C < 1$ and the statement-1 is True.

Example 80: Statement-1:

$$\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)} = \tan A$$

Statement-2: $\sin(A + B) + \sin(A - B) = \sin A$ and $\cos(A + B) + \cos(A - B) = \cos A$

Ans. (c)

Solution: L.H.S in statement-1

$$= \frac{\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B}{\cos A\cos B-\sin A\sin B+\cos A\cos B+\sin A\sin B}$$

$$= \frac{2\sin A\cos B}{2\cos A\cos B} = \tan A$$

$\Rightarrow$ statement-1 is True and statement-2 is false.

Example 81: Statement-1: If $2\sin^2((\pi/2)\cos^2 x) = 1 - \cos(\pi\sin 2x)$, $x \neq (2n + 1)\pi/2$, n is a integer, then $\sin 2x + \cos 2x$ is equal to 1/5.

Statement-2: $\sin 2x + \cos 2x = \dfrac{2-(\tan x-1)^2}{1+\tan^2 x}$.

Ans. (d)

Solution: $\sin 2x + \cos 2x =$

$$\frac{2\tan x+1-\tan^2 x}{1+\tan^2 x} = \frac{2-(\tan x-1)^2}{1+\tan^2 x}$$

$\Rightarrow$ statement-2 is True.

In statement-1, $2\sin^2\dfrac{\pi\cos^2 x}{2} = 2\sin^2\dfrac{\pi\sin 2x}{2}$

$\Rightarrow$ $\cos^2 x = \sin 2x$

$\Rightarrow$ $\cos x(\cos x - 2\sin x) = 0 \Rightarrow \tan x = 1/2$ as $\cos x \neq 0$.

From statement-2, $\sin 2x + \cos 2x = 7/5$

$\Rightarrow$ statement-1 is False.

Example 82: Statement-1: The system of linear equations

$x + (\sin\alpha)y + (\cos\alpha)z = 0$, $x + (\cos\alpha)y + (\sin\alpha)z = 0$ and $x - (\sin\alpha)y + (\cos\alpha)z = 0$ has a non-trivial solution for only one value of α lying between 0 and π

Statement-2: $\begin{vmatrix}\sin x & \cos x & \cos x\\ \cos x & \sin x & \cos x\\ \cos x & \cos x & \sin x\end{vmatrix} = 0$ has only one solution lying between 0 and $\pi/2$

Ans. (b)

Solution: In statement-1, equations have a non-trivial solution if

$$\begin{vmatrix}1 & \sin\alpha & \cos\alpha\\ 1 & \cos\alpha & \sin\alpha\\ 1 & -\sin\alpha & \cos\alpha\end{vmatrix} = 0$$

$\Rightarrow$ $2\sin\alpha(\cos\alpha - \sin\alpha) = 0 \Rightarrow \tan\alpha = 1$

[$\because$ $\sin\alpha \neq 0$ as $0 < \alpha < \pi$]

$\Rightarrow \alpha = \pi/4$ is the only solution $\Rightarrow$ statement-1 is True

In statements, $(\sin x + 2\cos x)\begin{vmatrix}1 & \cos x & \cos x\\ 1 & \sin x & \cos x\\ 1 & \cos x & \sin x\end{vmatrix} = 0$

$\Rightarrow$ $(\sin x + 2\cos x)(\cos x - \sin x)^2 = 0$

$\Rightarrow$ $\tan x = -2$ or $\tan x = 1$ which gives only one values of x. i.e $x = \pi/4$ as $0 < x < \pi/2$.

$\Rightarrow$ Statement-2 is also True but does not lead to statement-1.

LEVEL 2

Straight Objective Type Questions

Example 83: If f : R $\to$ S is given by $f(x) = \sin x - \sqrt{3}\cos x + 1$ is on to, then the interval of S is

(a) [0, 1] (b) [– 1, 1]
(c) [0, 3] (d) [– 1, 3]

Ans. (d)

Solution: We have

$$-\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}$$

So $-\sqrt{1+3} \le \sin x - \sqrt{3}\cos x \le \sqrt{1+3}$

$\Rightarrow$ $-2 \le \sin x - \sqrt{3}\cos x \le 2$

$\Rightarrow$ $-2 + 1 \le \sin x - \sqrt{3}\cos x + 1 \le 2 + 1$

$\Rightarrow$ $-1 \le f(x) \le 3$.

Example 84: If $0 < x < \pi$, and $\cos x + \sin x = 1/2$, then $\tan x$ is

(a) $(1 + \sqrt{7})/4$ (b) $(1 - \sqrt{7})/4$
(c) $(4 - \sqrt{7})/3$ (d) $-(4 + \sqrt{7})/3$

Ans. (d)

Solution: Clearly $x \neq \pi/2$ so $\cos x \neq 0$ and we have $2(1 + \tan x) = \sec x$.

Squaring,

$$4(1 + \tan x)^2 = 1 + \tan^2 x$$

$$\Rightarrow \quad 3\tan^2 x + 8\tan x + 3 = 0$$

$$\Rightarrow \quad \tan x = \frac{-8 \pm \sqrt{64-36}}{6} = \frac{-4 \pm \sqrt{7}}{3}$$

(Take the negative sign)

Example 85: If cos A = 3/4 then value of 32 sin (A/2) sin (5 A/2) is equal to

(a) $\sqrt{11}$ (b) $-\sqrt{11}$
(c) 11 (d) – 11

Ans. (c)

Solution: 32 sin (A/2) sin (5A/2)

$$= 16 [\cos 2A - \cos 3A]$$

$$= 16 [2\cos^2 A - 1 - 4\cos^3 A + 3\cos A]$$

$$= 16\left[2 \times \frac{9}{16} - 1 - 4 \times \frac{27}{64} + 3 \times \frac{3}{4}\right]$$

$$= 18 - 16 - 27 + 36 = 11.$$

Example 86: $\dfrac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°}$ is equal to

(a) tan 55° (b) cot 55°
(c) – tan 35° (d) – cot 35°

Ans. (a)

Solution: $\dfrac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \dfrac{1 + \tan 10°}{1 - \tan 10°}$

$$= \tan(45° + 10°) = \tan 55°.$$

Example 87: If $\tan x = \dfrac{b}{a}$ then $\sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{a-b}{a+b}}$ is equal to

(a) $\dfrac{2\sin x}{\sqrt{\sin 2x}}$ (b) $\dfrac{2\cos x}{\sqrt{\cos 2x}}$

(c) $\dfrac{2\cos x}{\sqrt{\sin 2x}}$ (d) $\dfrac{2\sin x}{\sqrt{\cos 2x}}$

Ans. (b)

Solution: $\sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{a-b}{a+b}}$

$$= \frac{a+b+a-b}{\sqrt{a^2-b^2}} = \frac{2a}{a\sqrt{1-\dfrac{b^2}{a^2}}}$$

$$= \frac{2}{\sqrt{1-\tan^2 x}} = \frac{2\cos x}{\sqrt{\cos 2x}}.$$

Example 88: If $\dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} = x$, then $\dfrac{\cos\alpha}{1+\sin\alpha}$ is equal to

(a) $1/x$ (b) x
(c) $1 + x$ (d) $1 - x$

Ans. (d)

Solution: $x = \dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} \times \dfrac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha-\cos\alpha}$

$$= \frac{2\sin\alpha(1+\sin\alpha-\cos\alpha)}{(1+\sin\alpha)^2-(1-\sin^2\alpha)}$$

$$= \frac{2\sin\alpha(1+\sin\alpha-\cos\alpha)}{(1+\sin\alpha)\,2\sin\alpha}$$

$$= 1 - \frac{\cos\alpha}{1+\sin\alpha}$$

$$\Rightarrow \quad \frac{\cos\alpha}{1+\sin\alpha} = 1 - x$$

Example 89: If $\sin x + \cos y = a$ and $\cos x + \sin y = b$, then $\tan\dfrac{x-y}{2}$ is equal to

(a) $a + b$ (b) $a - b$

(c) $\dfrac{a+b}{a-b}$ (d) $\dfrac{a-b}{a+b}$

Ans. (d)

Solution: From the given relations we have

$\sin x + \sin\left((\pi/2) - y\right) = a$ and $\cos x + \cos\left((\pi/2) - y\right) = b$

$$\Rightarrow \quad 2\sin\frac{x+(\pi/2)-y}{2}\cos\frac{x-(\pi/2)+y}{2} = a$$

and $$2\cos\frac{x+(\pi/2)-y}{2}\cos\frac{x-(\pi/2)+y}{2} = b$$

Dividing we get,

$$\tan\left(\frac{\pi}{4}+\frac{x-y}{2}\right) = \frac{a}{b} \quad \Rightarrow \quad \frac{1+\tan\dfrac{x-y}{2}}{1-\tan\dfrac{x-y}{2}} = \frac{a}{b}$$

or $$\tan\frac{x-y}{2} = \frac{a-b}{a+b}.$$

Example 90: $\dfrac{\sin 3\alpha}{\cos 2\alpha} < 0$ if α lies in

(a) $(13\pi/48, 14\pi/48)$ (b) $(14\pi/48, 18\pi/48)$
(c) $(18\pi/48, 23\pi/48)$ (d) any of these intervals

Ans. (a)

Solution: $\dfrac{\sin 3\alpha}{\cos 2\alpha} < 0$ if $\sin 3\alpha > 0$ and $\cos 2\alpha < 0$

or $\sin 3\alpha < 0$ and $\cos 2\alpha > 0$

i.e. if $3\alpha \in (0, \pi)$ and $2\alpha \in (\pi/2, 3\pi/2)$

or $3\alpha \in (\pi, 2\pi)$ and $2\alpha \in (-\pi/2, \pi/2)$

i.e. if $\alpha \in (0, \pi/3)$ and $\alpha \in (\pi/4, 3\pi/4)$

or $\alpha \in (\pi/3, 2\pi/3)$ and $\alpha \in (-\pi/4, \pi/4)$

i.e. if $\alpha \in (\pi/4, \pi/3)$

since $(13\pi/48, 14\pi/48) \subset (\pi/4, \pi/3)$, (a) is correct

Example 91: If $\cos\alpha + \cos\beta = a$, $\sin\alpha + \sin\beta = b$ and θ is the arithmetic mean between α and β then $\sin 2\theta + \cos 2\theta$ is equal to

(a) $(a + b)^2/(a^2 + b^2)$ (b) $(a - b)^2/(a^2 + b^2)$

(c) $(a^2 - b^2)/(a^2 + b^2)$ (d) none of these

Ans. (d)

Solution: From the given relations we have

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = a \text{ and } 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = b$$

By dividing we get $\tan\dfrac{\alpha+\beta}{2} = \dfrac{b}{a} \Rightarrow \tan\theta = \dfrac{b}{a}$

so that $\cos 2\theta = \dfrac{1 - b^2/a^2}{1 + b^2/a^2} = \dfrac{a^2 - b^2}{a^2 + b^2}$ and $\sin 2\theta = \dfrac{2ab}{a^2 + b^2}$.

$$\therefore \sin 2\theta + \cos 2\theta = \frac{a^2 - b^2 + 2ab}{a^2 + b^2}$$

Example 92: If $\dfrac{1}{\cos\alpha\cos\beta} + \tan\alpha\tan\beta = \tan\gamma$, $0 < \alpha, \beta < \pi$ then $1 - \tan^2\gamma < 0$ for

(a) all values of α and β

(b) no values of α and β

(c) finite number of values of α and β

(d) infinite number of values of α and β

Ans. (a)

Solution: We have $1 - \tan^2\gamma =$

$$\frac{\cos^2\alpha\cos^2\beta - (1 + \sin\alpha\sin\beta)^2}{\cos^2\alpha\cos^2\beta}$$

$$= \frac{(1-\sin^2\alpha)(1-\sin^2\beta) - (1 + 2\sin\alpha\sin\beta + \sin^2\alpha\sin^2\beta)}{\cos^2\alpha\cos^2\beta}$$

$$= \frac{-(\sin\alpha + \sin\beta)^2}{\cos^2\alpha\cos^2\beta} < 0 \quad (\sin\alpha + \sin\beta \neq 0, \text{ as } 0 < \alpha, \beta < \pi)$$

Example 93: If $x + y = z$, then $\cos^2 x + \cos^2 y + \cos^2 z - 2\cos x\cos y\cos z$ is equal to

(a) $\cos^2 z$ (b) $\sin^2 z$

(c) 0 (d) 1

Ans. (d)

Solution: The given expression can be written as

$\cos^2 x + \cos^2 y + \cos^2 z - \cos z\,[\cos(x + y) + \cos(x - y)]$

$= \cos^2 x + \cos^2 y + \cos^2 z - \cos^2 z - \cos(x + y)\cos(x - y)$

$= \cos^2 x + \cos^2 y - (1/2)[\cos 2x + \cos 2y]$

$= (1/2)\,[2\cos^2 x + 2\cos^2 y - \cos 2x - \cos 2y]$

$= (1/2)\,[2\cos^2 x + 2\cos^2 y - 2\cos^2 x + 1 - 2\cos^2 y + 1] = 1$

Example 94: If $\sin 2\theta = k$, then the value of $\dfrac{\tan^3\theta}{1 + \tan^2\theta} + \dfrac{\cot^3\theta}{1 + \cot^2\theta}$ is equal to

(a) $\dfrac{1 - k^2}{k}$ (b) $\dfrac{2 - k^2}{k}$

(c) $k^2 + 1$ (d) $2 - k^2$

Ans. (b)

Solution: We have $\dfrac{\tan^3\theta}{1 + \tan^2\theta} + \dfrac{\cot^3\theta}{1 + \cot^2\theta}$

$$= \frac{\sin^3\theta\cdot\cos^2\theta}{\cos^3\theta} + \frac{\cos^3\theta}{\sin^3\theta}\cdot\sin^2\theta = \frac{\sin^3\theta}{\cos\theta} + \frac{\cos^3\theta}{\sin\theta}$$

$$= \frac{\sin^4\theta + \cos^4\theta}{\sin\theta\cos\theta} = \frac{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1 - k^2/2}{k/2} = \frac{2 - k^2}{k}, \qquad \left[\because \sin\theta\cos\theta = \frac{k}{2}\right].$$

Example 95: If $\sin^2 A = x$, then $\sin A \sin 2A \sin 3A \sin 4A$ is a polynomial in x, the sum of whose coefficients is

(a) 0 (b) 40

(c) 168 (d) 336

Ans. (a)

Solution: We have $\sin A \sin 2A \sin 3A \sin 4A$

$= \sin A\,(2\sin A\cos A)\,(3\sin A - 4\sin^3 A) \times 2\sin 2A\cos 2A$

$= 2\sin^2 A\cos A \times \sin A\,(3 - 4\sin^2 A) \times 2 \times 2\sin A\cos A\,(1 - 2\sin^2 A)$

$= 8\sin^4 A\cos^2 A\,(3 - 4\sin^2 A)\,(1 - 2\sin^2 A)$

$= 8x^2(1 - x)(3 - 4x)(1 - 2x)$

$= 24x^2 - 104x^3 + 144x^4 - 64x^5.$

The required sum $= 24 - 104 + 144 - 64 = 0$.

Example 96: If $\dfrac{\sin A}{\sin B} = \dfrac{\sqrt{3}}{2}$ and $\dfrac{\cos A}{\cos B} = \dfrac{\sqrt{5}}{2}$, $0 < A, B < \pi/2$, then $\tan A + \tan B$ is equal to

(a) $\sqrt{3}/\sqrt{5}$ (b) $\sqrt{5}/\sqrt{3}$

(c) 1 (d) $(\sqrt{3} + \sqrt{5})/\sqrt{5}$

Ans. (d)

◎ **Solution:** From the given relation we have

$\Rightarrow \quad \dfrac{\tan A}{\sqrt{3}} = \dfrac{\tan B}{\sqrt{5}} = k$ (say), (clearly $k > 0$)

Also $2 \sin A = \sqrt{3} \sin B$.

$\Rightarrow \dfrac{2\tan A}{\sqrt{1+\tan^2 A}} = \dfrac{\sqrt{3}\tan B}{\sqrt{1+\tan^2 B}} \Rightarrow \dfrac{2\sqrt{3}k}{\sqrt{1+3k^2}} = \dfrac{\sqrt{3}\times\sqrt{5}k}{\sqrt{1+5k^2}}$

$\Rightarrow 4(1 + 5k^2) = 5(1 + 3k^2)$

$\Rightarrow k^2 = 1/5 \Rightarrow k = 1/\sqrt{5}$

so that $\tan A = \dfrac{\sqrt{3}}{\sqrt{5}}$, $\tan B = 1 \Rightarrow \tan A + \tan B = \dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{5}}$.

◉ **Example 97:** If α, β, γ, δ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the values of $4 \sin \dfrac{\alpha}{2} + 3 \sin \dfrac{\beta}{2} + 2 \sin \dfrac{\gamma}{2} + \sin \dfrac{\delta}{2}$ is equal to

(a) $2\sqrt{1-k}$ (b) $2\sqrt{1+k}$

(c) $2\sqrt{k}$ (d) none of these

Ans. (b)

◎ **Solution:** $\alpha < \beta < \gamma < \delta$ and

$\sin \alpha = \sin \beta = \sin \gamma = \sin \delta = k$

$\Rightarrow \quad \beta = \pi - \alpha, \gamma = 2\pi + \alpha, \delta = 3\pi - \alpha.$

so that the given expression is equal to

$$4 \sin \frac{\alpha}{2} + 3 \sin \frac{\pi-\alpha}{2} + 2 \sin \frac{2\pi+\alpha}{2} + \sin \frac{3\pi-\alpha}{2}$$

$$= 4 \sin \frac{\alpha}{2} + 3 \cos \frac{\alpha}{2} - 2 \sin \frac{\alpha}{2} - \cos \frac{\alpha}{2}$$

$$= 2\left(\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2}\right)$$

$$= 2\sqrt{1+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = 2\sqrt{1+k}.$$

◉ **Example 98:** If $(a - b) \sin (\theta + \phi) = (a + b) \sin (\theta - \phi)$ and $a \tan (\theta/2) - b \tan (\phi/2) = c$, then the value of $\sin \phi$ is equal to

(a) $2ab/(a^2 - b^2 - c^2)$

(b) $2bc/(a^2 - b^2 - c^2)$

(c) $2bc/(a^2 - b^2 + c^2)$

(d) $2ab/(a^2 - b^2 + c^2)$

Ans. (b)

◎ **Solution:** From the first relation we have

$a [\sin (\theta + \phi) - \sin (\theta - \phi)] = b[\sin (\theta - \phi) + \sin (\theta + \phi)]$

$\Rightarrow \quad 2a \sin \phi \cos \theta = 2b \sin \theta \cos \phi$

$\Rightarrow \quad a \tan \phi = b \tan \theta$

$\Rightarrow \quad \dfrac{2a\tan(\phi/2)}{1-\tan^2(\phi/2)} = \dfrac{2b\tan(\theta/2)}{1-\tan^2(\theta/2)}$

From the second relation we have

$$\tan (\theta/2) = (b \tan (\phi/2) + c)/a$$

so that $\quad \dfrac{a\tan(\phi/2)}{1-\tan^2(\phi/2)} = \dfrac{b(b\tan(\phi/2)+c)/a}{1-(b\tan(\phi/2)+c)^2/a^2}$

$\Rightarrow \quad \tan (\phi/2) (a^2 - b^2 - c^2) = bc (1 + \tan^2 (\phi/2))$

$\Rightarrow \quad \sin\phi = \dfrac{2\tan(\phi/2)}{1+\tan^2(\phi/2)} = \dfrac{2bc}{a^2-b^2-c^2}$

◉ **Example 99:** If $\dfrac{\cos x - \cos\alpha}{\cos x - \cos\beta} = \dfrac{\sin^2\alpha\cos\beta}{\sin^2\beta\cos\alpha}$ then $\cos x$ is equal to

(a) $\dfrac{\cos\alpha-\cos\beta}{1+\cos\alpha\cos\beta}$ (b) $\dfrac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$

(c) $\dfrac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$ (d) none of these

Ans. (c)

