Chapter 3

Friction, Centre of Gravity, Moment of Inertia

CHAPTER HIGHLIGHTS

- *Introduction*
- *Laws of friction*
- *Force determinations for different scenarios*
- *Cone of friction*
- *Virtual work*
- *Lifting machine*
- *Reversible and irreversible machine*
- *Screw jack*
- *Differential screw jack*
- *Centre of gravity*
- *Theorems of Pappus–Guldinus*
- *Area moment of inertia*
- *Centroid of solids*
- *Mass moment of inertia*

Introduction

Friction Definitions

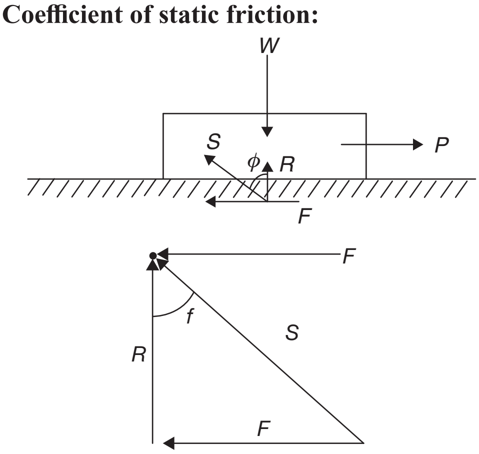
1. **Static friction:** It is the friction between two bodies which is a tangential force and opposes the sliding of one body relative to the other.
2. **Limiting friction:** It is the maximum value of the static friction that occurs when motion is impending.
3. **Kinetic friction:** It is the tangential force between two bodies after motion begins. Its value is less than the corresponding static friction.
4. **Angle of friction:** It is the angle between the action line of the total reaction of one body on another and the normal to the common tangent between the bodies when motion is impending.

   It is also defined as the angle made by the resultant (S) of the normal reaction (R) and the limiting force of friction (F) with the normal reaction R (see the following figure). It is denoted by ϕ. From the figure, we have:

$$\boxed{\tan\phi = \frac{F}{R} = \frac{\mu R}{R}}$$

   μ = Coefficient of friction

5. **Coefficient of static friction:**

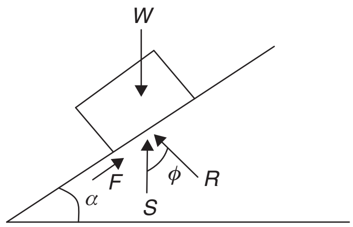


It is defined as the ratio of the limiting force of friction (F) to the normal reaction (R) between two bodies (see the above figure, where a solid body rests on a horizontal plane). It is denoted by μ.

$$\mu = \frac{\text{Limiting force of friction}}{\text{Normal reaction}} = \frac{F}{R}$$

$$\boxed{\therefore F = \mu R}$$

6. Angle of repose:



The above figure shows a block of weight W on a rough inclined plane which is inclined at an angle α with the horizontal. Let R be the normal reaction and F be the force of friction. From applying the condition of equilibrium, algebraic sum of the forces resolved along the plane:

$$\boxed{W\sin\alpha = F} \qquad (1)$$

Algebraic sum of the forces resolved perpendicular to the plane:

$$\boxed{W\cos\alpha = R} \qquad (2)$$

From Eqs. (1) and (2):

$$\tan\alpha = \frac{F}{R}$$

But, $\tan\phi = \dfrac{F}{R}$

$$\boxed{\therefore \text{Angle of plane} = \text{Angle of friction}}$$

Suppose the angle of the plane α is increased to a value ϕ, so that the block is at the point of sliding or the state of impending motion occurs, then at this angle:

$$\mu = \tan\lambda = \tan\alpha$$

$$\therefore \lambda = \alpha$$

Hence, the angle of repose is defined as the angle to which an inclined plane may be raised before an object resting on it will move under the action of the force of gravity and the reaction of the plane.

Hence, Angle of repose = Angle of plane.

Laws of Friction

First Law

Friction always opposes motion and comes into play only when a body is urged to move. Frictional force always acts in a direction opposite to that in which the body tends to move.

Second Law

The magnitude of the frictional force is just sufficient to prevent the body from moving. That is, only as much resistance as required to prevent motion is offered as friction.

Third Law

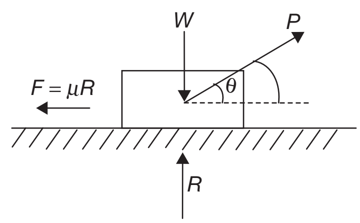
The limiting frictional force or resistance bears a constant ratio with the normal reaction. This ratio depends on the nature of the surfaces in contact. The limiting frictional resistance is independent of the area of contact.

Fourth Law

When motion takes place as one body slides over the other, the magnitude of the frictional force or resistance will be slightly less than the offered force at that condition of limiting equilibrium. The magnitude of the frictional force will depend only on the nature of the sliding surfaces and independent of the shape or extent of the contact surfaces.

Force Determinations for Different Scenarios

Least force is required to drag a body on a rough horizontal plane:

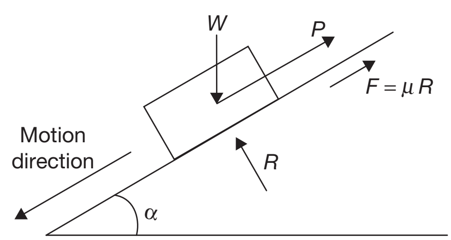


Force 'P' is applied, at an angle θ to the horizontal, on a block of weight W, such that the motion impends or the block tends to move.

$$\text{Force,}\quad P = \frac{W\sin\phi}{\cos(\theta-\phi)}$$

Least force required, $P_{\text{least}} = W\sin\phi$

Force 'P' acting on a block (weight = W) along a rough inclined plane:



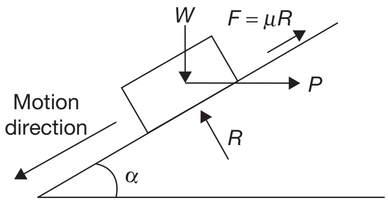
$$\boxed{\text{For motion down the plane, } P = \frac{W\sin(\alpha-\phi)}{\cos\phi}}$$

$$\boxed{\text{For motion up the plane, } P = \frac{W\sin(\alpha-\phi)}{\cos\phi}}$$

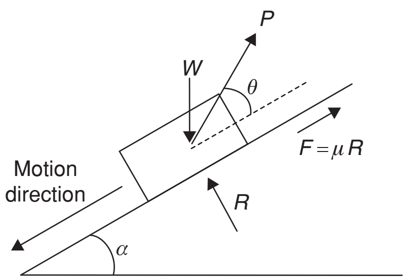
Force 'P' acting horizontally on a block (weight = W) resting on a rough inclined plane:

For motion down the plane, $P = W\tan(\alpha-\phi)$

For motion up the plane, $P = W\tan(\alpha + \phi)$



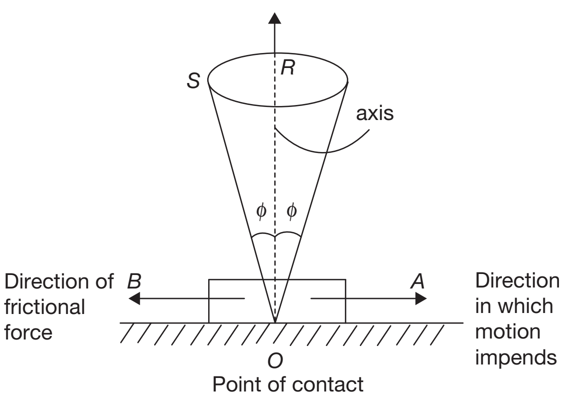
Force 'P' acting, at an angle θ to the plane, on a block (weight = W) resting on a rough inclined plane:



$$\text{For motion down the plane, } P = \frac{W\sin(\alpha - \phi)}{\cos(\theta + \phi)}$$

$$\text{For motion up the plane: } P = \frac{W\sin(\alpha + \phi)}{\cos(\theta - \phi)}$$

Cone of Friction

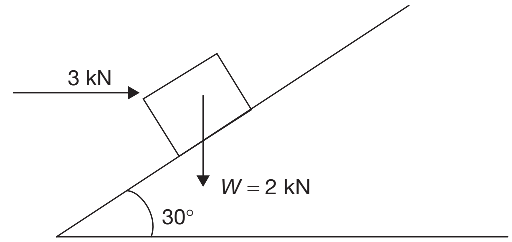


Let OR represent the normal reaction offered by a surface on a body, and let the direction of impending motion be along OA while the direction in which the frictional force acts is in the opposite direction, i.e., along OB. Assuming that the body is in a state of limiting equilibrium, the resultant reaction S makes an angle of ϕ with the normal OR. If the body slides in any other direction, the resultant reaction S will still make the same angle ϕ with the normal. It is, therefore, seen that when limiting equilibrium is maintained, then the line of action of the resultant reaction should always lie on the surface of an inverted right circular cone whose semi-vertical angle is ϕ. This cone is known as the cone of friction.

SOLVED EXAMPLE

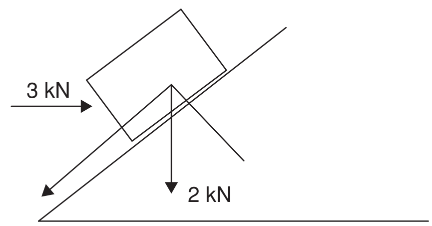
Example 1

Determine whether the 2 kN block (shown in the figure below) will be held in equilibrium by a horizontal force of 3 kN? The coefficient of static friction is 0.3 and the maximum value of frictional force is:



(A) 0.96 kN (B) 0.86 kN
(C) 0.75 kN (D) 0.65 kN

Solution



Applying the conditions of equilibrium and summing the force parallel and perpendicular to the plane, we have:

$\Sigma F(\text{parallel to the plane}) = 0$

$$-F - 2\sin 30° + 3\cos 30° = 0$$

$$F = -2 \times \frac{1}{2} + 3 \times 0.866$$

$$= -1 + 2.598 = 1.598 \text{ kN}$$

$$\Sigma F(\text{perpendicular to the plane}) = 0$$

$$R - 2\cos 30° - 3\sin 30° = 0$$

$$R = 2 \times 0.866 + 3 \times 0.5 = 1.732 + 1.5$$

$$= 3.232 \text{ kN}$$

This indicates that the value of F necessary to hold the block from moving up the plane is 1.598 kN. However, the maximum value obtainable as the frictional force,

$$F_f = \mu R = 0.3 \times 3.232 = 0.9696 \text{ kN}$$

This means that the block will move up the plane. Hence, the correct answer is option (A).

