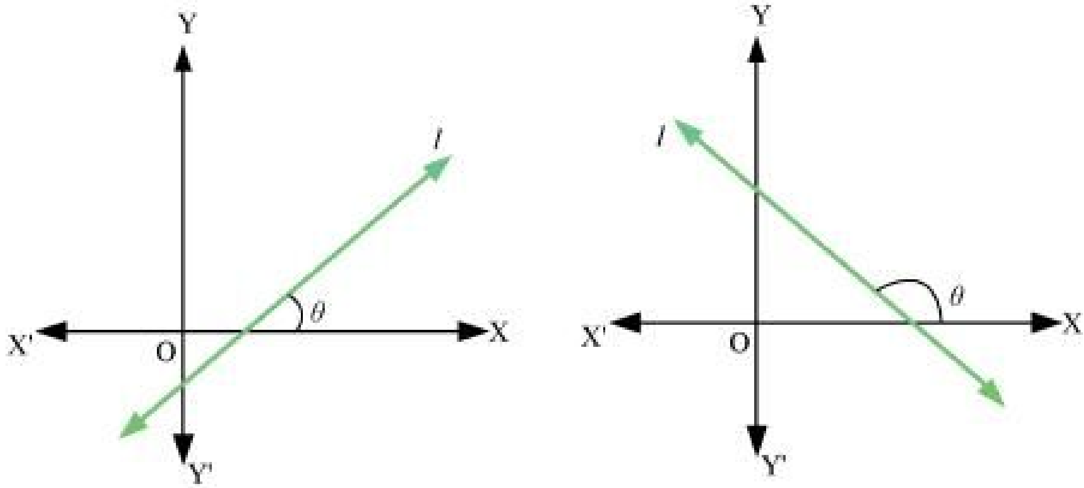
# Straight Lines

- **Slope of a line:** If θ is the inclination of a line *l* (the angle between positive *x*-axis and line *l*), then $m = \tan \theta$ is called the slope or gradient of line *l*.




    - The slope of a line whose inclination is 90º is not defined. Hence, the slope of the vertical line, *y*-axis is undefined.
    - The slope of the horizontal line, *x*-axis is zero.

  For example, the slope of a line making an angle of 135° with the positive direction of *x*-axis is $m = \tan 135° = \tan(180° - 45°) = -\tan 45° = -1$

- **Slope of line passing through two given points:**

  The slope ($m$) of a non-vertical line passing through the points $(x_1, y_1)$ and $(x_2, y_2)$ is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$.

  For example, the slope of the line joining the points (–1, 3) and (4, –2) is given by,
  $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2) - 3}{4 - (-1)} = -\frac{5}{5} = -1$

- **Conditions for parallelism and perpendicularity of lines:**

  Suppose $l_1$ and $l_2$ are non-vertical lines having slopes $m_1$ and $m_2$ respectively.

- $l_1$ is parallel to $l_2$ if and only if $m_1 = m_2$ i.e., their slopes are equal.
- $l_1$ is perpendicular to $l_2$ if and only if $m_1 m_2 = -1$ i.e., the product of their slopes is $-1$.

**Example:**
Find the slope of the line which makes an angle of 45° with a line of slope 3.
**Solution:**
Let *m* be the slope of the required line.

$$\therefore \tan 45° = \left|\frac{m-3}{1+3m}\right|$$

$$\Rightarrow \left|\frac{m-3}{1+3m}\right| = 1$$

$$\Rightarrow \left|\frac{m-3}{1+3m}\right| = \pm 1$$

$$\Rightarrow \frac{m-3}{1+3m} = 1 \quad \text{or} \quad \frac{m-3}{1+3m} = -1$$

$$\Rightarrow m-3 = 1+3m \quad \text{or} \quad m-3 = -1-3m$$

$$\Rightarrow -2m = 4 \quad \text{or} \quad 4m = 2$$

$$\Rightarrow m = -2 \quad \text{or} \quad m = \frac{1}{2}$$

- **Collinearity of three points:** Three points A, B and C are collinear if and only if slope of AB = slope of BC

- **Angle between two lines:** An acute angle, θ, between line $l_1$ and $l_2$ with slopes $m_1$ and $m_2$ respectively is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, \; 1 + m_1 m_2 \neq 0$$

**Example 1:** Two lines AB and CB, intersect at point B. The coordinates of end points are A(−4, −3), B(0, 5), and C(10, 5). Find the measures of angles between AB and CB.