◎ **Solution:** The given relation can be written as

$$\cos x \,(\sin^2 \beta \cos \alpha - \sin^2 \alpha \cos \beta) = \cos^2 \alpha \sin^2 \beta - \sin^2 \alpha \cos^2 \beta$$

$$\Rightarrow \quad \cos x = \frac{\cos^2\alpha(1-\cos^2\beta)-(1-\cos^2\alpha)\cos^2\beta}{\cos\alpha(1-\cos^2\beta)-\cos\beta(1-\cos^2\alpha)}$$

$$= \frac{\cos^2\alpha-\cos^2\beta}{(\cos\alpha-\cos\beta)(1+\cos\alpha\cos\beta)}$$

$$= \frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}.$$

◉ **Example 100:** If $0 < \alpha, \beta < \pi$ and $\cos \alpha + \cos \beta - \cos (\alpha + \beta) = 3/2$ then $\sin \alpha + \cos \beta$ is equal to

(a) 0 (b) 1

(c) $(\sqrt{3}+1)/2$ (d) $\sqrt{3}$

Ans. (c)

◎ **Solution:** From the given equation we have

$$2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} - 2 \cos^2 \frac{\alpha+\beta}{2} + 1 = \frac{3}{2}$$

$$\Rightarrow 4 \cos^2 \frac{\alpha+\beta}{2} - 4 \cos \frac{\alpha-\beta}{2} \cos \frac{\alpha+\beta}{2} + 1 = 0$$

$$\Rightarrow \left(2\cos\frac{\alpha+\beta}{2} - \cos\frac{\alpha-\beta}{2}\right)^2 = \cos^2\frac{\alpha-\beta}{2} - 1 \qquad \text{(i)}$$

so the only possibility is $\cos^2 \dfrac{\alpha-\beta}{2} - 1 = 0$

since $\cos^2\left(\dfrac{\alpha-\beta}{2}\right) \le 1$

As $0 < \alpha, \beta < \pi$, we have $\alpha = \beta$

From (i) we get $\cos\alpha = 1/2 = \cos\beta$

and $\sin\beta = \sin\alpha = \frac{\sqrt{3}}{2}$. so that $\sin\alpha + \cos\beta = \frac{\sqrt{3}+1}{2}$

Example 101: If $\sin\alpha, \sin\beta, \sin\gamma$ are in A.P.and $\cos\alpha$, $\cos\beta, \cos\gamma$ are in G.P. then $\frac{\cos^2\alpha + \cos^2\gamma - 4\cos\alpha\cos\gamma}{1-\sin\alpha\sin\gamma} =$

(a) – 2 (b) – 1
(c) 0 (d) 2

Ans. (a)

Solution: From the given conditions we have

$2\sin\beta = \sin\alpha + \sin\gamma$ (i)

$\cos^2\beta = \cos\alpha\cos\gamma$ (ii)

Squaring (i), $4\sin^2\beta = \sin^2\alpha + \sin^2\gamma + 2\sin\alpha\sin\gamma$

Using (ii), $4(1 - \cos\alpha\cos\gamma) = 1 - \cos^2\alpha + 1 - \cos^2\gamma + 2\sin\alpha\sin\gamma$

$\Rightarrow \cos^2\alpha + \cos^2\gamma - 4\cos\alpha\cos\gamma = 2(\sin\alpha\sin\gamma - 1)$

$$\Rightarrow \frac{\cos^2\alpha + \cos^2\gamma - 4\cos\alpha\cos\gamma}{1-\sin\alpha\sin\gamma} = -2.$$

Example 102: If $x = \frac{\sin^3 p}{\cos^2 p}$, $y = \frac{\cos^3 p}{\sin^2 p}$; and $\sin p + \cos p = 1/2$, then $x + y =$

(a) 75/18 (b) 44/9
(c) 79/18 (d) 48/9

Ans. (c)

Solution: $x + y = \frac{\sin^5 p + \cos^5 p}{\sin^2 p\cos^2 p}$

$$= \frac{\sin^3 p(1-\cos^2 p) + \cos^3 p(1-\sin^2 p)}{\sin^2 p\cos^2 p}$$

$$= \frac{\sin^3 p + \cos^3 p - \sin^2 p\cos^2 p(\sin p + \cos p)}{\sin^2 p\cos^2 p}$$

$$= \frac{(\sin p + \cos p)^3 - 3\sin p\cos p(\sin p + \cos p) - \sin^2 p\cos^2 p(\sin p + \cos p)}{\sin^2 p\cos^2 p}$$

Now $2\sin p\cos p = (\sin p + \cos p)^2 - 1 = -3/4$

$$\therefore \quad x + y = \frac{\left(\frac{1}{2}\right)^3 - 3\left(-\frac{3}{8}\right)\left(\frac{1}{2}\right) - \left(-\frac{3}{8}\right)^2\left(\frac{1}{2}\right)}{\left(-\frac{3}{8}\right)^2} = \frac{79}{18}.$$

Example 103: $\sin x + 2\sin 2x = 3 + \sin 3x$, $0 \le x \le 2\pi$ has

(a) 2 solutions in I quadrant
(b) one solution in II quadrant
(c) no solution in any quadrant
(d) one solution in each quadrant

Ans. (c)

Solution: From the given relation we have

$\sin x - \sin 3x + 2\sin 2x = 3$

$\Rightarrow 2\sin x\cos 2x - 2\sin 2x + 3 = 0$

$\Rightarrow (\sin x + \cos 2x)^2 + (\sin 2x - 1)^2 + 3$
$= \sin^2 x + \cos^2 2x + \sin^2 2x + 1$

$\Rightarrow (\sin x + \cos 2x)^2 + (\sin 2x - 1)^2 + \cos^2 x = 0$

which is possible only if

$\sin x + \cos 2x = 0$, $\sin 2x = 1$ and $\cos x = 0$

which is not possible for any value of x.

Example 104: The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$ where x is a variable, has real roots if p lies in the interval

(a) $(0, 2\pi)$ (b) $(-\pi, 0)$
(c) $(-\pi/2, \pi/2)$ (d) $(0, \pi)$

Ans. (d)

Solution: The equation has real roots if

$\cos^2 p - 4(\cos p - 1)\sin p \ge 0$

or $\cos^2 p - 4\cos p\sin p + 4\sin p \ge 0$

or $(\cos p - 2\sin p)^2 - 4\sin^2 p + 4\sin p \ge 0$

or $(\cos p - 2\sin p)^2 + 4\sin p(1 - \sin p) \ge 0$

since $(\cos p - 2\sin p)^2 \ge 0$, $1 - \sin p \ge 0$ for all values of p, and for $p \in (0, \pi)$, $\sin p \ge 0$ so that the discriminant is non-negative.

Example 105: The general solution of the equation

$$\frac{1 - \sin x + \ldots + (-1)^n\sin^n x + \ldots}{1 + \sin x + \ldots + \sin^n x + \ldots} = \frac{1-\cos 2x}{1+\cos 2x}, \; x \ne (2n+1)\pi/2, n \in \mathbf{I} \text{ is}$$

(a) $(-1)^n(\pi/3) + n\pi$
(b) $(-1)^n(\pi/6) + n\pi$
(c) $(-1)^{n+1}(\pi/6) + n\pi$
(d) $(-1)^{n-1}(\pi/3) + n\pi$, $(n \in \mathbf{I})$

Ans. (b)

Solution: The equation

$$\frac{1 - \sin x + \ldots + (-1)^n\sin^n x + \ldots}{1 + \sin x + \ldots + \sin^n x + \ldots} = \frac{1-\cos 2x}{1+\cos 2x}$$

$\Rightarrow \quad \dfrac{1}{1+\sin x} \times \dfrac{1-\sin x}{1} = \dfrac{2\sin^2 x}{2\cos^2 x}$

as $-1 < \sin x < 1$

$\Rightarrow \quad 1 - \sin x = \dfrac{\sin^2 x\,(1+\sin x)}{1-\sin^2 x}$

$\Rightarrow \quad (1 - \sin x)^2 = \sin^2 x \quad \Rightarrow \quad 1 - 2\sin x = 0$

$\Rightarrow \quad \sin x = 1/2 = \sin (\pi/6)$

$\Rightarrow \quad x = n\pi + (-1)^n\, \pi/6.$

Example 106: If $\sin^4 x + \cos^4 y + 2 = 4 \sin x \cos y$, $0 \le x, y \le \pi/2$ then $\sin x + \cos y =$

(a) -2 (b) 0
(c) 2 (d) none of these

Ans. (c)

Solution: The given equation can be written as

$\sin^4 x + \cos^4 y + 2 - 4 \sin x \cos y = 0$

$\Rightarrow \quad (\sin^2 x - 1)^2 + (\cos^2 y - 1)^2 + 2 \sin^2 x + 2 \cos^2 y - 4 \sin x \cos y = 0$

$\Rightarrow \quad (\sin^2 x - 1)^2 + (\cos^2 y - 1)^2 + 2 (\sin x - \cos y)^2 = 0$

which is true if $\sin^2 x = 1$, $\cos^2 y = 1$ and $\sin x = \cos y$, so $\sin x + \cos y = 2$ as $0 \le x, y \le \pi/2$.

Example 107: A solution (x, y) of the system of equation $x - y = 1/3$ and $\cos^2 (\pi x) - \sin^2 (\pi y) = 1/2$ is given by

(a) (2/3, 1/3) (b) (7/6, 1/6)
(c) (13/6, 11/6) (d) (1/6, 5/6)

Ans. (c)

Solution: $\cos^2 (\pi x) - \sin^2 (\pi y) = 1/2$

$\Rightarrow \quad \cos \pi (x + y) \cos \pi (x - y) = 1/2$

$\Rightarrow \quad \cos \pi (x + y) \cos (\pi/3) = 1/2 \qquad [\because x - y = 1/3]$

$\Rightarrow \quad \cos \pi (x + y) = 1$

$\Rightarrow \quad \pi (x + y) = 2n\pi \Rightarrow x + y = 2n$

Now $x + y = 2n$ and $x - y = 1/3$

$\Rightarrow \quad x = n + 1/6,\ y = n - 1/6, \qquad (n \in I)$

$\therefore (x, y) = \left(n+\dfrac{1}{6}, n-\dfrac{1}{6}\right)$ which is satisfied by (c) for $n = 2$

Example 108: $\cos (x - y) - 2 \sin x + 2 \sin y = 3$ if

(a) $\sin x = \sin y$
(b) $x + y = 2n\pi$, $x - y = (2k - 1)\pi/2$
(c) $x = 2k\pi - \pi/2$, $y = 2n\pi + \pi/2$
(d) $\cos (x - y) = -1 \qquad (n, k \in \mathbf{I})$

Ans. (c)

Solution: $\cos (x - y) - 2 \sin x + 2 \sin y = 3$

$\Rightarrow \quad 1 - 2 \sin^2 \dfrac{x-y}{2} - 4 \sin \dfrac{x-y}{2} \cos \dfrac{x+y}{2} - 3 = 0$

$\Rightarrow \quad \sin^2 \dfrac{x-y}{2} + 2 \sin \dfrac{x-y}{2} \cos \dfrac{x+y}{2} + 1 = 0$

$\Rightarrow \quad \sin \dfrac{x-y}{2} = \dfrac{-2\cos\dfrac{x+y}{2} \pm \sqrt{4\cos^2\dfrac{x+y}{2} - 4}}{2}$

For real values of $\sin \dfrac{x-y}{2}$, we have $\cos^2 \dfrac{x+y}{2} = 1$

$\Rightarrow \quad \sin^2 \dfrac{x+y}{2} = 0$ or $\sin \dfrac{x+y}{2} = 0$

$\Rightarrow \quad x + y = 2n\pi$

and then $\sin \dfrac{x-y}{2} = \pm 1 \quad \Rightarrow \quad \dfrac{x-y}{2} = (2k-1)\dfrac{\pi}{2}$

$\Rightarrow \quad x - y = (2k - 1)\, \pi$

so (b) is not correct.

Also if $\sin x = \sin y$ then the given equation becomes $\cos (x - y) = 3$ which is not correct; (a) is not correct.

Next if $x = 2k\pi - \pi/2$ and $y = 2n\pi + \pi/2$.

Then $\cos (x - y) - 2 \sin x + 2 \sin y = -1 + 2 + 2 = 3$, so (c) is correct.

Finally if $\cos (x - y) = -1$, the given equation becomes $\sin x - \sin y = -2$ which is not true for any real values of x and y so (d) is not correct.

Example 109: If $6 \cos 2\theta + 2 \cos^2 (\theta/2) + 2 \sin^2 \theta = 0$, $-\pi < \theta < \pi$, then θ is equal to

(a) $\pi/3$ (b) $\pi/3, \cos^{-1} (3/5)$
(c) $\cos^{-1} (3/5)$ (d) $\pi/3, \pi - \cos^{-1} (3/5)$

Ans. (d)

Solution: The given equation can be written as

$6(2 \cos^2 \theta - 1) + (1 + \cos \theta) + 2(1 - \cos^2 \theta) = 0$

$\Rightarrow \quad 10 \cos^2 \theta + \cos \theta - 3 = 0$

$\Rightarrow \quad (5 \cos \theta + 3) (2 \cos \theta - 1) = 0$

$\Rightarrow \quad \cos \theta = 1/2$ or $\cos \theta = -3/5$

$\Rightarrow \quad \theta = \pi/3$ or $\theta = \pi - \cos^{-1} (3/5)$ as $-\pi < \theta < \pi$

$\Rightarrow \quad \theta = \pi/3, \pi - \cos^{-1} (3/5).$

Example 110: The number of integral values of a for which the equation $\cos 2x + a \sin x = 2a - 7$ possesses solution is

(a) 2 (b) 3
(c) 4 (d) 5

Ans. (d)

Solution: The given equation can be written as

$1 - 2 \sin^2 x + a \sin x = 2a - 7$

$\Rightarrow \quad 2 \sin^2 x - a \sin x + 2a - 8 = 0$

$\Rightarrow \quad \sin x = \dfrac{a \pm \sqrt{a^2 - 8(2a-8)}}{4} = \dfrac{a \pm (a-8)}{4}$

$\Rightarrow \quad \sin x = \dfrac{a-4}{2}$ which is possible if $-1 \le \dfrac{a-4}{2} \le 1$ or

$2 \le a \le 6$.

so the required values of a are 2, 3, 4, 5, 6 and hence the required number is 5.

Example 111: The least difference between the roots of the equation

$4 \cos x\,(2 - 3 \sin^2 x) + (\cos 2x + 1) = 0$ $(0 \le x \le \pi/2)$ is

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

Ans. (a)

Solution: The given equation can be written as

$$4 \cos x\,(3 \cos^2 x - 1) + 2 \cos^2 x = 0$$

$$\Rightarrow \quad 2 \cos x\,(6 \cos^2 x + \cos x - 2) = 0$$

$$\Rightarrow \quad 2 \cos x\,(3 \cos x + 2)\,(2 \cos x - 1) = 0$$

$\Rightarrow$ either $\cos x = 0$ which gives $x = \pi/2$

or $\cos x = -2/3$, which gives no value of x for $0 \le x \le \pi/2$

or $\cos x = 1/2$, which gives $x = \pi/3$.

so the required difference $= \pi/2 - \pi/3 = \pi/6$

Example 112: The solution of $|\cos x| = \cos x - 2 \sin x$ is

(a) $x = n\pi$

(b) $x = n\pi + \pi/4$

(c) $x = n\pi + (-1)^n (\pi/4)$ $(n \in \mathbf{I})$

(d) $x = (2n + 1)\pi + \pi/4$

Ans. (d)

Solution: $|\cos x| = \cos x - 2 \sin x$

$\Rightarrow$ $\cos x = \cos x - 2 \sin x$ if $\cos x \ge 0$

$\Rightarrow$ $\sin x = 0 \Rightarrow x = 2n\pi$ (as $\cos x \ge 0$) $n \in \mathbf{I}$ and $x \ne n\pi$

(as $x = \pi \Rightarrow \cos x < 0$, for $n = 1$)

Next $|\cos x| = \cos x - 2 \sin x$

$\Rightarrow$ $-\cos x = \cos x - 2 \sin x$ if $\cos x < 0$

$\Rightarrow$ $\cos x - \sin x = 0 \Rightarrow \tan x = 1$

Now $\cos x < 0$ and $\tan x = 1 \Rightarrow \tan x = \tan (5\pi/4)$

$\Rightarrow$ $x = 2n\pi + (5\pi/4) = (2n + 1)\pi + \pi/4$

and $x \ne 2n\pi + \pi/4$ (as $x = \pi/4 \Rightarrow \cos x > 0$, for $n = 0$).

Example 113: If $0 \le a, b \le 3$ and the equation $x^2 + 4 + 3\cos(ax + b) = 2x$ has at least one solution, then the value of $a + b$ is

(a) $\pi/4$ (b) $\pi/2$

(c) π (d) 2π

Ans. (c)

Solution: We have $x^2 - 2x + 4 = -3\cos(ax + b)$

$\Rightarrow \quad (x - 1)^2 + 3 = -3 \cos(ax + b)$ (1)

As $(x - 1) \ge 0$ and $-1 \le \cos(ax + b) \le 1$

Equation (1) s possible if $\cos(ax + b) = -1$

and $x - 1 = 0$

$\Rightarrow \quad a + b = \pi, 3\pi, 5\pi, \ldots\ldots$

Since $a + b \le 6$ and $3\pi > 6$, $a + b = \pi$

Example 114: If $0 < \theta < \pi/2$, number of solutions of

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$ is

(a) 0 (b) 1

(c) 2 (d) 4

Ans. (c)

Solution: We can write

$$\frac{1}{\sqrt{2}} \sum_{m=1}^{6} \frac{1}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right) \sin\left(\theta + \frac{m\pi}{4}\right)} = 4$$

$$\Rightarrow \quad \sum_{m=1}^{6} \frac{\sin\left[\left(\theta + \frac{m\pi}{4}\right) - \left(\theta + \frac{(m-1)\pi}{4}\right)\right]}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right) \sin\left(\theta + \frac{m\pi}{4}\right)} = 4$$

$$\Rightarrow \quad \sum_{m=1}^{6} \left[\cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right)\right] = 4$$

$$\Rightarrow \quad \cot\theta - \cot\left(\theta + \frac{6\pi}{4}\right) = 4$$

$\Rightarrow \quad \cot\theta + \tan\theta = 4$

$\Rightarrow \quad \sin 2\theta = 1/2$

$\Rightarrow \quad 2\theta = \pi/6,\ 5\pi/6 \Rightarrow \theta = \pi/12,\ 5\pi/12$

EXERCISE

Concept-based Straight Objective Type Questions

1. Let $f(\theta) = \dfrac{\cot\theta}{1 - \tan\theta} + \dfrac{\tan\theta}{1 - \cot\theta}, \pi < \theta < \dfrac{3\pi}{4}$, then $f\left(\dfrac{9\pi}{8}\right)$ is equal to:

(a) $1 + \sqrt{2}$ (b) $1 + 2\sqrt{2}$

(c) $\sqrt{2} - 1$ (d) $2\sqrt{2} - 1$

2. Let $f(\theta) = \dfrac{\tan^3\theta}{1+\tan^2\theta} - \dfrac{\cot^3\theta}{1+\cot^2\theta}, 0 < \theta < \dfrac{\pi}{4}$.
   Then $f(\theta)$ is equal to
   (a) $\tan\theta + \cot\theta$ (b) $2\sin(2\theta)$
   (c) $-2\cot(2\theta)$ (d) 0

3. If $0 < \theta < \dfrac{\pi}{2}$, then $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$ is equal to:
   (a) $1 + \sin\theta + \cos\theta$ (b) $\dfrac{1+\sin\theta}{\cos\theta}$
   (c) $\dfrac{1-\cos\theta}{\sin\theta}$ (d) $\tan\theta - \sec\theta$

4. $\tan 15° + \tan 75°$ is equal to:
   (a) 1 (b) 2
   (c) 3 (d) 4

5. $\sin^2\left(\dfrac{\pi}{8}+\dfrac{A}{2}\right) - \sin^2\left(\dfrac{\pi}{8}-\dfrac{A}{2}\right)$ is equal to:
   (a) $(1/\sqrt{2})\sin A$ (b) $(1/\sqrt{2})\cos A$
   (c) $\sqrt{2}\sin A$ (d) $\sqrt{2}\cos A$

6. $\dfrac{\cos 11° - \sin 11°}{\cos 11° + \sin 11°}$ is equal to:
   (a) cot 56° (b) tan 11°
   (c) tan 56° (d) cot 11°

7. Suppose $\alpha, \beta > 0$ and $\alpha + 2\beta = \pi/2$, then $\tan(\alpha + \beta) - 2\tan\alpha - \tan\beta$ is equal to:
   (a) 0 (b) $\tan\beta$
   (c) $\cot\beta$ (d) $\tan\alpha - \cot\beta$