Example 2

An effort of 2 kN is required just to move a certain body up an inclined plane of angle 15°, the force acting parallel to the plane. If the angle of inclination of the plane is made 20°, the effort required, again applied parallel to the plane, is found to be 2.3 kN. Find the weight of the body and the coefficient of friction.

(A) 3.9 kN, 0.258 (B) 4.5 kN, 0.26
(C) 3.8 kN, 0.24 (D) 3.8 kN, 0.268

Solution

Let W be the weight of the body, μ be the coefficient of friction and P be the effort when the inclination of the plane is α.

Applying the conditions of equilibrium and summing the forces parallel and perpendicular to the plane, we have:

ΣF(parallel to the plane) = 0

$$P - \mu R - W\sin\alpha = 0 \quad (1)$$

ΣF(perpendicular to the plane) = 0

$$R - W\cos\alpha = 0 \quad (2)$$

liminating R from Eqs. (1) and (2), we have:

$P = \mu W\cos\alpha + W\sin\alpha$ or

$$P = W(\mu\cos\alpha + \sin\alpha) \quad (3)$$

When $\alpha = 15°$, $P = 2$ kN, and when $\alpha = 20°$, $P = 2.3$ kN. Substituting in Eqs. (3), we have:

$$2 = W(\mu\cos\alpha + \sin\alpha)$$

$$2 = W(\mu\cos 15° + \sin 15°) \quad (4)$$

$$2.3 = W(\mu\cos 20° + \sin 20°) \quad (5)$$

Dividing Eq. (5) by Eq. (4), we have:

$$\frac{2}{2.3} = \frac{\mu\cos 15° + \sin 15°}{\mu\cos 20° + \sin 20°}$$

$$\frac{2}{2.3} = \frac{\mu \times 0.966 + 0.258}{\mu \times 0.939 + 0.342}$$

or $$\mu[(2.3 \times 0.969) - (2 \times 0.939)] = [(2 \times 0.342) - (2.3 \times 0.258)]$$

or $$0.3507\mu = 0.0906$$

$$\mu = \frac{0.0906}{0.3507} = 0.258$$

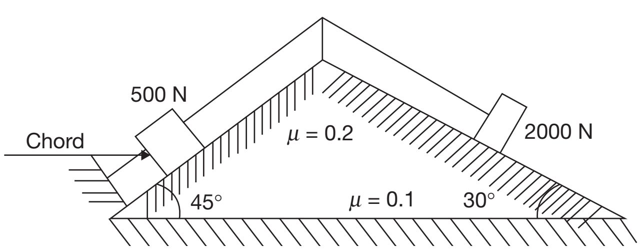
From Eq. (5),

$$2.3 = W[0.258 \times 0.939 + 0.342]$$
$$= W(0.242 + 0.342) = 0.584\ W$$
$$W = \frac{2.3}{0.584} = 3.938 \text{ kN}$$

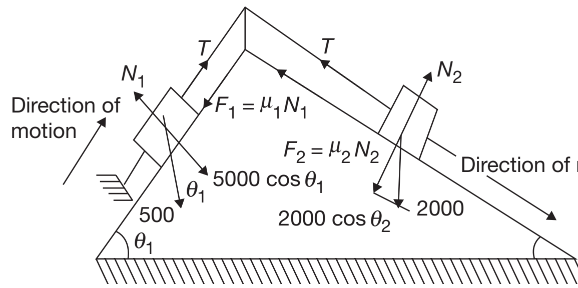
Hence, the correct answer is option (A).

Example 3

Find the maximum tension in the chord shown in the figure, if the bodies have developed full friction.



Solution



For mass 1,

$$\Sigma H = 0 \text{ gives } T = F_1 + 500\sin\theta_1$$
$$= \mu_1 N_1 + 500\sin\theta_1$$
$$\Sigma V = 0 \text{ gives } N_1 = 500\cos\theta_1$$
$$\Rightarrow T = 500\sin\theta_1 + \mu_1 \times 500\cos\theta_1$$
$$= 500\sin 45 + 0.2 \times 500\cos$$
$$45 = 424.3 \text{ N.}$$

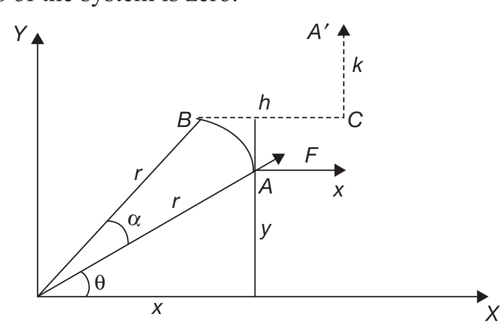
Virtual Work

Virtual displacement: Virtual displacement is defined as an infinitesimal (exceedingly small) displacement given hypothetically to a particle or to a body or to a system of bodies in equilibrium consistent with the constraints. The displacement is only imagined and it does not have to take place. For this reason it is called 'virtual displacement'.

Virtual work: Virtual work is defined as the work done by a force on a body due to a small virtual (i.e., imaginary) displacement of the body.

Principle of Virtual Work

If a system of forces acting on a body or a system of bodies be in equilibrium and if the system be assumed to undergo a small displacement consistent with the geometrical conditions, then the algebraic sum of the virtual work done by the forces of the system is zero.



To illustrate the principle of work, let us consider a body at equilibrium at a point A. A force 'F' acts on the body and displaces it to the point A', where the displacement consisting:

1. Very small rotation through angle α about the origin of the rectangular 2-D coordinate system, say origin O in the xy plane.
2. Very small displacement h along the X-axis, and
3. A very small displacement k along the Y-axis.

If the components of the force F along the X-axis and Y-axis are F_x and F_y respectively, then work done by the force 'F' when its point of application is displaced from point A to A'.

$$= hF_x + kF_y + \alpha(xF_y - yF_x)$$

If a system of forces act on the body where h, k, and α are the same for every force, then work done by all the forces,

$$= h\sum F_x + k\sum F_y + \alpha\sum(xF_y - yF_x)$$

where ΣF_x and ΣF_y are the sums of the resolved parts of the forces along the X-axis and Y-axis respectively, and $\Sigma(xF_y - yF_x)$ is the moments of the forces about origin O.

Since the system is in equilibrium, all the three terms in the above expression, for the work done by all the forces, is zero. Hence, the sum of the virtual works done by the forces is zero.

Lifting Machine

Lifting machines are defined as those appliances or machines which are used for lifting heavy loads. They are also called 'simple machines'. Some commonly used machines are:

1. Lever
2. Inclined plane
3. Wedge
4. Wheel and axle
5. Winch crab
6. A pulley and system of pulleys
7. Screw jack

Screw jack is the most important among all the above mentioned simple machines.

Load or Resistance

A machine has to either lift a load or overcome a resistance. It is usually denoted by W and its unit is N.

Example: A lifting device lifts a load or heavy weight, whereas a bicycle overcomes the frictional resistance between the wheels and the road.

Efforts

It is the force which is applied to a machine to lift a load or to overcome resistance against a movement. It is usually denoted by P and its unit is N.

Example: Force applied on the pedals of a bicycle or on the handle of a screw jack.

Input of a Machine

It is defined as the amount of total work done on the machine. This is measured by the product of the effort and the distance through which it moves.

$$\text{Input} = \text{Effort} \times \text{Distance moved by the effort}$$
$$= P \times y$$

It has the unit of Nm.

Output of a Machine

It is defined as the amount of work got out of a machine or the actual work done by the machine.

Output of the machine

$$= \text{Load} \times \text{Distance through which load is lifted}$$
$$= W \times x$$

It has the unit of Nm.

Velocity Ratio (VR)

It is defined as the ratio of the distance moved by the effort to the distance moved by the load during the same interval of time.

$$\text{VR} = \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}} = \frac{y}{x}$$

NOTE

In all machines $y > x$.

Mechanical Advantage (MA)

It is defined as the ratio of the load or weight lifted to the effort applied.

$$\text{MA} = \frac{\text{Weight lifted}}{\text{Effort applied}} = \frac{W}{P}$$

NOTE

In all machines $W > P$.

Ideal Machine

It is defined as the machine which is absolutely free from frictional resistances. In such a machine, input = output.

For an ideal machines, VR = MA.

Efficiency of a Machine

It is the ratio of output of the machine to the input of the machine.

$$\eta = \frac{\text{Output of the machine} \times 100}{\text{Input of the machine}}$$

$$= \frac{\text{Useful work done by the machine} \times 100}{\text{Energy supplied to the machine}}$$

$$= \frac{W \times x}{P \times y} \times 100$$

For an ideal machine, $\eta = 100\%$. For an actual machine:

$$\eta = \frac{\text{Ideal effort}}{\text{Actual effort}} = \frac{\text{Actual load}}{\text{Ideal load}}.$$

Relation between MA, VR, and η

$$\boxed{\eta = \frac{W \times x}{P \times y} = \frac{\frac{W}{P}}{\frac{y}{x}} = \frac{\text{MA}}{\text{VR}}}$$

Frictional Losses

Output = Input − Losses due to friction

$$\boxed{\text{Effort lost in friction} = P - \frac{W}{\text{VR}}}$$

Loss in load lifted due to friction $= P \times \text{VR} - W$

Here, P is the actual effort required to overcome resistance W or lift load W.

Reversible and Irreversible Machine

A machine is said to be reversible when the load W gets lowered on the removal of the effort. In such a case, work is done by the machine in reverse direction.

A machine is said to be irreversible when the load W does not fall down on the removal of the effort. In such a case, work is not done in the reverse direction.

The condition of irreversibility or self-locking of a machine is that its efficiency should be less than 50%.

Compound Efficiency

It is defined as the overall efficiency of the combination of machines and it is the product of the efficiencies of the individual machines.

The overall efficiency η of n machines coupled together is $\eta = \prod_{i=1}^{n} \eta_i$, where η_i is the efficiency of the ith machine.