**Solution:** Let the angle between the lines AB and BC be θ.

Slope of line $AB = \frac{5-(-3)}{0-(-4)} = \frac{8}{4} = 2$

Slope of line $BC = \frac{5-5}{10-0} = 0$

We know that the angle between two lines with slopes $m_1$ and $m_2$ is given by
$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$.

Therefore, $\tan\theta = \left|\frac{2-0}{1+2\times 0}\right| = 2$

$\Rightarrow \theta = \tan^{-1}(2)$.

- The equation of a horizontal line at distance $a$ from the $x$-axis is either $y = a$ (above $x$-axis) or $y = -a$ (below $x$-axis).
- The equation of a vertical line at distance $b$ from the $y$-axis is either $x = b$ (right of $y$-axis) or $x = -b$ (left of $y$-axis).

- **Point-slope form of the equation of a line**

The point $(x, y)$ lies on the line with slope $m$ through the fixed point $(x_0, y_0)$ if and only if its coordinates satisfy the equation. This means $y - y_0 = m(x - x_0)$.

**Example:**Find the equation of the line passing through (4, 5) and making an angle of 120° with the positive direction of $x$-axis?
**Solution:**Slope of the line, $m = \tan 120° = \tan\left(180° - 60°\right) = -\tan 60° = -\sqrt{3}$
Equation of the required line is,

$$y - 5 = -\sqrt{3}(x - 4)$$

$$\Rightarrow y - 5 = -\sqrt{3}x + 4\sqrt{3}$$

$$\Rightarrow \sqrt{3}x + y - \left(5 + 4\sqrt{3}\right) = 0$$

- **Two-point form of the equation of a line**

The equation of the line passing through the points $(x_1, y_1)$ and $(x_2, y_2)$ is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

**Example:** Find the equation of the line passing through the points (–5, 2) and (1, 6).
**Solution:** Equation of the line passing through points (–5, 2) and (1, 6) is

$$y-2=\frac{6-2}{1-(-5)}(x-(-5))$$

$$\Rightarrow y-2=\frac{4}{6}(x+5)$$

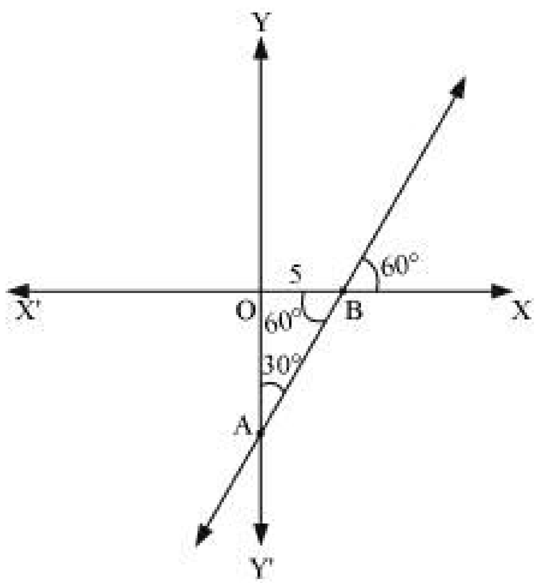
$$\Rightarrow y-2=\frac{2}{3}(x+5)$$

$$\Rightarrow 3y-6=2x+10$$

$$\Rightarrow 2x-3y+16=0$$

- **Slope-intercept form of a line**
  - The equation of the line, with slope $m$, which makes $y$-intercept $c$ is given by $y = mx + c$.
  - The equation of the line, with slope $m$, which makes $x$-intercept $d$ is given by $y = m(x - d)$.

**Example:**
Find the equation of the line which cuts off an intercept 5 on the $x$-axis and makes an angle of 30° with the $y$-axis.
**Solution:**




Slope of the line, $m = \tan 60° = \sqrt{3}$
OB = 5
Intercept on the x-axis, $c = –OB = –5$ and $\tan 60° = -5\sqrt{3}$
Equation of the required line is $y=\sqrt{3}x+\left(-5\sqrt{3}\right)$.

- **General equation of line**

Any equation of the form $Ax + By + C = 0$, where $A$ and $B$ are not zero simultaneously is called the general linear equation or general equation of line.