8. If $\sin\theta + \text{cosec}\,\theta = 2$, then $\cos^{2015}\theta + \text{cosec}^{2015}\theta$ is equal to:
   (a) -1 (b) 0
   (c) 1 (d) 2015

9. If $(1 - \sin A)(1 - \sin B)(1 - \sin C) = (1 + \sin A)(1 + \sin B)(1 + \sin C)$, then each side is equal to:
   (a) $\pm\cos A\cos B\cos C$ (b) 0
   (c) $\pm\sin A\sin B\sin C$ (d) 1

10. $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13} + \cos\dfrac{3\pi}{13} + \cos\dfrac{5\pi}{13}$ is equal to:
    (a) -1 (b) 0
    (c) 1 (d) $\sqrt{3}/2$

11. If $\text{cosec}\,A + \sec A = \text{cosec}\,B + \sec B$, then $\tan A\tan B$ is equal to:
    (a) $\tan\left(\dfrac{A+B}{2}\right)$ (b) $\cot\left(\dfrac{A+B}{2}\right)$
    (c) $\cot\left(\dfrac{A-B}{2}\right)$ (d) $\tan\left(\dfrac{A-B}{2}\right)$

12. If $0 < \theta < \pi/8$, then $\sqrt{2+\sqrt{2+2\cos(4\theta)}}$ is equal to:
    (a) $2\cos\theta$ (b) $-2\cos\theta$
    (c) $2\sin\theta$ (d) $-2\sin\theta$

13. Let $f(\theta) = \cos\theta\cos 2\theta\cos 4\theta\cos 7\theta$, then $f(\pi/15)$ is equal to:
    (a) 1/4 (b) 1/8
    (c) 1/16 (d) 1/32

14. Let $f(\theta) = \sin\theta\sin 3\theta\sin 5\theta$, then $f(\pi/14)$ is equal to:
    (a) 1/8 (b) 1/4
    (c) 1/7 (d) 1/14

15. If $A + B + C = \pi/2$, then
    $\sin^2 A + \sin^2 B + \sin^2 C + 2\sin A\sin B\sin C$
    is equal to
    (a) 0 (b) 1
    (c) -1 (d) none of these

16. Number of values of x lying in the interval $[0, 4\pi]$ and satisfying the equation $\tan 5x = \cot 3x = 0$ is:
    (a) 2 (b) 4
    (c) 6 (d) 8

17. Number of values of x lying in the interval $[0, 5\pi]$ and satisfying the equation $\sin x = \tan x$ is:
    (a) 5 (b) 6
    (c) 9 (d) 12

18. Number of values of $x \in [0, 4\pi]$ and satisfying the equation $\sin x + \cos x = 3/2$ is:
    (a) 0 (b) 1
    (c) 2 (d) 4

19. Number of values of $x \in [0, 5\pi]$ and satisfying $\sin^2 x = \cos^2 x$ is:
    (a) 5 (b) 10
    (c) 20 (d) 40

20. For $0 < \theta < \pi/4$, $\dfrac{\sec(8\theta)-1}{\sec(4\theta)-1}\cdot\dfrac{\tan(2\theta)}{\tan(8\theta)}$ is equal to
    (a) -1 (b) 0
    (c) 1 (d) none of these

LEVEL 1

Straight Objective Type Questions

21. If 2 tan α + cot β = tan β, then the value of tan $(\beta - \alpha)$ is
(a) tan α (b) cot α
(c) tan β (d) cot β

22. If cos $(x - y) = a$ cos $(x + y)$, then cot x cot y is equal to
(a) $\frac{a-1}{a+1}$ (b) $\frac{a+1}{a-1}$
(c) $a - 1$ (d) $a + 1$

23. $\sin^2 A + \sin^2 (A - B) + 2 \sin A \cos B \sin (B - A)$ is equal to
(a) $\sin^2 A$ (b) $\sin^2 B$
(c) $\cos^2 A$ (d) $\cos^2 B$

24. If $\frac{3 \sin 2\theta}{5 + 4\cos 2\theta} = 1$, then the value of $\tan\theta$ is equal to
(a) 1 (b) 1/3
(c) 3 (d) none of these

25. If $x = a \sec^3 \theta \tan \theta$, $y = b \tan^3 \theta \sec \theta$, then $\sin^2 \theta$ is equal to
(a) $\frac{x}{a} - \frac{y}{b}$ (b) $\frac{x}{a} + \frac{y}{b}$
(c) $\frac{xy}{ab}$ (d) $\frac{ay}{bx}$

26. $\cot \theta - \cot 3\theta$ is equal to
(a) $2 \sin \theta \sin 3\theta$ (b) $2 \cos \theta \cos 3\theta$
(c) $2 \cos \theta \operatorname{cosec} 3\theta$ (d) $2 \sin \theta \operatorname{cosec} 3\theta$

27. If $0 < x, y < 2\pi$, the number of solutions of the system of equations
$\sin x \sin y = 3/4$ and $\cos x \cos y = 1/4$ is
(a) 0 (b) 1
(c) 2 (d) infinite

28. If A and B be acute positive angles satisfying
$3 \sin^2 A + 2 \sin^2 B = 1$, $3 \sin 2A - 2 \sin 2B = 0$ then
(a) $B = \pi/4 - A/2$ (b) $A = \pi/4 - 2B$
(c) $B = \pi/2 - A/4$ (d) $A = \pi/4 - B/2$

29. If tan α, tan β, tan γ are the roots of the equation $x^3 - px^2 - r = 0$, then the value of $(1 + \tan^2\alpha)(1 + \tan^2\beta)(1 + \tan^2\gamma)$ is equal to
(a) $(p - r)^2$ (b) $1 + (p - r)^2$
(c) $1 - (p - r)^2$ (d) none of these

30. If A, B, C are the angles of a triangle such that angle A is obtuse then
(a) $\tan A \tan B < 1$ (b) $\tan B \tan C < 1$
(c) $\tan C \tan A < 1$ (d) $\tan A \tan B \tan C < 1$

31. If $\tan (\theta/2) = \operatorname{cosec} \theta - \sin \theta$, then $\cos^2 (\theta/2)$ is equal to
(a) sin 18° (b) cos 36°
(c) sin 36° (d) cos 18°

32. If $a \sin^2\theta + b \cos^2\theta = a \cos^2\phi + b \sin^2\phi = 1$ and $a \tan \theta = b \tan \phi$ $(a \neq b)$ then
(a) $a + b = 2ab$ (b) $a - b = 2ab$
(c) $a - b + 2ab = 0$ (d) $a + b + 2ab = 0$

33. The equation $\tan^2 x + \cot^2 x = 4a \operatorname{cosec}^2 2x$ has a real solution if
(a) $0 < a \leq 1$ (b) $1/2 \leq a \leq 1$
(c) $1/4 \leq a \leq 1/2$ (d) $-1 \leq a \leq 1$

34. For $n \in \mathbf{I}$, the line $x = n\pi + \pi/2$ does not intersect the graph of
(a) $\cot (x + \pi)$ (b) $\cos (x - \pi)$
(c) $\sin x$ (d) $\tan x$

35. If $\frac{\tan x}{2} = \frac{\tan y}{3} = \frac{\tan z}{5}$ and $x + y + z = \pi$, then the value of
$\tan^2 x + \tan^2 y + \tan^2 z$ is
(a) 38/3 (b) 38
(c) 114 (d) none of these

36. If the angles A, B, C of a triangle are in A.P. such that $\sin (2A + B) = 1/2$ then $\sin (B + 2C) =$
(a) $-1/2$ (b) $1/2$
(c) $\sqrt{3}/2$ (d) $1/\sqrt{2}$

37. If $\tan \theta + \tan \phi = a$, $\cot \theta + \cot \phi = b$, $\theta - \phi = \alpha$ $(\neq 0)$ then
(a) $ab < 4$ (b) $ab = 4$
(c) $ab > 4$ (d) $ab = 0$

38. If $\frac{x}{y} = \frac{\cos A}{\cos B}$ then $\frac{x \tan A + y \tan B}{x + y} =$
(a) $\frac{\sin A + \cos B}{\cos A + \sin B}$ (b) $\frac{\sin A + \sin B}{\cos A \cos B}$
(c) $\tan \frac{A + B}{2}$ (d) $\cot \frac{A - B}{2}$

39. If $x = a$ (cos θ + θ sin θ), $y = a$(sin θ – θ cos θ) then $a\theta$ =
   (a) $x + y - a$ (b) $\sqrt{x^2 + y^2 - a^2}$
   (c) $\sqrt{x^2 - y^2 + a^2}$ (d) $x - y + a$

40. (1 + cos ($\pi/8$)) (1 + cos ($3\pi/8$)) (1 + cos ($5\pi/8$)) (1 + cos ($7\pi/8$)) =
   (a) 1/2 (b) cos ($\pi/8$)
   (c) 1/8 (d) $(1 + \sqrt{2})/2\sqrt{2}$

41. If 2 cos x + 2 cos $3x$ = cos y, 2 sin x + 2 sin $3x$ = sin y then the value of cos $2x$ is
   (a) – 7/8 (b) 1/8
   (c) – 1/8 (d) 7/8

42. $\dfrac{\cos A}{3} = \dfrac{\cos B}{4} = \dfrac{1}{5}$, $-\dfrac{\pi}{2} < A < 0$, $0 < B < \dfrac{\pi}{2}$, then 3 sin A + 4 sin B =
   (a) 0 (b) – 1
   (c) 24/5 (d) 1

43. The value of $\log_3$ tan 1° + $\log_3$ tan 2° + … + $\log_3$ tan 89° is
   (a) 3 (b) 1
   (c) 2 (d) 0

44. If $\dfrac{\tan^3((\pi/2)-\theta)}{\sec^2\theta} \cdot \dfrac{\cot^2\theta}{\sec((\pi/2)-\theta)} \cdot \dfrac{\sin((\pi/2)-\theta)}{\sin^4\theta}$ = $\cot^n \theta$ then n =
   (a) 2 (b) 4
   (c) 6 (d) 8

45. The number of solutions of
   sin θ + 2 sin 2θ + 3 sin 3θ + 4 sin 4θ = 10, $0 < \theta < \pi$ is
   (a) 0 (b) 1
   (c) 2 (d) 4

46. cos 11° – cos 2° is
   (a) a positive integer
   (b) a negative integer
   (c) a positive rational number
   (d) a negative rational number

47. If sin A, cos A and tan A are in G.P., then $\cot^6 A - \cot^2 A$ =
   (a) – 1 (b) 0
   (c) 1 (d) none of these

48. If cos x – sin x = 1/2, then tan $2x$ =
   (a) $\sqrt{7}/3$ (b) $\sqrt{7}/4$
   (c) $3/\sqrt{7}$ (d) $2/\sqrt{7}$

49. Which of the following gives the least value of A
   (a) cos $2A$ = sin $3A$ (b) cos $3A$ = sin $7A$
   (c) tan A = cot $3A$ (d) cot A = tan $2A$

50. If $n \in \mathbf{N}$ and $\sin\left(\dfrac{\pi}{2n}\right) + \cos\left(\dfrac{\pi}{2n}\right) = \dfrac{\sqrt{n}}{2}$, then a possible value of n is
   (a) 4 (b) 6
   (c) 8 (d) 12

51. If $4n\alpha = \pi$, then the value of tan α tan 2α tan 3α … tan $(2n - 1)\alpha$ is
   (a) – 1 (b) 0
   (c) 1 (d) none of these

52. If $x > 0$ and $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = 0$ then
   (a) $x < \sqrt{2}$ (b) $x = \sqrt{2}$
   (c) $x > \sqrt{2}$ (d) none of these

53. If 2 tan ($\pi/3$) cos ($2\pi x$) = $\sqrt{3}$, the general solution of the equation is
   (a) $2n\pi \pm \pi/3$ (b) $n \pm 1/3$
   (c) $n \pm 1/6$ (d) $n \pm 1/2$ ($n \in \mathbf{I}$)

54. 2 $\cos^2 x$ + 4 cos x = 3 $\sin^2 x$ if
   (a) $\cos x = \dfrac{-2+\sqrt{14}}{5}$ (b) $\cos x = \dfrac{-2+\sqrt{19}}{5}$
   (c) $\sin x = \dfrac{-2+\sqrt{14}}{5}$ (d) $\sin x = \dfrac{-2+\sqrt{19}}{5}$

55. 6 $\tan^2 x$ – 2 $\cos^2 x$ = cos $2x$ if
   (a) cos $2x$ = – 1 (b) cos $2x$ = 1
   (c) cos $3x$ = – 1/2 (d) cos $2x$ = 1/2

56. The greatest value of cos θ for which cos 5θ = 0 is
   (a) 0 (b) $(1 + \sqrt{5})/4$
   (c) $\sqrt{\dfrac{5+\sqrt{5}}{8}}$ (d) $\sqrt{\dfrac{\sqrt{5}+1}{4}}$

57. If tan $p\theta$ = tan $q\theta$, then the values of θ form an A.P. with common difference
   (a) $\pi/(p + q)$ (b) π/p
   (c) π/q (d) $\pi/(p - q)$

58. The equation $\int_0^x (t^2 - 8t + 13)\, dt = x \sin(a/x)$ has a solution if sin (a/x) =
   (a) 0 (b) 1
   (c) 3 (d) 6

59. The smallest positive root of the equation $\sqrt{\sin(1-x)} = \sqrt{\cos x}$ is
   (a) $1/2 + \pi/4$ (b) $1/2 + 3\pi/4$
   (c) $1/2 + 5\pi/4$ (d) $1/2 + 7\pi/4$

60. The sum of the roots of the equation $4\cos^3 x - 4\cos^2 x - \cos(\pi + x) - 1 = 0$ in the interval [0, 315] is $p\pi$, where p is equal to
(a) 2500 (b) 2550
(c) 2600 (d) 2651

61. A solution (x, y) of $x^2 + 2x \sin xy + 1 = 0$ is
(a) (1, 0) (b) $(1, 7\pi/2)$
(c) $(-1, -7\pi/2)$ (d) $(-1, 0)$

62. $e^{\sin x} - e^{-\sin x} = 4$ for
(a) all real values of x (b) some $x \in [0, \pi/2]$
(c) some $x \in (-\pi/2, \pi/2)$ (d) no real value of x

63. The number of values of x satisfying the condition $\sin x + \sin 5x = \sin 3x$ in the interval $[0, \pi]$ is
(a) 6 (b) 2
(c) 10 (d) 0

64. $\sin x$, $\sin 2x$, $\sin 3x$ are in A.P. if
(a) $x = 165°$ (b) $x = 195°$
(c) $x = 15°$ (d) $x = 2n\pi \quad n \in \mathbf{I}$.

65. $2\sin^2((\pi/2)\cos^2 x) = 1 - \cos(\pi \sin 2x)$ if
(a) $x = (2n + 1)\pi/2$ (b) $x = n\pi/2$
(c) $x = 2n\pi$ (d) none of these, $n \in \mathbf{I}$

66. Number of solutions of the equation $4\sin^2 x + \tan^2 x + \cot^2 x + \operatorname{cosec}^2 x = 6$ in $[0, \pi]$ is
(a) 0 (b) 2
(c) 4 (d) 8.

67. Sum of integral values of n such that $\sin x(2\sin x + \cos x) = n$, has at least one real solution is
(a) 1 (b) 2
(c) 3 (d) 0.

68. If $0 < x < \pi/2$ and $\sin(2\sin x) = \cos(2\cos x)$ then $\tan x + \cot x = \dfrac{a}{\pi^C - b}$ where $a + b + c =$
(a) 32 (b) 16
(c) 50 (d) 64

69. If $15\sin^4\alpha + 10\cos^4\alpha = 6$ then the value of $8\operatorname{cosec}^6\alpha - 27\sec^6\alpha$ is
(a) 0 (b) 1
(c) –1 (d) 5

70. If $\sum_{r=1}^{\infty} \tan^{-1}\dfrac{1}{2r^2} = t$, then $\tan t$ is equal to
(a) 0 (b) 1
(c) $\sqrt{3}$ (d) $1/\sqrt{3}$.

71. If $\theta \in [-2011\pi, 2011\pi]$, then the number of solutions of the equation.
$[\sec\theta - 1] + \{\tan\theta \tan\frac{\theta}{2}\}$
$= [\{\sec\theta - 1 + \tan\theta \tan\frac{\theta}{2}\}]$
(where $\{\cdot\}$ denotes the fractional part function and $[\cdot]$ denotes the greatest integer function) is
(a) 0 (b) 2010
(c) 2011 (d) 2012.

72. If $x = \sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} + \sin\dfrac{8\pi}{7}$
and $y = \cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{8\pi}{7}$, then $x^2 + y^2$ is
(a) 1 (b) 2
(c) 3 (d) 4.

73. Number of solutions of the equation
$\sin\left(\dfrac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}x + 4$ is
(a) 0 (b) 1
(c) 2 (d) 4

Assertion-Reason Type Questions

74. **Statement-1:** $\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7} = -\dfrac{1}{8}$
Statement-2: $\cos\theta\cos 2\theta\cos 4\theta \ldots \cos 2^{n-1}\theta = -\dfrac{1}{2^n}$ if $\theta = \dfrac{\pi}{2^n - 1}$.

75. **Statement-1:** Let $f(x) = \dfrac{\sin 2x}{1 + \cos 2x}$ and $g(x) = \dfrac{1 + \sin x - \cos x}{1 + \sin x + \cos x}$
If $\beta, \gamma, \in (0, \pi)$ such that
$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + \beta + \gamma) = 0$
and $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + \beta + \gamma) = 0$
Then $f(\beta) + g(\gamma) = 0$
Statement-2: For all values of x, $f(x) + g(x) = 0$, $f(x)$ and $g(x)$ are functions defined in statement-1.

76. **Statement-1:** If x and y are real number such that $x^2 + y^2 = 27$, then the maximum possible value of $x - y$ is $3\sqrt{6}$.

Statement-2: $-1 \le \cos(\theta + \pi/4) \le 1$

77. **Statement-1:** If $p = 7 + \tan\alpha \tan\beta$, $q = 5 + \tan\beta \tan\gamma$ and $r = 3 + \tan\gamma \tan\alpha$, then the maximum value of $\sqrt{p} + \sqrt{q} + \sqrt{r}$ is $4\sqrt{3}$; $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = \pi/2$
Statement-2: If $\alpha + \beta + \gamma = \pi/2$, $\alpha, \beta, \gamma > 0$, then $\cot\alpha \cot\beta + \cot\beta \cot\gamma + \cot\gamma \cot\alpha = 1$.