Law of a Machine

It is defined as the relationship which exists between the effort applied and the load lifted.

$$\boxed{P = mW + C}$$

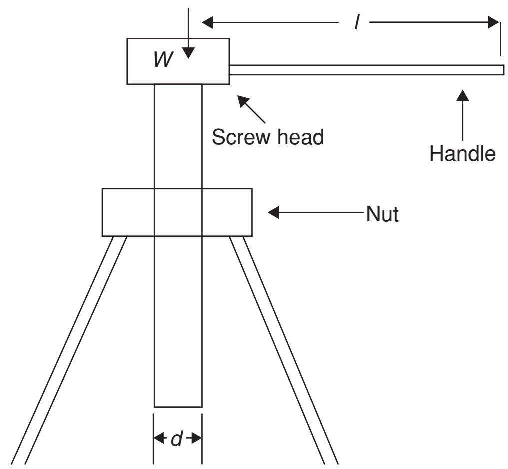
P is the effort applied, W is the corresponding load, m and C are coefficients which are determined in any machine after conducting a series of tests and plotting the W versus P graph.

The expression for maximum mechanical advantage is given by $(\text{MA})_{\max} = \frac{1}{m}$.

The expression for maximum efficiency is given by $\eta_{\max} = \frac{1}{m \times (\text{VR})}$.

Screw Jack

It is a device for lifting heavy loads by applying comparatively a smaller effort at the end of the handle. The screw jack works on the principle of inclined plane.



It mainly consists of a nut which forms the body of the jack and a screw is fitted into it. The threads are generally square. The load 'W' is placed on the head of the screw. By rotating the screw with a handle the load is lifted or lowered. Let W be the load lifted, α be the angle of helix of the screw and ϕ be the angle of friction.

Here, Efficiency $= \frac{\tan\alpha}{\tan(\alpha+\phi)}$, which shows that efficiency is independent of the load lifted or lowered.

Assuming that the effort is applied at the end of the handle, let us consider the following two cases.

Case 1: Let the weight W be lifted.

Let, P_E be the effort applied at the end of the handle. Let l be the length of the handle, and let d be the mean diameter of the screw.

Σm about the axis is zero.

Let p be the pitch and μ be the coefficient of friction, then:

$$\tan\alpha = \frac{p}{\pi d}$$

$$\tan\phi = \mu$$

$$\boxed{p_E = \frac{Wd}{2l} \cdot \frac{p + \mu\,\pi d}{\pi d - \mu p}}$$

Case 2: Let the weight W be lowered.

Let Q be the effort applied at the circumference of the screw, and let Q_E be the actual effort applied at the end of the handle.

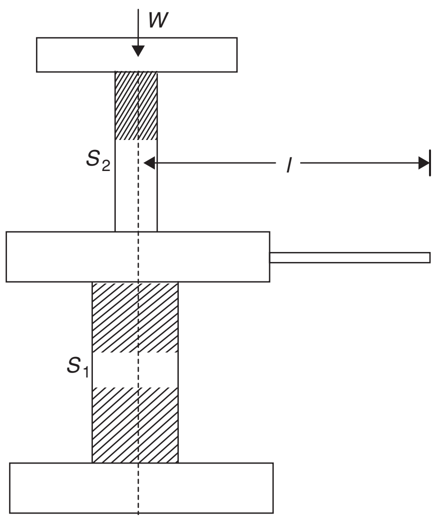
$$Q = W\tan(\phi - \alpha)$$

$$Q_E = \frac{Wd}{2l}\cdot\frac{\mu\pi d - p}{\pi d + \mu p}$$

For an n-threaded screw, $\tan\alpha = \dfrac{np}{\pi d}$.

Differential Screw Jack

Instead of only one threaded spindle as in the case of a simple screw jack it has two threaded spindles S_1 and S_2. The spindle S_1 is screwed to the base which is fixed.



This spindle carries both internal as well as external threads. The spindle S_2 is engaged to spindle S_1 by means of an internal thread. When spindle S_1 ascends, spindle S_2 descends. This is also known as 'differential screw' jack. The principle of working of this jack is similar to the one as described in the given figure.

Let p_{s_1} = Pitch of the threads on S_1

p_{s_2} = Pitch of the threads on S_2

Let the lever length be 'l' and the effort be applied at the end of this lever.

When the lever is moved by one revolution, the distance covered by the effort P is $2\pi l$, and correspondingly, the load distance is equal to $p_{s_1} - p_{s_2}$.

Then, Velocity ratio $(\text{VR}) = \dfrac{2\pi l}{p_{s_1} - p_{s_2}}$.

NOTE

p_{s_1} is always greater than p_{s_2}. Due to this difference, the mechanical advantage as well as the velocity ratio will be more.

Direction for solved examples 4 and 5:
A screw jack has a pitch of 12 mm with a mean radius of thread equal to 25 mm a lever 500 mm long is used to raise a load of 1500 kg. The coefficient of friction is 0.10.

Example 4

Find the helix angle α and θ (i.e., friction angle).

(A) 6.2°, 4.5° (B) 4.85°, 5.7°
(C) 4.85°, 5.7° (D) 4.36°, 5.7°

Solution

Given, $P = 12$ mm, $d = 2r = 25 \times 2 = 50$ mm,

$$l = 500 \text{ mm}$$
$$W = 1500 \text{ kg}, \mu = 0.10, \tan\phi = \mu = 0.10,$$
$$\phi = 5.71°$$
$$\tan\alpha = \frac{P}{\pi d} = \frac{12}{\pi \times 50} = 0.076$$
$$\alpha = 4.36°.$$

Example 5

What force is necessary when applied normal to the lever at its free end?

(A) 13.319 kg (B) 12.8 kg
(C) 14.5 kg (D) 18.3 kg

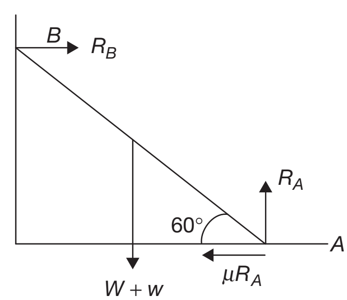
Solution

$$P = \frac{wd}{2l}\tan(\alpha + \phi)$$
$$= \frac{1500 \times 50}{2 \times 500} \times \tan(4.36 + 5.71)$$
$$P = 13.319 \text{ kg}.$$

Direction for solved examples 6 to 8:
A uniform ladder of weight 500 N and the length 8 m rests on a horizontal ground and leans against a smooth vertical wall. The angle made by the ladder with the horizontal is 60°. When a man of weight 500 N stands on the ladder at a distance of 4 metre from the top of the ladder, the ladder is at the point of sliding.

Example 6

Find the coefficient of friction in terms of R_B.



(A) $\mu = \frac{R_B}{1000}$ (B) $\mu = 1400\ R_B$

(C) $\mu = 500\ R_B$ (D) $\mu = \frac{R_B}{500}$

Solution

Resolving all the forces $R_B = \mu R_A$:

$$R_A = W + w = 500 + 500 = 1000$$

$$R_B = \mu \times R_A = \mu \times 1000$$

$$\mu = \frac{R_B}{1000}.$$

Example 7

Find the reaction at B (i.e., R_B).

(A) 289 (B) 300

(C) 350 (D) 400

Solution

Taking moment at A, $M_A = 0$:

$$R_B \times 8 \times \frac{\sqrt{3}}{2} = 500 \times \frac{8}{2} \times \frac{1}{2} + 500 \times 4 \times \frac{1}{2}$$

$$R_B = \frac{500 \times 2 + 1000}{6.92} = 289.$$

Example 8

Find the value of coefficient of friction.

(A) 0.370 (B) 0.486

(C) 0.289 (D) 0.355

Solution

From equation $\mu = \frac{R_B}{1000} = \frac{289}{1000} = 0.289$

CENTRE OF GRAVITY

The centre of gravity of a body is the point, through which the whole weight of the body acts, irrespective of the position in which the body is placed. This can also be defined as the centre of the gravitational forces acting on the body. It is denoted by G or CG.

Centroid

It is defined as that point at which the total area of a plane figure (i.e., rectangle, square, triangle, quadrilateral, circle, etc.,) is assumed to be concentrated. The centroid and the centre of gravity are one and the same point. It is also denoted by G or CG.

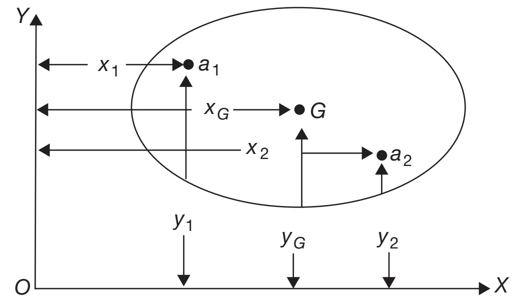
Centroidal Axis

It is defined as that axis which passes through the centre of gravity of a body or through the centroid of an area.

Lamina

A very thin plate or sheet of any cross-section is known as lamina. Its thickness is so small that it can be considered as a plane figure or area having no mass.

Determination of the Centre of Gravity of a Thin Irregular Lamina



The above figure shows an irregular lamina of total area 'A' whose centre of gravity is to be determined. Let the lamina be composed of small areas $a_1, a_2, \ldots$, etc., such that:

$$A = a_1 + a_2 + \ldots = \Sigma a_i$$

Let the distances of the centroids of the areas $a_1, a_2, \ldots$, etc., from the X-axis be $y_1, y_2, \ldots$, etc., respectively, and from the Y-axis be $x_1, x_2, \ldots$, etc. The sum of moments of all the small areas about the Y-axis

$$= a_1 x_1 + a_2 x_2 + \ldots = \Sigma a_i x_i$$

Let x_G and y_G be the coordinates of the centre of gravity G from the Y-axis and X-axis, respectively. From the principle of moments, it can be written that:

$$A x_G = \Sigma a_i x_i$$

or

$$x_G = \frac{\sum a_i x_i}{A} = \frac{\sum a_i x_i}{\sum a_i}$$

Similarly, it can be shown that:

$$y_G = \frac{\sum a_i y_i}{\sum a_i}.$$

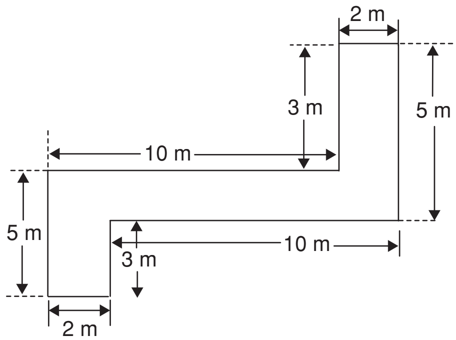
NOTES

1. The axis of reference of a plane figure is generally taken as the bottommost line of the figure for determining y_G and the leftmost line of the figure for calculating x_G.
2. If the figure is symmetrical about the X-axis or Y-axis, then the centre of gravity will lie on the axis of symmetry.
3. For solid bodies, elementary masses m_1, m_2, etc., are considered instead the areas a_1, a_2, etc., and the centre of gravity's coordinates are given as follows:

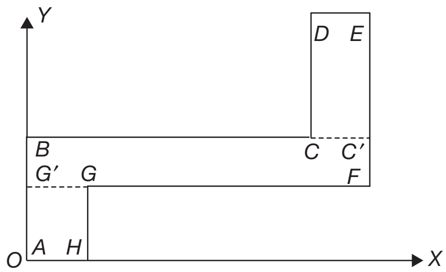
$$x_G = \frac{\sum m_i x_i}{\sum m_i}, y_G = \frac{\sum m_i y_i}{\sum m_i}$$

Example 9

Determine the position of the centre of gravity for the following figure.