Slope of the line $= -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{A}{B}$

$y$- intercept $= -\frac{C}{B}$

**Example**:
Find the slope and the $y$-intercept of the line $2x - 3y = -16$.
**Solution**:
The equation of the given line can be rewritten as $2x - 3y + 16 = 0$.
Here, $A = 2$, $B = -3$ and $C = 16$.
Slope of the line $= -\frac{A}{B} = -\frac{2}{(-3)} = \frac{2}{3}$
Intercept on the $y$-axis $= -\frac{C}{B} = -\frac{16}{(-3)} = \frac{16}{3}$

- **Intercept form**

The equation of the line making intercepts $a$ and $b$ on $x$-axis and $y$-axis respectively is
$\frac{x}{a} + \frac{y}{b} = 1$

**Example:**
If a line passes through (3, 2) and cuts off intercepts on the axes in such a way that the product of the intercepts is 24, then find the equation of the line.
**Solution:**
The equation of a line in intercept form is
$\frac{x}{a} + \frac{y}{b} = 1 \quad \ldots(1)$
Where, $a$ and $b$ are the intercepts on $x$ and $y$ axes respectively.
Since the line passes through (3, 2), we obtain
$\frac{3}{a} + \frac{2}{b} = 1$
$\Rightarrow 3b + 2a = ab$
$\Rightarrow 2a + 3b = 24 \quad \ldots(2)$ (Since product of intercepts is given as 24)
Now,
$(2a - 3b)^2 = 24ab$
$= (24)^2 - 24(24) \quad$ [From equation (2)]
$= 0$
$\therefore 2a - 3b = 0 \quad \ldots(3)$

On adding equations (2) and (3), we obtain
$4a = 24 \Rightarrow a = 6$
$\therefore 3b = 2a = 2\times 6 = 12$
$\Rightarrow b = 4$

Hence, from (1), the required equation of line is

$$\frac{x}{6} + \frac{y}{4} = 1$$

$$\Rightarrow 4x + 6y = 24$$

$$\Rightarrow 2x + 3y = 12$$

- **Normal form of the equation of a line**

The equation of the line at normal distance $p$ from the origin and angle $\omega$, which the normal makes with the positive direction of the $x$-axis is given by $x\cos\omega + y\sin\omega = p$

**Example:** Reduce the equation $x - \sqrt{3y} - 6 = 0$ to normal form and hence find the length of perpendicular to the line from the origin. Also find angle between the normal and positive direction of the $x$-axis.

**Solution:** The given equation is $x - \sqrt{3y} - 6 = 0$.

$$\Rightarrow x - \sqrt{3y} = 6 \quad \ldots(1)$$

On dividing (1) by $\sqrt{\left(\sqrt{1}\right)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$, we obtain

$$\frac{1}{2}x - \frac{\sqrt{2}}{2}y = 3$$

$$\Rightarrow x\cos 300^\circ + y\sin 300^\circ = 3 \quad \ldots (2)$$

On comparing equation (2) with $x\cos\omega + y\sin\omega = p$, we obtain $\omega = 300^\circ$ and $p = 3$

Therefore, the length of perpendicular to the line from the origin is 3 units and the angle between the normal and the positive $x$-axis is 300°.

- **Distance of a Point From a Line**

The perpendicular distance ($d$) of a line $Ax + By + C = 0$ from a point $(x_1, y_1)$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

**Example:** Find the distance of point (1, –2) from the line $8x - 6y - 12 = 0$.

**Solution:** On comparing the equation of the given line i.e., $8x - 6y - 12 = 0$ with $Ax + By + C = 0$, we obtain

$A = 8$, $B = -6$, $C = -12$

The distance ($d$) of point (1, –2) from line $8x - 6y - 12 = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|8\times 1 + (-6)(-2) + (-12)|}{\sqrt{8^2 + 6^2}} = \frac{|8 + 12 - 12|}{\sqrt{100}} = \frac{8}{10} = \frac{4}{5}$$

- **Distance between parallel lines**

The distance ($d$) between two parallel lines i.e., $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by, $d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$

**Example:** Find the distance between the lines $4x + 3y = 11$ and $4x + 3y = 8$.
**Solution:** The given lines are $4x + 3y - 11 = 0$ and $4x + 3y - 8 = 0$
Slope of the line $4x + 3y - 11 = 0$ is $-\frac{4}{3}$.
Slope of the line $4x + 3y - 8 = 0$ is $-\frac{4}{3}$.
Since the slopes of the given lines are equal, the lines are parallel.
Here, $A = 4$, $B = 3$, $C_1 = -11$ and $C2 = -8$

Distance between the lines $= \left|\frac{-11-(-8)}{\sqrt{4^2+3^2}}\right| = \left|\frac{-11+8}{\sqrt{16+9}}\right| = \left|\frac{-3}{\sqrt{25}}\right| = \frac{3}{5}$