78. **Statement-1:** If $x \sin^3\theta + y \cos^3\theta = \sin\theta \cos\theta$. and $x \sin\theta = y \cos\theta$, then $x^2 + y^2 = 1$
Statement-2: $4 \sin^3\theta = 3 \sin\theta - \sin 3\theta$
$4 \cos^3\theta = 3 \cos\theta + \cos 3\theta$

79. **Statement-1:** $2 \sin^2 x + 3 \sin x - 2 > 0$ and $x^2 - x - 2 < 0$, if $-1 < x < 2$.
Statement-2: $x^2 - x - 2 < 0$ if $-1 < x < 2$

80. **Statement-1:** $\cos^7 x + \sin^4 x = 1$ has only two non-zero solution in the interval $-\pi < x < \pi$.
Statement-2: $\cos^5 x + \cos^2 x - 2 = 0$ is possible only when $\cos x = 1$

LEVEL 2

Straight Objective Type Questions

81. The acute angle of a rhombus whose side is a mean proportional between its diagonals is
(a) 15° (b) 20°
(c) 30° (d) 80°

82. Given the height h and the angle bisector l drawn from the vertex of the right angle of a triangle, then cosine of an acute angle of the triangle is given by
(a) $\dfrac{h + \sqrt{l^2 - h^2}}{\sqrt{2}\, h}$ (b) $\dfrac{h - \sqrt{l^2 - h^2}}{\sqrt{2}\, h}$
(c) $\dfrac{h}{l}$ (d) $\dfrac{h - \sqrt{l^2 - h^2}}{\sqrt{2}\, l}$

83. If $2 \sin^2(x + \pi/4) + \sqrt{3} \cos 2x > 0$, then
(a) $\cos(2x - \pi/6) > -1/2$ (b) $\sin(2x - \pi/6) < -1/2$
(c) $\sin(2x - \pi/6) > -1/2$ (d) $\cos(2x - \pi/6) < -1/2$

84. $x = \sum_{n=0}^{\infty} \cos^{2n}\theta,\ y = \sum_{n=0}^{\infty} \sin^{2n}\theta,\ z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\theta,$
$|\cos\theta| < 1$, $|\sin\theta| < 1$ then $x + y + z$ is equal to
(a) xy (b) yz
(c) zx (d) xyz

85. The least positive value of x satisfying
$$\frac{\sin^2 2x + 4\sin^4 x - 4\sin^2 x \cos^2 x}{4 - \sin^2 2x - 4\sin^2 x} = \frac{1}{9} \text{ is}$$
(a) $\pi/3$ (b) $\pi/6$
(c) $2\pi/3$ (d) $5\pi/6$

86. $\cos 7.5° =$
(a) $\sqrt{\dfrac{2 + \sqrt{2} + \sqrt{6}}{8}}$ (b) $\sqrt{\dfrac{4 + \sqrt{2} + \sqrt{6}}{8}}$
(c) $\dfrac{2\sqrt{2} + \sqrt{3} + 1}{2\sqrt{2}}$ (d) $\sqrt{\dfrac{4 + \sqrt{2} + \sqrt{6}}{4}}$

87. If $x = X\cos\theta - Y\sin\theta$, $y = X\sin\theta + Y\cos\theta$; and $ax^2 + 2bxy + cy^2 = AX^2 + 2HXY + BY^2$, then
(a) $H = 0$ If $\theta = 0$ (b) $H = b$ if $\theta = \pi/2$
(c) $A + B = a + c$ (d) $H = c - a$ if $\theta = \pi/4$

88. If $\sin 5\theta = a \sin^5\theta + b \sin^3\theta + c \sin\theta + d$, then
(a) $a + b + c = 0$ (b) $a + b + c + d = 0$
(c) $5a + 3b - 4c = 0$ (d) $a - 3c + d = 0$

89. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then the value of $\cot(A - B)$ is
(a) $\dfrac{x - y}{xy}$ (b) $\dfrac{1}{x^2} + \dfrac{1}{y^2}$
(c) $\dfrac{x + y}{xy}$ (d) xy

90. $\dfrac{\cos 3\theta}{\cos^3\theta} + \dfrac{\sin 3\theta}{\sin^3\theta}$ is equal to
(a) $3 \cos 2\theta \operatorname{cosec} 2\theta$ (b) $3 \cot 2\theta \operatorname{cosec} 2\theta$
(c) $12 \cot 2\theta \operatorname{cosec} 2\theta$ (d) $12 \tan 2\theta \sec 2\theta$

91. $(x \tan\alpha + y \cot\alpha)(x \cot\alpha + y \tan\alpha) - 4xy \cot^2 2\alpha =$
(a) $x^2 + y^2$ (b) $4xy$
(c) $(x + y)^2$ (d) none of these

92. If $\tan A$, $\tan B$, $\tan C$ satisfy the equation $3\tan^3\theta - 4\tan^2\theta + 3\tan\theta + 1 = 0$, then $A + B + C =$
(a) 0 (b) $\pi/2$
(c) $3\pi/4$ (d) 2π

93. If $x \sin\theta + y \sin 2\theta + z \sin 3\theta = \sin 4\theta$, $(\theta \neq n\pi)$ then $8\cos^3\theta - 4z\cos^2\theta - 2(y+2)\cos\theta$ is equal to
(a) $x-y$ (b) $x-z$
(c) $y-z$ (d) none of these

94. The number of values of $\sin x$ satisfying $\sin 5x = 5\sin x$ is
(a) 0 (b) 1
(c) 2 (d) 3

95. If $\sin\alpha$, $\sin\beta$ are the roots of the equation $a\sin^2\theta + b\sin\theta + c = 0$ and $\sin\alpha + 2\sin\beta = 1$ then $a^2 + 2b^2 + 3ab + ac =$
(a) -1 (b) 0
(c) 1 (d) $a+b+c$

96. If $\sin(\theta/2) = a$, $\cos(\theta/2) = b$, then $(1+\sin\theta)(3\sin\theta + 4\cos\theta + 5) =$
(a) $(a+b)^2(a+3b)^2$ (b) $(a+b)^2(3a+b)^2$
(c) $(a-b)^2(a-3b)^2$ (d) $(a-b)^2(3a-b)^2$

97. If $\sin A = \sin B$ and $\cos A = \cos B$; $A \neq B$, then
(a) $\tan\dfrac{A-B}{2} = 0$ (b) $\cos(A+B) = 1$
(c) $\tan\dfrac{A+B}{2} = 0$ (d) $\sin(A-B) = 1/2$

98. $\cos 22° + \cos 78° + \cos 80° =$
(a) $4\sin 11°\sin 39°\sin 40°$
(b) $1 + 4\cos 11°\cos 39°\cos 40°$
(c) $1 + 4\sin 11°\sin 39°\sin 40°$
(d) $4\cos 11°\cos 39°\cos 40°$

99. $\tan x + \dfrac{1}{2}\tan\dfrac{x}{2} + \dfrac{1}{2^2}\tan\dfrac{x}{2^2} + \ldots + \dfrac{1}{2^{n-1}}\tan\dfrac{x}{2^{n-1}}$ is equal to
(a) $\dfrac{1}{2^n}\cot\dfrac{x}{2^n} - 2\cot 2x$
(b) $\dfrac{1}{2^{n-1}}\cot\left(\dfrac{x}{2^{n-1}}\right) - 2\cot 2x$
(c) $\tan\dfrac{2^n-1}{2^{n-1}}x$
(d) $2\cot 2x - \dfrac{1}{2^{n-1}}\cot\dfrac{x}{2^{n-1}}$

100. The value of $\dfrac{3+\cot 76°\cot 16°}{\cot 76° + \cot 16°}$ is
(a) $\cot 44°$ (b) $\cot 46°$
(c) $\tan 2°$ (d) $\cot 92°$

101. If $x\cos\theta = y\cos(\theta + 2\pi/3) = z\cos(\theta + 4\pi/3)$ then $xy + yz + zx =$
(a) $\cos^2\theta$ (b) $\sin^2\theta$
(c) 1 (d) 0

102. If $A > 0$, $B > 0$ and $A + B = \pi/3$ then the maximum value of $\tan A \tan B$ is
(a) $1/\sqrt{3}$ (b) $1/3$
(c) $\sqrt{3}$ (d) 3

103. If $\tan\theta$, $2\tan\theta + 2$, $3\tan\theta + 3$ are in G.P., then the value of $\dfrac{7-5\cot\theta}{9-4\sqrt{\sec^2\theta - 1}}$ is
(a) 12/5 (b) $-33/28$
(c) 33/100 (d) 12/13

104. If $\sin\theta + \cos\theta = a$ and $\cos\theta - \sin\theta = b$, then $\sin\theta(\sin\theta - \cos\theta) + \sin^2\theta(\sin^2\theta - \cos^2\theta) + \sin^3\theta(\sin^3\theta - \cos^3\theta) + \ldots$ is equal to
(a) $\dfrac{1-ab}{1+ab}$ (b) $\dfrac{1-a^2}{3-a^2}$
(c) $\dfrac{1-ab}{1+ab} + \dfrac{1-a^2}{3-a^2}$ (d) $\dfrac{1+ab}{1-ab} - \dfrac{a^2-1}{3-a^2}$

105. If $\left(1+\sqrt{1+x}\right)\tan\alpha = \left(1-\sqrt{1-x}\right)$ then $x =$
(a) $\sin\alpha$ (b) $\sin 2\alpha$
(c) $\sin 4\alpha$ (d) $\cos 4\alpha$

106. If $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$, then $f(\theta)$
(a) ≥ 0 only when $\theta \geq 0$ (b) ≤ 0 for all real θ
(c) ≥ 0 for all real θ (d) ≤ 0 only when $\theta \leq 0$

107. In a right angled triangle, the hypotenuse is $2\sqrt{2}$ times the length of the perpendicular drawn from the opposite vertex on its hypotenuse then the other two angles are
(a) $\pi/3, \pi/6$ (b) $\pi/4, \pi/4$
(c) $\pi/8, 3\pi/8$ (d) $\pi/12, 5\pi/12$

108. $\sqrt{\cos 2x} + \sqrt{1+\sin 2x} = 2\sqrt{\sin x + \cos x}$ if
(a) $\sin x + \cos x = 1$ (b) $x = 2n\pi$
(c) $x = n\pi + \pi/4$ (d) $\sin x - \cos x = 0$

109. $\sin x + 2\sin 2x = 3 + \sin 3x$
(a) if $\sin x + \cos 2x = 0$
(b) if $\sin 2x - 1 = 0$
(c) if $\cos x = 0$
(d) for no real value of x

110. The number of pairs (x, y) satisfying the equation $\sin x + \sin y = \sin(x+y)$ and $|x| + |y| = 1$ is
(a) 2 (b) 4
(c) 6 (d) infinite

Previous Years' AIEEE/JEE Main Questions

1. The numbers of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$ is
 (a) 2 (b) 3
 (c) 0 (d) 1 **[2003]**

2. Let α, β be such that $\pi < \alpha - \beta < 3\pi$. If $\sin\alpha + \sin\beta = -21/65$ and $\cos\alpha + \cos\beta = -27/65$ then the values of $\cos\frac{\alpha-\beta}{2}$ is
 (a) 6/65 (b) $3/\sqrt{130}$
 (c) $-3/\sqrt{130}$ (d) $-6/65$ **[2004]**

3. If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$, then the difference between the maximum and minimum values of u^2 is given by
 (a) $(a+b)^2$ (b) $2\sqrt{a^2+b^2}$
 (c) $2(a^2+b^2)$ (d) $(a-b)^2$ **[2004]**

4. If $f: R \to S$ is defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$ is onto, then the interval of S is
 (a) [0, 1] (b) [– 1, 1]
 (c) [0, 3] (d) [– 1, 3] **[2004]**

5. The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is
 (a) 2 (b) 4
 (c) 6 (d) 1 **[2006]**

6. If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15°, then the value of $2 + q - p$ is
 (a) 1 (b) 2
 (c) 3 (d) 0. **[2006]**

7. If $0 < x < \pi$, and $\cos x + \sin x = 1/2$, then $\tan x$ is
 (a) $(1+\sqrt{7})/4$ (b) $(1-\sqrt{7})/4$
 (c) $(4-\sqrt{7})/3$ (d) $-(4+\sqrt{7})/3$ **[2006]**

8. Let A and B denote the statements
 $A : \cos\alpha + \cos\beta + \cos\gamma = 0$
 $B : \sin\alpha + \sin\beta + \sin\gamma = 0$
 If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -3/2$ then
 (a) both A and B are true
 (b) both A and B are false
 (c) A is ture and B is false
 (d) A is false and B is true. **[2009]**

9. For a regular polygon, let r, R be the radii of the inscribed and the circumscribed circles. A **false** statement among the following is
 (a) There is a regular polygon with $r/R = 3/3$
 (b) There is a regular polygon with $r/R = \sqrt{3}/2$
 (c) There is a regular polygon with $r/R = 1/2$
 (d) There is a regular polygon with $r/R = 1/\sqrt{2}$ **[2010]**

10. Let $\cos(\alpha + \beta) = 4/5$ and let $\sin(\alpha - \beta) = 5/13$ where $0 \le \alpha, \beta \le \pi/4$. Then $\tan 2\alpha =$
 (a) 19/12 (b) 20/7
 (c) 25/16 (d) 56/33 **[2010]**

11. If $A = \sin^2 x + \cos^4 x$, then for all real x
 (a) $3/4 \le A \le 13/16$ (b) $3/4 \le A \le 1$
 (c) $13/16 \le A \le 1$ (d) $1 \le A \le 2$ **[2011]**

12. The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are:
 (a) $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{12}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$
 (b) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{35\pi}{36}$
 (c) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$
 (d) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$ **[2011]**

13. In a ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to
 (a) $\pi/4$ (b) $3\pi/4$
 (c) $5\pi/6$ (d) $\pi/6$ **[2012]**

14. $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to
 (a) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$ (b) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$
 (c) $\frac{(p^2+q^2)\cos\theta}{(p\cos\theta+q\sin\theta)^2}$ (d) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ **[2013]**

15. The expression $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}$ can be written as
    (a) $\sec A \operatorname{cosec} A + 1$ (b) $\tan A + \cot A$
    (c) $\sec A + \operatorname{cosec} A$ (d) $\sin A \cos A + 1$ **[2013]**

16. **Statement-1:** The number of common solutions of the trigonometric equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is two.
    Statement-2: The number of solutions of the equation, $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, \pi]$ is two **[2013, online]**

17. The number of solutions of the equation $\sin 2x - 2\cos x + 4\sin x = 4$ in the interval $[0, 5\pi]$ is:
    (a) 3 (b) 4
    (c) 5 (d) 6 **[2013, online]**

18. Let $f_k(x) = (1/k)\left(\sin^k x + \cos^k x\right)$ where $x \in \mathbf{R}$ and $k \geq 1$. Then $f_4(x) - f_6(x)$ equals:
    (a) 1/6 (b) 1/3
    (c) 1/4 (d) 1/12 **[2014]**

19. The number of values of α in $[0, 2\pi]$ for which $2\sin^3\alpha - 7\sin^2\alpha + 7\sin\alpha = 2$, is:
    (a) 6 (b) 4
    (c) 3 (d) 1 **[2014, online]**

20. If $\operatorname{cosec}\theta = \dfrac{p+q}{p-q}$ $(p \neq q \neq 0)$, then $\left|\cot\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)\right|$ is equal to:
    (a) $\sqrt{\dfrac{p}{q}}$ (b) $\sqrt{\dfrac{q}{p}}$
    (c) $\sqrt{pq}$ (d) pq **[2014, online]**

21. If $2\cos\theta + \sin\theta = 1$, $(\theta \neq \pi/2)$, then $7\cos\theta + 6\sin\theta$ is equal to:
    (a) 1/2 (b) 2
    (c) 11/2 (d) 46/5 **[2014, online]**

22. If $f(\theta) = \begin{vmatrix} 1 & \cos\theta & 1 \\ -\sin\theta & 1 & -\cos\theta \\ -1 & \sin\theta & 1 \end{vmatrix}$ and A and B are respectively the maximum and the minimum values of $f(\theta)$, then (A, B) is equal to
    (a) $(3, -1)$ (b) $\left(4, 2-\sqrt{2}\right)$
    (c) $\left(2+\sqrt{2}, 2-\sqrt{2}\right)$ (d) $\left(2+\sqrt{2}, -1\right)$
    [2014, online]

23. If $\cos\alpha + \cos\beta = \dfrac{3}{2}$ and $\sin\alpha + \sin\beta = \dfrac{1}{2}$ and θ is the arithmetic mean of α and β, then $\sin 2\theta$ is equal to:
    (a) $\dfrac{3}{5}$ (b) $\dfrac{4}{5}$
    (c) $\dfrac{7}{5}$ (d) $\dfrac{8}{5}$ **[2015, online]**

24. In a ΔABC, $\dfrac{a}{b} = 2+\sqrt{3}$ and $\underline{|c} = 60°$. Then the ordered pair $(\underline{|A}, \underline{|B})$ is equal to:
    (a) (15°, 105°) (b) (105°, 15°)
    (c) (45°, 75°) (d) (75°, 45°)
    [2015, online]

25. If $0 \leq x < 2\pi$, then the number of real values of x which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is:
    (a) 3 (b) 5
    (c) 7 (d) 9 **[2016]**

26. If m and M are the minimum and the maximum values of $4 + \dfrac{1}{2}\sin^2 2x - 2\cos^4 x$, $x \in \mathbf{R}$, then $M - m$ is equal to:
    (a) $\dfrac{9}{4}$ (b) $\dfrac{15}{4}$
    (c) $\dfrac{7}{4}$ (d) $\dfrac{1}{4}$ **[2016, online]**

27. The number of $x \in [0, 2\pi]$ for which $\left|\sqrt{2\sin^4 x + 18\cos^2 x} - \sqrt{2\cos^4 x + 18\sin^2 x}\right| = 1$ is:
    (a) 2 (b) 6
    (c) 4 (d) 8 **[2016, online]**

28. The number of distinct real roots of the equation,
    $$\Delta = \begin{vmatrix} \cos x & \sin x & \sin x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$$
    in the interval $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ is:
    (a) 1 (b) 4
    (c) 2 (d) 3 **[2016, online]**

29. If $A > 0$, $B > 0$ and $A + B = \dfrac{\pi}{6}$, then the minimum value of $\tan A + \tan B$ is:
    (a) $\sqrt{3} - \sqrt{2}$ (b) ü– √
    (c) $\dfrac{2}{\sqrt{3}}$ (d) $2 - \sqrt{3}$ **[2016, online]**

30. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then:
    (a) $P \subset Q$ and $Q - P \neq \phi$
    (b) $Q \not\subset P$
    (c) $P = Q$ (d) $P \not\subset Q$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. The number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is
   (a) 1 (b) 2
   (c) 3 (d) 0 **[2006]**

2. For $\theta \neq 0$, if $\cos\theta + \sec\theta = 2$, then $\cos^n\theta + \sec^n\theta$ equals
   (a) 2^n (b) $(-2)^n$
   (c) 2^{n+1} (d) 2 **[2006]**

3. If for real p the equation $\sin^6 x + \cos^6 x = p$ always has a real solutions, then p lies in the interval:
   (a) $]1, 2]$ (b) $\left[\frac{1}{4}, 1\right[$
   (c) $\left[\frac{1}{4}, 1\right]$ (d) $[1, 2]$ **[2007]**

4. The value of $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$ is
   (a) 3/2 (b) 1/2
   (c) 3/4 (d) 1/4 **[2008]**

5. The value of $\sin\frac{\pi}{14}\cdot\sin\frac{3\pi}{14}\cdot\sin\frac{5\pi}{14}$ is
   (a) 1/4 (b) 1/8
   (c) 1/2 (d) 1/16 **[2008]**

6. Let $0 < \alpha, \beta \leq \pi/4$. If $\cos(\alpha + \beta) = 4/5$ and $\sin(\alpha - \beta) = 5/13$ then the value of $\tan 2\alpha$ is:
   (a) 19/12 (b) 56/33
   (c) 20/7 (d) 25/16 **[2009]**

7. The sides of a triangle are $\sin\beta, \cos\beta$ and $\sqrt{1 + a\sin 2\beta}$, for some $a > 0$, $0 < \beta < \frac{\pi}{2}$. If the greatest angle of this triangle is 120°, then a equals
   (a) 1 (b) 3/2
   (c) 2/3 (d) 1/2 **[2010]**

8. If $\cos 25° + \sin 25° = K$, then $\cos 50°$ equals
   (a) $-K\sqrt{2-K^2}$ (b) $K\sqrt{1-K^2}$
   (c) $-K\sqrt{1-K^2}$ (d) $K\sqrt{2-K^2}$ **[2010]**

9. $\tan 9° - \tan 27° - \tan 63° + \tan 81°$ is equal to:
   (a) 4 (b) 0
   (c) 1 (d) –1 **[2011, 12]**

10. If a and b are such that $a\sin\theta = b\cos\theta$ for $0 \leq \theta < \frac{\pi}{4}$, then $\sqrt{\frac{a-b}{a+b}} + \sqrt{\frac{a+b}{a-b}}$ equals
    (a) $\frac{2}{\sqrt{\cos 2\theta}}$ (b) $2\cos\theta$
    (c) $\frac{2\cos\theta}{\sqrt{\cos 2\theta}}$ (d) $\frac{2\sin\theta}{\sqrt{\cos 2\theta}}$

11. The value of $\tan 9° - \tan 27° - \tan 63° + \tan 81°$ is
    (a) 4 (b) 3
    (c) 2 (d) 0 **[2012]**

12. In a ΔABC, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, then $\sin^2 A + \sin^2 B + \sin^2 C$ is:
    (a) 9/4 (b) 5/4
    (c) 2 (d) $3\sqrt{3}/2$ **[2014]**

13. Let $\frac{3\pi}{4} < \theta < \pi$ and $\sqrt{2\cot\theta + \frac{1}{\sin^2\theta}} = K - \cot\theta$, then K equals
    (a) – 1 (b) 0
    (c) 1/2 (d) 1 **[2014]**