Solution



The X-axis and Y-axis of reference are chosen as shown in the above figure such that origin O coincides with point A of the figure and the axes coincide with the leftmost and bottommost lines of the figure, respectively. The position of the centre of gravity is determined with respect to origin O.

The figure is broken down into three areas $AHGG'$, $G'FC'B$, and $CC'ED$

For rectangle $AHGG'$,

Area, $A_1 = 3 \times 2 = 6\ \text{m}^2$

CG coordinates, $x_1 = \dfrac{2}{2} = 1$ m

$$y_1 = \frac{3}{2} = 1.5 \text{ m}$$

For rectangle $G'FC'B$,

Area, $A_2 = (2 + 10) \times (5 - 3) = 24\ \text{m}^2$

CG coordinates, $x_2 = \dfrac{(2+10)}{2} = 6$ m

$$y_2 = 3 + \frac{(5-3)}{2} = 4 \text{ m}$$

For rectangle $CC'ED$,

Area, $A_3 = 3 \times 2 = 6\ \text{m}^2$

CG coordinates, $x_3 = 10 + \dfrac{2}{2} = 11$ m

$$y_3 = 5 + \frac{3}{2} = 6.5 \text{ m}$$

CG of the figure coordinates,

$$X_G = \frac{A_1x_1 + A_2x_2 + A_3x_3}{A_1 + A_2 + A_3} = 6 \text{ m}$$

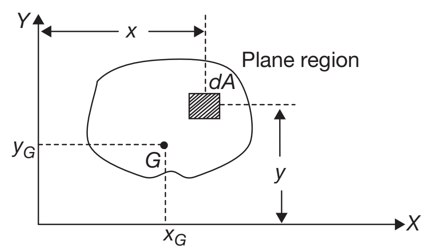
$$Y_G = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A_1 + A_2 + A_3} = 4 \text{ m.}$$

Integration Method for Centroid Determination in a Thin Lamina or Solid

In this method, the given figure is not split into shapes of figures of known centroid as done in the previous section. The centroid is directly found out by determining $\Sigma a_i y_i$ or $\Sigma a_i x_i$ and Σa_i by direct integration.

First Moment of Area

Consider a plane region of area A as shown in the following figure.



Let dA be a differential (i.e., infinitesimal) area located at the point (x, y) in the plane region area A.

Here, $$A = \int_A dA$$

First moments of the area about the X-axis and Y-axis are respectively:

$$M_x = \int_A y\,dA$$

$$M_y = \int_A x\,dA$$

The coordinates (x_G, y_G) of the centre of gravity of the plane region is given by:

$$X_G = \frac{M_y}{A} = \frac{\int_A x\,dA}{\int_A dA}$$

$$Y_G = \frac{M_x}{A} = \frac{\int_A y\,dA}{\int_A dA}$$

NOTES

1. If the X-axis passes through the centre of gravity, then $M_x = 0$. Similarly, $M_y = 0$, when the Y-axis passes through the centre of gravity.
2. If the plane region is symmetric about the Y-axis, then $M_y = 0$ and $x_G = 0$, i.e., the centre of gravity would lie somewhere on the Y-axis. Similarly, $M_x = 0$ and $y_G = 0$, if the plane region is symmetric about the X-axis, i.e., the centre of gravity would lie somewhere on the X-axis.

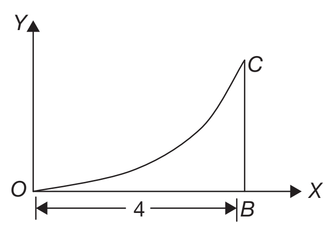
   If instead of a plane region, we have a plane curve of length L and on which a differential length dL is considered which is located at the point (x, y) on the curve, then the coordinates of the centre of gravity for the planar curve is given as follows:

$$X_G = \frac{M_y}{L} = \frac{\int_L x\,dL}{\int_L dL}$$

$$Y_G = \frac{M_x}{A} = \frac{\int_L y\,dL}{\int_L dL}$$

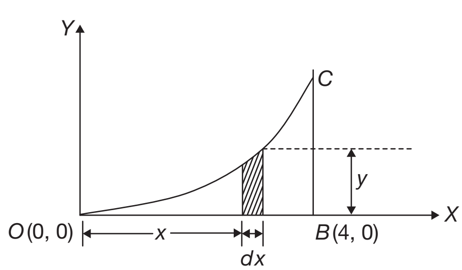
Example 10

The centre of gravity of the following shown area OBC, where the curve OC is given by the equation $y = 0.625x^2$, with respect to the point $O(0, 0)$ is



(A) (6, 5)
(B) (6, 3)
(C) (3, 5)
(D) (3, 3)

Solution



Let us consider an elementary rectangular area of height y and width dx as shown in the given figure.

Area of the elementary rectangle, $dA = y\,dx = 0.625x^2\,dx$

Area of OBC, $A = \int_0^4 dA = \int_0^4 0.625x^2 dx$

$$= 0.625 \times \frac{4^3}{3}$$

Moment of area about X-axis:

$$M_x = \int_0^4 dA\frac{y}{2} = \int_0^4 0.625x^2 dx\,\frac{0.625x^2}{2}$$

$$= \frac{0.625^2}{2} \times \frac{4^5}{5}$$

Moment of area about Y-axis:

$$M_y = \int_0^4 dAx = \int_0^4 0.625x^2 dxx = 0.625 \times \frac{4^4}{4}$$

Let x_G and y_G be the x and y coordinates of the centre of gravity of OBC with respect to the point O.

Then, $M_x = Ay_G$ and $M_y = Ax_G$

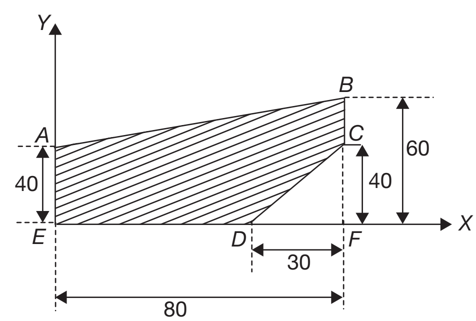
$$y_G = \frac{0.625^2}{2} \times \frac{4^5}{5} \times \frac{3}{0.625 \times 4^3} = 3$$

$$x_G = 0.625 \times \frac{4^4}{4} \times \frac{3}{0.625 \times 4^3} = 3$$

Hence, the correct answer is option (D).

Example 11

The centre of gravity of the following hatched figure with respect to the point E is



(A) (20, 30)
(B) (37.84, 27.45)
(C) (20, 27.45)
(D) (37.84, 30)

Solution

For ΔABC, area $A_1 = \frac{1}{2}80 \times (60 - 40) = 800$

CG coordinates, $x_1 = \frac{2}{3} \times 80 = \frac{160}{3}$

$$y_1 = 40 + \frac{1}{3} \times (60 - 40) = \frac{140}{3}$$

For $\Delta ACFE$, area $A_2 = 40 \times 80 = 3200$

CG coordinates, $x_2 = \frac{80}{2} = 40$

$$y_2 = \frac{40}{2} = 20$$

For ΔCFD, area $A_3 = \frac{1}{2}30 \times 40 = 600$

CG coordinates, $x_3 = 50 + \frac{2}{3} \times 30 = 70$

$$y_3 = \frac{1}{3} \times 40 = \frac{40}{3}$$

Since ΔCFD is cut out from figure *ABFE* to obtain the hatched figure, the area of ΔCFD is assigned a negative sign.

$$\therefore A_3 = -600$$

Let x_G and y_G be the x and y coordinates of the centre of gravity of the hatched figure with respect to the point E, then:

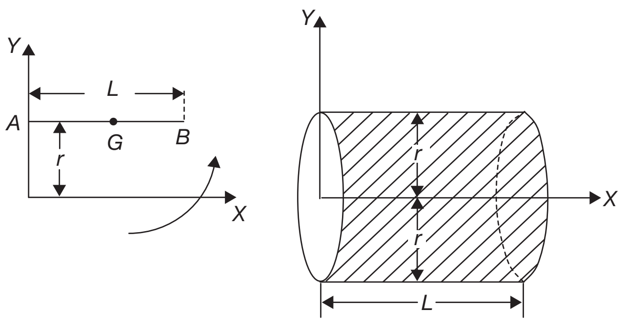
$$x_G = \frac{A_1x_1 + A_2x_2 + A_3x_3}{A_1 + A_2 + A_3} = 37.84$$

$$y_G = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A_1 + A_2 + A_3} = 27.45$$

Hence, the correct answer is option (B).

Theorems of Pappus–Guldinus

A surface of revolution is a surface which can be generated by rotating a plane curve about a fixed axis.



For example, in the above figure, the curved surface of a cylinder is obtained by rotating the line *AB* about the *X*-axis.

Theorem I

The area of a surface of revolution is equal to the product of the length of the generating curve and the distance travelled by the centroid of the curve while the surface is being generated.

NOTE

The generating curve must not cross the axis about which it is rotated.

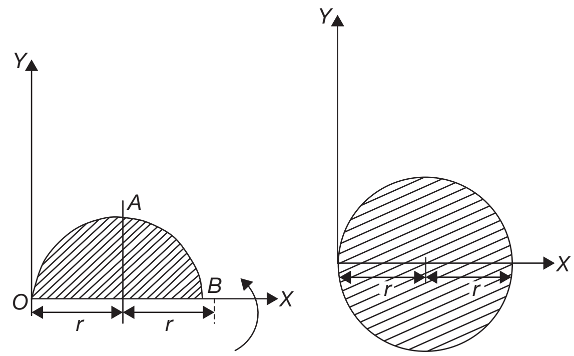
In the given figure:

Length of the generating curve = L

Distance travelled by the centroid while the surface is being generated = $2\pi r$ (circumference of a circle of radius r)

$\therefore$ Area of the surface of the cylinder generated $= L \times 2\pi r = 2\pi rL$

A body of revolution is a body which can be generated by rotating a plane area about a fixed axis.