14. The maximum value of $f(x) = 2\sin x + \sin 2x$, in the interval $\left[0, \frac{3\pi}{2}\right]$ is:
    (a) $\sqrt{2}+1$ (b) $2\sqrt{3}$
    (c) $\frac{3\sqrt{3}}{2}$ (d) $\sqrt{3}$ **[2015]**

15. For all values of $\theta \in \left(0, \frac{\pi}{2}\right)$, the determinant of the matrix $\Delta = \begin{bmatrix} -2 & \tan\theta + \sec^2\theta & 3 \\ -\sin\theta & \cos\theta & \sin\theta \\ -3 & -4 & 3 \end{bmatrix}$ always lies in the interval:
    (a) $[3, 5]$ (b) $(4, 6)$
    (c) $\left(\frac{5}{2}, \frac{19}{4}\right)$ (d) $\left[\frac{7}{2}, \frac{21}{4}\right]$ **[2016]**

Answers

Concept-based

1. (b)	**2.** (c)	**3.** (b)	**4.** (d)
5. (a)	**6.** (a)	**7.** (a)	**8.** (c)
9. (a)	**10.** (b)	**11.** (b)	**12.** (a)

13. (c)	**14.** (a)	**15.** (b)	**16.** (d)
17. (b)	**18.** (a)	**19.** (b)	**20.** (c)

Level 1

21. (d)	**22.** (b)	**23.** (b)	**24.** (c)
25. (d)	**26.** (c)	**27.** (c)	**28.** (a)
29. (b)	**30.** (b)	**31.** (b)	**32.** (a)
33. (b)	**34.** (d)	**35.** (a)	**36.** (a)
37. (c)	**38.** (c)	**39.** (b)	**40.** (c)
41. (a)	**42.** (a)	**43.** (d)	**44.** (d)
45. (a)	**46.** (d)	**47.** (c)	**48.** (c)
49. (b)	**50.** (b)	**51.** (c)	**52.** (d)
53. (c)	**54.** (b)	**55.** (d)	**56.** (c)
57. (d)	**58.** (b)	**59.** (d)	**60.** (b)
61. (b)	**62.** (d)	**63.** (a)	**64.** (d)
65. (a)	**66.** (b)	**67.** (c)	**68.** (c)
69. (a)	**70.** (b)	**71.** (c)	**72.** (b)
73. (c)	**74.** (a)	**75.** (c)	**76.** (a)
77. (c)	**78.** (b)	**79.** (d)	**80.** (a)

Level 2

81. (c)	**82.** (d)	**83.** (a)	**84.** (d)
85. (b)	**86.** (b)	**87.** (c)	**88.** (c)
89. (c)	**90.** (c)	**91.** (c)	**92.** (b)
93. (b)	**94.** (b)	**95.** (b)	**96.** (a)
97. (a)	**98.** (c)	**99.** (b)	**100.** (a)
101. (d)	**102.** (b)	**103.** (c)	**104.** (c)
105. (c)	**106.** (c)	**107.** (c)	**108.** (b)
109. (d)	**110.** (c)		

Previous Years' AIEEE/JEE Main Questions

1. (a)	**2.** (c)	**3.** (d)	**4.** (d)
5. (b)	**6.** (c)	**7.** (d).	**8.** (d)
9. (a)	**10.** (d)	**11.** (b)	**12.** (d)
13. (d)	**14.** (d)	**15.** (a)	**16.** (b)
17. (a)	**18.** (a)	**19.** (c)	**20.** (a)
21. (b)	**22.** (c)	**23.** (a)	**24.** (b)
25. (c)	**26.** (a)	**27.** (d)	**28.** (c)
29. (b)	**30.** (c)		

Previous Years' B-Architecture Entrance Examination Questions

1. (b)	**2.** (d)	**3.** (c)	**4.** (a)
5. (b)	**6.** (b)	**7.** (d)	**8.** (d)
9. (a)	**10.** (c)	**11.** (a)	**12.** (a)
13. (a)	**14.** (c)	**15.** (a)	

Hints and Solutions

Concept-based

1. $f(\theta) = \dfrac{1}{\tan\theta}\dfrac{1}{(1-\tan\theta)} + \dfrac{\tan\theta\cdot\tan\theta}{\tan\theta - 1}$

$$= \frac{1-\tan^3\theta}{\tan\theta(1-\tan\theta)} = \frac{1+\tan\theta+\tan^2\theta}{\tan\theta}$$

$$= 1 + \frac{2}{\sin 2\theta}$$

$$\therefore f\left(\frac{9\pi}{8}\right) = 1 + \frac{2}{\sin\left(\frac{9\pi}{4}\right)} = 1 + 2\sqrt{2}$$

2. $f(\theta) = \dfrac{\sin^3\theta}{\cos\theta} - \dfrac{\cos^3\theta}{\sin\theta}$

$$= \frac{\sin^4\theta - \cos^4\theta}{\cos\theta\sin\theta} = \frac{-(\cos^2\theta - \sin^2\theta)(1)}{\cos\theta\sin\theta}$$

$$= -2\cot(2\theta)$$

3. $\dfrac{\tan\theta + \sec\theta - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$

$$= \frac{(\sec\theta + \tan\theta)(1 - \sec\theta + \tan\theta)}{\tan\theta - \sec\theta + 1}$$

$$= \frac{1+\sin\theta}{\cos\theta}$$

4. $\tan 15° + \tan 75° = \tan 15° + \cot 15°$

$$= \frac{\tan^2 15° + 1}{\tan 15°}$$

$$= 2\left(\frac{1}{\sin 30°}\right) = 4$$

5. Use : $\sin^2 A - \sin^2 B = \sin(A - B)\sin(A + B)$

6. $\dfrac{\cos 11° - \sin 11°}{\cos 11° + \sin 11°} = \dfrac{\cos 11° + \cos(90° + 11°)}{\cos 11° - \cos(90° + 11°)}$

$$= \frac{2\cos 56°\cos 45°}{2\sin 56°\sin 45°} = \cot 56°$$

7. $\tan(\alpha + \beta) = \tan(\pi/2 - \beta)$

$\Rightarrow \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \cot\beta$

$\Rightarrow \tan\alpha + \tan\beta = \cot\beta - \tan\alpha$

$\Rightarrow 2\tan\alpha + \tan\beta = \cot\beta = \tan(\alpha + \beta)$

8. $\sin\theta + \operatorname{cosec}\theta = 2 \Rightarrow \sin\theta + \dfrac{1}{\sin\theta} = 2$

$\Rightarrow \sin\theta = 1 = \operatorname{cosec}\theta$

$\therefore \cos\theta = 0$. Thus, $\cos^{2015}\theta + \operatorname{cosec}^{2015}\theta = 1$

9. Multiply both the sides by $(1 - \sin A)(1 - \sin B)(1 - \sin C)$ to obtain $(1 - \sin A)^2 (1 - \sin B)^2 (1 - \sin C)^2 = \cos^2 A \cos^2 B \cos^2 C$
$\Rightarrow (1 - \sin A)(1 - \sin B)(1 - \sin C) = \pm \cos A \cos B \cos C$

10. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$

$= 2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + 2\cos\frac{\pi}{13}\cos\frac{4\pi}{13}$

$= 2\cos\frac{\pi}{13}\left[\cos\left(\frac{9\pi}{13}\right) + \cos\left(\frac{4\pi}{13}\right)\right]$

$= 2\cos\left(\frac{\pi}{13}\right)\left[2\cos\frac{\pi}{2}\cos\frac{5\pi}{26}\right] = 0$

11. $\sec A - \sec B = \operatorname{cosec} B - \operatorname{cosec} A$

$\Rightarrow \frac{\cos B - \cos A}{\cos A \cos B} = \frac{\sin A - \sin B}{\sin A \sin B}$

$\Rightarrow \frac{\sin A \sin B}{\cos A \cos B} = \frac{\sin A - \sin B}{\cos B - \cos A}$

$\Rightarrow \tan A \tan B = \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$

$= \cot\left(\frac{A+B}{2}\right)$

12. Use $1 + \cos 2A = 2\cos^2 A$.

13. $f(\theta) = \frac{1}{2\sin\theta}(2\sin\theta\cos\theta)\cos 2\theta\cos 4\theta\cos 7\theta$

$= \frac{1}{2\times 2\sin\theta}(2\sin 2\theta\cos 2\theta)\cos 4\theta\cos 7\theta$

$= \frac{1}{2\times 4\sin\theta}(2\sin 4\theta\cos 4\theta)\cos 7\theta$

$= \frac{1}{2\times 8\sin\theta}[2\sin 8\theta\cos 7\theta]$

$= \frac{1}{16\sin\theta}[\sin(15\theta) + \sin\theta]$

When $\theta = \frac{\pi}{15}$, $f(\theta) = \frac{1}{16}$.

14. As $\frac{\pi}{14} = \frac{\pi}{2} - \frac{3\pi}{7}$, $\frac{3\pi}{14} = \frac{\pi}{2} - \frac{2\pi}{7}$

and $\frac{5\pi}{14} = \frac{\pi}{2} - \frac{\pi}{7}$, we can write

$f\left(\frac{\pi}{14}\right) = \cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}$.

$\Rightarrow 2\sin\frac{\pi}{7} f\left(\frac{\pi}{14}\right) = \sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$

$\Rightarrow 4\sin\frac{\pi}{7} f\left(\frac{\pi}{14}\right) = \sin\frac{4\pi}{7}\cos\frac{3\pi}{7}$

$\Rightarrow 8\sin\left(\frac{\pi}{7}\right) f\left(\frac{\pi}{14}\right) = \sin\left(\frac{7\pi}{7}\right) + \sin\left(\frac{\pi}{7}\right) = \sin\frac{\pi}{7}$

$\Rightarrow f\left(\frac{\pi}{14}\right) = \frac{1}{8}$

15. $\sin^2 A + \sin^2 B + \sin^2 C + 2\sin A \sin B \sin C$
$= 1 - (\cos^2 A - \sin^2 B) + \sin C[\sin C + 2\sin A \sin B]$
$= 1 - \cos(A + B)\cos(A - B) + \sin C[\cos(A + B) + 2\sin A \sin B]$

$[\because C = \pi/2 - A - B]$

$= 1 - \cos(\pi/2 - C)\cos(A - B) + \sin C\cos(A - B)$
$= 1$

16. $\tan 5x = -\cot 3x = \tan(\pi/2 + 3x)$

$\Rightarrow 5x = n\pi + \frac{\pi}{2} + 3x$, $n \in \mathbf{I}$

$\Rightarrow 2x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{I}$

$\Rightarrow x = (2n + 1)\frac{\pi}{4}$, $n \in \mathbf{I}$.

Now, $x \in [0, 4\pi]$

$\Rightarrow 0 \le (2n + 1)\frac{\pi}{4} \le 4\pi$

$\Rightarrow -\frac{1}{2} \le n \le \frac{15}{2} \Rightarrow 0 \le n \le 7$

$\therefore$ There are 8 values of x.

17. $\sin x = \tan x \Rightarrow \sin x(\cos x - 1) = 0$
$\Rightarrow \sin x = 0$ or $\cos x = 1$.
However, when $\cos x = 1$, then $\sin x = 0$
$\therefore \sin x = 0 \Rightarrow x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$

18. $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \le \sqrt{2} < \frac{3}{2}$

Thus, no values of x.

19. $\tan^2 x = 1 \Rightarrow \tan x = \pm 1$

$\tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}$

and $\tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}$

Thus, there are 10 values of x.

20. $\frac{\sec(8\theta) - 1}{\sec(4\theta) - 1}\cdot\frac{\tan(2\theta)}{\tan(8\theta)}$

$= \frac{1-\cos(8\theta)}{1-\cos(4\theta)} \cdot \frac{\cos(4\theta)}{\cos(8\theta)} \cdot \frac{\sin(2\theta)}{\cos(2\theta)} \cdot \frac{\cos(8\theta)}{\sin(8\theta)}$

$= \frac{2\sin^2(4\theta)}{2\sin^2(2\theta)} \frac{\cos(4\theta)\sin(2\theta)}{\cos(2\theta)\sin(8\theta)} = 1$

Level 1

21. $\tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha} = \frac{\tan\alpha + \cot\beta}{1+\tan\beta\tan\alpha}$

$= \frac{\tan\alpha\tan\beta + 1}{\tan\beta(1+\tan\alpha\tan\beta)}$

$= \cot\beta.$

22. $\frac{\cos(x-y)}{\cos(x+y)} = a$

$\Rightarrow \frac{\cos(x-y)+\cos(x+y)}{\cos(x-y)-\cos(x+y)} = \frac{a+1}{a-1}$

$\Rightarrow \cot x \cot y = \frac{a+1}{a-1}$

23. $\sin^2 A + \sin^2(A - B) + [\sin(A + B) + \sin(A - B)] \sin(B - A)$

$= \sin^2 A + \sin^2(A - B) + \sin(A + B)\sin(B - A) - \sin^2(A - B)$

$= \sin^2 A + \sin^2 B - \sin^2 A = \sin^2 B.$

24. $\frac{6\sin\theta\cos\theta}{8\cos^2\theta + 1} = \frac{6\tan\theta}{9+\tan^2\theta} = 1$

$\Rightarrow \tan^2\theta - 6\tan\theta + 9 = 0$

$\Rightarrow \tan\theta = 3$

25. $\frac{x}{a} = \frac{\sin\theta}{\cos^4\theta}, \frac{y}{b} = \frac{\sin^3\theta}{\cos^4\theta} \Rightarrow \sin^2\theta = \frac{y}{b} \times \frac{a}{x} = \frac{ay}{bx}$

26. $\cot\theta - \cot 3\theta = \frac{\cos\theta\sin 3\theta - \sin\theta\cos 3\theta}{\sin\theta\sin 3\theta}$

$= \frac{\sin 2\theta}{\sin\theta\sin 3\theta}$

$= 2\cos\theta \operatorname{cosec} 3\theta.$

27. $\cos(x - y) = 1 \Rightarrow x - y = 0$ (Note $x - y \neq 2\pi$)

$\Rightarrow x = y.$

$\cos(x + y) = -1/2 \Rightarrow x + y = 120°$ or $240° \Rightarrow x = y = 60°$

or $x = y = 120°$.

28. $\sin 2B = (3/2)\sin 2A$, $\cos 2B = 3\sin^2 A$.

$\Rightarrow \tan 2B = \cot A \Rightarrow \tan A \tan 2B = 1$

$\Rightarrow A + 2B = \pi/2$

$\Rightarrow B = \pi/4 - A/2$

29. $x^3 - px^2 - r = (x - \tan\alpha)(x - \tan\beta)(x - \tan\gamma)$

$\Rightarrow i^3 - pi^2 - r = (i - \tan\alpha)(i - \tan\beta)(i - \tan\gamma)$

and $-i^3 - pi^2 - r = (-i - \tan\alpha)(-i - \tan\beta)(-i - \tan\gamma)$

Multiplying we get

$(p - r)^2 - (i)^2 = -[((i)^2 - \tan^2\alpha)(i^2 - \tan^2\beta)(i^2 - \tan^2\gamma)$

$\Rightarrow (p - r)^2 + 1 = (1 + \tan^2\alpha)(1 + \tan^2\beta)(1 + \tan^2\gamma)$

30. $A > \pi/2 \quad B + C < \pi/2 \Rightarrow \tan(B + C) > 0$

$\Rightarrow \frac{\tan B + \tan C}{1 - \tan B\tan C} > 0$

$\Rightarrow \tan B \tan C < 1$ (as $\tan B, \tan C > 0$)

31. $\frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{1-\sin^2\theta}{\sin\theta} \Rightarrow 2\sin^2(\theta/2) = \cos^2\theta$

$\Rightarrow \cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \cos\theta = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1\pm\sqrt{5}}{2}$

But $\cos\theta \neq \frac{-1-\sqrt{5}}{2}$ so $\cos\theta = \frac{\sqrt{5}-1}{2}$

$\Rightarrow 2\cos^2(\theta/2) = \frac{\sqrt{5}-1}{2} + 1 = \frac{\sqrt{5}+1}{2}$

$\Rightarrow \cos^2(\theta/2) = \frac{\sqrt{5}+1}{4} = \cos 36°.$

32. $a\tan^2\theta + b = 1 + \tan^2\theta \Rightarrow (a - 1)\tan^2\theta = 1 - b$

and $(b - 1)\tan^2\phi = 1 - a \Rightarrow \frac{\tan^2\theta}{\tan^2\phi} = \frac{(1-b)^2}{(1-a)^2} = \frac{b^2}{a^2}$

$\Rightarrow (a - b)(a + b - 2ab) = 0$

$\Rightarrow a + b = 2ab$ [as $a \neq b$]

33. $a = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$\Rightarrow \sin^2 2x = -2(a - 1) \Rightarrow 1 - \cos 4x = -4(a - 1)$

$\Rightarrow \cos 4x = 4a - 3$ so $-1 \le 4a - 3 \le 1$

$\Rightarrow 1/2 \le a \le 1$

34. For $x = n\pi + \pi/2$, $\tan x$ is infinite and all other graphs have finite values.

35. $x + y + z = \pi \Rightarrow \tan x + \tan y + \tan z = \tan x \tan y \tan z$

$\Rightarrow 2k + 3k + 5k = 2k \times 3k \times 5k \Rightarrow k^2 = 1/3.$

$\tan^2 x + \tan^2 y + \tan^2 z = (1/3)(4 + 9 + 25) = 38/3.$

36. $B = 60°$, $\sin(2A + B) = 1/2 \Rightarrow 2A + B = 150°$

$\Rightarrow A = 45°, C = 75°$

so that $B + 2C = 210° = 180° + 30°$

$\Rightarrow \sin(B + 2C) = -\sin 30° = -1/2.$

37. $\frac{a}{b} = \tan\theta\tan\phi$, $\tan^2\alpha = \tan^2(\theta - \varphi)$

$= \left(\frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}\right)^2$

$$= \frac{a^2 - 4(a/b)}{(1 + (a/b)^2} = \frac{ab\,(ab - 4)}{(a+b)^2}$$

Since $\tan^2 a > 0$, $ab > 4$.

38. $\frac{x}{\cos A} = \frac{y}{\cos B}$ so $\frac{x \tan A + y \tan B}{x + y}$

$$= \frac{\cos A \tan A + \cos B \tan B}{\cos A + \cos B}$$

$$= \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}}$$

$$= \tan \frac{A+B}{2}.$$

39. $x^2 + y^2 = a^2[\cos^2 \theta + \sin^2 \theta + \theta^2 \sin^2 \theta + \theta^2 \cos^2 \theta]$ $= a^2(1 + \theta^2)$

$\Rightarrow \quad a\theta = \sqrt{x^2 + y^2 - a^2}$

40. We can write $(1 + \cos(\pi/8))(1 + \cos(3\pi/8))$ $(1 - \cos(3\pi/8))(1 - \cos(\pi/8))$

$= (1 - \cos^2(\pi/8))(1 - \cos^2(3\pi/8))$

$= \sin^2 \pi/8 \sin^2 3\pi/8$

$= \sin^2 \pi/8 \cos^2 \pi/8 = (1/4) \sin^2 \pi/4) = 1/8.$

41. $\cos y = 2 \times 2 \cos 2x \cos x$

$\sin y = 2 \times 2 \sin 2x \cos x$

$\Rightarrow 1 = \cos^2 y + \sin^2 y = 16 \cos^2 x$

So $\cos 2x = 2 \cos^2 x - 1 = -7/8$.

42. $\cos A = 3/5 \Rightarrow \sin A = -4/5$, $\cos B = 4/5$

$\Rightarrow \sin B = 3/5$.

43. $\log_3 \tan 1° \tan 2° \ldots \tan 44° \tan 45° \cot 44° \ldots \cot 1° = \log_3 1 = 0$

44. $\cot^n \theta = \frac{\cos^5 \theta}{\sin^3 \theta} \times \frac{\cos^2 \theta}{\sin \theta} \times \frac{\cos \theta}{\sin^4 \theta} = \cot^8 \theta$

$\Rightarrow n = 8$

45. Given equation is possible if $\sin \theta = \sin 2\theta = \sin 3\theta = \sin 4\theta = 1$ for some θ, $0 < \theta < \pi$ which is not possible.

46. $\cos 11° < \cos 2° \Rightarrow \cos 11° - \cos 2° < 0$, $\cos 11°$ and $\cos 2°$ are rational numbers. $\cos 11° - \cos 2°$ is a negative rational number.

47. $\cos^2 A = \sin A \tan A = \sin^2 A \,.\, \sec A$

$\Rightarrow \cot^2 A = \sec A$.

$\therefore \quad \cot^6 A - \cot^2 A = \cot^2 A(\cot^4 A - 1)$

$= \cot^2 A(\sec^2 A - 1)$

$= \cot^2 A \times \tan^2 A = 1$

48. $(\cos x - \sin x)^2 = 1/4 \Rightarrow 1 - \sin 2x = 1/4 \Rightarrow \sin 2x = 3/4 \Rightarrow \cos 2x = \sqrt{7}/4 = \tan 2x = 3/\sqrt{7}$.

49. (a) $\cos 2A = \sin 3A = \cos(\pi/2 - 3A)$

$\Rightarrow 5A = \pi/2 = A = \pi/10$

(b) $\Rightarrow 10A = \pi/2 \Rightarrow A = \pi/20$,

(c) $\Rightarrow A = \pi/8$, (d) $\Rightarrow A = \pi/6$

Least value of A is $\pi/20$ given by (b).

50. $\frac{n}{4} = \sin^2\left(\frac{\pi}{2n}\right) + \cos^2\left(\frac{\pi}{2n}\right) + \sin\left(\frac{2\pi}{2n}\right)$

$\Rightarrow \quad \sin\left(\frac{\pi}{n}\right) = \frac{n-4}{4}$

For $n \in \mathbf{N}$, $\sin\left(\frac{\pi}{n}\right) > 0$

$\therefore \quad \frac{n-4}{4} > 0$ or $n > 4$

For $n > 4$, $\frac{\pi}{n} < \frac{\pi}{2} \Rightarrow \sin\left(\frac{\pi}{n}\right) < 1$

$\Rightarrow \quad 0 < \frac{1}{4}(n-4) < 1 \Rightarrow 4 < n < 8$

Thus, a possible value of $n = 6$.

In fact, for $n = 6$

$$\sin\left(\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)$$

$$= \sqrt{2} \sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = \sqrt{2} \sin\left(\frac{\pi}{3}\right)$$

$$= \sqrt{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}}{4} = \frac{\sqrt{n}}{4}$$

51. $\tan(2n - 1)\alpha = \tan(\pi/2 - \alpha) = \cot \alpha$.

$\tan(2n - 2)\alpha = \cot 2\alpha$ and so on. So the required value is 1.

52. $x(x^2 - 1) + \sin \theta(x \sin \theta - \cos \theta) + \cos \theta(\sin \theta - x \cos \theta) = 0$

$\Rightarrow \quad x^3 - x + x(\sin^2 \theta - \cos^2 \theta) = 0$

$\Rightarrow \quad x^2 - (1 + \cos 2\theta) = 0$ as $x \neq 0$

$\Rightarrow \quad x^2 = 2 \cos^2 \theta \leq 2$

53. $2 \times \sqrt{3} \cos(2\pi x) = \sqrt{3}$

$\Rightarrow \cos(2\pi x) = 1/2 = \cos(2n\pi \pm \pi/3)$

$\Rightarrow x = n \pm 1/6$.

54. $2 \cos^2 x + 4 \cos x = 3(1 - \cos^2 x)$

$\Rightarrow 5 \cos^2 x + 4 \cos x - 3 = 0$

$\Rightarrow \cos x = \frac{-4 \pm \sqrt{16 + 60}}{10} = \frac{-2 \pm \sqrt{19}}{5}$

But $\cos x \neq \frac{-2 - \sqrt{19}}{5}$

55. $6 \sin^2 x - 2 \cos^4 x = \cos^2 x(2 \cos^2 x - 1)$

$\Rightarrow 4 \cos^4 x + 5 \cos^2 x - 6 = 0$

$\Rightarrow \cos^2 x = 3/4 = \cos 2x = 1/2$

56. $\cos 5\theta = 0 \Rightarrow 5\theta = 2n\pi \pm \pi/2$
As $\cos\theta$ decreases in $[0, \pi/2]$, it is greatest when θ is least,
i.e. $\theta = \pi/10 = 18°$ and $\cos\theta = \cos 18° = \sqrt{\dfrac{5+\sqrt{5}}{8}}$

57. $p\theta = n\pi + q\theta \Rightarrow (p - q)\theta = n\pi \Rightarrow \theta = \dfrac{n\pi}{p-q}$.
So the values of θ form an A.P. with common difference $\pi/(p - q)$.

58. $\dfrac{x^3}{3} - \dfrac{8x^2}{2} + 13x = x \sin(a/x)$
$\Rightarrow x^2 - 12x + 3(13 - \sin(a/x)) = 0$
which has a solution if
$144 - 12(13 - \sin(a/x)) \geq 0$
or $\sin(a/x) \geq 1$ but $\sin(a/x) \leq 1$
so $\sin(a/x) = 1$.

59. $\sin(1 - x) \geq 0$, $\cos x \geq 0$ and $\sin(1 - x) = \cos x$
$\Rightarrow 1 - x = n\pi + (-1)^n (\pi/2 - x)$ where $n \in \boldsymbol{I}$.
For $n = 2m$ we get no values of x, $m \in \boldsymbol{I}$.
$\therefore\ 1 - x = (2m + 1)\pi - \pi/2 + x$
$\Rightarrow x = \dfrac{1}{2} - \dfrac{4m+1}{4}\pi$
For $m \geq 0$, $x < 0$
For $m = -1$, $x = \dfrac{1}{2} + \dfrac{3\pi}{4}$
and $\sin(1-x) = \sin\left(\dfrac{1}{2} - \dfrac{3\pi}{4}\right) = -\sin\left(\dfrac{1}{2} + \dfrac{\pi}{4}\right) < 0$
so $x \neq \dfrac{1}{2} + \dfrac{3\pi}{4}$.
For $m = -2$, $x = \dfrac{1}{2} + \dfrac{7\pi}{4}$ and
$\sin(1 - x) = \sin\left(\dfrac{1}{2} + \dfrac{\pi}{4} - 2\pi\right) > 0$ and
$\cos x = \cos\left(\dfrac{1}{2} + 2\pi - \dfrac{\pi}{4}\right) > 0$.
Hence $x = \dfrac{1}{2} + \dfrac{7\pi}{4}$ is the smallest positive root of the equation.

60. $4\cos^3 x - 4\cos^2 x + \cos x - 1 = 0$
$\Rightarrow (4\cos^2 x + 1)(\cos x - 1) = 0$
$\Rightarrow \cos x = 1 \Rightarrow x = 2n\pi$
$100\pi < 315 < 101\pi$
Required sum $= 2\pi + 4\pi + \ldots 100\pi$
$= 2(1 + 2 + \ldots 50)\pi$
$= \dfrac{2 \times 50 \times 51}{2}\pi = 2550\pi$

61. When $x = 1$, $2 + 2\sin y = 0 \Rightarrow \sin y = -1$
and $\sin\dfrac{7\pi}{2} = \sin\left(3\pi + \dfrac{\pi}{2}\right) = -1$
So $(1, 7\pi/2)$ is a solution.
For $x = -1$, $\sin y = 1$ which is not satisfied by $y = 7\pi/2$ or 0.

62. $e^{\sin x} = 4 + \dfrac{1}{e^{\sin x}}$ (1)
$-1 \leq \sin x \leq 1$ and $e > 1$
$e^{-1} \leq e^{\sin x} \leq e < 3$.
$\Rightarrow e^{\sin x} < 3$, also $e^{\sin x} > 0$.
$\therefore\ \dfrac{1}{e^{\sin x}} + 4 > 4$.
So two sides of (1) cannot be equal for any real value of x.

63. $2\sin 3x \cos 2x = \sin 3x$
$\Rightarrow \sin 3x(2\cos 2x - 1) = 0$
$\Rightarrow \sin 3x = 0$ or $2\cos 2x = 1$,
$\Rightarrow x = 0, \pi/3, 2\pi/3, \pi$
or $x = \pi/6, 5\pi/6$ as $x \in [0, \pi]$.

64. $\sin x, \sin 2x, \sin 3x$ are in A.P.
If $2\sin 2x = \sin x + \sin 3x = 2\sin 2x\cos x$
or if $2\sin 2x(1 - \cos x) = 0$
or if either $\sin 2x = 0$ or $\cos x = 1$
which is satisfied by $x = 2n\pi$ for which $\cos x = 1$.

65. $2\sin^2((\pi/2)\cos^2 x) = 2\sin^2((\pi/2)\sin 2x)$
or if $(\pi/2)\cos^2 x = \pm(\pi/2)\sin 2x$
or if either $\cos x = 0$ or $\cos x = \pm 2\sin x$.
and $\cos x = 0$ if $x = (2n + 1)\pi/2$.

66. $\tan x = t \Rightarrow \dfrac{4t^2}{1+t^2} + t^2 + \dfrac{2}{t^2} = 5$
$\Rightarrow t^6 - 3t^2 + 2 = 0 \Rightarrow (t^2 - 1)(t^4 + t^2 - 2) = 0$
$\Rightarrow t^2 = 1,\ t^2 = -2$ so $t = \pm 1$, $x = \pi/4,\ 3\pi/4$

67. $2\sin^2 x + \dfrac{2\sin x\cos x}{2} = n$
$\Rightarrow \sin 2x + 2(1 - \cos 2x) = 2n$
$\Rightarrow \sin 2x - 2\cos 2x = 2(n - 1)$
$\Rightarrow -\sqrt{5} \leq 2n - 2 \leq \sqrt{5}$
$\Rightarrow 1 - \dfrac{\sqrt{5}}{2} \leq n \leq 1 + \dfrac{\sqrt{5}}{2}$
Integral values of n are 0, 1, 2.

68. $\sin(2\sin x) = \sin(\pi/2 - 2\cos x)$
$\Rightarrow 2\sin x = \pi/2 - 2\cos x$
$\Rightarrow \sin x + \cos x = \pi/4$
$\Rightarrow 1 + \sin 2x = \pi^2/16$

$$\tan x + \cot x = \frac{2}{\sin 2x} = \frac{32}{\pi^2 - 16}$$

$$a = 32, b = 16, c = 2$$

69. $15 \tan^4\alpha + 10 = 6(1 + \tan^2\alpha)^2$
$\Rightarrow \quad 9 \tan^4\alpha - 12\tan^2\alpha + 4 = 0$
$\Rightarrow \quad \tan^2\alpha = 2/3.$
so $8 \operatorname{cosec}^6 \alpha - 27 \sec^6 \alpha$
$= 8(1 + 3/2)^3 - 27(1 + 2/3)^3 = 0$

70. $$\sum_{r=1}^{\infty} \tan^{-1}\frac{1}{2r^2} = \sum_{1}^{\infty}\left[\tan^{-1}(2r+1) - \tan^{-1}(2r-1)\right]$$
$$= \tan^{-1}3 - \tan^{-1}1 + \tan^{-1}5 - \tan^{-1}3 + \dots$$
$$= \pi/2 - \pi/4 = \pi/4$$

71. $[x] + \{x\} = x$ and $[\{x\}] = 0.$
$$\sec\theta - 1 = \frac{1-\cos\theta}{\cos\theta} = \frac{2\sin^2(\theta/2)\times\sin\theta}{\cos\theta\times 2\sin(\theta/2)\cos(\theta/2)}$$
$$= \tan\theta \tan(\theta/2)$$
Given equation reduces to $\cos\theta = 1$
$\Rightarrow \quad \theta = 2n\pi, \quad n \in I$
Number of solutions in $[-2011\pi, 2011\pi]$ is 2011.

72. $$x^2 + y^2 = 3 + 2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right)$$
$$= 3 + 2(-1/2) = 2$$
(Ref. Ex 49,)

73. $\sin\dfrac{\pi x}{2\sqrt{3}} = \left(x - \sqrt{3}\right)^2 + 1 \quad \Rightarrow \quad x = \sqrt{3}.$

74. $\cos\theta \cos 2\theta \dots \cos 2^{n-1}\theta$
$$= \frac{1}{2\sin\theta}\left[2\sin\theta\cos\theta\cos 2\theta\dots\cos 2^{n-1}\theta\right]$$
$$= \frac{1}{2^n\sin\theta}\left(\sin 2^n\theta\right) = \frac{1}{2^n\sin\theta}\sin(\pi+\theta)$$
$$= \frac{1}{2^n\sin\theta}(-\sin\theta) = -\frac{1}{2^n}.$$
So statement-2 is true which implies statement-1 is also true.

75. $f(x) = \tan x$, $g(x) = \tan(x/2)$
$\Rightarrow f(x) + g(x) \neq 0$ for all x.
so statement-2 is False.
we have $[\cos\alpha + \cos(\alpha + \beta)]^2 + [\sin\alpha + \sin(\alpha + \beta)]^2 = 1$
$\Rightarrow \quad 2 + 2\cos(\alpha - \alpha - \beta) = 1$
$\Rightarrow \quad \cos\beta = -1/2 \quad \Rightarrow \quad \beta = 2\pi/3.$
Similarly $\gamma = 2\pi/3$.
$\Rightarrow \quad f(\beta) + g(\gamma) = \tan(2\pi/3) + \tan(\pi/3) = 0$
$\Rightarrow$ statement-1 is true.

76. Statement-2 is True. In statement-1, $x = 3\sqrt{3}\cos\theta$, $y = 3\sqrt{3}\sin\theta$ then $x - y = 3\sqrt{3}(\cos\theta - \sin\theta) = 3\sqrt{3}\times\sqrt{2}\cos(\theta + \pi/4) \leq 3\sqrt{6}$ follows from statement-2

77. If $\alpha + \beta + \gamma = \pi/2$ $\tan(\alpha + \beta) = \tan(\pi/2 - \gamma)$
$= \dfrac{1}{\tan\gamma}.$
$$\Rightarrow \quad \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{1}{\tan\gamma}$$
$\Rightarrow \quad \tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = 1$
$\Rightarrow$ statement-2 is False.
In statement-1, $\left(\sqrt{p} + \sqrt{q} + \sqrt{r}\right)^2$
$$= p + q + r + 2\left(\sqrt{pq} + \sqrt{qr} + \sqrt{rp}\right)$$
$$\leq p + q + r + 2(p + q + r)$$
$$= 3(p + q + r) = 48$$

78. Statement-2 is True.
Statement-1 $\Rightarrow y\cos\theta \cdot \sin^2\theta + y\cos^3\theta$
$= \sin\theta\cos\theta \Rightarrow y\cos\theta = \sin\theta\cos\theta$
$\Rightarrow \quad y = \sin\theta \quad \Rightarrow \quad x = \cos\theta$
$\Rightarrow x^2 + y^2 = 1 \Rightarrow$ statement-1 is also true but does not follow from statement-1.

79. $x^2 - x - 2 = (x - 2)(x + 1) < 0$
$\Rightarrow -1 < x < 2 \Rightarrow$ Statement-2 is True
$2\sin^2 x + 3\sin x - 2$
$= (2\sin x - 1)(\sin x + 2) > 0 \Rightarrow \sin x > 1/2$
which is true if $\pi/6 < x \leq \pi/2$
i.e. $\dfrac{11}{21} < x < \dfrac{11}{7}$, so statement-1 is false.

80. Statement-2 is True as $-1 \leq \cos x \leq 1$.
In statement-1, $\cos^7 x + \sin^4 x = 1$
$\Rightarrow \cos^7 x + (1 - \cos^2 x)^2 = 1$
$\Rightarrow \cos^7 x + \cos^4 x - 2\cos^2 x = 0$
$\Rightarrow \cos^2 x(\cos^5 x + \cos^2 x - 2) = 0$
$\Rightarrow$ Either $\cos x = 0$ or $\cos^5 x + \cos^2 x - 2 = 0$
$\Rightarrow \cos x = 1$ from statement-2
$\Rightarrow x = -\pi/2, \pi/2$ as $\cos x = 1 \Rightarrow x = 0$

Level 2

81. Let the length of the diagonals of the rhombus be a and b and x be its side then

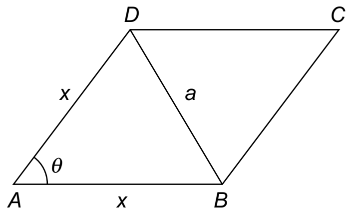

Fig. 25.5

$\frac{x}{a} = \frac{b}{x} \Rightarrow x^2 = ab$

Now $\cos\theta = \frac{x^2 + x^2 - a^2}{2x \cdot x} = \frac{2ab - a^2}{2ab} = 1 - \frac{a}{2b}$

Similarly $\cos(180° - \theta) = 1 - \frac{b}{2a}$

$\Rightarrow \frac{a}{2b} = 1 - \cos\theta,\ \frac{b}{2a} = 1 + \cos\theta$

$\Rightarrow 1 - \cos^2\theta = \frac{1}{4} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = 30°$

82. Let $\underline{|ACB} = \theta$

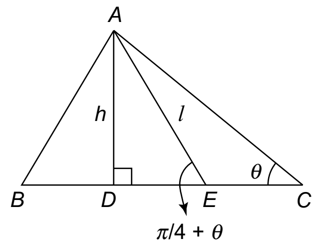

Fig. 25.6

$\underline{|BAE} = \underline{|EAC} = \pi/4$

$\Rightarrow \underline{|AED} = \pi/4 + \theta\ l\ \underline{|EAD} = \pi/4 - \theta$

From $\Delta\ ADE$

$\frac{h}{l} = \cos(\pi/4 - \theta) = \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)$

$\frac{\sqrt{l^2 - h^2}}{l} = \sin(\pi/4 - \theta) = \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)$

$\Rightarrow \frac{h - \sqrt{l^2 - h^2}}{l} = \frac{1}{\sqrt{2}} \times 2\sin\theta = \sqrt{2}\sin C$

$\Rightarrow \frac{h - \sqrt{l^2 - h^2}}{\sqrt{2}l} = \cos B.$

83. $2\sin^2(x + \pi/4) + \sqrt{3}\cos 2x > 0$

$\Rightarrow 1 - \cos(2x + \pi/2) + \sqrt{3}\cos 2x > 0$

$\Rightarrow \frac{1}{2}\sin 2x + \frac{\sqrt{3}}{2}\cos 2x > -\frac{1}{2}$

$\Rightarrow \cos(2x - \pi/6) > -\frac{1}{2}$

84. $x = \frac{1}{1 - \cos^2 x} = \frac{1}{\sin^2 x},\ y = \frac{1}{\cos^2 x}$

and $z = \frac{1}{1 - \sin^2 x \cos^2 x} = \frac{1}{1 - \frac{1}{x} \cdot \frac{1}{y}} = \frac{xy}{xy - 1}$

$\Rightarrow z(xy - 1) = xy = x + y$

$\Rightarrow x + y + z = xyz.$

85. $\frac{4\sin^4 x}{4\cos^2 x - 4\sin^2 x \cos^2 x} = \frac{1}{9}$

$\Rightarrow \frac{\sin^4 x}{\cos^4 x} = \frac{1}{9} \Rightarrow \tan^4 x = \frac{1}{9} \Rightarrow \tan x = \frac{1}{\sqrt{3}}$

$\Rightarrow x = \pi/6$

86. Let $\theta = 7.5° \Rightarrow 2\theta = 15° = 45° - 30°$

$\Rightarrow \cos 2\theta = \frac{1}{2\sqrt{2}}\left(\sqrt{3} + 1\right)$

$\Rightarrow 2\cos^2\theta = \frac{1}{2\sqrt{2}}\left(\sqrt{3} + 1\right) + 1$

$\Rightarrow \cos^2\theta = \frac{\sqrt{3} + 1 + 2\sqrt{2}}{2 \times 2\sqrt{2}} = \frac{\sqrt{6} + \sqrt{2} + 4}{8}$

87. We have

$A = a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta$

$B = a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta$

$H = (c - a)\sin 2\theta + 2b\cos 2\theta$

So $A + B = a + c$

88. $\text{Sin}5\theta = \sin(2\theta + 3\theta)$

$= \sin 2\theta\cos 3\theta + \cos 2\theta\sin 3\theta$

$= 2\sin\theta\cos\theta \times \cos\theta(4\cos^2\theta - 3) + (1 - 2\sin^2\theta)(3\sin\theta - 4\sin^3\theta)$

$= 2\sin\theta(1 - \sin^2\theta)(1 - 4\sin^2\theta) + (1 - 2\sin^2\theta)(3\sin\theta - 4\sin^3\theta)$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

So $a = 16,\ b = -20,\ c = 5,\ d = 0$

$\Rightarrow 5a + 3b - 4c = 0$

89. $\cot(A - B) = \frac{\cot A\cot B + 1}{\cot B - \cot A} = \frac{y/x + 1}{y} = \frac{x + y}{xy}$

90. $\frac{\cos 3\theta}{\cos^3\theta} + \frac{\sin 3\theta}{\sin^3\theta}$

$= \frac{4\cos^3\theta - 3\cos\theta}{\cos^3\theta} + \frac{3\sin\theta - 4\sin^3\theta}{\sin^3\theta}$

$= 4 \times 3\ \frac{\left(\cos^2\theta - \sin^2\theta\right)}{4\sin^2\theta\cos^2\theta} = \frac{12\cos 2\theta}{\sin 2\theta} \times \frac{1}{\sin 2\theta}$

91. The given expression is equal to
$x^2 + y^2 + xy\ [(\tan^2\alpha + \cot^2\alpha) - 4\cot^2 2\alpha]$
$= x^2 + y^2 + xy$
$$\left[\frac{1-2\sin^2\alpha\cos^2\alpha}{\sin^2\alpha\cos^2\alpha} - 4\frac{(\cos^2\alpha - \sin^2\alpha)^2}{4\sin^2\alpha\cos^2\alpha}\right]$$
$= x^2 + y^2 + xy$
$$\left[\frac{1}{\sin^2\alpha\cos^2\alpha} - 2 - \frac{1-4\sin^2\alpha\cos^2\alpha}{\sin^2\alpha\cos^2\alpha}\right]$$
$= x^2 + y^2 + 2xy = (x + y)^2$

92. $S_1 = \tan A + \tan B + \tan C = 4/3$
$S_2 = \Sigma \tan A \tan B = 1,$
$S_3 = \tan A \tan B \tan C = -1/3$
$$\tan(A + B + C) = \frac{S_1 - S_3}{1 - S_2} \to \infty$$
$\Rightarrow A + B + C = \pi/2$

93. We have $\sin\theta\,[x + 2y\cos\theta + z(3 - 4 + 4\cos^2\theta)]$
$= 4\cos\theta\,(2\cos^2\theta - 1)\sin\theta$
$\Rightarrow 8\cos^3\theta - 4z\cos^2\theta - 2(y + 2)\cos\theta = x - z$
$(\because \sin\theta \neq 0)$

94. We have $16\sin^5 x - 20\sin^3 x + 5\sin x = 5\sin x$ (see Ex 68)
$\Rightarrow \sin^3 x\,(16\sin^2 x - 20) = 0 \Rightarrow \sin x = 0$, only one value

95. $\sin\alpha + \sin\beta = -b/a$, $\sin\alpha\sin\beta = c/a$,
$\sin\alpha + 2\sin\beta = 1$
$\Rightarrow \sin\alpha = \dfrac{c}{a+b}$ which satisfies the given equation (Note $c \neq 0$) and the required value is 0.