(a) **(b)**

For example, in the above figure the volume of a sphere is obtained by rotating the semi-circle *OAB* about the *X*-axis.

Theorem II

The volume of a body of revolution is equal to the product of the generating area and the distance travelled by the centroid of the area while the body is being generated.

NOTE

The theorem does not apply if the axis of rotation intersects the generating area.

In the above figure:

Generating area $= \frac{1}{2}\pi r^2$

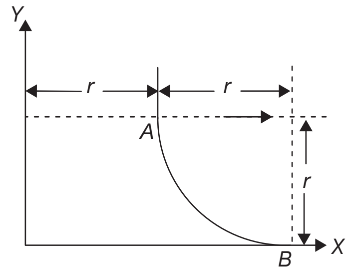
Distance travelled by the centroid of the area while the body is being generated $= 2\pi \times \frac{4r}{3\pi}$ (circumference of a circle of radius $\frac{4r}{3\pi}$)

$\therefore$ Volume of the sphere generated $= \frac{1}{2}\pi r^2 \times 2\pi \times \frac{4r}{3\pi}$

$$= \frac{4\pi r^3}{3}.$$

Example 12

A quartered-circular arc *AB* when rotated about the *Y*-axis generates a surface of area A_y. The same arc when rotated about the X-axis generates a surface of area A_x. If the ratio $A_y : A_x$ is related to length r by equation $\frac{A_y}{A_x} = kr^n$, where k, n are constants, then the value of k and n are



(A) 0.27 and 0 (B) 0.27 and 1
(C) 3.75 and 0 (D) 3.75 and 1

Solution

Length of the arc $= \frac{1}{2}\pi r$

x coordinate of the centroid of the arc $= 2r - \frac{2r}{\pi}$

Distance travelled by the centroid when the arc is rotated about the Y-axis $= \frac{2\pi \times 2r(\pi - 1)}{\pi}$

Using Pappus–Guldinus theorem I,

$$A_y = \left(2r - \frac{2r}{\pi}\right) \times \frac{2\pi \times 2r(\pi - 1)}{\pi} = 2r^2\pi(\pi - 1)$$

y coordinate of the centroid of the arc $= r - \frac{2r}{\pi}$

Distance travelled by the centroid when the arc is rotated about the X-axis $= \frac{2\pi \times r(\pi - 2)}{\pi}$

Using Pappus–Guldinus theorem I,

$$A_x = \left(r - \frac{2r}{\pi}\right) \times \frac{2\pi \times r(\pi - 2)}{\pi} = r^2\pi(\pi - 2)$$

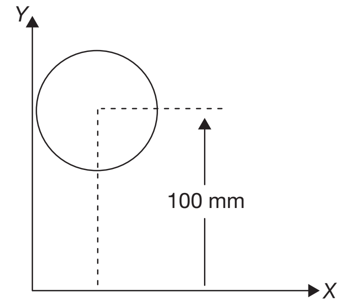
$$\therefore \frac{A_y}{A_x} = kr^n = \frac{2(\pi - 1)}{\pi - 2}$$

$$\Rightarrow n = 0 \text{ and } k = \frac{2(\pi - 1)}{\pi - 2}.$$

Hence, the correct answer is option (C).

Example 13

A solid ring (torus) of circular cross-section is obtained by rotating a circle of radius 25 mm about the X-axis as shown in the following figure.



If the density of the material making up the circular cross-section is 7800 kg/m^3, the weight of the ring generated is

(A) 82.6 N (B) 94.4 N
(C) 123.4 N (D) 90.6 N

Solution

y coordinate of the centroid of the circle = 100 mm = 0.1 m

Area of the circle $= \pi \times (0.025)^2$

Distance travelled by the centroid of the circle while generating the ring $= 2\pi \times (0.1)$ (circumference of a circle of radius 0.1 m)

Using Pappus–Guldinus theorem II,

Volume of the ring generated

$$= \pi \times (0.025)^2 \times 2\pi \times (0.1) = 0.001233 \text{ m}^3$$

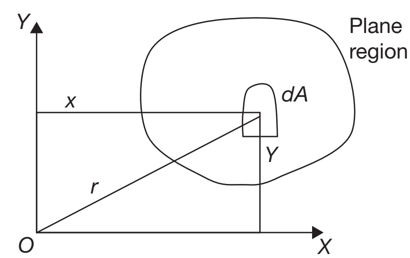
Weight of the generated ring

$$= 7800 \times 0.001233 \times 9.81 = 94.4 \text{ N}.$$

Hence, the correct answer is option (B).

Area Moment of Inertia

In a plane region of area A, a differential area dA located at point (x, y) is considered as shown in the following figure.



The moment of inertia of the area about the X-axis and Y-axis are

$$I_x = \int_A y^2 dA \text{ and } I_y = \int_A x^2 dA$$

I_x and I_y are also called 'the second moments of the area'.

Polar Moment of Inertia

In the above figure, the polar moment of inertia of the area about the point O (actually, about an axis through the point O, perpendicular to the plane of the area) is

$$J_o = \int_A r^2 dA$$

$$J_o = I_x + I_y$$

The above equation states that the polar moment of inertia of an area about a point O is the sum of the moments of inertia of the area about two perpendicular axes that intersect at O.

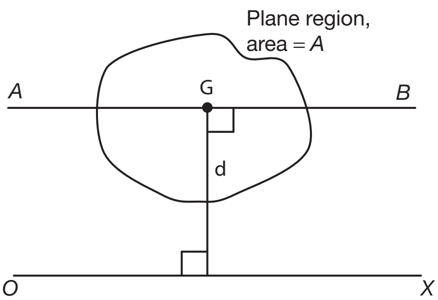
Radius of Gyration

In the given figure, the radii of gyration of an area about the X-axis, Y-axis and the origin O are:

$$k_x = \sqrt{\frac{I_x}{A}},\ k_y = \sqrt{\frac{I_y}{A}} \text{ and } k_o = \sqrt{\frac{J_o}{A}}$$

Parallel Axis Theorem

The moment of inertia of a plane region area about an axis, say AB, in the plane of area through the centre of gravity of the plane region area be represented by I_G, then the moment of inertia of the given plane region area about a parallel axis, say OX, in the plane of the area at a distance d from the centre of gravity of the area is $I_X = I_G + Ad^2$,



Where

I_X = Moment of inertia of the given area about the OX axis

I_G = Moment of inertia of the given area about AB axis

A = Area of the plane region

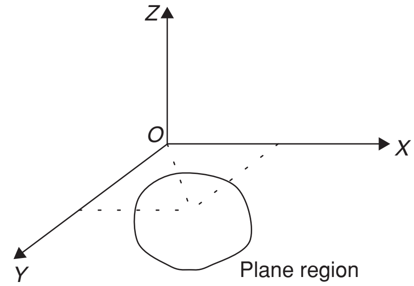
d = Perpendicular distance between the parallel axes AB and OX

G = Centre of gravity of the plane region

Perpendicular Axis Theorem

If I_{OX} and I_{OY} are the moments of inertia of a plane region area about two mutually perpendicular axes OX and OY in the plane of the area, then the moment of inertia of the plane region area I_{OZ} about the axis OZ, perpendicular to the plane and passing through the intersection of the axes OX and OY is

$$\boxed{I_{OZ} = I_{OX} + I_{OY}}$$

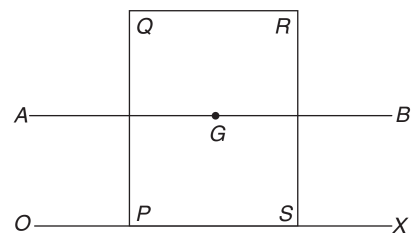


NOTE

I_{OZ} is also called as the polar moment of inertia and the axis OZ is called as the polar axis.

Example 14

In the following figure, the axes AB and OX are parallel to each other. If the moments of inertia of the rectangle $PQRS$ along the axis AB, which passes through the centroid of the rectangle, and the axis OX are I_G and I_X respectively, then the value of I_X/I_G is



(A) 4 (B) 12
(C) 3 (D) 0.25

Solution

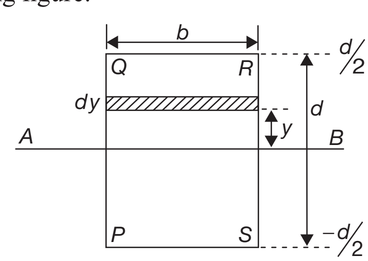
From the parallel axis theorem, we have $I_X = I_G + A$ (perpendicular distance between axes)^2.

Let $PQ = d$ and $QR = b$, then the perpendicular distance between the axes $= \frac{d}{2}$.

$$\therefore I_X = I_G + A\frac{d^2}{4} = I_G + bd\frac{d^2}{4}$$

$$\boxed{\frac{I_X}{I_G} = 1 + \frac{bd^3}{4I_G}}$$

To determine I_G, let us consider a rectangular strip of thickness dy at a distance y from the axis AB as shown in the following figure.



Area of the rectangular strip = bdy

Moment of inertia of the strip about the axis,

$$AB = (bdy)y^2$$

Moment of inertia of the rectangle $PQRS$ about the axis AB,

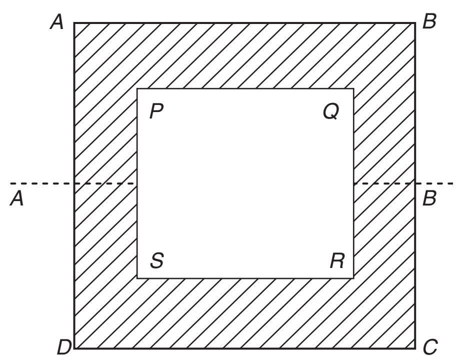
$$I_G = \int_{-d/2}^{d/2} by^2 dy = \frac{bd}{12}$$

$$\therefore \frac{I_X}{I_G} = 4$$

Hence, the correct answer is option (A).