96. $(1 + \sin\theta)(3\sin\theta + 4\cos\theta + 5)$
$= (\cos(\theta/2) + \sin(\theta/2))^2\,[6\sin(\theta/2)\cos(\theta/2)$
$+ 4(2\cos^2\theta/2 - 1) + 5]$
$= (a + b)^2\,[6ab + 8b^2 + a^2 + b^2]$
$= (a + b)^2\,(a + 3b)^2$

97. $\sin A = \sin B$ and $\cos A = \cos B$
$\Rightarrow A = 2n\pi + B$ where n is an integer, $n \neq 0$
$$\Rightarrow \frac{A-B}{2} = n\pi \Rightarrow \tan\frac{A-B}{2} = 0$$

98. Apply the identity, if $A + B + C = 180°$
then $\cos A + \cos B + \cos C$
$= 1 + 4\sin(A/2)\sin(B/2)\sin(C/2)$

99. We can write $\tan x = \dfrac{\tan^2 x - 1 + 1}{\tan x}$
$= \cot x - 2\cot 2x$

so $\tan x + \dfrac{1}{2}\tan\dfrac{x}{2} + \dfrac{1}{2^2}\tan\dfrac{x}{2^2} + \cdots$
$+ \dfrac{1}{2^{n-1}}\tan\dfrac{x}{2^{n-1}}$
$= \cot x - 2\cot 2x + \dfrac{1}{2}\left(\cot\dfrac{x}{2} - 2\cot x\right)$
$+ \dfrac{1}{2^2}\left(\cot\dfrac{x}{2^2} - 2\cot\dfrac{x}{2}\right)$
$+ \cdots + \dfrac{1}{2^{n-1}}\left(\cot\dfrac{x}{2^{n-1}} - 2\cot\dfrac{x}{2^{n-2}}\right)$
$$= \frac{1}{2^{n-1}}\cot\left(\frac{x}{2^{n-1}}\right) - 2\cot 2x$$

Note Remember $\tan x + 2\cot 2x = \cot x$

100. We can write $\dfrac{4 + \cot 76°\cot 16° - 1}{\cot 76° + \cot 16°}$
$$= \frac{4\sin 76°\sin 16°}{\sin 92°} + \cot 92°$$
$$= \frac{2(\cos 60° - \cos 92°) + \cos 92°}{\sin 92°} = \frac{1 - \cos 92°}{\sin 92°}$$
$= \cot 46° = \tan 44°$

101. $K\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right) = \cos\theta + \cos(\theta + 2\pi/3)$
$+ \cos(\theta + 4\pi/3)$
$= \cos\theta + 2\cos(\theta + \pi)\cos\pi/3$
$= \cos\theta - \cos\theta = 0$
$\Rightarrow \quad xy + yz + zx = 0$

102. Let $\tan A\tan B = x$
so $\tan(A + B) = \tan(\pi/3) = \sqrt{3}$
$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A\tan B} = \sqrt{3}$$
$\Rightarrow \sqrt{3}\,(1 - x) = \tan A + \tan B \le 2\sqrt{\tan A\tan B} = 2\sqrt{x}$
$\Rightarrow 3(1 - x)^2 \le 4x$
$\Rightarrow 3x^2 - 10x + 3 \le 0$
$\Rightarrow (3x - 1)(x - 3) \le 0$
$\Rightarrow \dfrac{1}{3} \le x \le 3$

103. $\tan\theta\,(3\tan\theta + 3) = (2 + 2\tan\theta)^2$
$\Rightarrow (1 + \tan\theta)(4 + \tan\theta) = 0$
$\Rightarrow \tan\theta = -4$ as $\tan\theta \neq -1$
$$\text{So } \frac{7 - 5\cot\theta}{9 - 4\sqrt{\sec^2\theta - 1}} = \frac{7 + 5/4}{9 + 16} = \frac{33}{100}$$

104. We have
$\sin^2\theta + \sin^4\theta + \sin^6\theta + \cdots$
$- (\sin\theta\cos\theta + \sin^2\theta\cos^2\theta + \sin^3\theta + \cos^3\theta + \cdots)$

$= \dfrac{\sin^2\theta}{1-\sin^2\theta} - \dfrac{\sin\theta\cos\theta}{1-\sin\theta\cos\theta}$

$= \dfrac{1-\cos 2\theta}{1+\cos 2\theta} - \dfrac{\sin 2\theta}{(2-\sin 2\theta)}$

$= \dfrac{1-ab}{1+ab} + \dfrac{1-a^2}{3-a^2}$

105. $\left(1+\sqrt{1+x}\right)\ \tan\alpha = 1-\sqrt{1-x}$

$\Rightarrow \dfrac{1+\sqrt{1+x}}{1-\sqrt{1-x}} = \dfrac{\cos\alpha}{\sin\alpha} = \dfrac{2\cos\alpha\,(\cos\alpha+\sin\alpha)}{2\sin\alpha\,(\cos\alpha+\sin\alpha)}$

$= \dfrac{2\cos^2\alpha + 2\sin\alpha\cos\alpha}{2\sin\alpha\cos\alpha + 2\sin^2\alpha}$

$= \dfrac{1+\cos 2\alpha + \sin 2\alpha}{\sin 2\alpha + 1 - \cos 2\alpha}$

$= \dfrac{1+\sqrt{(\cos 2\alpha + \sin 2\alpha)^2}}{1-\sqrt{(\cos 2\alpha - \sin 2\alpha)^2}}$

$= \dfrac{1+\sqrt{(1+\sin 4\alpha)}}{1-\sqrt{1-\sin 4\alpha}}$

106. $f(\theta) = \sin\theta\,(\sin\theta + 3\sin\theta - 4\sin^3\theta)$

$= 4\sin^2\theta\,(1-\sin^2\theta) = \sin^2 2\theta \geq 0$ for all real θ.

107. $l\cot B + l\cot C = h = 2\sqrt{2}\,l$

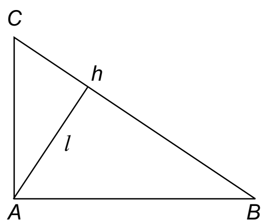

Fig. 25.7

$\Rightarrow \cot B + \cot C = 2\sqrt{2}$

$B + C = \pi/2 \Rightarrow \cot B \cot C = 1$

$\Rightarrow \cot B = \sqrt{2}-1,\ \cot C = \sqrt{2}+1$

The required angles are $\pi/8$ and $3\pi/8$

108. $\sqrt{\cos^2 x - \sin^2 x} + \cos x + \sin x = 2\sqrt{\sin x + \cos x}$

$\Rightarrow \sqrt{\cos x + \sin x}\left[\sqrt{\cos x - \sin x} + \sqrt{\cos x + \sin x} - 2\right]$

$= 0$

$\Rightarrow$ either $\sin x + \cos x = 0$

or $\sqrt{\cos x - \sin x} + \sqrt{\cos x + \sin x} = 2$

$\Rightarrow 2\cos x + 2\sqrt{\cos 2x} = 4$

$\Rightarrow 2\cos^2 x - 1 = (2-\cos x)^2 = 4 - 4\cos x + \cos^2 x$

$\Rightarrow \cos^2 x + 4\cos x - 5 = 0$

$\Rightarrow \cos x = 1 \Rightarrow x = 2n\pi$

Note The answer could be obtained by trial method, taking $x = 2n\pi$

109. $\sin x + 2\sin 2x - \sin 3x = 3$

$\Rightarrow \sin 3x \geq 0 \Rightarrow 0 \leq 3x \leq \pi$

$\Rightarrow 0 \leq x \leq \pi/3 \Rightarrow 0 \leq \sin x \leq \sqrt{3}/2$

and $0 \leq 2x \leq 2\pi/3 \Rightarrow 0 \leq \sin 2x \leq 1$

$-1 \leq -\sin 3x \leq 0$

So $-1 \leq \sin x + \sin 2x - \sin 3x \leq \sqrt{3}/2 + 1 < 3$

So the given equation is not satisfied for any real value of x

110. $\sin x + \sin y = \sin (x + y)$

$\Rightarrow 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2} = 2\sin\dfrac{x+y}{2}\cos\dfrac{x+y}{2}$

$\Rightarrow \sin\dfrac{x+y}{2}\left(\cos\dfrac{x+y}{2} - \cos\dfrac{x-y}{2}\right) = 0$

$\Rightarrow$ either $\sin\dfrac{x+y}{2} = 0$ or $\cos\dfrac{x+y}{2} = \cos\dfrac{x-y}{2}$

Now $\sin\dfrac{x+y}{2} = 0$ and $|x| + |y| = 1$

$(x, y) = \left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$ or $\left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$

and $\cos\dfrac{x+y}{2} = \cos\dfrac{x-y}{2}$

$\Rightarrow \sin\dfrac{x}{2}\sin\dfrac{y}{2} = 0$

$\Rightarrow \sin\dfrac{x}{2} = 0$ or $\sin\dfrac{y}{2} = 0$

$\Rightarrow (x, y) = (1, 0), (-1, 0), (0, 1), (0, -1)$

So the required number of pairs is 6.

Previous Years' AIEEE/JEE Main Questions

1. $\tan x + \sec x = 2\cos x$

$\Rightarrow\ 1 + \sin x = 2\cos^2 x = 2(1-\sin^2 x)$

$\Rightarrow\ (1+\sin x)(1 - 2 + 2\sin x) = 0$

$\Rightarrow\ \sin x = -1$ or $\sin x = \dfrac{1}{2}$

But $\sin x = -1 \Rightarrow \cos x = 0$. This is not possible

Thus $\sin x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6} \in [0, 2\pi]$

2. $\sin\alpha + \sin\beta = \dfrac{-21}{65}, \cos\alpha + \cos\beta = \dfrac{-27}{65}$

Squaring and adding we get

$$2 + 2\cos(\alpha - \beta) = \frac{441+729}{(65)^2} = \frac{1170}{(65)^2} = \frac{18}{65}$$

$$\Rightarrow \quad 2 \times 2\cos^2\frac{\alpha-\beta}{2} = \frac{18}{65}$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2} = \pm\frac{3}{\sqrt{130}}$$

$\pi < \alpha - \beta < 3\pi$

$$\Rightarrow \quad \frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{3\pi}{2}$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2} = \frac{-3}{\sqrt{130}}$$

3. We have

$$u = \sqrt{\frac{a^2}{2}(1+\cos 2\theta) + \frac{b^2}{2}(1-\cos 2\theta)} + \sqrt{\frac{a^2}{2}(1-\cos 2\theta) + \frac{b^2}{2}(1+\cos 2\theta)}$$

$$= \sqrt{\frac{a^2+b^2}{2} + \frac{a^2-b^2}{2}\cos 2\theta} + \sqrt{\frac{a^2+b^2}{2} - \frac{a^2-b^2}{2}\cos 2\theta}$$

Squaring we get

$$u^2 = a^2 + b^2 + 2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2 - \left(\frac{a^2-b^2}{2}\right)^2\cos^2 2\theta}$$

Thus max $(u^2) = a^2 + b^2 + a^2 + b^2 = 2(a^2 + b^2)$

and min $(u^2) = a^2 + b^2 + 2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2 - \left(\frac{a^2-b^2}{2}\right)^2}$

$$= a^2 + b^2 + 2ab = (a+b)^2$$

So the required difference $= 2(a^2 + b^2) - (a+b)^2 = (a-b)^2$.

4. $f(x) = \sin x - \sqrt{3}\cos x + 1$

$$= 2\left[\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right] + 1$$

$$= 2\sin(x - \pi/3) + 1$$

$$-2 \le 2\sin(x - \pi/3) \le 2$$

$$\Rightarrow \quad -2 + 1 \le 2\sin(x - \pi/3) + 1 \le 2 + 1$$

$$\Rightarrow \quad -1 \le f(x) \le 3$$

$$\Rightarrow \quad S = [-1, 3]$$

5. $2\sin^2 x + 5\sin x - 3 = 0$

$$\Rightarrow \quad (2\sin x - 1)(\sin x + 3) = 0$$

$$\Rightarrow \quad \sin x = 1/2 \qquad [\because \sin x \ne -3]$$

$$\Rightarrow \quad x = \pi/6,\ 5\pi/6,\ 13\pi/6,\ 17\pi/6.$$

6. Let $\alpha = \tan 30°$, $\beta = \tan 15°$, then $\alpha = \frac{1}{\sqrt{3}}$ and $\beta = 2 - \sqrt{3}$.

$$\therefore \quad -p = \frac{1}{\sqrt{3}} + (2 - \sqrt{3})$$

and $$q = \frac{1}{\sqrt{3}}(2-\sqrt{3}) = \frac{2}{\sqrt{3}} - 1$$

Thus $$q - p = \frac{3}{\sqrt{3}} + 2 - \sqrt{3} - 1 = 1$$

$$\therefore \quad 2 + q - p = 3$$

7. $2\sin x\cos x = (\sin x + \cos x)^2 - 1$

$$= 1/4 - 1 = -3/4$$

As $0 < x < \pi$, $\sin x > 0$.

Thus, $\cos x < 0$

$$\Rightarrow \quad \pi/2 < x < \pi$$

Also, $(\sin x - \cos x)^2$

$$= \sin^2 x + \cos^2 x - 2\sin x\cos x$$

$$= 1 + 3/4 = 7/4$$

$$\Rightarrow \quad \sin x - \cos x = \frac{\sqrt{7}}{2}$$

Since $\sin x + \cos x = 1/2$

We get

$$2\sin x = (\sqrt{7} + 1)/2$$

and $$2\cos x = (1 - \sqrt{7})/2$$

$$\therefore \quad \tan x = -\frac{\sqrt{7}+1}{\sqrt{7}-1} = -\frac{(\sqrt{7}+1)^2}{7-1}$$

$$= -\frac{1}{3}(4 + \sqrt{7})$$

8. $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -3/2$

$\Rightarrow 2[\cos\beta\cos\gamma + \cos\gamma\cos\alpha + \cos\alpha\cos\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha + \sin\alpha\sin\beta] + (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + (\cos^2\gamma + \sin^2\gamma) = 0$

$\Rightarrow (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$

$\Rightarrow \cos\alpha + \cos\beta + \cos\gamma = 0$

and $\sin\alpha + \sin\beta + \sin\gamma = 0$

Thus, both A and B are true.

9. We have

$$\frac{r}{R} = \cos\left(\frac{\pi}{n}\right)$$

When

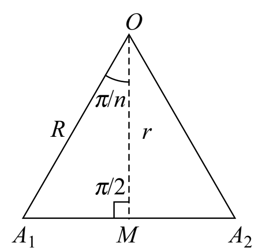

Fig. 25.8

$$\frac{r}{R} = \frac{\sqrt{3}}{2}, \text{ we get}$$

$$\cos\left(\frac{\pi}{n}\right) = \cos\left(\frac{\pi}{6}\right)$$

$\Rightarrow \quad n = 6$

when $\frac{r}{R} = \frac{3}{3} = \cos\frac{\pi}{n}$

there is no value of n

10. $0 \le \alpha, \beta \le \pi/4$

$\Rightarrow 0 \le \alpha + \beta \le \pi/2$ and $-\pi/4 \le \alpha - \beta \le \pi/4$

Now, $\cos(\alpha + \beta) = 4/5$

$\Rightarrow \quad \tan(\alpha + \beta) = 3/4$

and $\sin(\alpha - \beta) = 5/13 \quad \Rightarrow \quad \tan(\alpha - \beta) = \pm 5/12$

We have

$$\tan(2\alpha) = \tan[(\alpha + \beta) + (\alpha - \beta)]$$

$$= \frac{\tan(\alpha+\beta) + \tan(\alpha-\beta)}{1 - \tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \left(\frac{3}{4}\right)\left(\frac{5}{12}\right)} = \frac{\frac{14}{12}}{\frac{33}{48}} = \frac{56}{33}$$

Note $\tan(\alpha - \beta) = -\frac{5}{12}$ does not satisty any of the choices.

11. $A = \sin^2 x + \cos^4 x \le \sin^2 x + \cos^2 x = 1$

$[\because 0 \le \cos^2 x \le 1 \Rightarrow \cos^4 x \le \cos^2 x]$

Also, $A = 1 - \cos^2 x + \cos^4 x$

$= (\cos^2 x - 1/2)^2 + 3/4 \ge 3/4$

Thus, $\frac{3}{4} \le A \le 1$

12. We have

$\sin\theta + \sin(4\theta) + \sin(7\theta) = 0$

$\Rightarrow \quad 2\sin(4\theta)\cos(3\theta) + \sin(4\theta) = 0$

$\Rightarrow \quad \sin(4\theta)[2\cos(3\theta) + 1] = 0$

$\Rightarrow \quad \sin(4\theta) = 0$ or $\cos(3\theta) = -1/2 = \cos(2\pi/3)$

$\Rightarrow \quad \theta = \pi/4, \pi/2, 3\pi/4$ or $3\theta = 2\pi/3, 2\pi \pm 2\pi/3$

$\Rightarrow \quad \theta = \frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{4\pi}{9}, \frac{3\pi}{4}, \frac{8\pi}{9}$

13. We have

$(3\sin P + 4\cos Q)^2 + (3\cos P + 4\sin Q)^2 = 6^2 + 1$

$\Rightarrow \quad 9(\sin^2 P + \cos^2 P) + 16(\cos^2 Q + \sin^2 Q)$

$+ 24(\sin P\cos Q + \cos P\sin Q) = 37$

$\Rightarrow \quad \sin(P + Q) = 1/2 \quad \Rightarrow \quad \sin(\pi - R) = 1/2$

$\Rightarrow \quad \sin R = 1/2 \quad \Rightarrow \quad R = \pi/6$ or $5\pi/6$

$R = 5\pi/6 \quad \Rightarrow \quad P < \pi/6 \quad \Rightarrow \quad 3\sin P < 3/2$

$\Rightarrow \quad 3\sin P + 4\cos Q < 6$, so $R = \pi/6$.

14. Let $\angle BDC = \phi$, then $\angle DBA = \phi$. We have

$$\sin\phi = \frac{p}{BD} = \frac{p}{\sqrt{p^2 + q^2}}$$

and $$\cos\phi = \frac{q}{\sqrt{p^2 + q^2}}$$

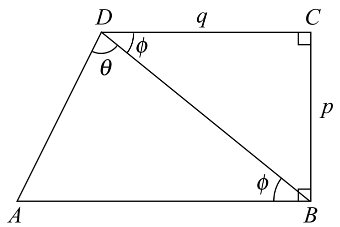

Fig. 25.9

By the law of sines

$$\frac{\sin(\pi - \theta - \phi)}{BD} = \frac{\sin\theta}{AB}$$

$$AB = \frac{\sin\theta}{\sin(\theta + \varphi)} BD$$

$$= \frac{\sqrt{p^2 + q^2}\sin\theta}{\sin\theta\cos\varphi + \cos\theta\sin\varphi}$$

$$= \frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

15. $$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$$

$$= \frac{\tan^2 A}{\tan A - 1} - \frac{1}{\tan A(\tan A - 1)}$$

$$= \frac{\tan^3 A - 1}{\tan A(\tan A - 1)} = \frac{\tan^2 A + \tan A + 1}{\tan A}$$

$$= \frac{\sec^2 A}{\tan A} + 1 = \sec A \operatorname{cosec} A + 1$$

16. $2\cos^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad 2\sin^2\theta + 3\sin\theta - 2 = 0$

$\Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow$ Statement-2 is True.

Next, $2\sin^2\theta - \cos 2\theta = 0$

$\Rightarrow \quad 4\sin^2\theta = 1 \quad \Rightarrow \quad \sin\theta = \pm\frac{1}{2}$

Which gives four values of θ in $[0, 2\pi]$

$2\cos^2\theta - 3\sin\theta = 0$ gives two values of θ in $[0, 2\pi]$

So, Statement-1 is also true but does not follow from Statement-2.

17. $\sin 2x - 2\cos x + 4\sin x = 4$

$\Rightarrow \quad (2\cos x + 4)(\sin x - 1) = 0$

$\Rightarrow \quad \sin x = 1 \quad \Rightarrow \quad x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

18. $f_k(x) = \frac{1}{k}\left(\sin^k x + \cos^k x\right)$

$\Rightarrow f_4(x) - f_6(x) = \frac{1}{4}\left(\sin^4 x + \cos^4 x\right) - \frac{1}{6}\left(\sin^6 x + \cos^6 x\right)$

$= \frac{1}{4}\left(1 - 2\sin^2 x \cos^2 x\right) - \frac{1}{6}\left(1 - 3\sin^2 x \cos^2 x\right)$

$= \frac{1}{12}$

19. $2\sin^3\alpha - 7\sin^2\alpha + 7\sin\alpha = 2$

$\Rightarrow \quad 2(\sin^3\alpha - 1) - 7\sin\alpha(\sin\alpha - 1) = 0$

$\Rightarrow \quad (\sin\alpha - 1)[2(\sin^2\alpha + \sin\alpha + 1) - 7\sin\alpha] = 0$

$\Rightarrow \quad (\sin\alpha - 1)[2\sin^2\alpha - 5\sin\alpha + 2] = 0$

$\Rightarrow \quad (\sin\alpha - 1)(2\sin\alpha - 1)(\sin\alpha - 2) = 0$

$\Rightarrow \quad \alpha = \pi/6, 5\pi/6, \pi/2 \in [0, 2\pi]$

$\therefore$ There are 3 solutions.

20. $\cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{\cot(\theta/2) - 1}{\cot(\theta/2) + 1}$

$= \frac{\cos(\theta/2) - \sin(\theta/2)}{\cos(\theta/2) + \sin(\theta/2)}$

$= \frac{\cos^2(\theta/2) - \sin^2(\theta/2)}{1 + 2\sin(\theta/2)\cos(\theta/2)} = \frac{\cos\theta}{1 + \sin\theta}$

$\Rightarrow \quad \cot^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 - \sin\theta}{1 + \sin\theta} = \frac{\operatorname{cosec}\theta - 1}{\operatorname{cosec}\theta + 1} = \frac{q}{p}$

$\left|\cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right| = \sqrt{\frac{q}{p}}$

21. $2\cos\theta + \sin\theta = 1$ (1)

$\Rightarrow \quad 4\cos^2\theta = (1 - \sin\theta)^2$

$\Rightarrow \quad 4(1 - \sin^2\theta) = (1 - \sin\theta)^2$

$\Rightarrow \quad 4(1 + \sin\theta) = 1 - \sin\theta \quad (\because \theta \neq \pi/2)$

$\Rightarrow \quad \sin\theta = -\frac{3}{5}$ (2)

From (1) and (2) $\cos\theta = 4/5$

Thus, $7\cos\theta + 6\sin\theta$

$= 7\left(\frac{4}{5}\right) + 6\left(-\frac{3}{5}\right) = 2$

22. $f(\theta) = 1 + \cos^2\theta - \sin^2\theta + 1 + \sin 2\theta$

$= 2 + \cos(2\theta) + \sin(2\theta)$

$= 2 + \sqrt{2}\sin\left(2\theta + \frac{\pi}{4}\right)$

$A = \text{Max}(f(\theta)) = 2 + \sqrt{2}$

$B = \text{Min}(f(\theta)) = 2 - \sqrt{2}$

$\therefore \ (A, B) = \left(2 + \sqrt{2}, 2 - \sqrt{2}\right)$

23. $\cos\alpha + \cos\beta = 3/2$

$\Rightarrow \quad 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = \frac{3}{2}$ (1)

and $\sin\alpha + \sin\beta = \frac{1}{2}$

$\Rightarrow \quad 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = \frac{1}{2}$ (2)

From (1) and (2), we get

$\tan\theta = \frac{1}{3}$, where $\theta = \frac{1}{2}(\alpha + \beta)$

$\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{2\left(\frac{1}{3}\right)}{1 + \left(\frac{1}{3}\right)^2} = \frac{2}{3} \times \frac{9}{10} = \frac{3}{5}$

24. $2 + \sqrt{3} = \frac{a}{b} = \frac{\sin A}{\sin B}$

$\Rightarrow \quad 2 + \sqrt{3} = \frac{\sin(120° - B)}{\sin B}$

$\Rightarrow \quad (2 + \sqrt{3})\sin B = \sin 120° \cos B - \cos 120° \sin B$

$\Rightarrow \quad (2 + \sqrt{3} + \cos(180° - 60°))\sin B$

$= \sin(180° - 60°)\cos B$

$\Rightarrow \quad \left(2 + \sqrt{3} - \frac{1}{2}\right)\sin B = \frac{\sqrt{3}}{2}\cos B$

$\Rightarrow \quad \tan B = \frac{\sqrt{3}}{3 + 2\sqrt{3}} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$

$\Rightarrow \quad B = 15°$ [use $\tan 15° = \tan(60° - 45°)$]

$\therefore A = 105°$

$\Rightarrow \quad (A, B) = (105°, 15°)$

25. $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$

$\Rightarrow (\cos 4x + \cos x) + (\cos 3x + \cos 2x) = 0$

$\Rightarrow 2\cos\frac{5x}{2}\cos\frac{3x}{2} + 2\cos\frac{5x}{2}\cos\frac{x}{2} = 0$

$\Rightarrow 2\cos\frac{5x}{2}\left[\cos\frac{3x}{2} + \cos\frac{x}{2}\right] = 0$

$\Rightarrow 2\cos\frac{5x}{2}\left[2\cos x \cdot \cos\frac{x}{2}\right] = 0$

$\Rightarrow\ \cos\frac{x}{2}=0, \cos x = 0 \text{ or } \cos\frac{5x}{2}=0$

$\Rightarrow\ \frac{x}{2}=\frac{\pi}{2}; x=\frac{\pi}{2},\frac{3\pi}{2}; \frac{5x}{2}=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\frac{9\pi}{2}$

$\Rightarrow\ x=\pi,\frac{\pi}{2},\frac{3\pi}{2},\frac{\pi}{5},\frac{3\pi}{5},\frac{7\pi}{5},\frac{9\pi}{5}$

26. Let $E = 4+\frac{1}{2}\sin^2 2x - 2\cos^4 x$

$= 4+\frac{1}{2}\sin^2 2x - \frac{1}{2}(2\cos^2 x)^2$

$= 4+\frac{1}{2}\left(1-\cos^2 2x\right)-\frac{1}{2}(1+\cos 2x)^2$

$= 4+\frac{1}{2}(1+\cos 2x)[1-\cos 2x - 1 - \cos 2x]$

$= 4-\frac{1}{2}(1+\cos 2x)(2\cos 2x)$

$= 4 - \cos 2x - \cos^2 2x$

$= \frac{17}{4}-\left(\frac{1}{2}+\cos 2x\right)^2$

Note that

$m = \frac{17}{4}-\frac{9}{4}=2$ when $\cos 2x = 1$

and $M = \frac{17}{4}$ when $\cos 2x = -1/2 \Rightarrow M - m = \frac{9}{4}$

27. Let $a=\sqrt{2\sin^4 x + 18\cos^2 x},\ \ b=\sqrt{2\cos^4 x+18\sin^2 x}$

Now, $2a^2 = (2\sin^2 x)^2 + 18(2\cos^2 x)$

$= (1 - \cos 2x)^2 + 18(1 + \cos 2x)$

$= 19 + 16 \cos 2x + \cos^2 2x$

and $2b^2 = (2\cos^2 x)^2 + 18(2\sin^2 x)$

$= (1 + \cos 2x)^2 + 18(1 - \cos 2x)$

$= 19 - 16\cos 2x + \cos^2 2x$

We have $|a - b| = 1$

and $2(a^2 - b^2) = 32\cos 2x$

If $a - b = 1$, $2(a - b)(a + b) = 32 \cos 2x$

$\Rightarrow a + b = 16 \cos 2x$

$\therefore\ 2a = 1 + 16 \cos 2x$

$\Rightarrow 4a^2 = (1 + 16 \cos 2x)^2$

$= 1 + 32 \cos 2x + 256 \cos^2 2x$

$\Rightarrow 38 + 32 \cos 2x + 2 \cos^2 2x$

$= 1 + 32 \cos 2x + 256 \cos^2 2x$

$\Rightarrow\ 37 = 254\cos^2 2x \Rightarrow \cos 2x = \pm\sqrt{\frac{37}{254}}$

which gives us 8 solutions in $[0, 2\pi]$.

When $a - b = -1$, we get the same 8 solutions in $[0, 2\pi]$

28. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (2\sin x+\cos x)\begin{vmatrix} 1 & \sin x & \sin x \\ 1 & \cos x & \sin x \\ 1 & \sin x & \cos x \end{vmatrix}$$

Using $C_2 \to C_2 - (\sin x)C_1$

and $C_3 \to C_3 - (\sin x)C_1$,

we get

$$\Delta = (2\sin x+\cos x)\begin{vmatrix} ü & & \\ 1 & \cos x - \sin x & 0 \\ 1 & 0 & \cos x - \sin x \end{vmatrix}$$

$= (2\sin x + \cos x)(\cos x - \sin x)^2$

$\therefore\ \ \Delta = 0 \Rightarrow \tan x = -\frac{1}{2}, 1$

As $-\pi/4 \le x \le \pi/4$, $-1 \le \tan x \le 1$

and $\tan x$ is one-to-one in the interval $[-\pi/4, \pi/4]$.

Thus, there are two values of x.

29. Let $A = \frac{\pi}{12}+\theta$, $B = \frac{\pi}{12}-\theta$

where $-\pi/12 < \theta < \pi/12$

We have

$S = \tan A + \tan B$

$= \frac{\sin(A+B)}{\cos A \cos B} = \frac{2\sin(\pi/6)}{\cos(A+B)+\cos(A-B)}$

$= \frac{2(1/2)}{\left(\sqrt{3}/2\right)+\cos(2\theta)}$

$= \frac{2}{\sqrt{3}+\cos(2\theta)}$

As $-\pi/6 < 2\theta < \pi/6$

$\Rightarrow\ \sqrt{3}/2 < \cos(2\theta) \le 1$

$\Rightarrow\ \sqrt{3} < 2\cos(2\theta) \le 2$

$\Rightarrow\ \sqrt{3}+\sqrt{3} < \sqrt{3}+2\cos(2\theta) \le \sqrt{3}+2$

$\Rightarrow\ \frac{1}{\sqrt{3}+2\cos(2\theta)} \ge \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3}$

30. $\theta \in P$

$\Leftrightarrow\ \sin\theta - \cos\theta = \sqrt{2}\cos\theta$

$\Leftrightarrow\ \sin\theta = (\sqrt{2}+1)\cos\theta$

$\Leftrightarrow\ \sin\theta = \frac{1}{\sqrt{2}-1}\cos\theta$

$\Leftrightarrow\ (\sqrt{2}-1)\sin\theta = \cos\theta$

$\Leftrightarrow\ \sqrt{2}\sin\theta = \sin\theta + \cos\theta$

$\Leftrightarrow\ \theta \in Q$

$\therefore\ P = Q$

Previous Years' B-Architecture Entrance Examination Questions

1. See Question No. 1 in AIEEE/JEE Main Questions
2. $\cos\theta + \sec\theta = 2 \Rightarrow \cos^2\theta - 2\cos\theta + 1 = 0$

   $\Rightarrow \cos\theta = \sec\theta = 1 \Rightarrow \cos^n\theta + \sec^n\theta = 2$
3. $(\sin^2x + \cos^2x)^3 = \sin^6x + \cos^6x + 3\sin^2x\cos^2x(\sin^2x + \cos^2x)$

   $$\Rightarrow\quad 1 = p + \frac{3}{4}\sin^2 2x$$

   $$\Rightarrow\quad \sin^2 2x = \frac{4}{3}(1-p)$$

   $$\Rightarrow\quad 0 \le \frac{4}{3}(1-p) \le 1$$

   $$\Rightarrow\quad \frac{1}{4} \le p \le 1 \Rightarrow p \in \left[\frac{1}{4}, 1\right]$$
4. $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$

   $$= 2\left(\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right)$$

   $$= \frac{1}{2}\left[\left(1+\cos\frac{\pi}{4}\right)^2 + \left(1+\cos\frac{3\pi}{4}\right)^2\right]$$

   $$= \frac{1}{2}\left[\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right] = \frac{1}{2}\times 2\left(1+\frac{1}{2}\right) = \frac{3}{2}$$
5. $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$

   $$= \sin\left(\frac{\pi}{2}-\frac{3\pi}{7}\right)\sin\left(\frac{\pi}{2}-\frac{2\pi}{7}\right)\sin\left(\frac{\pi}{2}-\frac{\pi}{7}\right)$$

   $$= \cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}$$

   $$= \frac{1}{2\sin\frac{\pi}{7}}\cdot\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$$

   $$= \frac{1}{4\sin\frac{\pi}{7}}\sin\frac{4\pi}{7}\cos\frac{3\pi}{7}$$

   $$= \frac{1}{8\sin\frac{\pi}{7}}\left[\sin\frac{7\pi}{7} + \sin\frac{\pi}{7}\right] = \frac{1}{8}$$
6. $0 \le \alpha, \beta \le \pi/4$

   $\Rightarrow 0 \le \alpha + \beta \le \pi/2 \Rightarrow -\pi/4 \le \alpha - \beta \le \pi/4$

   Now $\cos(\alpha + \beta) = 4/5 \Rightarrow \tan(\alpha + \beta) = 3/4$

   and $\sin(\alpha - \beta) = 5/13 \Rightarrow \tan(\alpha - \beta) = 5/12$

   we have $\tan 2\alpha = \tan[(\alpha + \beta) + (\alpha - \beta)]$

   $$= \frac{\tan(\alpha+\beta) + \tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$$

   $$= \frac{(3/4)+(5/12)}{1-(3/4)(5/12)} = \frac{14/12}{33/48} = \frac{56}{33}$$

   Same as Q10 AIEEE/JEE Main
7. $1 + a\sin 2\beta = \sin^2\beta + \cos^2\beta - 2\sin\beta\cos\beta\cos A$

   $\Rightarrow \cos A = -a$

   $\Rightarrow A$ is obtuse.

   $\Rightarrow A = 120°$

   $\Rightarrow a = -\cos 120° = \frac{1}{2}$

   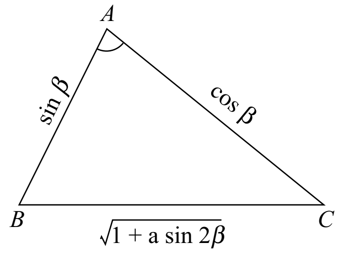

   Fig. 25.10
8. $\cos 25° + \sin 25° = k$

   Squaring we get

   $1 + \sin 50° = k^2$

   $$\Rightarrow\quad \cos 50° = \sqrt{1-\left(k^2-1\right)^2} = k\sqrt{2-k^2}$$
9. $\tan 9° - \tan 27° - \tan 63° + \tan 81°$

   $= \tan 9° + \tan(90° - 9) - (\tan 27° + \tan(90° - 27°))$

   $= \tan 9° + \cot 9° - (\tan 27° + \cot 27°)$

   $$= \frac{2}{\sin 18°} - \frac{2}{\sin 54°} = \frac{2\times 4}{\sqrt{5}-1} - \frac{2\times 4}{\sqrt{5}+1} = 4$$
10. $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$

    $$= \frac{a+b+a-b}{\sqrt{a^2-b^2}} = \frac{2a}{a\sqrt{1-\frac{b^2}{a^2}}}$$

    $$= \frac{2}{\sqrt{1-\tan^2\theta}} = \frac{2\cos\theta}{\sqrt{\cos 2\theta}}$$
11. Same as Question No. 9.
12. $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$

    $\Rightarrow a^2 + b^2 + c^2 - (ab + bc + ca) = 0$

    $\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

    $\Rightarrow a = b = c \Rightarrow A = B = C = 60°$

$\Rightarrow \sin^2 A + \sin^2 B + \sin^2 C = 3\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4}$

13. L.H.S. $= \frac{\sqrt{2\cos\theta\sin\theta + 1}}{\sin\theta} = \frac{\pm(\cos\theta + \sin\theta)}{\sin\theta}$

$= \pm [\cot \theta + 1] = k - \cot \theta$

$\Rightarrow k = -1$

14. $f(x) = 2 \sin x + \sin 2x = 2 \sin x + 2 \sin x \cos x$

$= 2 \sin x (1 + \cos x)$

$f'(x) = 2 \cos x (1 + \cos x) - 2 \sin x \sin x$

$= 2 [\cos x + \cos^2 x - (1 - \cos^2 x)]$

$= 2 [2\cos^2 x + \cos x - 1]$

$= 2 (\cos x + 1) (2 \cos x - 1)$

Now, $f'(x) = 0 \Rightarrow x = \pi/3, \pi$ ($\because 0 \le x \le 3\pi/2$)

We have $f(0) = 0$, of $\left(\frac{3\pi}{2}\right) = -2$

$f\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \frac{3}{2}\sqrt{3},$

and $f(\pi) = 0, f\left(\frac{3\pi}{2}\right) = -2$

Thus, maximum value is $\frac{3}{2}\sqrt{3}$.

15. Using $C_1 \to C_1 + C_3$, we get

$$\Delta = \begin{vmatrix} 1 & \tan\theta + \sec^2\theta & 3 \\ 0 & \cos\theta & \sin\theta \\ 0 & -4 & 3 \end{vmatrix}$$

$= 3\cos \theta + 4\sin \theta$

$\frac{d\Delta}{d\theta} = -3\sin\theta + 4\cos\theta$

$\frac{d\Delta}{d\theta} = 0 \Rightarrow \frac{\sin\theta}{4} = \frac{\cos\theta}{3} = \frac{1}{5}$

$\Rightarrow \sin \theta = 4/5, \cos \theta = 3/5.$

Max $\Delta = \max \{\Delta(0), \Delta(\sin^{-1} (4/5)), \Delta\left(\frac{\pi}{2}\right)\}$

$= \max\left\{3, \frac{24}{5}, 4\right\} = \frac{24}{5}$

min $\Delta = 3$

$\therefore \Delta \in [3, 5]$