Example 15

The moment of inertia for the following hatched figure about the axis AB (which passes through the centroid of the figure), where $AB = DC = 30$ m, $PQ = SR = 20$ m, $BC = AD = 20$ m and $QR = PS = 10$ m, is



(A) 6.78×10^4 m^4

(B) 5.41×10^3 m^4

(C) 1.83×10^4 m^4

(D) 2.6×10^5 m^4

Solution

Moment of inertia of the hatched figure = moment of inertia of □$ABCD$ – Moment of inertia of □$PQRS$

$$= \frac{1}{12} \times (DC \times AD^3 - SR \times QR^3)$$

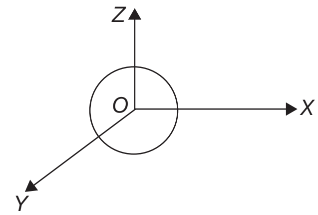
$$= \frac{1}{12} \times (30 \times 20^3 - 20 \times 10^3)$$

$$= 18333.33 \text{ m}^4$$

Hence, the correct answer is option (C).

Example 16

A circular section of diameter d is lying on the xy plane where the centre of the circular section coincides with the origin O as shown in the following figure.

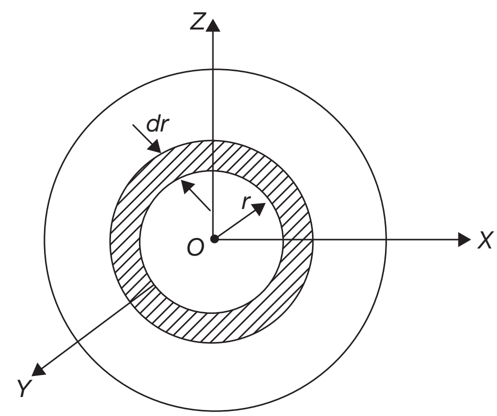


If the moments of inertia of the circular section along the x, y, and z axes are I_X, I_Y, and I_Z respectively, then which of the following is NOT correct?

(A) $I_X = \frac{\pi d^4}{32}$ (B) $I_X = I_Y$

(C) $I_Z = \frac{\pi d^4}{32}$ (D) $I_Y = \frac{\pi d^4}{64}$

Solution



Let us consider an elementary ring of thickness dr and located at a distance r from the origin O.

Area of the elementary ring = $2\pi r dr$

Moment of inertia of the elementary ring about the Z-axis $= 2\pi r dr \times r^2 = 2\pi r^3 dr$.

Moment of inertia of the whole circular section about the

$$Z\text{-axis} = \int_0^{D/2} 2\pi r^3 dr = \frac{\pi d^4}{32}$$

From the symmetry of the circular section, it can be written that $I_X = I_Y$.

From the perpendicular axis theorem, we have,

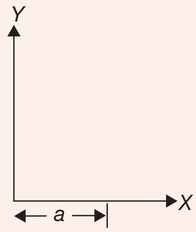
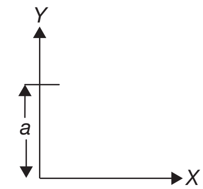
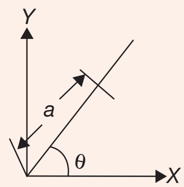
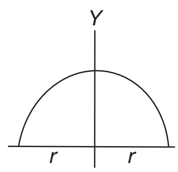
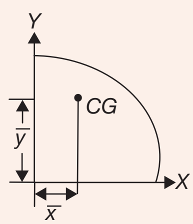
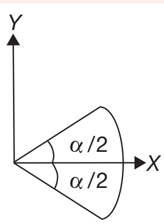
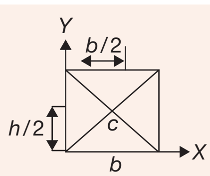
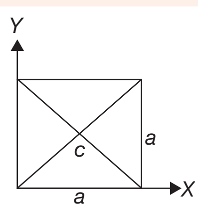
$$I_Z = I_X + I_Y$$

i.e.,
$$I_Z = 2I_X$$

$$\therefore I_X = \frac{\pi d^4}{64} = I_Y.$$

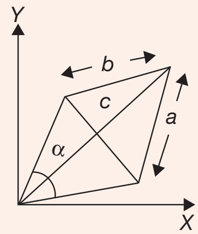
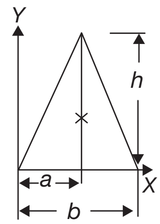
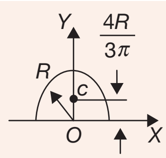
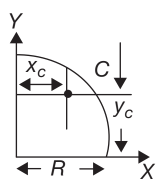
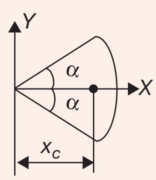
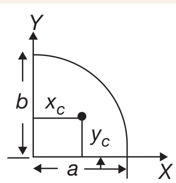
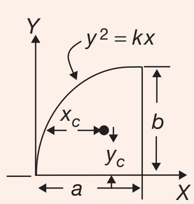
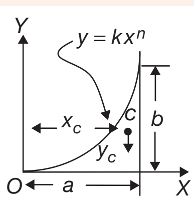
Hence, the correct answer is option (A).

Centre of Gravity

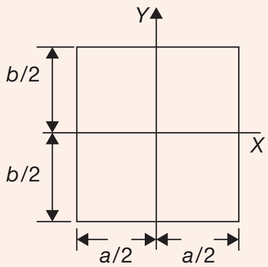
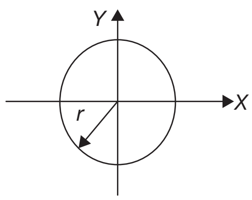
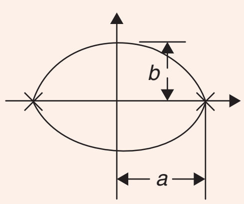
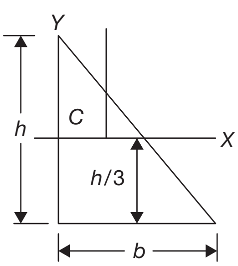
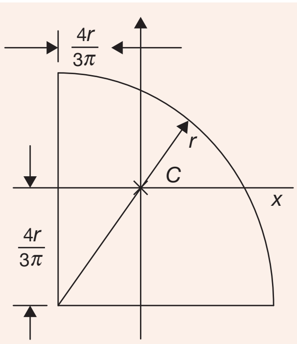
Description	Shape	L	$\overline{x_c}$	$\overline{y_c}$
Horizontal line		a	$\frac{a}{2}$	0
Vertical line		a	0	$\frac{a}{2}$
Inclined line with θ		a	$\left(\frac{a}{2}\right)\cos\theta$	$\left(\frac{a}{2}\right)\sin\theta$
Semicircular arc		πr	0	$\frac{2r}{\pi}$
Quarter circular arc		$\frac{\pi r}{2}$	$\frac{2r}{\pi}$	$\frac{2r}{\pi}$
Circular arc		αr	$\frac{2r\sin\frac{\alpha}{2}}{\alpha}$	0
Rectangle		bh	$\frac{b}{2}$	$\frac{h}{2}$
Square		a^2	$\frac{a}{2}$	$\frac{a}{2}$

(Continued)

(Continued)

Description	Shape	L	$\overline{x_c}$	$\overline{y_c}$
Parallelogram		$ab \sin\alpha$	$\frac{b+a\cos\alpha}{2}$	$\frac{a\sin\alpha}{2}$
Triangle		$\frac{bh}{2}$	$\frac{a+b}{3}$	$\frac{h}{3}$
Semi circle		$\frac{\pi R^2}{2}$	0	$\frac{4R}{3\pi}$
Quarter circle		$\frac{\pi r^2}{2}$	$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$
Sector of a circle		$R^2\alpha$	$\frac{2}{3}\frac{R\sin\alpha}{\alpha}$	0
Quarter ellipse		$\frac{\pi ab}{4}$	$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$
Quarter parabola		$\frac{\pi ab}{3}$	$\frac{3a}{5}$	$\frac{3b}{5}$
General spandrel		$\frac{ab}{3}$	$\frac{3a}{4}$	$\frac{3b}{4}$

Moment of Inertia

Figure	$\overline{I_x}$	$\overline{I_y}$
 Rectangle	$\frac{ab^3}{12}$	$\frac{ba^3}{12}$
 Circle	$\frac{\pi r^4}{4}$	$\frac{\pi r^4}{4}$
 Ellipse	$\frac{\pi ab^3}{4}$	$\frac{\pi ba^3}{4}$
 Triangle	$\frac{bh^3}{36}$	$\frac{hb^3}{36}$
 Quadrant circle	$0.0549\, r^4$	$0.0549\, r^4$

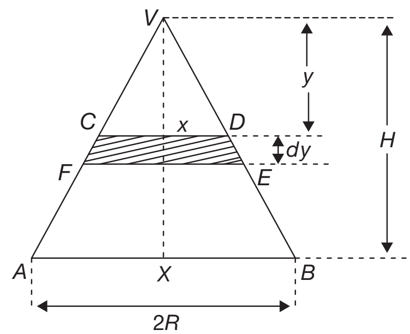
Centroid of Solids

If dm is an elemental mass in a body of mass M and x_G, y_G are the coordinates of the center of gravity of the body from the reference axes Y-axis and X-axis respectively, then

$$x_G = \frac{\int x\,dm}{\int dm} = \frac{\int x\,dm}{M}, \quad y_G = \frac{\int y\,dm}{\int dm} = \frac{\int y\,dm}{M}$$

Let us consider a right-circular solid cone whose centre of gravity is to be determined. Let the diameter of the base of the right circular solid cone be $2R$, and its height H as shown in the following figure.

Since the cone is symmetric about the VX axis, its centre of gravity will lie on this axis. The cone can be imagined to be consisting of an infinite number of circular discs with different radii, parallel to the base.



Consider one such disc of radius x, thickness dy and at a depth y from the vertex of the cone, i.e., from V.

From the geometry of the above figure:

$$\frac{x}{R} = \frac{y}{H} \text{ or } x = \frac{yR}{H}$$

$$\text{Volume of disc } = \pi x^2 dy = \pi \frac{y^2 R^2}{H^2} dy$$

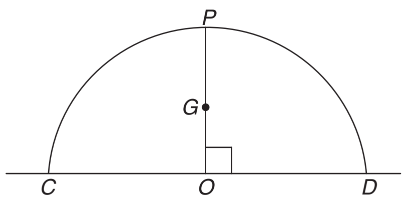
If ρ is the density of the material making up the cone, then $dm = \rho \frac{\pi y^2 R^2}{H^2} dy$

$$\therefore\ y_G = \frac{\int y\,dm}{dm} = \frac{\int_0^H \rho \frac{\pi y^3 R^2}{H^2} dy}{\int_0^H \rho \frac{\pi y^2 R^2}{H^2} dy} = \frac{3}{4}[y]_0^H = \frac{3}{4} H$$

$\therefore$ Centroid or centre of gravity of a right circular cone is situated at a distance of $\frac{3}{4} H$ from its vertex V and lies on its axis VX.

Example 17

In the homogenous hollow hemisphere, shown in the following figure, $OP = 10$ cm = The radius of the hemisphere. The points P, G, and O lie on a straight line that is perpendicular to the base CD. If G is the centroid of the hollow hemisphere, then which one of the following statements is NOT correct?



(A) $OG = 5$ cm (B) $OG = \frac{3}{8}OP$

(C) $CO = 10$ cm (D) $OD = 2 \times OG$

Solution

The centre of gravity of a hollow hemisphere with respect to the X-axis would lie on a perpendicular axis along which the homogeneous hemisphere is symmetrical.

Since G is the centre of gravity, then the hemisphere should be symmetrical along OP, i.e., $CO = OD$.

It can also be deciphered that $CO = OD$ = radius of the hemisphere = OP = 10 cm.

Now, OG will be equal to $R/2$, where R is the radius of the hollow hemisphere.

$$\therefore OG = 0.5\text{ OP} = 5\text{ cm}$$

It can be written as $OP = CO = OD = 2OG$. Hence option (B) is NOT correct.

Hence, the correct answer is option (B).

NOTE

Option (B) would be right if the hemisphere had been a homogeneous solid hemisphere.

Mass Moment of Inertia

The moment of inertia of an element of mass is the product of the mass of the element and the square of the distance of the element from the axis.

The mass moment of inertia of the body with respect to Cartesian frame xyz is given by:

$$I_{XX} = \int (y^2 + z^2)dm = \int_v (y^2 + z^2)\rho dv$$

$$I_{YY} = \int (x^2 + z^2)dm = \int_v (x^2 + z^2)\rho dv$$

$$I_{ZZ} = \int (x + y^2)dm = \int_v (x^2 + y^2)\rho dv,$$

where, I_{XX}, I_{YY} and I_{ZZ} are the axial moments of inertia of mass with respect to X, Y and Z axes.

For thin plates, essentially in the X–Y plane, the following relations hold.

$$I_{XX} = \int y^2 dm$$

$$I_{YY} = \int x^2 dm$$

$$I_{ZZ} = \int z^2 dm = \int (x^2 + y^2)dm$$

$$= I_{XX} + I_{YY}$$

I_{ZZ} is also called the polar moment of inertia.

Mass Moment of Inertia and Radius of Gyration

$$I_{XX} = K_x^2 m$$
$$I_{YY} = K_y^2 m$$
$$I_{ZZ} = K_z^2 m$$
$$K_x = \sqrt{\frac{I_{XX}}{m}}$$
$$K_y = \sqrt{\frac{I_{YY}}{m}}$$
$$K_z = \sqrt{\frac{I_{ZZ}}{m}}$$

The parallel-axis theorem for the mass moment of inertia states that the mass moment of inertia with respect to any axis is equal to the moment of inertia of the mass with respect to a parallel axis through the centre of mass plus the product of the mass and the square of the perpendicular distance between the axes.

Mathematically, $I_{AB} = I_G + md^2$

For a thin plate,

$$I_{XX(\text{mass})} = \rho t I_{XX(\text{area})}$$

$$I_{YY(\text{mass})} = \rho t I_{YY(\text{area})}$$

$$I_{ZZ(\text{mass})} = \rho t I_{ZZ(\text{area})}$$

where, t is the uniform thickness and ρ is the mass of the thin plate.

$$\boxed{I_{ZZ} = I_{XX} + I_{YY}}$$

The mass moment of inertia about a centroidal axis perpendicular to a uniform thin rod of length l, mass m and small cross section is given by:

$$I_{YY} = \frac{1}{12}ml^2$$

Radius of gyration about a centroidal axis perpendicular to a uniform thin rod of length l, mass m and a small cross-section is given by:

$$K_y = \frac{l}{\sqrt{12}}$$

The mass moment of inertia about the longitudinal and transverse axes passing through the centre of mass of a rectangular prism (block) of cross-section (axb), uniform density ρ and length l is given by:

$$I_{XX} = \frac{1}{12}m(l^2 + b^2)$$

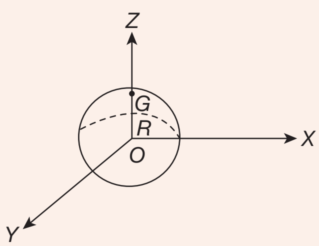
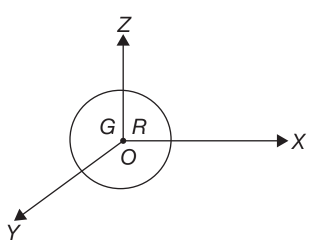
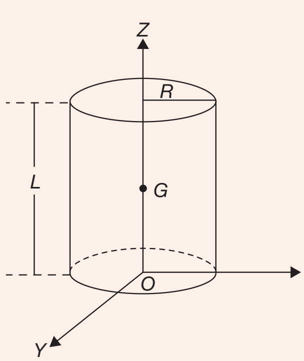
$$I_{YY} = \frac{1}{12}m(a^2 + b^2)$$
$$I_{ZZ} = \frac{1}{12}m(a^2 + l^2)$$

In the above case, if the three axes were chosen through a corner instead the centre of mass, the results are:

$$I_{XX} = \frac{1}{3}m(l^2 + b^2)$$
$$I_{YY} = \frac{1}{3}m(a^2 + b^2)$$
$$I_{ZZ} = \frac{1}{3}m(a^2 + l^2)$$

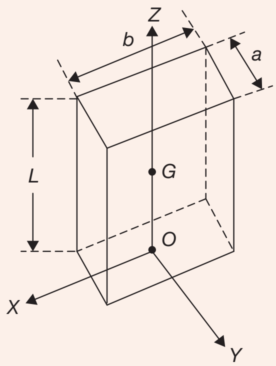
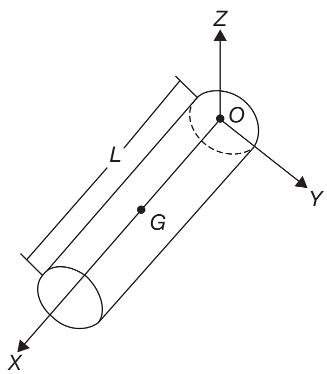
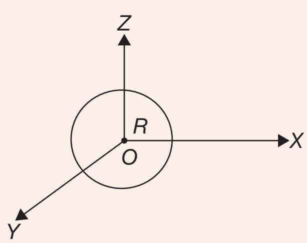
For a right-circular cylinder of radius R, length or height l and mass m, the mass moment of inertia about the centroidal X-axis is given by:

$$I_{XX} = m\left[\frac{R^4}{4} + \frac{l^2}{12}\right]$$

Solid Body	Centroid	Mass Moment of Inertia
Solid hemisphere 	$x_G = y_G = 0$ $z_G = \frac{3}{8}R$	$I_{XX} = I_{YY} = I_{ZZ} = \frac{2}{5}mR^2$
Solid sphere 	$x_G = y_G = z_G = 0$	$I_{XX} = I_{YY} = I_{ZZ} = \frac{2}{5}mR^2$ $K_y = \sqrt{\frac{2}{5}}R$
Solid cylinder 	$x_G = y_G = 0$ $z_G = \frac{L}{2}$	$I_{XX} = I_{YY} = \frac{1}{4}mR^2 + \frac{1}{3}mL^2$ $I_{ZZ} = \frac{1}{2}mR^2$

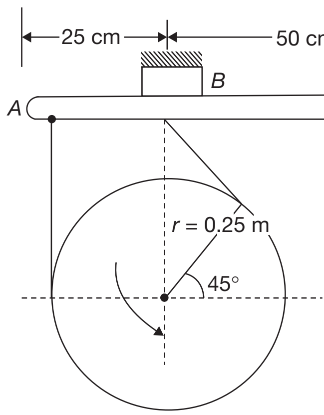
(Continued)

(Continued)

Solid Body	Centroid	Mass Moment of Inertia
Rectangular Block (cuboid) 	$x_G = y_G = 0$ $z_G = \frac{L}{2}$	$I_{XX} = \frac{1}{12}ma^2 + \frac{1}{3}mL^2$ $I_{YY} = \frac{1}{12}mb^2 + \frac{1}{3}mL^2$ $I_{ZZ} = \frac{1}{12}m(a^2 + b^2)$
Slender rod (thin cylinder) 	$x_G = \frac{L}{2}$ $y_G = z_G = 0$	$I_{XX} = 0$ $I_{YY} = I_{ZZ} = \frac{mL^2}{3}$
Solid disk 	$x_G = y_G = z_G = 0$	$I_{YY} = I_{YY} = \frac{mR^2}{4}$ $I_{ZZ} = \frac{mR^2}{2}$ $k_Z = \frac{r}{\sqrt{2}}$

Exercises

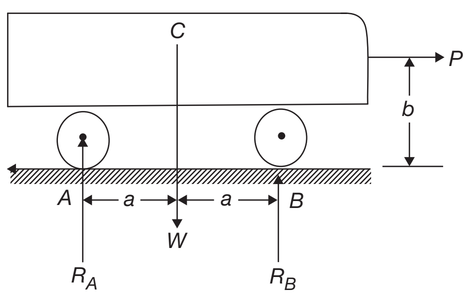
1.



A rotating wheel is braked by a belt *AB* attached to the lever *ABC* hinged at *B*. The coefficient of friction between the belt and the wheel is 0.5. The braking moment exerted by the vertical weight $W = 200$ N is

(A) 98.23 Nm (B) 85.96 Nm
(C) 95.00 Nm (D) 93.24 Nm

2. A locomotive of weight *W* is at rest. The reactions at *A* and *B* are



(A) $\frac{W}{2}$ N (B) 2W N
(C) $\frac{2}{3}$W N (D) $\sqrt{3}$ W N

3. When it is pulling a wagon, the draw bar pull *P* is just equal to the total friction at the points of contact, *A* and *B*. The new magnitudes of the vertical reactions at *A* and *B* respectively are

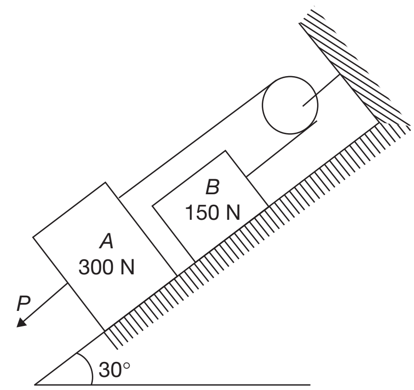
(A) $\frac{Wa - Pb}{2a}, \frac{Wa + Pb}{2a}$

(B) $\frac{W}{2}, \frac{W}{2a}$

(C) $\frac{W}{2}, \frac{W}{3}$

(D) $\frac{W}{2}, \frac{2}{3}W$

4.



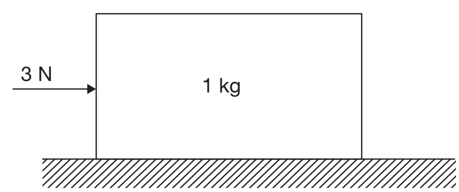
Two blocks *A* and *B* weighing 300 N and 150 N respectively are placed on a rough inclined plane of angle 30° and connected through a string over a pulley as shown in the figure. Coefficient of friction of the contact surfaces are 0.25. Force *P* required on block *A* for impending motion of the blocks is

(A) 22.43 N (B) 25.24 N
(C) 28.62 N (D) 30.14 N

5. Angle of the inclined plane is increased to 20° and the connecting string is removed. If the coefficients of kinetic friction for blocks *A* and *B* are 0.1 and 0.35 respectively, the frictional forces on *A* and *B* are

(A) 46.1 N, 369 N
(B) 49.44 N, 398 N
(C) 52.14 N, 404 N
(D) 56.48 N, 410 N

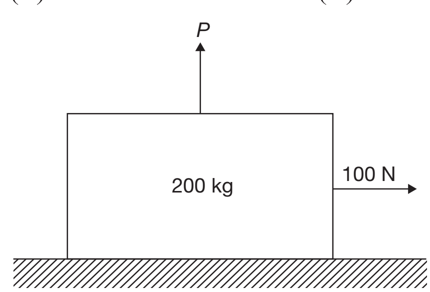
6.



A body of mass 1 kg is resting on a plane surface as shown in the figure. A force of 3 N is gradually applied on one side as shown. Coefficient of static friction is 0.35 and coefficient of kinetic friction is 0.3. The friction force acting is

(A) 3.4335 N (B) 2.943 N
(C) 3 N (D) 0 N

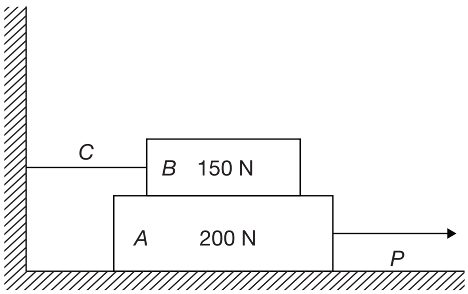
7.



A body of mass 200 kg rests on a horizontal surface as shown in the figure. Coefficient of friction between the body and surface is 0.2. If a horizontal pull of 100 N can be exerted on the body, the vertical force P required to move the body is

(A) 1462 N (B) 1418 N
(C) 1360 N (D) 1322 N

Direction for questions 8 and 9:



Block A weighing 200 N is placed on plane floor and block B weighing 150 N is placed over block A. Block B is constrained by a string *C* and a force *P* is applied on block A as shown in figure. For the contact surfaces, coefficient of static friction is 0.3 and coefficient of kinetic friction is 0.25.

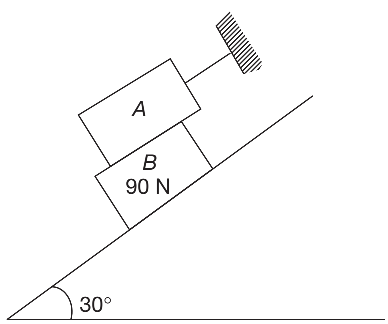
8. The smallest force *P* required to start block *A* moving is

(A) 143 N (B) 150 N
(C) 156 N (D) 160 N

9. If a force *P* of 160 N is applied, the resultant friction forces exerted on block *A* is

(A) 110 N (B) 120 N
(C) 125 N (D) 150 N

10.



Refering to the figure given above, coefficient of friction for all surfaces of contact is 0.3. The minimum weight of block *A* required to keep block *B* in position is

(A) 35.6 N (B) 38.4 N
(C) 41.6 N (D) 44.5 N

11. If the momentum of a given body is tripled, its kinetic energy will

(A) increase by 3 times (B) increase by 9 times
(C) decrease by 3 times (D) decrease by 9 times

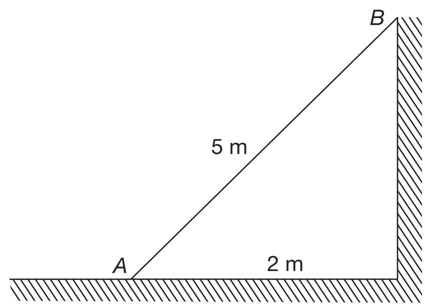
12. The rate of change of velocity and the rate of change of momentum of a moving body respectively are

(A) acceleration and impulse
(B) acceleration and force
(C) displacement and force
(D) force and displacement

13. In the equation of virtual work, which of the following force is neglected?

(A) Reaction at any smooth surface with which the body is in contact.
(B) Reaction of rough surface of a body which rolls on it without shipping.
(C) Reaction at a point on an axis fixed in space, around which a body is constrained to turn.
(D) All of these

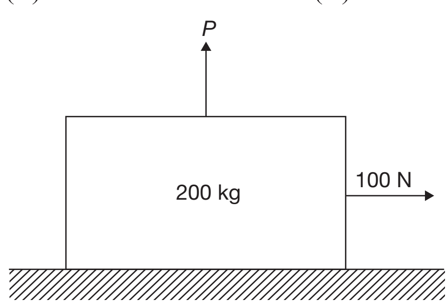
14.



A uniform ladder of length 5 m and weight 250 N is placed against a smooth vertical wall with its lower end 2 m from the 00wall. Coefficient of friction between floor and ladder is 0.25. Friction at the wall may be neglected. Frictional force at A, the bottom of ladder is

(A) 54.56 N (B) 62.54 N
(C) 68.36 N (D) 72.43 N

15.

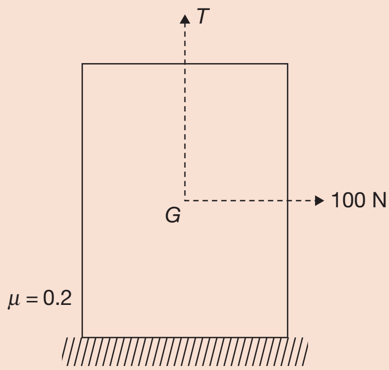


A body of mass 200 kg rests on a horizontal surface as shown in the figure. Coefficient of friction between the body and surface is 0.2. If a horizontal pull of 100 N can be exerted on the body, the vertical force *P* required to move the body is

(A) 1462 N (B) 1418 N
(C) 1360 N (D) 1322 N

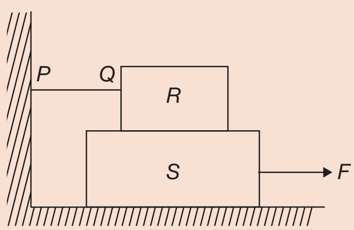
Previous Years' Questions

1. A block weighing 981 N is resting on a horizontal surface. The coefficient of friction between the block and the horizontal surface is $\mu = 0.2$. A vertical cable attached to the block provides partial support as shown. A man can pull horizontally with a force of 100 N. What will be the tension, T (in N) in the cable if the man is just able to move the block to the right? **[GATE, 2009]**



(A) 176.2 (B) 196.0
(C) 481.0 (D) 981.0

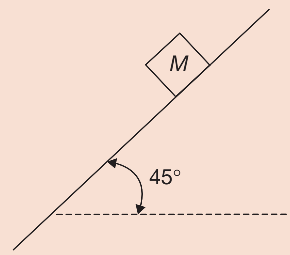
2. A block R of mass 100 kg is placed on a block S of mass 150 kg as shown in the figure. Block R is tied to the wall by a massless and inextensible string PQ. If the coefficient of static friction for all surfaces is 0.4, the minimum force F (in kN) needed to move the block S is **[GATE, 2014]**



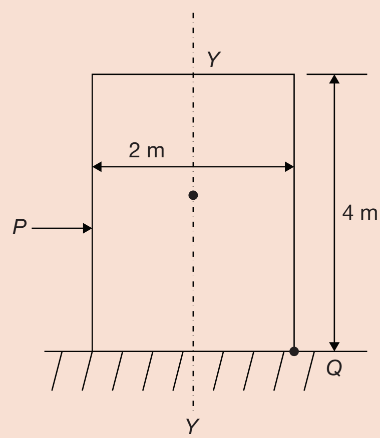
(A) 0.69 (B) 0.88
(C) 0.98 (D) 1.37

3. A block weighing 200 N is in contact with a level plane whose coefficients of static and kinetic friction are 0.4 and 0.2 respectively. The block is acted upon by a horizontal force (in newton) $P = 10t$, where t denotes the time in seconds. The velocity (in m/s) of the block attained after 10 seconds is __________. **[GATE, 2014]**

4. A body of mass (M) 10 kg is initially stationary on a 45° inclined plane as shown in figure. The coefficient of dynamic friction between the body and the plane is 0.5. The body slides down the plane and attains a velocity of 20 m/s. The distance travelled (in metre) by the body along the plane is ________. **[GATE, 2014]**



5. A wardrobe (mass 100 kg, height 4 m, width 2 m, depth 1 m), symmetric about the Y-Y axis, stands on a rough level floor as shown in the figure. A force P is applied at mid-height on the wardrobe so as to tip it about point Q without slipping. What are the minimum values of the force (in newton) and the static coefficient of friction μ between the floor and the wardrobe, respectively? **[GATE, 2014]**



(A) 490.5 and 0.5 (B) 981 and 0.5
(C) 1000.5 and 0.15 (D) 1000.5 and 0.25

Answer Keys

Exercises

1. B	**2.** A	**3.** A	**4.** A	**5.** A	**6.** C	**7.** A	**8.** B	**9.** C	**10.** C
11. B	**12.** B	**13.** D	**14.** A	**15.** A					

Previous Years' Questions

1. C	**2.** D	**3.** 4.8 to 5	**4.** 56 to 59	**5.** A